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Blending of mathematics and physics

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Chapter 5

Making the structural role of mathematics in physics explicit for students: a tutorial in the context of the heat equation

Abstract

The general aim of this study was to find ways to scaffold the blending of mathematics and physics in instructional materials. In this chapter, we propose blended encapsulation as a teaching approach with the intention to support students in making the structural role of mathematics in physics explicit in the case of the relation between the partial derivative $\frac{\partial T}{\partial x}$ and heat flow in the context of the heat equation. We extended the framework for the concept of derivative (Zandieh, 2000) to incorporate the relation between $\frac{\partial T}{\partial x}$ and heat flow. This framework forms the basis of our developed tutorial. To test the developed tutorial, we conducted teaching-learning interviews with three groups of three students from the second year undergraduate program of Physics or Mathematics at KU Leuven and used the framework to analyse student reasoning. We found that, generally, the blended encapsulation approach has the potential to guide students towards formulating the relation between $\frac{\partial T}{\partial x}$ and heat flow. However, we identify some additional difficulties that were not tackled by the current implementation of blended encapsulation in our tutorial and some points of improvement.

5.1 Framing the problem

To study advanced physics topics, being able to blend physical and mathematical knowledge is crucial. In this dissertation, we have studied the blending of mathematics and physics in the context of boundary conditions for the heat equation. As we established that there are many problems in student reasoning on this topic (Chapter 2), it is important to develop instructional materials focusing on these boundary conditions.

In the previous chapters, we focused on boundary conditions of the form $T(x_b, t) = c$ and $\frac{\partial T}{\partial x}(x_b, t) = c$ with x_b the position at the boundary and c a constant. These boundary conditions express a constant temperature or a constant heat flow respectively, at the boundary at all times t. In Chapter 4, we found that students often constructed graphs in the process of attaching physical meaning to the partial derivative $\frac{\partial T}{\partial x}$ and established that this graphical reasoning fosters the blending of mathematical and physical knowledge. However, we also established that often more scaffolding is needed for students to be able to explain in detail how $\frac{\partial T}{\partial x}$ relates to heat flow. Therefore, in this chapter, we build on these findings and find ways to provide that necessary scaffolding in instructional materials.

We developed a tutorial, where we use the term ‘tutorial’ here in the sense of worksheet-based group learning activities created to help students make sense of physics (McDermott, Shaffer, & Somers, 1994; Scherr, 2009). In tutorial sessions, students typically work in small groups on sequences of tasks that lead them to make predictions and compare various lines of reasoning in order to build an understanding of basic concepts. A teaching assistant (TA) serves as a facilitator rather than as a lecturer. Tutorials are designed to replace traditional problem-solving recitations, and to complement large-group lectures.

In this tutorial, we focus on the different boundary conditions that we explored in the other studies in this dissertation. We also focus explicitly on interpreting both $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial t}$ in the context of heat transfer. With this tutorial, we address the following learning goals:

After completion of the tutorial, the students can…

1. interpret mathematical boundary conditions physically in the context of the heat equation.
2. formulate boundary conditions mathematically based on a description of a physical situation in the context of the heat equation.
3. explain how $\frac{\partial T}{\partial t}$ gives information about the change of the temperature over time at a certain position.
4. explain how $\frac{\partial T}{\partial x}$ gives information about the heat flow through a certain position at a certain moment in time.

All four learning goals are aimed at establishing the blending of mathematical and physical knowledge. In this chapter, we focus on the fourth learning goal: explaining the relation between $\frac{\partial T}{\partial x}$ and heat flow. This is a topic that is well suited to investigate how students can be guided to understand the structural role of mathematics in physics (Uhden et al., 2012) because it allows students to experience how the structure of the partial derivative fits the physical context, and thus can describe heat flow. This explicit focus on the structural interplay between mathematics and physics is very important because we know from literature that it is not often taught to students even though we know it is not self-evident to them (Greca & de Ataíde, 2019). We test the tutorial in a teaching-learning interview setting with three small groups of undergraduate students majoring in mathematics or physics. This allows us to see in detail what reasoning is induced by the tasks in this tutorial.

In section 5.2, we describe the theoretical frameworks that guided our design. Specifically, we introduce the blended partial derivative framework and argue how this fits within the conceptual blending perspective. Next, in section 5.3, we discuss the development of the tutorial and the design principles in detail. In section 5.4 we discuss the teaching-learning interviews, the interview protocol, the setting and the participants. The data analysis is discussed in section 5.5. We explain how we used the framework for the concept of partial derivative to analyze the students' reasoning induced by our tutorial tasks. In the results section (section 5.6), we present the reasoning of the three participating groups as three case studies. These cases show how the tutorial induces different reasoning in the different groups. We end with a discussion of the findings in section 5.7.

5.2 Theoretical frameworks

The concept of (partial) derivative is crucial in understanding the heat equation, its boundary conditions, and the relation between $\frac{\partial T}{\partial x}$ and heat flow. Therefore, we start from the framework for the concept of derivative as it was developed in mathematics education research (MER) by Zandieh (2000). We extend this framework using the conceptual blending framework (Fauconnier & Turner, 2003b) to account for the relation between the mathematical concept of the partial derivative $\frac{\partial T}{\partial x}$ and the physical concept of heat flow. Using this *blended partial derivative framework*, we will propose an instructional approach that guides students to make the structural role of mathematics in physics explicit in the context of the relation between $\frac{\partial T}{\partial x}$ and heat flow.

In this section, we first introduce Zandieh's framework for the concept of derivative. Next, we explain how the conceptual blending perspective helped us in extending this framework to the blended partial derivative framework. Last, we introduce the construct of encapsulation, which will form the basis for the instructional approach in the tutorial.

5.2.1 Zandieh's framework for the concept of derivative

Zandieh (2000) developed a theoretical framework to clarify, describe and organize the different aspects of the understanding of the concept of derivative. The framework is built on two main components. *Multiple representations* form the columns of the table shown in Figure 5.1. The rows consist of different *layers*, each representing a *process-object pair*. We explain both components of the framework below.

Multiple representations Zandieh states that the concept of derivative can be expressed using different representations (or contexts). She distinguishes between symbolic (i.e. in terms of difference quotient), graphic (i.e. in terms of slope), verbal (i.e. in terms of rate of change), and physical representations (e.g. velocity). However, she stresses that the derivative may refer to many other physical concepts. The 'other' column accounts for all other contexts or representations in which there is a functional relationship for which one may discuss the concept of derivative. Roorda (2012) mentions for example that the concept of derivative is important for many physical or

	Contexts/representations				
	Graphical	Verbal	Paradigmatic physical	Symbolic	Other
Process-object layer	Slope	Rate	Velocity	Difference quotient	
Ratio					
Limit					
Derivative function					

Figure 5.1: Outline of the framework for the concept of derivative as presented in Zandieh (2000).

economical concepts, like radioactive decay, or marginal cost. Each of these examples can be represented as a column in the framework.

The concept of derivative has the same underlying structure in each representation. This structure is described using *layers of process-object pairs*, which form the rows of the table in Figure 5.1.

Layers of process-object pairs We return to the formal symbolic definition of the derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

This derivative of function $f(x)$ is a <u>function</u> whose value at any point is defined as the <u>limit</u> of a <u>ratio</u>. These three underlined aspects of the concept of derivative form the layers of the framework. Zandieh deliberately calls these aspects 'layers', based on the work of Sfard (1991). Sfard's research on the historical and psychological evolution of mathematical concepts suggests a transition from a process or operational conception to a static structural conception. According to Sfard, processes are operations on previously established objects. Each process is reified into an object to be acted on by other processes. This forms a chain of what Zandieh calls 'process-object pairs'. Each layer can be seen as both a dynamic process and a static object. Zandieh explains this with the example of the ratio of integer numbers, which may be considered operationally as division, but also statically as a pair of integers within a multiplicative structure. The three process-object pairs in the framework are linked in a chain, where the obtained object is used by the next process to arrive at the next object. Therefore, the layers induce a type of hierarchy or order. However, it is possible that one has knowledge about an object without understanding the underlying process. This means that students might be able to understand and work meaningfully with the derivative at a point (e.g. in terms of slope) without understanding how a limit process gives rise to that derivative. Another option is that a student has compartmentalized the knowledge about the underlying structure, i.e. that he has this knowledge but does not evoke it in an appropriate context. Sfard refers to this as a pseudo-structural conception or a pseudo-object.

The two components form a matrix like the one shown in Figure 5.1. Each empty box in the matrix represents an aspect of the concept of derivative. For example, the box in the ratio row and the graphical column represents the slope of a secant line of a graph of the function. We will discuss the content of the matrix in more detail in section 5.2.3.

Zandieh emphasizes that the framework does not map out a specific learning trajectory and that the learning process of students can be different from the suggested hierarchy in the scheme. For instance, a student may understand derivative in terms of slope *before* learning about the derivative as a limit of a difference quotient.

5.2.2 Conceptual blending

In this dissertation, we approach the interplay of mathematics and physics in student reasoning from a conceptual blending perspective. Conceptual blending was originally introduced by Fauconnier and Turner (1998) in linguistics and has been used in various contexts to explore human information integration (e.g. Bain et al., 2018; Gregorcic & Haglund, 2021; Zandieh et al., 2014). It provides a way to describe individuals' knowledge construction through the integration or blending of ideas from different mental spaces. A mental space is comprised of conceptual packets or knowledge elements that tend to be activated together, and has an organizing frame that specifies the relationships, or connections between the elements. The blended space is constructed through selective projection from the input spaces. When used in the context of the interplay of mathematics and physics in student reasoning, the conceptual blending framework provides a language to discuss how students draw from physical and mathematical knowledge in their reasoning. We opted for this framework because it matches well with the way we view the interplay of mathematics and physics at the advanced undergraduate level. Concepts and structures from mathematics have to be combined with physical knowledge and intuition, which ideally results in an enhanced understanding of both. In order to understand the relation between $\frac{\partial T}{\partial x}$ and heat flow, blending of mathematical and physical knowledge is indispensable.

Zandieh's framework for the concept of derivative offers a way to describe the different aspects of the understanding of the concept of derivative. In order to describe the relation between $\frac{\partial T}{\partial x}$ and heat flow, we extend it and change its focus from the mathematical concept of derivative to the blend between mathematics and physics formed in articulating the relation between $\frac{\partial T}{\partial x}$ and heat flow. In the next section, we discuss how we extended the framework, resulting in *the blended partial derivative framework*.

5.2.3 The blended partial derivative framework

Our aim in this section is to show how we started from Zandieh's framework in MER and extended it with columns focusing on physics to be able to represent the blending that has to take place in order to formulate the relation between $\frac{\partial T}{\partial x}$ and heat flow. Our adaptation was inspired by other studies that adopted Zandieh's framework before. Roorda (2012) adopted the framework to study students' mathematical proficiency with respect to the concept of derivative. He investigated how secondary students develop their knowledge about the concept of derivative and how they applied this knowledge to other school subjects, like physics or economics. Roundy, Dray, Manogue, Wagner, and Weber (2015) also adopted the framework and extended it with a focus on physical context.

Tables 5.2 and 5.3 show our blended partial derivative framework in the context of the relation between $\frac{\partial T}{\partial x}$ and heat flow. In what follows, we first discuss the two components: multiple representations and layers of process-object pairs. Next, we discuss the content of the matrix in more detail. There are some important differences between our adaptation and the original implementation of Zandieh. First, Zandieh uses both 'representations' and 'contexts' to label the columns. In what follows, we will always use 'representations' because we already refer to heat transfer in a one-dimensional system as the context of our investigation. Second, Zandieh developed her framework originally for derivatives and we adopt it for partial derivatives. Students' understanding of the concept of partial derivative relies heavily on their understanding of the concept of derivative.

Multiple representations In order to represent all aspects of the relation between $\frac{\partial T}{\partial x}$ and heat flow, we introduce some new columns. We adopt the symbolic, graphic and verbal column as formulated in the framework by Zandieh. Based on the adaptations of both Roorda (2012) and Roundy et al. (2015), we add a numerical column. This column is especially important for the use of the derivative in the sciences and in numerical analysis. Because we aim to represent the necessary reasoning steps to explain the relation between $\frac{\partial T}{\partial x}$ and heat flow, the physical meaning is of great importance. The temperature gradient is an essential concept that ties $\frac{\partial T}{\partial x}$ and heat flow together. Therefore, we introduce two physics columns: the 'physical meaning' column and the 'physical implication' column. The first one refers to the direct physical meaning of $\frac{\partial T}{\partial x}$ as the temperature gradient and the latter represents the way the temperature gradient is related to heat flow.

Layers of process-object pairs We extend the process-object layers as defined by Zandieh with two extra layers. We precede the ratio, limit and derivative function layers with a *function* and a *difference* layer. The function layer is inspired by the implementation of the framework by Roorda (2012). He adds this first layer as a foundation for the rest of the framework. As a second step, we add the difference layer. In order to understand the relation between the mathematical concept of the partial derivative $\frac{\partial T}{\partial x}$ and the physical concept of heat flow, thinking in terms of temperature differences between points in the system is essential. Moreover, also from a mathematical point of view, it makes sense to consider the forming of a difference quotient as being composed of two steps: first making differences and next, performing division. As these differences do not have an explicit place in the original form of the framework, we add a corresponding layer in our adaptation.

