Blending of mathematics and physics

van den Eynde, Sofie

DOI:
10.33612/diss.177955315

Chapter 2

Difficulties with boundary conditions for the diffusion equation

This chapter is accepted as:

Van den Eynde, S., Deprez, J., Goedhart, M., & De Cock, M. (2020). Undergraduate student's difficulties with boundary conditions for the diffusion equation. *International Journal of Mathematical Education in Science and Technology.*

Abstract

Combining mathematical and physical understanding in reasoning is difficult, and a growing body of research shows that students experience problems with the combination of physics and mathematics in reasoning beyond the introductory level. We investigated students' reasoning about boundary conditions for the diffusion equation by conducting exploratory task-based, think-aloud interviews with twelve undergraduate students majoring in physics or mathematics. We identified several difficulties students experienced while solving the interview task and categorized them using the conceptual blending framework. This framework states that in reasoning, people draw from separate input spaces, in this case the mathematics and the physics input space, to form a blended space, where they make connections between elements from these spaces. To identify difficulties, we used open coding techniques. We observed few difficulties in the physics space. In the mathematics space, we identified several difficulties that we clustered in two main groups: findings about the mathematical meaning of boundary conditions, and findings about reasoning with functions of two variables. Finally, we identified four ways in which blending failed. Starting from our findings, we formulate recommendations for teaching and future research.

2.1 Introduction

In physics, we describe the world in mathematical structures. This is true for all levels of physics, but at an advanced level, the role of mathematics becomes even more important. As such, proficiency in mathematics is required to understand physical phenomena, and being able to combine the different fields is a prerequisite to become more proficient in physics. Understanding an equation in physics is not just connecting the symbols to physical variables and being able to perform calculations and operations with that equation, but also being able to connect the mathematical equation to its physical meaning and integrating the equation with its implications in the physical world (Redish & Kuo, 2015). This requires more than the sum of mathematics and physics and has proven to be difficult for students. It is therefore not surprising that the relation between mathematics and physics is an active research area in physics education research (PER) (e.g. Bollen et al., 2016; Karam, 2015; Redish & Kuo, 2015; Uhden et al., 2012) and plays a prominent role in this study. Identifying and understanding student difficulties with the role of mathematics in physics is an important part of this research area. Most of the work on student difficulties has focused on introductory physics. However, a growing body of research suggests that intermediate and upper-division students continue to struggle with reasoning and problem solving even in the advanced physics courses (e.g. Caballero, Wilcox, Doughty, & Pollock, 2015; Ryan et al., 2018). Student difficulties in upper-division problem solving originate, in part, from the more complicated math and more sophisticated physics characteristic for upper-division content. However, difficulties in the upper-division might also relate to the cyclic nature of the physics curriculum, in which some physics topics appear several times in different contexts across the undergraduate curriculum (Manogue et al., 2001; Zwolak & Manogue, 2015). Ryan et al. (2018) state that for these recurring topics, difficulties that are not addressed in early courses can persist, and become worse, when the topic appears again in a more advanced course.

We study the relation between mathematics and physics in the context of particle diffusion. This is a promising context because the diffusion equation is a good example where mathematics (a partial differential equation) and physics (the physical process of diffusion) come together. Specifically, we focus on the mathematical description of boundary conditions for phenomena described by the diffusion equation. Generally, boundary conditions appear several times throughout the physics and mathematics undergraduate curriculum (Ryan et al., 2018), which make them interesting for investigating difficulties as these can persist in other contexts later on in the curriculum. Boundary conditions define the conditions physical quantities must satisfy at the boundary between two regions. They are particularly critical because they are necessary to reduce general and abstract mathematical expressions to physically meaningful solutions that have descriptive and predictive power within a particular physical system (Boas, 2006). Therefore, boundary conditions are a powerful topic to investigate the blending of mathematics and physics.

There is not a lot of research on student reasoning or difficulties in the context of partial differential equations in general, and, as far as we know, none in the context of the diffusion equation. Ryan et al. (2018) recently investigated student difficulties with boundary conditions in the context of electromagnetic waves. Their data sources were student responses to traditional exam questions, conceptual survey questions, and think-aloud interviews. They arranged difficulties in the four phases that the ACER framework considers to be part of the problem solving process: Activation of the tools, Construction of the model, Execution of the mathematics, and Reflection on the results. Some of the difficulties they reported are the following: not using appropriate boundary conditions, even when asked explicitly to use boundary conditions and difficulties determining the correct direction for the wave vectors $\vec{k}$, $\vec{E}$, and/or $\vec{B}$. Even though the topic of boundary conditions is the same in their and our study, the context of electromagnetic waves differs from the diffusion context, which means that we expect many differences between their and our findings.