Now that we have explained the structure of the framework, we discuss in more detail the elements in the matrix, each of which represents an aspect of the concept of partial derivative. We formulated these elements explicitly in the context of the relation between $\frac{\partial T}{\partial x}$ and heat flow. The function $T(x,t)$ is a function of two variables, but in explaining the relation between $\frac{\partial T}{\partial x}$ and heat flow, it is most meaningful to discuss $\frac{\partial T}{\partial x}$ at a specific time, which is why we write everything in terms of $T(x,t_0)$ with t_0 being a constant. This is visible in the notations of all elements, but especially in the graphic column, where we opt for two-dimensional graphs of $T(x)$ at time t_0. We want to guide students to the insight that the sign and absolute value of the temperature gradient determine the direction and strength of the heat flow through that point. We think that a two-dimensional graphical representation serves this purpose best.

The physical meaning column represents the different layers of the concept of temperature gradient. Again, the same structure in terms of function, difference, ratio, limit and derivative function is identified. The physical implication column focuses on the relation between the concept of temperature gradient and the heat flow. Note that the first layer of the physical implication column is empty, because the physical implication in terms of heat flow only becomes relevant at the difference layer.

The content of the framework shows overlap with the work of Martinez-Planell et al. (2017) who developed a genetic decomposition of the concept of partial derivative. A genetic decomposition gives an overview of mental constructions students may make in order to understand a particular mathematical topic. In their genetic decomposition, Martinez-Planell and colleagues distinguish between a graphical and an analytical learning path, which coincide largely with the symbolic and graphic column in Figure 5.2. A genetic decomposition has the aim to propose a learning trajectory that can be used in curriculum materials. This approach differs from the framework as implemented by Zandieh, Roundy et al., and Roorda. They present their framework as a way to give an overview of all aspects of the concept of derivative in multiple contexts and representations, but with the emphasis that the framework does not map out a specific

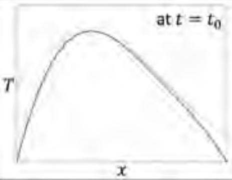
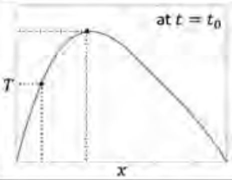
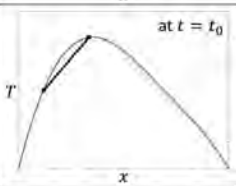
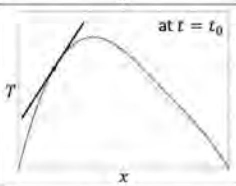
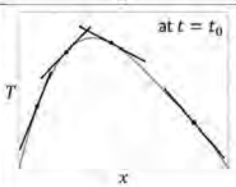
	Representations		
Process-object layer	**Symbolic**	**Graphic**	**Numeric**
Function	$T(x, t_0)$		$T(x, t_0)$ as an array of numbers, i.e. T values for a sequence of values for x
Difference	$T(x_2, t_0) - T(x_1, t_0)$		$T(x_2, t_0) - T(x_1, t_0)$ as a number
Ratio	$\frac{T(x_1 + h, t_0) - T(x_1, t_0)}{h}$		$\frac{T(x_1+h,t_0)-T(x_1,t_0)}{h}$ as a number
Limit	$\frac{\partial T}{\partial x}(x_1, t_0) = \lim_{h \to 0} \frac{T(x_1 + h, t_0) - T(x_1, t_0)}{h}$		$\frac{T(x_1+h,t_0)-T(x_1,t_0)}{h}$ values of the difference quotient for a sequence of decreasing values for h
Derivative function	$\frac{\partial T}{\partial x}(x, t_0)$		$\frac{\partial T}{\partial x}(x, t_0)$ as an array of numbers, i.e. numerical approximations for a sequence of values for x

Figure 5.2: The blended partial derivative framework (part 1).

	Representations		
Process-object layer	Verbal	Physical meaning	Physical implication
Function	Function value at every x at $t = t_0$	Temperature at every position in the system at time $t = t_0$	
Difference	Difference between function values at two positions at $t = t_0$	Difference between temperatures at positions x_1 and x_2 at time $t = t_0$	A temperature difference causes heat to flow from high to low temperatures
Ratio	Average rate of change	Temperature gradient between x_1 and x_2 at time $t = t_0$	The temperature gradient relates to the direction and strength of the heat flow between x_1 and x_2 at time $t = t_0$
Limit	Instantaneous rate of change	Temperature gradient in x_1 at time $t = t_0$	Temperature gradient in x_1 at time $t = t_0$ is proportional to the heat flow through x_1
Derivative function	Rate of change at any point	Temperature gradient at time $t = t_0$	Temperature gradient at time $t = t_0$ is proportional to heat flow

Figure 5.3: The blended partial derivative framework (part 2).

learning trajectory and that the learning or reasoning process of students can be different from the suggested hierarchy in the scheme.

5.2.4 Encapsulation and de-encapsulation

As stated in section 5.2.1, the process-object pairs in the framework for the concept of partial derivative form a chain where one can move within a layer from a process to an object, and where this resulting object is acted on in turn by the process at the next layer. The mechanism behind this concept construction has been described in different theories, e.g. ‘action, interiorization, process, encapsulation, object’ (Dubinsky, 1991); or ‘process, interiorization, condensation, reification, object’ (Sfard, 1991). These theories are based on Piaget’s notion of ‘reflective abstraction’, in which actions on existing or known objects become interiorized as processes and are then encapsulated as mental objects of thought (Pegg & Tall, 2005; Sriraman & English, 2009).

Dubinsky developed APOS theory, which stands for Action, Process, Object, and Schema (Arnon et al., 2014; Dubinsky, 1991). It is a theory of how mathematical concepts can be learned. APOS has been used as an analytical framework (e.g. Dubinsky, Arnon, & Weller, 2013), as a framework for the development of instructional materials (e.g. Breidenbach, Dubinsky, Hawks, & Nichols, 1992), and as a combination of both (e.g. Weller, Arnon, & Dubinsky, 2011). Dubinsky described *encapsulation* as part of the transformation of an action into a mental object in which actions are interiorized as *processes* and then thought of as *objects* within a wider schema. Encapsulation is very similar to *reification* from Sfard (1991), who stated that each *process* can be reified into an *object* to be acted on by other processes. Referring back to the framework for the concept of (partial) derivative, encapsulation allows one to move on to the next layer in the concept construction. Arnon et al. (2014) state that once a process has been encapsulated into a mental object, it can be *de-encapsulated*, when the need arises, back to its underlying process.

Projecting this onto our context of the partial derivative $\frac{\partial T}{\partial x}$, encapsulation happens when students learn how the concept of (partial) derivative is structured by its underlying processes and objects at the different layers. Once the (partial) derivative is encapsulated, the students can apply actions on this partial derivative without constantly acknowledging all the underlying concepts. However, they should be able to de-encapsulate the concept and access these separate underlying processes and objects.

In this study, we aim to make the structural role of mathematics in physics explicit for students and we do this in the context of the relation between $\frac{\partial T}{\partial x}$ and heat flow. We want students to match the structure of the concept of partial derivative and its physical implication explicitly. We hypothesize that encapsulation and de-encapsulation might help students in giving physical meaning to $\frac{\partial T}{\partial x}$ and then formulate the connection with heat flow.

5.3 Development of the tutorial

We developed a tutorial on boundary conditions for the heat equation based on three design principles. Even though in this chapter we focus on that part of the tutorial where we target the relation between $\frac{\partial T}{\partial x}$ and heat flow, we start this section by discussing the general design of the tutorial.

5.3.1 General design

The overall aim of the tutorial is to foster blending of mathematics and physics; all four learning goals that were formulated in section 5.1 aim at establishing the blending of mathematical and physical knowledge. In order to reach this aim, we formulate three *design principles* that guide the design of the tutorial. Design principles are a construct from (Educational) Design Research (Bakker, 2018). The aim of Design Research is to design and develop an intervention as a solution to a complex educational problem, and at the same time to advance the knowledge about the characteristics of these interventions (Plomp & Nieveen, 2013). Such knowledge is often summarized in the form of design principles, which are intermediaries between educational theory and practice.

The three design principles that guided our design are: (1) giving explicit attention to both the mathematical and the physical aspects, (2) stimulating graphical reasoning, and (3) guiding students to (de-)encapsulate the partial derivative in order to blend mathematical and physical meaning. These three design principles show some overlap and cannot be seen entirely separate from each other, as will become clear when we discuss the development of the specific tasks in the next section.

The first design principle is rooted in the conceptual blending perspective, which is central in this dissertation. We see mathematics and physics as two disciplines that are intertwined and have the potential to enhance each other. When mathematics and physics are blended, new ideas and inferences emerge, resulting in an enhanced understanding of both (Bing & Redish, 2007). The physics informs the mathematical thinking which in turn informs physics reasoning (Brahmia, 2017). Therefore, in the tutorial, we spend explicit attention to both the mathematical and the physical aspects of the reasoning. Several tasks ask students explicitly to make qualitative predictions about the physical situation, which is often kept implicit in traditional exercises. We developed tasks that explicitly ask to connect mathematical and physical lines of reasoning in order to foster blending.

The second design principle is based on the results presented in Chapter 4, where we concluded that graphical reasoning is a promising way to promote the blending of mathematical and physical knowledge. Specifically, our data showed that constructing graphs can trigger students to give physical meaning to the partial derivatives $\frac{\partial T}{\partial t}$ and/or $\frac{\partial T}{\partial x}$. Actively promoting graph construction and reasoning based on those graphs in teaching/learning activities offers perspectives to improve the blending of mathematics and physics. Therefore, as a second design principle, we promote graphical reasoning over the course of the tutorial to foster blending. In Chapter 4, the analysis was focused on $T(x)$-graphs at several times t. In Chapter 2, we established that students often have difficulties while reasoning with functions of two variables. In the tutorial, we use a graphical approach to explicitly acknowledge the role of these two variables in the temperature distribution. In particular, we also incorporate $T(t)$-graphs at several positions x and plots in three dimensions of the function $T(x,t)$.

We know from Chapter 4 that even though graph construction helped students to make progress in their reasoning, in most cases it was not sufficient to arrive at an understanding of all mathematical and physical aspects of the task. Therefore, in this tutorial, we added a third design principle to provide more scaffolding, which is the most important design principle for this chapter. We use the blended partial derivative framework and the ideas from encapsulation and de-encapsulation to develop an instructional approach that stimulates students to formulate the relation between the partial derivative $\frac{\partial T}{\partial x}$ and heat flow. In order to explain the third design principle, we start from observations from the study presented in Chapter 4 where we saw in several interviews that it is difficult for students to connect $\frac{\partial T}{\partial x}$ directly to heat flow. Translating this to the blended partial derivative framework, this means that blending mathematics and physics at the limit layer appears to be difficult. We observed one pair of students that explained the relation between $\frac{\partial T}{\partial x}$ and heat flow by connecting $\frac{\partial T}{\partial x}$ at a certain position to temperature differences between neighboring positions, which enabled them to connect this temperature difference to its physical implication that heat must flow from one position to the other. In terms of the blended partial derivative framework, this indicates that blending mathematics and physics at the difference layer might be more accessible. This inspired us to develop a teaching approach, which we will refer to as *blended encapsulation*, which forms the third design principle.

We hypothesize that guiding the students to blend at the difference layer and then encapsulate that blend to the limit layer might be a fruitful way to build the relation between $\frac{\partial T}{\partial x}$ and heat flow. Encapsulating the blend, i.e. encapsulating the concept of partial derivative while maintaining the connections between the mathematical and the physical concepts, is what we call blended encapsulation.

We visualize this design principle in Figure 5.4 (we will explain in the next section why we omitted the verbal and numeric columns and the function and derivative function layers). The aim of the tutorial is to blend the relevant mathematical and physical knowledge in such a way that the relation between $\frac{\partial T}{\partial x}$ and heat flow can be explained at the limit layer. In the tutorial, we guide students in making the blend at the difference layer and then foster blended encapsulation.

5.3.2 Introducing the relation between $\frac{\partial T}{\partial x}$ and heat flow

In this section, we zoom in on the part of the tutorial that focuses on the relation between $\frac{\partial T}{\partial x}$ and heat flow. We describe the development of the tasks focusing on this relation and discuss the role of the three design principles in these tasks. This sequence of tasks (tasks 1.j to 1.n) is part of exercise 1 in the tutorial that can be found

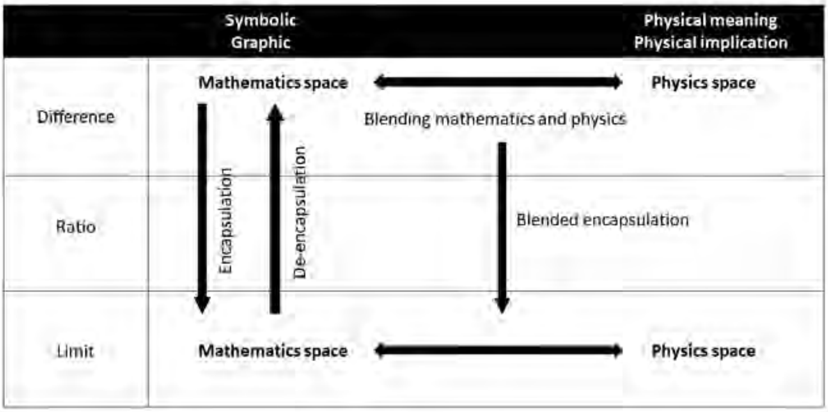


Figure 5.4: Visualization of the third design principle: blended encapsulation.

In the lectures, you have discussed the following partial differential equation. It models the time evolution of the temperature in a 1D system along the x-axis:

$$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial x^2}(x,t).$$

Herein, T is the temperature, x the position along the system ($0 < x < L$), and t the time ($0 < t < \infty$). α is the thermal diffusivity ($\alpha > 0$).

Figure 5.5: Introduction to the tutorial.

in Appendix C. The whole tutorial consists of three exercises and we will go deeper into exercises 2 and 3 in section 5.6.4. Here, we give a brief introduction to exercise 1 as a whole, but the analysis will solely focus on tasks 1.j to 1.n.

The start of the tutorial is shown in Figure 5.5. This general introduction is followed by exercise 1. The task provides the students with a graphical representation of the initial temperature distribution of the system (Figure 5.6). Along the way, we add a physical description of boundary conditions and we ask the students to formulate these boundary conditions in mathematical terms, and to construct $T(x)$-graphs at different times and $T(t)$-graphs at different positions (see tasks 1.a to 1.g in the tutorial in Appendix C). In task 1.h, we provide them with a plot in three dimensions of the temperature as a function of position and time (Figure 5.7), which they can rotate on a computer screen. One of the tasks asks to check if their own sketches match this 3D-plot, and from then

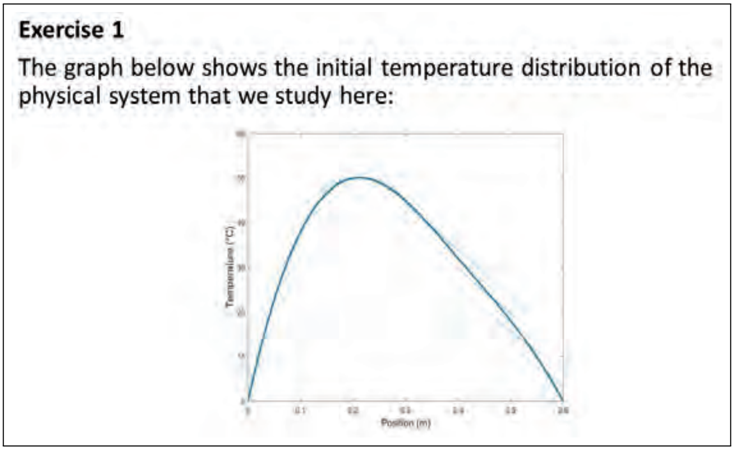


Figure 5.6: Initial temperature distribution of the physical system that is subject to exercise 1.

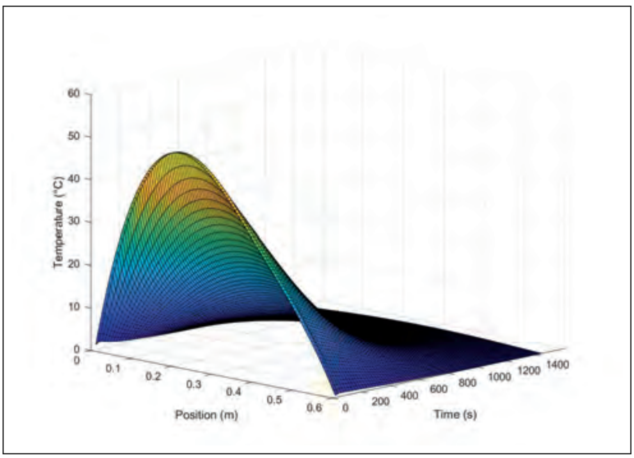


Figure 5.7: Plot in three dimensions of the temperature as a function of position and time for exercise 1.

on, they can use this 3D-plot to answer subsequent tasks.

Based on our observations in earlier work, we know that students find it easier to physically interpret $\frac{\partial T}{\partial t}$ than $\frac{\partial T}{\partial x}$. Therefore, in the tutorial, we first ask them to discuss $\frac{\partial T}{\partial t}$ (give a physical and a mathematical interpretation) in task 1.i. Next comes the sequence of tasks about $\frac{\partial T}{\partial x}$ (tasks 1.j-1.n). This way, we contrast both partial derivatives and

1.j) Use the plot of $T(x,t)$ and/or your own sketches of $T(x)$-graphs at different times to look at the initial temperatures at $x_1 = 0.1$ and $x_2 = 0.2$. Do these temperatures differ? What will happen in the system because of this?

1.k) Do the same for the initial temperatures at $x_1 = 0.1$ and $x_2 = 0.1 + h$, with $0 < h < 0.1$. Let h become smaller and smaller. Do these temperatures differ? What will happen in the system because of this temperature difference?

1.l) Use the plot of $T(x,t)$ and/or your own sketches of $T(x)$-graphs at different times to discuss the following partial derivative: $\frac{\partial T}{\partial x}(0.1,0)$. Connect your answer from task 1.k to this partial derivative.

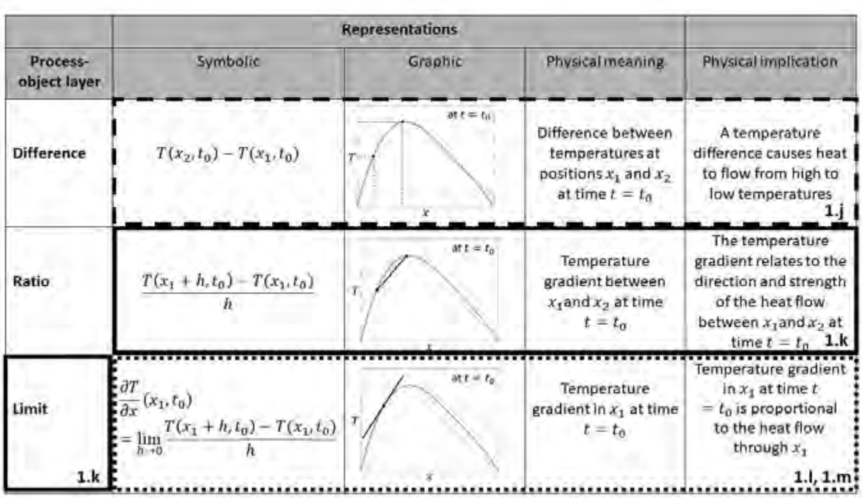
Figure 5.8: Tasks 1.j to 1.l of the tutorial. These tasks are designed with the intention to guide students through the different layers of the blended encapsulation framework.

highlight the differences between them in terms of graphical and physical interpretation.

Here, we explain in detail the development of tasks 1.j to 1.n, which focus on building the relation between $\frac{\partial T}{\partial x}$ and heat flow. Blended encapsulation is the primary design principle in this sequence of tasks, but the two other design principles also play a role. As explained and shown in Figure 5.4, we hypothesize that guiding the students into a learning trajectory that we refer to as blended encapsulation might be a promising strategy to foster the connections between the mathematical and physical concepts necessary to formulate the relation between $\frac{\partial T}{\partial x}$ and heat flow. Therefore, we start with a sequence of tasks primarily based on this blended encapsulation principle. Figure 5.8 shows the three tasks.

Figure 5.9 visualizes how each task aims to trigger a different layer of the framework for the concept of partial derivative. Task 1.j starts from a graphical perspective and explicitly asks the students to discuss the physical implication of these graphs. Using the blended partial derivative framework, we situate the expected answers to this task at the difference layer. The task asks about a temperature difference between two positions and ‘what will happen in the system because of this?’ With this last part of the question, we aim to trigger the physical implication column without explicitly mentioning the term ‘heat flow’.

Notice the role of the two other design principles in this task: we stimulate students to reason with the graphs and we stimulate explicit reasoning about physical meaning. This shows that the design principles show some overlap and cannot be seen entirely separate from each other. Graphical reasoning for example is a design principle on its own, but it also has an important role in the blended encapsulation, where it has a place



	Representations			
Process-object layer	Symbolic	Graphic	Physical meaning	Physical implication
Difference	$T(x_2, t_0) - T(x_1, t_0)$		Difference between temperatures at positions x_1 and x_2 at time $t = t_0$	A temperature difference causes heat to flow from high to low temperatures 1.j
Ratio	$\frac{T(x_1 + h, t_0) - T(x_1, t_0)}{h}$		Temperature gradient between x_1 and x_2 at time $t = t_0$	The temperature gradient relates to the direction and strength of the heat flow between x_1 and x_2 at time $t = t_0$ 1.k
Limit 1.k	$\frac{\partial T}{\partial x}(x_1, t_0) = \lim_{h \to 0} \frac{T(x_1 + h, t_0) - T(x_1, t_0)}{h}$		Temperature gradient in x_1 at time $t = t_0$	Temperature gradient in x_1 at time $t = t_0$ is proportional to the heat flow through x_1 1.l, 1.m

Figure 5.9: Selection from the blended partial derivative framework. The table shows how the developed tutorial tasks relate to the framework.

in the graphical column.

Task 1.k asks again about the difference in temperature between two points and its physical implication. Hence, the task starts at the difference layer, repeating the reasoning the students should have shown already in task 1.j. By letting h become smaller and smaller, we explicitly trigger the limit process. This way, we prepare them for what we want to see in the next question: the partial derivative at a point, i.e. the object resulting from the limit process. The ratio layer, which could be considered an intermediate step between the difference and limit layer, is not explicitly triggered in this task. However, we do expect that the students go through the ratio layer. Because we are working with students from the second year of the undergraduate program in physics or mathematics, we know that they have learned about the derivative as a limit of a ratio of differences in secondary education and in their Calculus course at university. Therefore, we assumed that the students are familiar enough with the concept of a derivative to take the step towards the ratio layer themselves, without explicit scaffolding.

In task 1.l, we take a different approach: we start in the symbolic representation at the limit layer by asking about the partial derivative $\frac{\partial T}{\partial x}$ at a specific point. Next, we ask students to make connections to their previous answers, which should be situated in the graphical and physical columns.

To conclude the sequence on blended encapsulation, we formulated the summarizing

> 1.m) Explain in words how the following concepts are related: heat flow, slope, temperature gradient, direction of the heat flow, $\frac{\partial T}{\partial x}$ and sign of the partial derivative.

Figure 5.10: Task 1.m of the tutorial. This task asks the students to formulate a summary of how all relevant concepts are related.

task 1.m, shown in Figure 5.10. In this task, we aim to see whether students can relate all concepts at the limit layer in an explicit way and explain their reasoning. This task takes two different roles in the design. First, it serves as a wrap-up, where students summarize their reasoning and are triggered to formulate a coherent conclusion. Second, it is built in as a 'safety' in case the sequence 1.j-1.l did not trigger the students to formulate a connection between $\frac{\partial T}{\partial x}$ and heat flow yet. We already mentioned that we opted not to use the term 'heat flow' explicitly in questions 1.j-1.l, so it is possible that up until this point students have not evoked this concept in their reasoning. In that case, this task will at least trigger them to think about the connections between these different concepts in an explicit way, and the intended blending process might eventually start.

In summary, tasks 1.j, 1.k, 1.l, and 1.m are developed to first guide the students to make the blend between mathematical and physical knowledge at the difference layer. Next, the tasks guide the students through the process of blended encapsulation, with the aim to eventually formulate the relation between $\frac{\partial T}{\partial x}$ and heat flow at the limit layer.

We mentioned before that we have made a selection in the representations that are relevant to this study: symbolic, graphic, and physical meaning (and implication). This is because we aim to induce semi-qualitative reasoning, i.e. without calculations, in which students blend mathematical structure with physical meaning. The numeric and verbal columns are less relevant for the reasoning we want to induce with the tutorial. The developed tasks are situated at the difference, ratio and limit layers. The function layer plays a role earlier on in the tutorial, in a part that is not subject to the analysis presented in this chapter. Therefore, we reduced the blended partial derivative framework from its complete form in Figures 5.2 and 5.3 to a reduced form, which only contains the relevant columns and rows (Figure 5.9).