Wilcox and Pollock (2015) investigated upper-division student difficulties with separation of variables in the context of the Laplace equation in electrostatics. The main focus was on applying the separation of variables technique, but part of their investigation dealt with boundary conditions. Their findings showed that, in particular in spherical coordinates, some students struggled to identify and express the appropriate boundary conditions when they were not explicitly provided in the problem statement. Research in the context of ordinary differential equations (ODEs) has shown difficulties to distinguish 'amount'- and 'rate of change of amount'-type thinking when reasoning in the context of physical processes (Rowland & Jovanoski, 2004). We expect this might also be a problem for students when reasoning about boundary conditions expressed as a derivative, which we investigate in this paper.

We report here on a study of student reasoning when mathematically formulating boundary conditions in the context of the one-dimensional diffusion equation (section 2.2). In particular, we focus on the role of mathematics and physics, and how students blend both fields in their reasoning while formulating these boundary conditions. To structure our investigation, we use the conceptual blending framework (Fauconnier & Turner, 1998) as an analytical lens (section 2.3). We conducted think-aloud interviews with undergraduate students to study their reasoning. Participants, interview protocol and content are discussed in section 2.4. We go over the data analysis procedure in detail (section 2.5) and give an overview of the identified student difficulties (section 2.6). Finally, we discuss our findings and their implications for teaching (section 2.7).

2.2 Boundary conditions in the context of the diffusion equation

In this section, we briefly review the diffusion equation and the boundary conditions we focus on. Particle diffusion processes in one dimension are modeled by the following partial differential equation:

$$\frac{\partial u}{\partial t}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t)$$

for $0 < x < L$ and $0 < t < \infty$, with $u(x,t)$ expressing the concentration in terms of position and time. α is the diffusion coefficient, which is a measure of the rate at which particles can spread in a specific medium. Note that this is the one-dimensional diffusion equation, which describes an idealised system in which diffusion is only possible in one spatial dimension x.

In this paper, we focus on a problem (see Figure 2.2, discussion later in the article) where this partial differential equation is used to model the particle flow in a one-dimensional tube of length L. It relates the quantities $\frac{\partial u}{\partial t}(x,t)$, the rate of change in concentration with respect to time, and $\frac{\partial^2 u}{\partial x^2}(x,t)$, the concavity of the concentration distribution $u(x,t)$, which essentially compares the concentration at one point to the concentration at neighboring points (Farlow, 1993).

The initial condition describes the state of the system at the beginning ($t = 0$). Boundary conditions refer to the conditions physical quantities must satisfy at the boundary of the system. In this study we focus on boundary conditions of the form $\frac{\partial u}{\partial x}(x,t) = g(t)$, which specify the flux, which in this context is the particle flow, through the boundary. More specifically, we consider a closed tube, so no particles can pass the boundary. Mathematically, this is described by the following boundary conditions: $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$.

2.3 Analytical framework

To investigate student reasoning when mathematically formulating boundary conditions for a diffusion process, we use the conceptual blending framework. This framework was originally introduced by Fauconnier and Turner to model how people create new meaning in linguistic contexts by selectively combining information from previous experiences (Fauconnier & Turner, 1998, 2003a, 2003b). It has repeatedly been used in PER to model the blending of physics and mathematics (Bing & Redish, 2007; Bollen et al., 2016; Hu & Rebello, 2013; Kuo, Hull, Gupta, & Elby, 2013; Podolefsky & Finkelstein, 2007).

In blending, elements from input spaces are selectively combined into a blended space. Generally, a mental space, which can be input spaces or the blended space, is composed of conceptual packets or knowledge elements that tend to be activated together, and has an organizing frame that specifies the relationships among the elements within the mental space (Bollen et al., 2016). According to the conceptual blending framework, two or more input mental spaces that share content or structure can be combined into a new, blended space. The language of conceptual blending provides a framework for analyzing students' combination of mathematics and physics. This framework emphasizes both the emergence of new relations and the different ways the combination itself can be constructed (Bing & Redish, 2007). It is also interesting that there is no hierarchical relation between the different input spaces. The input spaces ('mathematics' and 'physics' in the case of this paper) are considered of equal value, which joins our opinion that mathematics should not be seen as just a tool to be used in physics, nor physics as merely a context for mathematics.