As a final task in exercise 1, we ask the students to think about the special case when there is no heat flow in task 1.n (Figure 5.11). Task 1.n also triggers reasoning at the limit layer of the blended partial derivative framework. Prior to this task, we expect that students came to the conclusion that heat flow is related to $\frac{\partial T}{\partial x}$ and apply this now to the special case when there is no heat flow, which means that the temperature gradient is zero, the slope is zero and the partial derivative is also equal to zero.

> 1.n) Under which physical and mathematical conditions will there be no heat flow? Connect your insights from the previous tasks and the graphs, and explain your reasoning.

Figure 5.11: Task 1.n of the tutorial, which asks the students to think about the case when there is no heat flow.

5.4 Teaching-learning interviews

To investigate the reasoning induced by our tutorial tasks in detail, we set up a series of teaching-learning interviews with a small number of students. In these interviews, researchers create situations and ways of interacting with students that encourage them to modify their current thinking, which is in contrast with clinical interviews which intend to measure how students are thinking and reasoning in the moment without modifying their thinking (Engelhardt, Corpuz, Ozimek, & Rebello, 2004; Kelly & Lesh, 2000). In a teaching-learning interview, the aim is to model a natural learning environment while allowing more direct access to students' reasoning than would be possible in a real class-setting (Chini, Carmichael, Rebello, & Puntambekar, 2009). This method allows for the testing of new teaching/learning materials. The term teaching-learning implies that there is active instruction going on during the interview. This instruction is provided by the tutorial, which is developed to guide students in their reasoning. The interviewer takes on the roles of both researcher and TA, i.e. she not only observes the students but also facilitates group discussion and is available for questions in case there are any. In this section, we give an overview of the educational context and the participants and we explain how we implemented the interview.

5.4.1 Context and participants

We conducted interviews with second year undergraduate students at KU Leuven who completed a course on differential equations in the previous semester. The part of the course on partial differential equations entailed a chapter in which the heat equation was discussed in depth, i.e. the derivation of the heat equation from Newton's law of cooling combined with conservation of energy, different types of boundary conditions, physical systems described by this equation and the algorithmic technique of separation of variables are discussed in this chapter. The course is compulsory for all second year undergraduate students majoring in mathematics or physics. The interviewer and the research team were not involved in the course.

All students who participated in the study did so voluntarily. We contacted students through email and they could choose their own partners for working in groups of three. We found three groups willing to participate. All participating students had passed the exam and we had access to their grades before the interview (categories: A (16-17-18-19-20/20), B (13-14-15/20) and C (10-11-12/20)). The composition of the three groups was as follows: group 1 consisted of three physics students (grades A, B, C), group 2 had a physics student and two "twin" students (who take up extra credits to follow both the physics and mathematics program in an integrated way) (grades A, A, B), and group 3 consisted of three mathematics students (grades B, C, C). In the results section, we use pseudonyms for the names of the students to guarantee anonymity.

5.4.2 Implementation of the interviews

One month before the data collection, we conducted a pilot interview with two physics students to see whether the tasks were clear, whether the interview was not too long, etc. Following on this pilot interview, we made some small changes in the formulation of some of the tasks. The improved version of the tutorial is the one discussed in this chapter.

We interviewed each group separately and each interview lasted approximately one hour and a half. We conducted the interviews using a smart pen, which audio recorded the conversations and kept track of the students' notes, drawings and calculations. Furthermore, we also videotaped the interview. By performing the interviews with a group of students, we mimic the natural classroom environment where discussion is possible (and desirable). Each group completed one worksheet together, but each student had his own smart pen.

The interview tasks were in English but the students could respond in Dutch. The English interview tasks were no problem for the students as they were all used to English textbooks and English speaking teachers. In case they did not understand the task, the interviewer provided clarity.

At the start of the interview, the interviewer informed the students about the subject and purpose of the interview. The interviewer also emphasized that she would not give feedback about the correctness of responses during the interview. After this introduction, the students signed an informed consent. At the end of the interview, the students had the chance to discuss their answers and the aim of the research project in an informal way.

5.5 Data analysis

The interviews were transcribed verbatim and students' written answers and notes were added to the transcript. In this chapter, we treat each group as a case study. For each case, we study the students' reasoning in response to the tasks in detail. We provide a "thick description" (e.g. Ponterotto, 2018; Ryle, 1971) of the reasoning process of the group complemented with excerpts from the transcript in order to present how each group responded to the developed tasks.

Additionally, we use the blended partial derivative framework, as it was presented in section 5.3.2, to interpret the group's reasoning steps and situate their answers in terms of the different layers and representations. Hence, the framework plays a double role in this study. First, we used it in the design of the tutorial, guiding the way we scaffold the blended encapsulation process. Second, we use the framework in the analysis to visualize which elements (at a certain layer and in a certain representation) the group used in their reasoning in response to a certain task. This results in one table per group (e.g. Figure 5.12), which gives an overview of the elements used in each task. When the students discuss the element at a certain layer in a certain representation while answering a certain task, we place the task number at the corresponding location in the table. In some cases, the group needed a hint from the interviewer/TA in order to answer a task. This is indicated by adding the symbol '*' to the corresponding location in the table. It is important to note that this table gives an overview of the reasoning, but it always has to be interpreted in combination with the description of the reasoning and the excerpts from the transcripts. In the analysis, we consider the answers of the group as a whole and we will not distinguish between answers of separate students. We observe that generally, students discuss their ideas and opinions with each other and eventually come to one shared answer. Sometimes there is a need for extra context in order to understand a selected excerpt from the transcripts. In that case, we add this context between square brackets. This context is always based on the transcript or video recording.

5.6 Results: three case studies

In the following sections, we present the reasoning of the three groups. The groups respond to the tasks in very different ways. Group 1 reasons along the lines that we intended when designing the tutorial, without help of the TA. Group 2 also follows the design, showing extensive discussion, but they encounter some difficulties along the way in which they need help from the TA. Group 3 does not follow the intended trajectory based on blended encapsulation, but takes a different direction.

In this chapter, the main focus is on the reasoning of the groups in response to the tasks in (part of) exercise 1 where we aimed to introduce the relation between $\frac{\partial T}{\partial x}$ and heat flow. Furthermore, we want to see how the groups use this relation in other problems. Therefore, at the end of this section, we give some background on the design of the follow-up exercises 2 and 3 and discuss briefly how each group continued with their insights from exercise 1 in these exercises.

5.6.1 Group 1

Below, we give a description of the students' reasoning, illustrated with quotes from the interview. The analysis results in Table 1, which shows an overview of the used elements from the framework for the concept of partial derivative in the answer to each task.

Tasks 1.j to 1.l

We start by discussing the reasoning of group 1 in response to the designed sequence 1.j-1.l, which is based on the blended encapsulation of the partial derivative in combination with physical meaning. We use the structure of the framework to situate the students' reasoning in response to the tasks (see Figure 5.12). The sequence starts with task 1.j (Figure 5.8). Aron summarizes the group's answer as follows:

> *Aron: At every moment in time, the temperature in x_2 is greater than in x_1, so you always have heat flowing from x_2 in the direction of x_1.*

This answer is situated entirely at the difference layer. The students start reasoning in the physics space in terms of temperature differences, which is situated in the physical meaning column. From this, the student mentions the relationship between temperature differences and heat flow, including the direction of the heat flow, which shows that he can link it to its physical implication. This group spontaneously reasons in terms of 'heat flow', like we hoped when we chose to not include this term explicitly in the tasks. From the context of this sentence in the transcript, we know that the students read the temperature at the two positions from the 3D plot (Figure 5.7), which shows that they also connected their reasoning to the graphical representation.

The students move on to task 1.k (Figure 5.8). Here, the students build on their previous answer, but add some new insights:

> *Bert: So x_2 is still greater than x_1.*
> *Cameron: So yeah, in taking the limit, they will become equal.*

...
Cameron: Do these temperatures differ? Yes there is a difference. And what will happen in the system because of this?
Bert: Yes, so, heat will flow from x_2 to x_1.
...
Aron: The big difference is when h becomes equal to zero... Ah yes, then it goes towards a derivative.

In this excerpt, the students' reasoning in the graphical and physical representation, and the physical implication columns is mostly situated at the difference layer. Cameron mentions 'taking the limit', which indicates that the task is guiding him towards the limit layer. At the end, Aron makes the jump towards a derivative, but the ratio and limit processes are not explored explicitly in this answer. Filling this in in the table in Figure 5.12, the answer is situated in the graphical and physical representation at the difference layer and at the start of the limit layer. Task 1.k is not yet situated in the symbolic representation at the limit layer because the student only names the concept of a derivative, without specifying that it should be a partial derivative with respect to x.

Next, the students read and answer task 1.l (Figure 5.8). In the video recordings, we see that the students look at the plot of the initial temperature distribution and link this partial derivative to the slope of the $T(x)$-graph at $t = 0$ at point $x = 0.1$. Bert concludes:

Bert: Oh yes, so it is positive actually.

Next, the students make the connection to their answer to task 1.k by stating the relation between a difference quotient and the partial derivative.

Aron: Well, it's when h gets too close to zero, did we say it's colder in x_1 than in x_2?
Cameron: Yes
Aron: Hence, if you take a small step h, that [the temperature] gets bigger anyway.
Cameron: That is literally calculating the derivative.
Bert writes: $\lim_{h\to 0} \frac{T(x+h)-T(x)}{h}$

In this excerpt, the students show explicitly that they understand that the limit process gives rise to a derivative, which is situated at the limit layer of the symbolic representation. Furthermore, in this answer we see that Bert writes : $\lim_{h\to 0} \frac{T(x+h)-T(x)}{h}$, which is the limit of the difference quotient. The difference quotient takes into account not only the difference between the T-values, but also the distance between the points in the system and is situated at the ratio layer. Note that the students write this difference quotient for a derivative (i.e. without mentioning the argument t), and not for the partial derivative that is the focus here.

The students also show to be able to think in terms of temperature, but the physical implication in terms of heat is not stated explicitly in their answer. Completing the table in Figure 5.12, we observe that this answer starts at the limit layer in the symbolic and graphic representations. In terms of physical meaning, the students continue to think in terms of temperature differences, and not yet in terms of gradient. Therefore, this answer is situated at the difference layer in the physical meaning column. Last, the use of the difference quotient acknowledges the ratio layer in the symbolic representation.

At this point, the students have not yet explicitly stated the relation between heat flow and $\frac{\partial T}{\partial x}$, which was the aim of this sequence of tasks. Therefore, it is important to see how they respond to the summarizing task 1.m.

Task 1.m

In task 1.m, we ask the students to relate all concepts at the limit layer and explain their reasoning (Figure 5.10). The following two excerpts from the transcript illustrate the reasoning of this group:

> *Cameron: Yes, direction of the heat flow is really just the sign of the derivative.*
> *Aron: Wait, negative means to the right, positive to the left.*
> *Cameron: Ehm heat flow, if they mean the magnitude, than that is just the magnitude of the derivative. How fast it is actually flowing to that side.*
> *Cameron: Temperature gradient, higher gradient results in a higher flow.*
> *Aron: Well yes, but the temperature gradient and the slope are actually equivalent in this case.*
> *Bert: So it is actually the same as* $\frac{\partial T}{\partial x}$.
> *...*
> *Aron: The absolute value of* $\frac{\partial T}{\partial x}$ *determines the slope, gradient and heat flow. They are proportional with that.*
> *Cameron: Yes, but we have to write it in words: heat flow arises when there is a temperature gradient, which is equivalent with a non-zero* $\frac{\partial T}{\partial x}$ *here because it is a one-dimensional system, so that is equal to the slope.*

We observe that the students use and link concepts that stayed implicit in their answers to 1.j to 1.l, like temperature gradient and heat flow. In this answer, they mention the elements at the limit layer of all the representations (see Figure 5.12). Furthermore, they make connections between these elements and formulate a relation between heat flow and $\frac{\partial T}{\partial x}$, which was the aim of the task. They are also able to reason with these elements and infer new insights, which they prove with their reasoning about the relation between the sign of the derivative and the direction of the heat flow. We have to note that in the first excerpt, Cameron states that the magnitude of the heat flow is *equal* to the magnitude of the derivative, while Aron states in the second excerpt that $\frac{\partial T}{\partial x}$ is *proportional* to the heat flow. The group members do not discuss this aspect, so we cannot know for sure what their final opinion about this is.

These students answer task 1.m as we intended it. They are able to identify all connections between the concepts in the task and therefore to blend mathematical and physical knowledge.

Task 1.n

Task 1.n asks about the special case where there is no heat flow (Figure 5.11). The following excerpt from the transcript shows the reasoning of this group:

> *Aron: It's just, if the temperature's uniform. The mathematical condition is that the slope everywhere is equal to zero ehm.*
> *Bert: Yeah, so actually, only if you have a uniform distribution then you don't get heat flow. You might also say that there is no heat flow if for each point* $\frac{\partial T}{\partial x}$ *is equal to zero.*

Aron starts by identifying a state in which there is no heat flow in the system: when the temperature is uniform. Bert builds on this and concludes that there is no heat flow in the system "if for each point $\frac{\partial T}{\partial x}$ is equal to zero". Note that this shows that they consider the physical meaning at the macro level (i.e. in terms of the whole system) and not at the micro level, i.e. in terms of a single point where locally there is no heat flow. Situating this answer in Figure 5.12, they mention the symbolic and graphic representations at the limit layer and combine this with the physical meaning and implication at the difference layer ("if you have a uniform distribution then you don't get heat flow").