Figure 2.1 shows an example of a blending diagram from the work of Hu and Rebello (2013). They used the conceptual blending framework to make sense of the ways in which students combined their knowledge from calculus and physics to set up integrals in a physics context. Students calculated the total resistance of a cylinder with varying resistivity as a function of position. The authors distinguished three input spaces: the symbolic space (which refers to the technical role of mathematics), the math notion space (which refers to the mathematical concepts), and the physics space

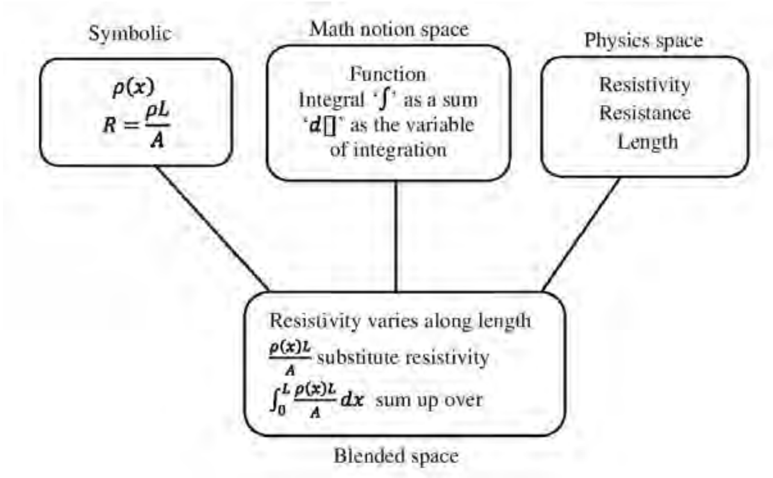

Figure 2.1: Example of a blending diagram from the work of Hu and Rebello (2013).

(which refers to physical quantities and concepts). Different blends were identified and graphically represented, all with the same three input spaces, but with different organizing structures. From the blending diagrams, the authors determined several blending mechanisms and ranked them from least to most productive. They concluded that the difficulties that students experience seem to originate not necessarily from a lack of prerequisite knowledge of mathematics, but rather from inappropriate blending of the knowledge of mathematics with their knowledge of physics concepts and the physical scenario at hand.

The conceptual blending framework has also been used in mathematics education research. Zandieh et al. (2014) investigated student reasoning when proving 'conditional implies conditional' statements. Gerson and Walter (2008) used conceptual blending as a lens to illuminate individual and collective understandings of calculus concepts as they emerge from sustained mathematical inquiry.

In this paper, we study the role of mathematics, physics, and their combination in student reasoning when formulating boundary conditions for diffusion in a one-dimensional tube with closed ends. We hypothesize that conceptual blending might be an adequate framework and therefore answer the following research questions:

1. What reasoning difficulties do we observe when students mathematically formulate boundary conditions based on a described physical situation?
2. How can we use the conceptual blending framework to characterize these reasoning difficulties?

2.4 Data collection

2.4.1 Participants

We conducted individual, semi-structured, task-based think-aloud interviews with twelve undergraduate students to gain insight into the thinking and reasoning mechanisms behind students' responses. The students came from two different universities and were interviewed after they finished a course in which the diffusion equation was discussed.

Six of the participants were second year undergraduate students majoring in physics or mathematics at KU Leuven, Belgium. They followed a course on differential equations, taught in Dutch. The part of the course on partial differential equations entailed a chapter in which the diffusion equation was discussed in depth, i.e. the derivation of the diffusion equation, boundary conditions, physical systems described by this equation and the algorithmic technique of separation of variables are discussed.

The other six participants were first year undergraduate physics majors at the University of Groningen, the Netherlands. They were interviewed a week after following a Mathematical Physics course, which was taught in English. In this course, the learning goals concerning partial differential equations were: being able to apply boundary conditions in complex form, being able to solve partial differential equations using the separation of variables technique, and understanding complex physical systems via partial differential equation analysis.

At both universities, we had access to students' grades for the respective courses dealing with partial differential equations. We aimed for a stratified sample (Hsieh & Shannon, 2005) with a mixture of low and high achieving students to ensure heterogeneity. All participating students passed their exam. Half of the sample consists of students scoring between 75% and 100% of the maximum score, the other half between 50% and 75%. Participation to the research was voluntarily.

2.4.2 Interview content

Initially, we designed and conducted broad interviews to explore instances of blending of mathematics and physics in reasoning about the diffusion equation. There is almost no literature about student understanding of the diffusion equation. Therefore, we identified four potentially interesting aspects to investigate this blending, and developed interview questions related to these aspects:

1. Conceptual understanding of the diffusion equation: Starting from the partial differential equation (mathematical form), explain the physical meaning of the

> In a tube with a length of one meter there are u_0 particles. At time $t = 0$ they are distributed as the function $f(x) = u_0(1 - \cos(2\pi x))$. The left and right end of the tube are closed so no particles can flow in or out of the tube. Write down the mathematical description of this physical situation (PDE, boundary and initial conditions). Also make a sketch of the initial distribution of the particles.