Summary

Figure 5.12 can be used to summarize the trajectory of this group. We observe that the sequence based on blended encapsulation (1.j to 1.l) partially instigated the intended reasoning in this group of students. They did not explicitly mention every step as we intended, but they arrived at the limit layer and briefly mentioned the ratio layer.

Group 1	Symbolic	Graphic	Physical meaning	Physical implication
Difference		1.j 1.k	1.j 1.k 1.l 1.n	1.j 1.k 1.n
Ratio	1.l			
Limit 1.k	1.l 1.m 1.n	1.l 1.m 1.n	1.m	1.m

Figure 5.12: Overview of the elements from the blended partial derivative framework used by group 1 in each task.

However, the physical implication column has not been mentioned explicitly in their answer to 1.l. This shows the importance of the summarizing task 1.m. In that task, the group was able to identify all connections between the concepts in the task and thus managed to blend mathematical and physical knowledge. In task 1.n, finally, the students reasoned about the system at the macro level and combined reasoning at the limit and difference layers. In conclusion, these students clearly formed a relation between $\frac{\partial T}{\partial x}$ and heat flow, were able to explain it, and to use it in the special case where heat flow is zero.

5.6.2 Group 2

Again, we give a description of the students' reasoning, illustrated with quotes from the interview. Figure 5.13 gives an overview of the used elements from the framework for the concept of partial derivative in the answer to each task.

Tasks 1.j to 1.l

In their answer to 1.j, the students start reasoning in the physics space.

> *Felix: The temperatures are different, and it is greater at x_2 than at x_1. This means that there will be a heat flow from x_2 to x_1, and in x_1 there is heat coming in, but it also gives off a lot of heat, so ehm...*
> *Eric: So heat will flow from higher temperatures to lower temperatures.*
> *Felix: This causes a heat flow at later times which, although not enough to compensate for the heat flow to the reservoir, so that the temperature drops at every point.*

The reasoning in this excerpt is situated at the difference layer in Figure 5.13. They read information from a graph, give physical meaning in terms of temperature, and formulate the physical implication in terms of heat flow. Like group 1, this group also did not need an explicit trigger to use the concept of heat flow in their reasoning. Note that Felix takes it a step further than just describing what happens between the two points x_1 and x_2; he discusses the heat flow in the whole system, where heat is flowing out from the sides into a reservoir. He takes into account that the system loses heat through the boundaries and that this results in an overall decrease in temperature at every point in the system.

The group's initial response to task 1.k is:

> *Felix: That's completely the same, only that heat flow is going to be a little less strong in that direction, to the left, so to speak.*
> *Eric: because the difference is smaller.*

This reasoning is situated again at the difference layer. The students are thinking in terms of absolute differences without incorporating the distance between the points. This indicates that they have not yet arrived at the ratio layer. However, only a few seconds later they add:

> *Eric: What happens when h gets smaller and smaller and smaller, uhm*
> *Dan: That locally, there will be small flows.*
> *Eric: If we let h become smaller and smaller...*
> *Felix: Then you will get a derivative with respect to that one, right? If you divide again by h?*
> *Dan: I would say analogous to task 1.j, and then maybe we can add something about that there will be local heat flows at any point in the system. If we have to interpret it physically.*

In this excerpt, we observe that the idea to let h become smaller prompted the recognition of the ratio and the limit layers by the students (Figure 5.13). This can be recognized in the statement that they should divide the difference by h, which will result in a derivative (without specifying the variable). Dan also formulates the physical implication as it describes a local heat flow at every point in the system.

In the following excerpt, the students specify how it will approach a partial derivative with respect to x:

> *Felix: I think what they're on to is that if you let h get smaller and smaller you get the first derivative with respect to position and that goes um, for fixed t.*

Eric: But the absolute value of the slope would reduce if you pulled a cord between x_1 and x_2.
Felix: Eh, or wait,
Eric: Or would it get bigger? Because if you look between 0.1 and 0.2 it is steep, but if you come closer to 0.1 it seems to become even steeper.
Felix: Ah yes, indeed.
Eric: So then the flow would be stronger in that direction.

The reasoning demonstrated in this excerpt is situated at the ratio and limit layers (Figure 5.13). Eric explicitly uses the idea of the secant line approaching the tangent line at x_1, which starts at the ratio layer, but expresses the limiting process. Eric expands on the physical implication in terms of heat flow by stating that the steeper the slope becomes, the stronger the heat flow will be. So also the ratio and limit layers of the physical implication are discussed in their answer.

The students conclude their answer to task 1.k as follows:

Felix: Ehm, the smaller h gets, the...
Dan: the bigger the heat flow? Or can't we define heat flow yet?
Felix: I think, the closer we approach the local heat flow. So... the better the heat flow is described or so.

This excerpt reflects how the students use the limit process in their reasoning in terms of the physical implication, as they also did in the symbolic and graphic representations in earlier excerpts.

In response to task 1.l, the students repeat their insights. Task 1.k already triggered them to reason at the limit layer. Therefore, this is not new to them:

Dan: Yes, that is just the slope.
Eric: And that gives the heat flow.

The students repeat how they associate $\frac{\partial T}{\partial x}$ to 'the slope' and how this is related to heat flow, which is the aim of the sequence of tasks. In Figure 5.13, this answer is situated at the limit layer, involving three columns: symbolic, graphic and physical implication.

Task 1.m

It is clear that these students formed a solid basis in tasks 1.j to 1.l to answer task 1.m, where we ask them to connect all relevant concepts in a summarizing answer. However, the next excerpts show that this basis was not solid enough to avoid difficulties.

Eric: How energy spontaneously goes from a warm temperature to a lower temperature.
Felix: But that's just what Fick's law, or Newton's, one of the two, says that the heat flow that then, uh that which is of course directly proportional to the temperature difference, the slope. But I just can't put my finger on which partial derivative that has to do with.
Dan: Yes indeed
Felix: But I find it so...
Dan: vague with the temperature, I mean with time, right? Yeah I think so too. Because in order to define heat flow you need to...
Felix: Yes you need something over time of course.

In this excerpt, the group starts reasoning from scratch: they are not sure which partial derivative is relevant for the description of heat flow. This indicates that they are not building on their insights from tasks 1.j to 1.l, where they already mentioned that "$\frac{\partial T}{\partial x}$ gives the heat flow". This might indicate that they did not transfer the connections they formed in tasks 1.j to 1.l to this task, or that they did not fully understand their prior answers.

Continuing their answer to task 1.m, the students turn to the concept of temperature gradient, which they have not used yet in their previous answers. Here, they turn to it in an attempt to describe the heat flow. However, the concept of gradient causes problems:

1. Dan: I think that the temperature gradient depends on the time anyway.
2. Felix: Yeah, normally. I think that in general, the heat flow goes in the direction of the gradient.
3. Dan: So the gradient depends on two variables.
4. Felix: But yes, that is, that takes into account the time and the uh, the position, but direction of the heat flow is purely along the x-axis, where temperature at a certain point on the x-axis goes to another point on the x-axis. And that's just, I think, uh, minus the uh partial derivative with respect to x. Because if that partial derivative is positive, it means that for x the temperature is higher here, so the heat flow goes in the other direction.
5. Eric: Yes and if it is here like this [slope is negative], it will go in the other direction.
6. Dan: So we can conclude that it is actually about equal to the slope, but yeah you actually have two types of slopes here, you also have a slope over time. The partial derivative of temperature with respect to any variable is definitely equal to a slope. And uhm I think the temperature gradient is depending on both variables.
7. Eric: Yes indeed, uhm... We can write that the heat flow goes in the direction of the temperature gradient.

8. Felix: wait, right, heat flow... yes heat flow is purely along the x-axis, so just the negative direction in which it will increase, so minus.
9. Dan: I find this hard. Time is a very different variable in physical terms than position, so that is just... But from a mathematical point of view...

In this excerpt, we observe that the students are on two different tracks. On the one hand, they state that heat flows in the direction of the temperature gradient, which according to them depends on both time and position (turns 1, 2, 3, 6, 7, and 9). On the other hand, Felix mentions twice that he thinks heat flow can only be "along the x-axis" (turns 4, 5, and 8). He also adds that heat will flow in the negative direction of $\frac{\partial T}{\partial x}$ (turn 8), which is correct. However, he cannot convince the group yet and they do not arrive at one joint conclusion at this point.

We identify two possible interpretations for what is happening in this excerpt. In the first one, there seems to be tension between the recalling of the mathematical concept of gradient and the way it is implemented here in a physical context. Turn 9 makes this tension explicit. Dan starts by mentioning that time and position are of a different nature from a physical perspective. However, then he adds "But from a mathematical point of view..." This might refer to the following: the students have learned about the concept of gradient in their mathematics courses as a vector whose components are the partial derivatives of the multivariate function. They have only used this concept in a truly multivariate setting (i.e. more than one dimension), which might explain why they are attracted to the idea of the gradient as the vector $(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial t})$ instead of the scalar $\frac{\partial T}{\partial x}$. A second way to interpret this excerpt, is that the students correctly interpret the temperature gradient as being $\frac{\partial T}{\partial x}$ and that the value of this gradient depends on the variable t. In that sense, it is correct that the gradient depends on both variables.

At this point, the interviewer/TA decided to interfere with some explanation: the system is one-dimensional in spatial terms. This hint is enough to cause a breakthrough:

Dan: Well yes, that is what I was saying before. Time is different from position.
Felix: Yes yes, so uhm, but then you have to ehm...
Dan: But actually, it is only the partial derivative...
Felix: Yeah, I guess we're confused with how we incorporate time because the heat flow is only going to be in one of those x directions. The physical heat flow of course only takes place on the x-axis. This then is dictated by euh $\frac{\partial T}{\partial x}$. So the physical heat flow [writes meanwhile] euh on the x-axis is given by $\frac{\partial T}{\partial x}$ in size and has as direction the opposite of $\frac{\partial T}{\partial x}$. And then the sign of the partial derivative, which means that if it is positive that the temperature increases, then of course there is heat flow in the opposite direction.
Eric: Indeed.

The fragment shows that the intervention of the interviewer resolved the insecurity about the definition of the temperature gradient in this context. It brings them back to the insight that the derivative of temperature with respect to position is relevant for the description of heat flow, which they already established before in questions 1.k and 1.l. Felix repeats his reasoning from before, but the difference with the previous excerpt is that this time the other students agree.

Adding this to Figure 5.13, we observe that in their answer to task 1.m, the students were able to formulate a coherent summary of the relation between the different concepts after a hint by the interviewer situated in the physical meaning column (represented by the asterisk in the table).

Task 1.n

In task 1.n, which asks about the special case where there is no heat flow, we see that, even though they have already connected heat flow to $\frac{\partial T}{\partial x}$ twice before, the group once again returns to the role of the derivative of temperature with respect to time in the description of heat flow:

> *Dan: There is, if the derivative with respect to time is zero, then it is not necessarily, then there can still be heat flow right? Suppose that function remains constant over time, is there still heat flow?*
> *Felix: There will be no net heat flow I guess.*
> *Dan: Uhm, then there is still, yes, then there is still heat flow, right?*
> *Eric: But physically there is no heat flow if the temperature is constant everywhere.*

The intervention of Eric is the first one in which they seem to change to the derivative in terms of x: remark the 'everywhere'. It is clear that Eric is thinking in terms of temperature being constant at a macro level (i.e. the whole system has the same temperature), but this is only one specific case in which heat flow is zero. Similar to group 1, he reasons at the macro level instead of the micro, i.e. considering the heat flow at a specific point of the system.

Next, the interviewer intervenes with the intention to orient the reasoning towards the micro level: "What mathematical condition would be needed to represent no heat flow locally?"

> *Eric: Then the partial derivative with respect to x is zero, isn't it?*
> *Dan: Uh yes.*
> *Eric: That there's no difference in temperature locally, so to speak. But*

yeah, not always, right? Because at that peak from before there was a maximum, so the partial derivative to x was zero there, and there you did have a flow.

From this excerpt, we observe that Eric connects the derivative $\frac{\partial T}{\partial x} = 0$ to “no difference in temperature locally” and thus to no heat flow locally. We interpret this as a link between the physical meaning and implication columns (at the difference layer) and the symbolic representation (at the limit layer) in the framework.

However, Eric experiences a contradiction in his reasoning. When he applies this reasoning to the specific case of the location of the maximum in the initial temperature distribution (Figure 5.6), he says: “the partial derivative to x was zero there, and there you did have a flow”. It is possible that Eric thinks that because the temperature at that maximum is changing over time, there should be heat flowing through that point, which is incorrect. With this sentence, he seems to say that $\frac{\partial T}{\partial x} = 0$ does not necessarily mean that there is no heat flow.

Felix decides to dig deeper in this confusion and therefore considers an example of a system in equilibrium with on each side a heat reservoir at a different temperature:

Felix: I'm again confused with net heat flow. So if you have a container at a certain temperature on the right, and one at zero on the left, for example. And in between, for all t, there is a linear function between those two points. Is there still a, because I am confused for a moment, is there still a net heat flow in each point? No, right? Because heat comes in but leaves too.
Dan: Yes there is, then there is a net heat flow, towards the heat reservoir, uh, with temperature zero. And if that is an infinite heat reservoir, then you can assume that that will probably be a constant heat flow over time.

Felix utters his confusion: he considers an equilibrium situation, i.e. where the temperature does not change anymore over time, and he formulates the implication for the heat flow at each point of the system. Felix concludes that there is no heat flow in that case. However, Dan has another opinion: there is a heat flow and it is constant over time. This way, Dan establishes that it is possible to have a non-zero heat flow while the temperature is not changing over time.

Dan concludes with the following statement:

Dan: So, uhm when will there be no heat flow? When actually just the derivative with respect to position is zero. And when the derivative to time is zero, that has nothing to do with it. We just have to separate heat flow from time.

Group 2	Symbolic	Graphic	Physical meaning	Physical implication
Difference		1.j 1.k	1.j 1.k 1.m 1.n	1.j 1.k 1.m 1.n
Ratio	1.k	1.k		1.k
Limit 1.k	1.k 1.l 1.m 1.n	1.k 1.l 1.m 1.n	1.m*	1.k 1.l 1.m 1.n

Figure 5.13: Overview of the elements from the blended partial derivative framework used by group 2 in each task.

These students have been considering the two partial derivatives $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial t}$ to describe heat flow. At different points in their answer, they discuss the idea that heat flow is connected to a change in temperature over time. In the end, they arrive at the conclusion that there can still be heat flow when $\frac{\partial T}{\partial t} = 0$. This supports their idea that heat flow should not be described in terms of $\frac{\partial T}{\partial t}$, but in terms of $\frac{\partial T}{\partial x}$. They have not further explored the problem that Eric stated about the maximum in the initial temperature distribution and the heat flow through that point. However, this does not stop them to arrive at the conclusion that there will be no heat flow when $\frac{\partial T}{\partial x} = 0$. In conclusion, the students strengthened their understanding of the relation between $\frac{\partial T}{\partial x}$ and heat flow, but there are still some loose ends.

It is indeed true that $\frac{\partial T}{\partial t}$ has no influence on the heat flow, as Dan concluded. However, heat flow is the amount of energy passing through a point per unit of time (as explained in section 1.2). This probably causes the confusion about the role of time in the description of heat flow in terms of temperature.

Summary

In conclusion, in tasks 1.j to 1.l, we observed that this group followed the trajectory based on blended encapsulation as it was intended in the design. The students started at the difference layer, then moved on to the ratio layer, and finally made the step to the limit layer where they acknowledged the relation between $\frac{\partial T}{\partial x}$ and heat flow. They explicitly acknowledged the ratio layer. Reasoning in terms of the temperature gradient was the only element that was not observed. This concept only made its way to their reasoning when we explicitly mentioned it in task 1.m (and there they needed a hint in order to be able to work with this concept).

When starting to answer tasks 1.m and 1.n, the students did not build on their reasoning in task 1.j to 1.l and seemed to start thinking about describing heat flow from scratch. This might imply that the relation between $\frac{\partial T}{\partial x}$ and heat flow formed in the blended encapsulation sequence was not as stable as we hoped. In their reasoning in answer to 1.m, we observed difficulties with the definition of the temperature gradient and with the role of the time dimension in the definition and description of heat flow. These issues are partly resolved by an intervention of the interviewer/TA, but the tutorial in itself did not offer enough support for the students to resolve the issues by themselves.

5.6.3 Group 3

Also for the third group, we give a description of the students' reasoning, illustrated with quotes from the interview. The analysis results in Figure 5.14, which shows an overview of the used elements from the framework for the concept of partial derivative in the answer to each task.

Tasks 1.j to 1.l

Initially, one of the students responds to task 1.j as follows:

> *Iris: If your point, if you have a colder point here than there, the temperature is going to spread there, logically.*

We interpret this as reasoning situated at the difference layer, where the students think in terms of temperature differences and how this results in the temperature "spreading". Note that this student talks about temperature that is spreading instead of heat. This also means that the group does not reason yet in terms of the physical implication (heat).

Shortly after that statement, they start thinking in terms of the time derivative at x_1 and x_2:

> *George: Isn't it just that if the temperature's lower, it's decreasing slower? That 0.1 has a lower temperature, so that means that the derivative [with respect to time] is less steep, so yeah [George writes 'At $x = 0.1$ the starting temperature is lower, so slower decrease in temperature than at $x = 0.2$'].*

George compares the derivative of temperature with respect to time at the two positions x_1 and x_2 instead of discussing the heat flow between these two points, as we intended with this task. A possible reason for this answer is that the previous task 1.i, in which we asked students to discuss the time derivative at a specific point, may have affected the way they interpreted task 1.j. Note that their statement that a lower temperature leads to a slower decrease may be correct in this specific case, but is not correct in general.

The students answer task 1.k in a similar way:

> *Harry: What did we say before? At x_1 the temperature starts lower, so slower decrease in temperature than at x_2.*

This answer is in accordance with the students' answer to question 1.j, but they seem to completely ignore the part of the question where we let h become smaller and smaller. Situating this reasoning, which we see both in response to tasks 1.j and 1.k, in Figure 5.14, we see that until now the students have not discussed any of the intended elements yet. This means that the tasks did not guide the students to discussing the temperature in function of position and eventually to let them arrive at the partial derivative of T with respect to x.

In task 1.l, we explicitly ask the students to discuss the partial derivative $\frac{\partial T}{\partial x}$ and to connect this to their answer in 1.k. The students respond very briefly to this:

> *Iris: Uh, you want to take the derivative of this with respect to position at position 0.1. So, that is positive.*

We know from the video recordings that Iris uses the 3D plot to look at the slope in the x-direction, which indicates that she is able to think in terms of slope at the limit layer (see Figure 5.14). Again, it seems like the group is only answering the first part of the task ("Use the plot of $T(x,t)$ and/or your own sketches of $T(x)$-graphs at different times to discuss the following partial derivative: $\frac{\partial T}{\partial x}$") and ignores the second part ("Connect your answer from task 1.k to this partial derivative"). This ignoring of prompts in the tasks might partly explain why the blended encapsulation sequence does not have the intended effect for this group.

As a final attempt, the interviewer/TA asks the students explicitly to return to task 1.k to see if the students' interpretation of the task changed after seeing $\frac{\partial T}{\partial x}$ in 1.l. However, we do not identify any new ideas in the transcript:

> *Iris: So that's temperature... it will distribute itself homogeneously, then you get... yes if that temperature is already lower there, then it needs to distribute itself less because it is already smaller.*

And this is again connected to the derivative of temperature with respect to time $\frac{\partial T}{\partial t}$:

Harry: Going slower means smaller time derivative.

The students' statements about the time derivative may be correct in this specific situation. However, the group does not answer the sub-tasks that are designed to guide them towards the partial derivative with respect to position. They are too focused on the partial derivative with respect to time and formulate all their answers in terms of this partial derivative, which finds no place in Figure 5.14. This indicates that the designed tasks 1.j to 1.l failed to guide this group's reasoning process the way we intended.

Task 1.m

Because the students cannot build on the preparatory tasks 1.j to 1.l, it is interesting to see how they respond to task 1.m. Remember that we implemented task 1.m as a chance to summarize the reasoning, but also as a 'safety' in case the students have not reasoned about these concepts in their previous answers. This latter function is important here. Heat flow is a first concept that this groups has not used in previous answers, and this causes problems:

Harry: But what do they mean by heat flow? Is that, uh, high heat flow is that it moves fast or what? So that the temperature goes away quickly.

Harry seems to interpret heat flow as some kind of velocity of temperature, which we interpret as $\frac{\partial T}{\partial t}$. This implies that these students think that heat flow and the change of temperature over time are connected. Like in their answer to 1.j, they talk in terms of temperature moving instead of heat. When they continue their answer, the students discuss both partial derivatives. For example, in the first of the following two excerpt, they discuss $\frac{\partial T}{\partial t}$ and how temperature changes over time:

Harry: If the temperature goes faster, the time derivative will...
Iris: Slope [writes slope]
Harry: So it is steeper, right? The slope is steeper? Because the tangent line is steeper when it decreases faster.

Further on in the transcript they discuss $\frac{\partial T}{\partial x}$:

Harry: Derivative, you mean the one with respect to x, right? Or? Because there are different derivatives now.
Iris: Ah yes. Derivative delta T over delta x [writes $\frac{\partial T}{\partial x}$], and then you have the sign of the partial derivative, uhm.

Harry: Yes, the temperature decreases, so that means it is negative, but what did you say before about it being positive?
Iris: But as a function of x the temperature is still increasing there, right?
Harry: Yes ok
Iris: So it is still positive there.
Harry: Ah yes as a function of x it is, but as a function of t it isn't, right?

From these excerpts, we conclude that the students can interpret the partial derivative with respect to time graphically and physically, while the interpretation of the derivative with respect to position at a point is limited to its graphical representation: the slope in the x-direction. This can be deducted from Iris's statements where she used the graph to state that the temperature is increasing as a function of x, so 'it' (the slope) is still positive there. This group does not link heat flow to the partial derivative $\frac{\partial T}{\partial x}$ in their answer.

Another concept that appears unknown to the students is temperature gradient:

Harry: Temperature gradient, isn't that the same as heat flow? Do we know this?
George: Yes, that is just a gradient, right?
Harry: Yes, and what is a gradient in two dimensions?

This response is similar to the one we observed in the reasoning of group 2. Harry mentions a gradient in two dimensions, so he thinks that the gradient depends on both variables in the function $T(x,t)$. The students ask for help:

Harry: I don't know what a temperature gradient is, so
Iris: Difficult
Harry: Yes, but I think we don't know the definition of a gradient, can you help?

The interviewer/TA helps the students in the same way as was done with group 2 by stating that the system considered in this problem only has one spatial dimension and therefore the gradient is also only in one dimension. This intervention seems to cause a breakthrough; the students connect the gradient to heat flow and to slope:

George: The gradient increases when the heat flow is higher.
Iris: So if you have a higher heat flow, you have a steeper slope.
Iris writes: high heat flow, steeper slope, greater derivative $\frac{\partial T}{\partial x}$, *temperature gradient also higher.*
Harry: Yes, I think we kind of have it now.

In this excerpt, the students formulate the relation between $\frac{\partial T}{\partial x}$, slope, temperature gradient, and heat flow. Situating this in the framework, we can interpret this as reasoning on the limit layer in all columns. The intervention of the interviewer in explaining the temperature gradient is represented with an asterisk. The connection between the four columns expressed in this excerpt is correct, but the group never substantiates this blend with elements from the difference or ratio layers in the framework. The explanation of why these concepts are related is also very limited to even non-existent. The end product of the blending process is present, but the blending process and substantiation of this conclusion did not go as intended in the design.

Again, the students did not answer all parts of the task. At this point, they have related heat flow, slope and temperature gradient, but the interviewer needs to ask explicitly about the direction of the heat flow:

> *Harry: Here is the highest temperature, so it goes towards the closest side with lower temperature, right?*
> *George: Yeah, the shortest distance it has to go to the coldest, I guess.*
> *Harry: Uh, when the sign is positive it's going left, right? And when the sign is negative it's definitely going right. Left of the maximum it's going to the left, and where the derivative is zero, the heat flow will change direction.*

As mentioned before, we do not know how the students substantiated the conclusion that temperature gradient, slope and heat flow are related. However, in this excerpt we do see that Harry builds on these formulated relations in a correct way and infers new insights, like here about the direction of the heat flow.

Task 1.n

In task 1.n, we ask about the special case where there is no heat flow. The students easily relate this to their insights about $\frac{\partial T}{\partial x}$ from the previous task.

> *Harry: So if there would be no heat flow, so if there is none, then the derivative must be zero.*
> *Iris: Yes. That is true. Otherwise it would definitely go somewhere.*
> *Harry: Yes indeed. That's all, right? That the position derivative should be equal to zero.*

Here again, we observe that Harry is able to use the connection between $\frac{\partial T}{\partial x}$ and heat flow to infer new insights, this time about the special case where heat flow is zero. The reasoning demonstrated here is situated at the limit layer in Figure 5.14 in the physical

Group 3	Symbolic	Graphic	Physical meaning	Physical implication
Difference		1.j	1.j	
Ratio				
Limit	1.l 1.m 1.n	1.l 1.m	1.m*	1.m 1.n

Figure 5.14: Overview of the elements from the blended partial derivative framework used by group 3 in each task.

implication column. It is not clear from the transcripts if these students reasoned at the micro or macro level.