Figure 2.2: Interview question which is the focus of the analysis.

partial differential equation and its different terms.

2. Conceptual interpretation of the diffusion process: Starting from a description of a physical situation combined with a graph representing the temperature profile at time zero and the partial differential equation and boundary conditions (mathematical form), reason qualitatively (so without solving the partial differential equation) about the evolution of the concentration over time.

3. Setting up the mathematical equations: Starting from a description of a physical situation/process, formulate the mathematical description (partial differential equation, initial condition and boundary conditions).

4. Interpreting the solution of the diffusion equation: Starting from a given analytic solution, describe the particle flow and link it to the physical process.

During the interview, we observed that mathematically formulating the boundary conditions was a topic where the blending of mathematics and physics showed to be difficult, which made it particularly interesting for analysis. We therefore focused on boundary conditions in our analysis and limited our analysis to these parts of the interview. This is situated in the part of the interview where students have to set up the mathematical equations for a described physical system (number 3 in the list above). The interview question which is the focus of the analysis in this paper is shown in Figure 2.2 [1].

2.4.3 Model solution to the interview question

As a reference, in this section we show how the interview question in Figure 2.2 can be answered.

[1] The whole interview can be found in Appendix A. In this paper, we focus on student answers to the interview question "Boundary and initial conditions" in the part "Concrete example: Diffusion".

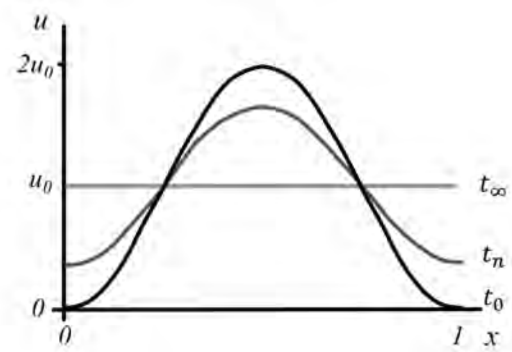

Figure 2.3: This figure shows a possible sketch of the way the graphical representation of the concentration distribution changes over time. The black graph shows the initial condition, which is provided in the problem statement. The two gray graphs show the distribution after some time t_n and after a long time t_∞.

The necessary information to answer this question is given in the sentence "The left and right end of the tube are closed so no particles can flow in or out of the tube." This means that at both sides of the tube, the particle flow through the boundary is zero. Physically, particles flow between neighbouring points when the concentration in these points differs. This concentration difference can mathematically be described by the gradient $\frac{\partial u}{\partial x}(x,t)$ of the concentration (which is the limit of the difference $\frac{\Delta u}{\Delta x}$). As in this problem no particles can flow through the boundaries, this is mathematically expressed as $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(0,t) = 0$.

The question can also be answered using a graphical approach. In the problem statement, we provide a function that describes the initial concentration in each point of the tube. Fig 2.3 shows a graph of this distribution. As both ends of the tube are closed, the distribution will evolve towards a constant distribution where the concentration is the same in every point of the tube. This is the equilibrium state of the system. If we zoom in on what happens at both ends of the graph, one can argue that the slope with respect to x has to stay zero over the whole process. Indeed, a non-zero slope would indicate a concentration difference between neighbouring positions and that would result in a particle flow that wants to even out that difference. The fact that we know that there is no particle flow through the boundary must therefore mean that $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(1,t) = 0$.

2.4.4 Interview protocol

Students were interviewed individually for approximately one hour. The participants were encouraged to ‘think aloud’ as they worked through the tasks. The interviewer was the first author of the paper, and was not involved in any of the respective courses on partial differential equations. We conducted the interviews using a smart pen, which audio recorded the conversations and kept track of the student’s notes, drawings and calculations. We also video taped the interview as a backup for the smart pen. The interview questions were written in English because the course in Groningen was taught in English. However, the students could respond orally in Dutch if their native language was Dutch (which was the case for all participants and the interviewer). The English interview questions were no problem for the students as they were all used to English textbooks and English speaking teachers. In case they did not understand the question, there was always the opportunity to ask the interviewer for clarification.