Summary

Using Figure 5.14, it seems that the students have eventually mentioned all elements in the limit layer in all columns and that they were able to formulate a relation between these elements. This happened after a hint from the interviewer/TA about the definition of temperature gradient and the one-dimensionality of the physical system. However, the blended encapsulation sequence of tasks 1.j-1.l did not contribute to this. In the responses to these tasks, the students were on a different track, ignored parts of the questions, and did not reason in terms of the derivative with respect to position.

The summarizing task 1.m played an important role here, as the group had not reasoned about heat flow in the system before. It fulfilled its role as a 'safety' and forced the students to discuss this important concept. In their response, we observed that the students formulated the relation between $\frac{\partial T}{\partial x}$ and heat flow, but without much explanation. We cannot judge if the students understood why these concepts are related because this was not made explicit in the interview. We did observe, however, that they were comfortable building on this relation and using it to arrive at new insights in task 1.n (e.g. $\frac{\partial T}{\partial x} = 0$ means that heat flow is zero).

5.6.4 Exercises 2 and 3

After having introduced the students to the relation between $\frac{\partial T}{\partial x}$ and heat flow in the exercise 1, we are interested to see if the acquired understanding of the different concepts and the connections between those concepts are stable enough to use it in a different problem. Therefore, we designed a second and a third exercise with other boundary conditions and we reduced the scaffolding, which is also referred to as fading in literature (Collins, Brown, & Holum, 1991; Lin et al., 2012; McNeill, Lizotte, Krajcik, & Marx, 2006). We give a brief overview of exercises 2 and 3 (the complete tutorial can be found in Appendix C).

Exercise 2 is about a physical system with boundary conditions

$$T(0,t) = 10 \text{ and } T(3,t) = 40 \text{ for } 0 < t < \infty.$$

We provide a graph with the initial condition of the physical system, like we did in exercise 1. We ask the students to discuss the physical meaning of the boundary conditions, reason about the evolution of the system and sketch $T(x)$-graphs at different times. At that point, we again provide a plot in three dimensions of $T(x,t)$. The students can use it when we ask them to discuss the heat flow in the system and at the boundaries.

Exercise 3 is about an isolated system and a graphical representation of the initial condition is provided. We start by asking to explain what an isolated system is. Then we ask the students to reason about the evolution of the system and sketch $T(x)$-graphs at different times. Next, we ask the students to express the boundary conditions mathematically (which makes this exercise different from exercises 1 and 2). Afterwards, we again provide a plot in three dimensions of $T(x,t)$. The students can use it when we ask them to discuss the heat flow in the system and at the boundaries.

These exercises are particularly interesting to investigate how the students are able to reason with and build on the relation between heat flow and $\frac{\partial T}{\partial x}$, which they should have formed in exercise 1. Especially the last part of both exercises, where we ask the students about the heat flow in the system and its direction, is relevant for our aim to see if students have formed a stable connection between $\frac{\partial T}{\partial x}$ and heat flow. A detailed analysis would bring us too far, but in this section, we give a brief overview of how each group reasoned with the concepts at the limit layer in Figure 5.9.

Group 1

This group encountered little to no difficulties in answering the tasks in exercise 1. In exercise 2, they reason primarily in the physics space, more specifically in terms of heat flow, and combine this with graphical reasoning, focusing on the information they can get from the slope in the x direction. The underlying connections with mathematics stay implicit and are only expressed in the graphical aspect (e.g. referring to the shape of the graph and the slope, but never explicitly mentioning the word ‘derivative’). They use the slope in the x-direction actively to support their physical reasoning. Only at the end of the exercise (task 2.d) they explicitly mention $\frac{\partial T}{\partial x}$. This shows that the connection between $\frac{\partial T}{\partial x}$ and heat flow formed in exercise 1 is evoked without the scaffolding like it was provided in exercise 1.

In exercise 3, the students find it difficult to formulate mathematical boundary conditions that express isolated boundaries. They explain an isolated system as "The energy of the system stays in the system. There is also no heat flow to the 'external world'", which refers to a macroscopic level. This interpretation suffices to sketch $T(x)$-graphs for different times. However, when asked to formulate the boundary conditions in a mathematical way, they have to switch to a microscopic formulation of an isolated system, i.e. in terms of heat flow through a single point, and they encounter problems.

First, they try reasoning in terms of temperature at the boundaries and link this to what they see in their graphs: the temperatures at the boundaries change over time. Furthermore, they stated in their explanation of an isolated system that there should be no heat flow to the "world outside the system". The combination of these ideas makes them answer that there is a heat reservoir at each side which temperature must vary along with the temperature at the boundaries.

These students had a physical understanding of what an isolated system means in terms of heat flow at the boundary: no heat flows through the boundary. However, instead of describing this literally, they formulate a physical condition (a heat reservoir with changing temperature) that does not need to be formulated in terms of heat flow, but in terms of temperature (which is easier). This might indicate that this group reasons in terms of "temperature is equal to..." when asked to formulate boundary conditions.

Afterwards, one of the students suggests to consider $\frac{\partial T}{\partial x} = 0$ as a boundary condition (without giving a clear reason for it). In the end, the students choose this option based on their earlier constructed $T(x)$-graphs: they see that the tangent line is always horizontal at the boundaries, which confirms their idea. It is important to notice that the students did not have a clear reason for the horizontal tangent lines either. The initial condition had horizontal tangents and they copied this feature in their own sketches without mentioning why.

The connection between heat flow and $\frac{\partial T}{\partial x}$ is not the problem here. The students discussed, for example, the direction of the heat flow and evolution of the heat flow in the system in a correct way. The problem seems to lie in the distinction between microscopic and macroscopic reasoning. Both in exercise 1 (1.n) and in exercise 3, the students reason in macroscopic terms. However, in order to construct boundary conditions for an isolated system, they need to translate the idea of 'no heat flow to the outside world' to what happens locally at the boundaries of the system (micro level).

Group 2

In exercise 1, this group went fluently through the blended encapsulation sequence, but in tasks 1.m and 1.n they did not build on the conclusion they formed in 1.j-1.l.

Even though they arrived three times at the correct conclusion that $\frac{\partial T}{\partial x}$ and heat flow are related, each time they restarted their reasoning instead of building on their prior answers.

In exercise 2, the students still connect $\frac{\partial T}{\partial x}$ and heat flow: they express no doubt anymore and they use it in the setting of exercise 2. Their reasoning is brief and condensed and much of the steps stay implicit. However, we can assume this group has an explanation for their answer as they were very explicit about their reasoning in exercise 1.

Their answers to exercise 3 are also brief and to the point. They interpret 'being isolated' as 'no heat flow', which immediately leads them to what they had already stated in task 1.n: $\frac{\partial T}{\partial x} = 0$. In summary, for this group, the discussions and insecurities in the first exercise have led to a connection between $\frac{\partial T}{\partial x}$ and heat flow, from which they can infer new insights in other tasks.

Group 3

In exercise 1, this group did not pick up on the blended encapsulation sequence in the tasks and was reasoning on a different 'track'. In task 1.m eventually, we observed that they formulated the relation between $\frac{\partial T}{\partial x}$ and heat flow without explaining why these concepts are related. However, we observed that they built on this relation to arrive at new insights in task 1.n ($\frac{\partial T}{\partial x} = 0$ means that heat flow is zero).

In their answers in exercise 2, this group only uses the concept of heat flow when the task asks about this explicitly (i.e. in task 2.d). Graphs are central in their reasoning: they use the tangent line in the x-direction and connect this to how strong the heat flow will be ("the steeper, the stronger"). The students are able to discuss the evolution and the direction of the heat flow in the system in a correct way.

In exercise 3, the students describe an isolated system as a system "without external influences and thus without heat reservoirs at the ends". In task 3.b, the students use this description to construct $T(x)$-graphs that show the evolution of the temperature distribution over time correctly. In task 3.c, when asked to mathematically formulate the boundary conditions, they neither use heat flow nor a partial derivative in their reasoning. They use their $T(x)$-graphs, which show that the temperatures at the sides change over time. They write down what they can describe: they know at which temperatures the boundaries of the system start and they know to which temperature the system will converge. Their initial response (they correct it later on) is "$T(0,0) = 60$", "$T(3,0) = 0$" and "$T(x,\infty) = 30$". We interpret this incomplete answer as showing that this group reasons in terms of "temperature is equal to..." when asked to formulate boundary conditions (like we observed in the reasoning of group 1 in answer to exercise 3). The concept of heat flow plays no role in this reasoning. We can combine this observation with what we observed in this group's answer to task 1.n. There, the students were able to connect an isolated system to no heat flow and thus to $\frac{\partial T}{\partial x} = 0$. However, it was not clear if the students formed this connection at the macro or micro level. Here, in exercise 3, we see that the students describe the system in macroscopic terms and not in terms of heat flow through the boundary (micro level). This indicates that the students might have difficulties reasoning at the micro level, which prevents them from finding the correct solution.

In their answer to task 3.e, the students discuss the heat flow within the system based on their graphs without any problem. However, when they are asked to discuss the heat flow at the boundaries they fall back to the limiting cases: when time approaches infinity there is no heat flow anymore, so $\frac{\partial T}{\partial x}(0, \infty) = 0$ and $\frac{\partial T}{\partial x}(3, \infty) = 0$. Next, they notice that at time zero the tangent lines at the boundaries are also horizontal, so $\frac{\partial T}{\partial x}(0, 0) = 0$ and $\frac{\partial T}{\partial x}(3, 0) = 0$. From this, they suspect that these partial derivatives might possibly be zero during the whole process. The provided plot in three dimensions confirms their assumption. This finally allows them to write down the boundary conditions for task 3.c: $\frac{\partial T}{\partial x}(0, t) = 0$ and $\frac{\partial T}{\partial x}(3, t) = 0$. However, when asked by the interviewer why this is true, they say "Good question. I don't know... This means that the heat flow should be constant there, right?" We conclude that these students arrive at the correct boundary conditions for an isolated system purely based on a graphical observation (like group 1 also did). They cannot explain in physical terms why $\frac{\partial T}{\partial x}$ should be zero at the boundaries. The students now say that this refers to a constant heat flow, while in task 1.n they linked $\frac{\partial T}{\partial x} = 0$ to a zero heat flow. This shows inconsistency between these answers. It is unclear when this shift from zero to constant happened. After the interview was completed, the interviewer asked the students why they changed their mind, and they did not notice that nor understood why they did this. This again indicates that this group did not fully understand the relation between $\frac{\partial T}{\partial x}$ and heat flow when they stated their answer to task 1.n.

5.7 Discussion

The general aim of this study was to find ways to scaffold the blending of mathematics and physics in instructional materials. We developed a tutorial focusing on the physical meaning of mathematical boundary conditions for the heat equation and the physical interpretation of the partial derivatives $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial t}$ based on three design principles: giving explicit attention to both the mathematical and the physical aspects, stimulating graphical reasoning, and guiding students to (de-)encapsulate the partial derivative in order to blend mathematical and physical meaning (blended encapsulation). In this chapter, we focused on the effect of blended encapsulation as a teaching approach with the intention to support students in making the structural role of mathematics in physics explicit.

In this section, we first evaluate the blended encapsulation approach. In addition, we discuss two topics within our tutorial that appeared to be conceptually difficult. The first one is the role of time in the description of the concept of heat flow and the second one is the tension between reasoning at micro and macro level about the physical system. We discuss both topics and formulate some suggestions to tackle these aspects in instruction.

5.7.1 Evaluating the blended encapsulation approach

We extended Zandieh's framework to the *blended partial derivative framework* in order to explicitly visualize the blending between mathematical and physical knowledge that has to take place when reasoning about the relation between $\frac{\partial T}{\partial x}$ and heat flow. The layers of process-object pairs offered a way to gradually guide students towards the limit layer, which contains the relation that we want to promote in student reasoning. This resulted in the blended encapsulation approach (see Figure 5.4) which led the design of the tutorial tasks (see Figure 5.9). If the students followed the intended trajectory, the pattern in the tables resulting from the data analysis (Figures 5.12, 5.13, and 5.14) is similar to the pattern in Figure 5.9.

We designed tasks 1.j to 1.l so as to trigger students to do some kind of 'preparation', following the blended encapsulation approach, before formulating their summary or conclusion in task 1.m. Overall, we see that all groups eventually formulated the relation between $\frac{\partial T}{\partial x}$ and heat flow at the limit layer in response to task 1.m. However, we observed that the preparation did not always follow the blended encapsulation trajectory as we intended.

Group 1 largely followed the intended trajectory. Task 1.j started them off at the difference layer. In 1.k, the students briefly discussed the ratio layer and made the step towards the limit layer at the mathematical level (limited to symbolic and graphic columns). Task 1.m elicited their physical understanding explicitly and made them formulate the relation between $\frac{\partial T}{\partial x}$ and heat flow the way we intended. Overall, we conclude that the task design prompted the intended reasoning with this group.

Group 2 also started by reasoning at the difference layer in response to task 1.j. Task 1.k was successful in eliciting reasoning at the ratio layer with this group. In response to task 1.l, they successfully formulated the relation between $\frac{\partial T}{\partial x}$ and heat flow the way we intended. However, the reasoning of this group shows that following the intended 'preparation' trajectory based on blended encapsulation in tasks 1.j-1.l does not guarantee a fluent conclusion in response to task 1.m. Group 2 formed the basis we anticipated in the design, but when asked to summarize their insights and once more relate all concepts, they did not use that formed basis. They started reasoning all over, and arrived at the correct relation between $\frac{\partial T}{\partial x}$ and heat flow again.

Group 3 did not follow the intended trajectory. The tasks failed to lead them towards $\frac{\partial T}{\partial x}$, but instead they reasoned in terms of $\frac{\partial T}{\partial t}$. This reasoning was possibly influenced by the preceding task 1.i, which was about $\frac{\partial T}{\partial t}$. Moreover, they did not answer parts of the tasks that were designed to foster the structure of the blended encapsulation sequence. Ignoring these prompts might partially explain why the blended encapsulation sequence did not have the intended effect for them. The reasoning of group 3 shows that it is also possible to formulate the intended conclusion in 1.m without following the blended encapsulation trajectory. However, the reasoning of group 3 was very brief and we cannot judge if they have understood the relation between the different concepts thoroughly. We do not see proof that their conclusion is based on understanding of the underlying layers.

Generally, we conclude that the blended encapsulation approach has the potential to help students in recognizing the way temperature differences lead to heat flow and how these temperature differences between positions can be described using the concept of a partial derivative of temperature with respect to position. However, there are some weaknesses in the current design that need to be optimized in order for the approach to reach its full potential.

First, the ratio layer needs more explicit triggering in order to make sure that the students make this step explicit in their reasoning. Group 2 was the only group that made the ratio layer explicit in response to task 1.k. Group 1, also acknowledged the ratio layer (symbolic representation) in their answer to 1.l by writing $\lim_{h\to 0}\frac{T(x+h)-T(x)}{h}$. This indicates that the concepts at the ratio layer were part of this group's reasoning, but that task 1.k did not trigger the students to make this explicit. In section 5.3.2 we explained that we opted not to trigger the ratio layer in an explicit way because we assumed that the students are familiar enough with the concept of a (partial) derivative to recognize the role of the ratio layer themselves. However, we observed that this assumption is wrong. In a redesign, there needs to be an extra task that specifically triggers the ratio layer. However, this is a tricky exercise as it is hard to find a balance between too little and too much scaffolding. We want to avoid 'giving away' too much information resulting in the effect that students do not have to think for themselves as much.

Second, with regard to the physical meaning column, we observed that 'temperature gradient' is a concept that was not used spontaneously by the students. They rather reasoned in terms of temperature differences (situated at the difference layer) and only used the concept of temperature gradient after it was introduced in task 1.m. Moreover, two groups needed hints about the concept of temperature gradient. Our interpretation is that the students had difficulties transferring their mathematical knowledge about the concept of gradient to this physics context of one-dimensional heat transfer, while the system is described using a function with two variables $T(x,t)$. The different nature of the physical variables position and time plays a crucial role here. It is clear that the transfer from mathematics to physics is not self-evident to the students in this case, which confirms the literature (e.g. Eichenlaub & Redish, 2019; Greca & de Ataíde, 2019). Therefore, we suggest to introduce the concept of temperature gradient in an explicit way. We assumed that students would be able to reason with this concept because it is used in their course notes, but apparently they did not have this prior knowledge as we expected.

Third, we want to stress the importance of the summarizing task 1.m. In this task, we asked the students to relate the different concepts at the limit layer in Figure 5.9. This task ensures that the students touched upon all crucial concepts in their answer, and it provides an opportunity to summarize the understanding they built up during the activity. We see that both roles have shown their worth: group 1 did not formulate the intended conclusion explicitly after task 1.j, so task 1.m ensured a summary of their insights; group 3 did not arrive at the intended insights in the sequence 1.j-1.l, so there 1.m worked as a safety. We recommend to include this type of task in learning activities in general.

5.7.2 The role of time in the description of heat flow

The role of the time dimension in the mathematical description of heat flow has been a recurrent theme over the course of this dissertation. In all chapters, we presented observations of students that tended to describe the concept of heat flow using a derivative of temperature with respect to time. In this chapter, we see this again in the reasoning of groups 2 and 3. We start from our observations and formulate some suggestions of how this could be tackeled in instruction.

In our tutorial design, we intended to address the known issue where students have difficulties deciding between $\frac{\partial T}{\partial t}$ and $\frac{\partial T}{\partial x}$ to describe heat flow (Chapter 2) by first focusing on $\frac{\partial T}{\partial t}$ (task 1.i, not included in the analysis) and then focus on $\frac{\partial T}{\partial x}$ (tasks 1.j-1.m). This way we expected that students would experience the difference in physical meaning between these two partial derivatives and thus resolve the issue. We observed in the interviews that this approach was insufficient. Two out of three groups still struggled with the role of time in the description of heat flow. However, these new observations helped us to define the problem more precisely. As stated in section 1.2 in the introduction, heat flow is the amount of energy passing a point per unit of time. Writing this symbolically, this means that heat flow can be referred to as $\frac{\partial Q}{\partial t}$ with Q being heat or energy [J]. It is clear that the time dimension plays a crucial role in this description and it is therefore not surprising that the term heat flow induces reasoning in terms of time, as we can observe in the reasoning of groups 2 and 3. However, the crucial point here is that it is correct to reason in terms of rate of change of *heat* (or energy) over time, but not in terms of rate of change of *temperature* over time.

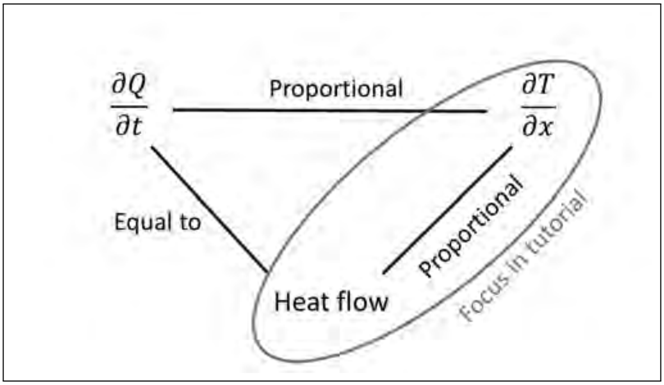


Figure 5.15: Visualization of the relation between heat flow and its descriptions in terms of time and position.

The law of thermal conduction, also known as Fourier's law, shows the relation between $\frac{\partial Q}{\partial t}$ and $\frac{\partial T}{\partial x}$. It states that the heat that flows through a point per unit of time is proportional to the negative gradient of the temperature. In the one-dimensional differential form, this law can be written as:

$$\frac{\partial Q}{\partial t}(x,t) = -kA_x \frac{\partial T}{\partial x}(x,t)$$

where $\frac{\partial Q}{\partial t}(x,t)$ is the heat flow, i.e. the amount of heat passing through a point per unit of time [W]. k is the thermal conductivity [$\mathrm{Wm^{-1}K^{-1}}$], which is a measure of how well the material conducts heat, and is related to the thermal diffusivity α as $\alpha = \frac{k}{\rho c_p}$ with ρ the density [$\mathrm{kgm^{-3}}$] and c_p the specific heat capacity [$\mathrm{Jkg^{-1}K^{-1}}$] (Farlow, 1993). A_x [$\mathrm{m^2}$] is the cross-sectional area perpendicular to the x-direction in which the heat can flow. The minus sign is introduced to make the heat flow a positive quantity in the positive coordinate direction (i.e., opposite of the temperature gradient) (Hahn & Özışık, 2012). We visualize this content knowledge in Figure 5.15 and reflect on the focus of our tutorial.

Currently, we focused the design solely on the difference between $\frac{\partial T}{\partial t}$ and $\frac{\partial T}{\partial x}$, and on the way that temperature differences between points in the system lead to a heat flow. We payed no attention to the description in terms of heat Q because this physical quantity is not part of the heat equation, which indicates that the term '*heat* equation' can be considered misleading. In Figure 5.15 this one-sided approach is visualized by the gray ellipse. We established that this approach is not entirely solving the problem with the role of the time dimension. The findings in this chapter support the idea that the description in terms of heat needs explicit attention too.

Fourier's law was discussed in their lecture and course notes, but we see that none of the students used this knowledge during our tutorial. Therefore, we suggest incorporating Fourier's law explicitly in the tutorial. It can be used to include the other half of the triangle in Figure 5.15 in the design. This way, the students apply this theoretical knowledge to gain insight in the description of the physical process.

Although most students had no problem identifying the partial derivative $\frac{\partial T}{\partial t}$ as describing the rate of change of temperature over time at a certain position, we also see opportunity to extend the learning trajectory in this direction.

In future research or in instruction, it would be interesting to also include the heat equation itself and investigate if we can guide students to reason about its physical and mathematical meaning in a qualitative way.

The heat equation,

$$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial x^2}(x,t),$$

relates the change of temperature over time at a certain position to the curvature of the temperature distribution at that position. In particular, it shows that the temperature at a certain position will increase or decrease if the temperature distribution has a positive or negative curvature , i.e. if either more heat is flowing in or out of that point respectively. It would be good to also develop a tutorial where students learn to reason about the heat equation. Moreover, the topic of equilibrium, i.e. when $\frac{\partial T}{\partial t} = 0$ at every point, has the potential to explore the meaning of the heat equation in a qualitative way and add physical meaning to both sides of the equation.

5.7.3 Microscopic versus macroscopic reasoning

In the analysis, we observed tension between reasoning about the physical system at the micro (i.e. in terms of specific points in the system) and macro level (i.e. in terms of the system as a whole). We did not anticipate on this finding in our design. This tension was especially observable in the responses to task 1.n and exercise 3, which discussed the cases where there was no heat flow through the boundaries of the system, i.e. an isolated system. We expected students to reason at the micro level to answer these tasks because the function $T(x,t)$ gives a local description in every point of the system. However, we observed that some students tended to reason at the macro level. Therefore, in this section, we propose some adjustments to the formulations of the relevant tasks in order to be more sensitive to this issue.

In response to task 1.n, group 1 linked 'no heat flow' to a uniform temperature distribution in the whole system and only then link this to $\frac{\partial T}{\partial x} = 0$. The answer of group 3 to task 1.n was very brief. They answered with "then $\frac{\partial T}{\partial x} = 0$" without explaining their reasoning. It is not clear from the transcripts if these students reasoned at the micro or macro level. We link these observations to some answers in exercise 3, which was about an isolated system (and thus without heat flowing through the boundaries). Group 1 described an isolated system as “a system in which the energy stays in the system. There is no heat flow to the ‘outside world’.” Group 3 described it as “no external influences, so no reservoirs.” These groups did not describe the isolated system in terms of local heat flow through the boundary points, but rather in terms of conservation of energy, which points towards macroscopic reasoning. In task 3.c, we asked the students to express the boundary conditions of this isolated system mathematically. Both group 1 and 3 were not able to link their descriptions of an isolated system to their insight that ‘no heat flow means $\frac{\partial T}{\partial x} = 0$’ from before (1.n). It seems that the difference between reasoning at the macro and micro level plays a role in this issue.

We observed that reasoning in terms of heat flow through a specific point or heat flow through the overall system is different for students. The current formulation of the tasks is not specific enough. We see that it induces macroscopic reasoning for some students, which is correct, but is not helpful in building towards a description of heat flow though the boundary, and thus towards a mathematical description of boundary conditions. In an optimized version, we therefore suggest to include explicit prompts to make students aware of the importance to think about heat flow through a specific point. Task 1.n was formulated as “Under which physical and mathematical conditions will there be no heat flow?” This question could be made more precise by adding “through a specific point?” Similarly, task 3.a could be changed from “Explain what an isolated system is.” to “Explain what an isolated boundary is.”

5.7.4 Concluding remarks

Our aim in this chapter was to investigate whether the implementation of blended encapsulation helped students to formulate the relation between $\frac{\partial T}{\partial x}$ and heat flow and also how they use this relation in other problems. The results discussed in this study show once more that the relation between $\frac{\partial T}{\partial x}$ and heat flow is difficult for students to formulate and understand. The proposed approach based on blended encapsulation showed to be promising to stimulate the blending of mathematics and physics, even though the design of the specific tasks in the tutorial needs to be optimized.

To conclude, we reflect on the applicability of the blended encapsulation approach in other topics. The blended partial derivative framework can be adapted for any physical concept that is modelled by a (partial) derivative. This way, blended encapsulation can guide the instruction of these concept, e.g. the more straightforward concepts of velocity and acceleration, but also more advanced concepts from electrodynamics or thermodynamics. By extension, the blended encapsulation approach could also be interesting for other mathematical concepts that have a similar layered structure, e.g. a definite integral. This opens the possibility of incorporating the approach for many more physical concepts.