At the start of the interview, the interviewer informed the students about the subject and purpose of the interview. The interviewer also emphasized that she would not give feedback about the correctness of the student’s responses during the interview. After this introduction, the student signed an informed consent. Generally, the interviewer tried not to interfere, except for the prepared follow-up questions, prompts to think out loud, and requests for further explanation of what the students did and why. After the interview, the students had the chance to discuss their answers and the aim of the research project in an informal way. In the rest of the article, we replaced student names by letters to guarantee anonymity.

2.5 Data analysis

As a first step, the interviews were transcribed verbatim and drawings, calculations, and notes from students were added to the transcripts. In the first phase of analysis, we looked at student reasoning through the lens of the conceptual blending framework. The unit of analysis here is a reasoning step. Each reasoning step is characterized as being part of the mathematics (M) space, the physics (P) space or the blended (B) space.

- **Mathematics space:** In general, the math space contains all steps of reasoning related to mathematical knowledge, without any new physical input (i.e. using only physical input that has already been mentioned before in the solution process). This broadly entails reasoning with mathematical concepts, functions, graphs, equations, derivatives, mathematical relations, etc.

- **Physics space:** Similarly, the physics space contains all steps of reasoning related to physics knowledge, without any new mathematical input, such as reasoning with physics concepts (e.g. concentration, particle flow); describing a physical system/process/relation in words without a mathematical description; and reasoning with experimentally known facts or relations between physical quantities or an established physical principle or law in the domain (e.g. particles flow from high to low concentration) (Sirnoorkar, Mazumdar, & Kumar, 2016).

- **Blended space:** The blended space contains all steps of reasoning that explicitly connect mathematical concepts/equations/graphs to physical concepts/situations/processes/quantities (e.g. stating that the particle flux is proportional to $\frac{\partial u}{\partial x}$).

The first author did the initial analysis (dividing in reasoning steps and categorizing them in the different spaces), after which two other authors independently checked every transcript. Where the opinions differed, interpretation and categorization were discussed until consensus.

During the analysis, it became clear that mathematically formulating the boundary conditions was quite challenging for the students and most of the reasoning was incorrect or incomplete. We observed that students experienced many types of difficulties. Therefore, in the second phase of analysis, we decided to focus on those parts in the interviews where difficulties arise. We performed a content analysis (Hsieh & Shannon, 2005) and used open coding techniques to develop a code book that encompasses the different difficulties students experienced when formulating the boundary conditions for the question in Figure 2.2. We structured the code book around the three categories from the first phase of analysis: difficulties situated in the mathematics space, the physics space, and the blended space. The code book, with examples for every type of difficulty that we observed, is presented in the results section. Because of the small number of participants and the exploratory nature of the interview, the aim is to give an overview of the observed difficulties without discussing the frequency of appearance of each difficulty, or linking them to students' achievement or curriculum.

2.6 Results

Overall, all students have some basic understanding related to the task: most students can explain the meaning of 'boundary conditions' and most students connect 'change' to 'derivative'. However, only four out of twelve participating students could formulate the correct solution for the task: boundary conditions $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(1,t) = 0$ (students D, E, F and G). Even among the students who solved the problem successfully, we observed a lot of doubt and incompleteness in the reasoning, which often obstructed productive blending.

In the following sections, we provide an overview of the identified difficulties for the reasoning in the three mental spaces: physics, mathematics, and blending. We describe the difficulties, and provide examples of student reasoning and interpretations for every difficulty.

In the rest of this chapter, we will use an abbreviated form of the boundary conditions, like $\frac{\partial u}{\partial x} = 0$, instead of the complete $\frac{\partial u}{\partial x}(0, t) = 0$ and $\frac{\partial u}{\partial x}(1, t) = 0$. We do this for readability, and because we observed that none of the students had problems identifying the values of x and t when formulating the boundary conditions.

2.6.1 Reasoning in the physics space

Students seem to have few problems with the purely physical aspect of the reasoning. They generally have a good understanding of what happens in the physical system, which is illustrated by the following quotes:

> *Student B: We know that nothing can go this way, those particles bump into the edge of the tube and cannot go further on to the left, but they can go back to the right.*

> *Student D: The change in concentration ehm, so you have a tube [draws a tube with particles] with particles that are moving randomly and here [draws thicker edges to the tube] they cannot pass.*

We only identified one difficulty in the reasoning in the physics space: pinpointing the correct physical quantity that is described by the diffusion equation. We distinguish between two types of confusion: concentration versus number of particles, and heat versus temperature (when students changed to the context of heat transfer on their own initiative). The first example shows a student using number of particles to formulate the boundary conditions:

> *Student H: The boundary conditions and hm... If we call this just x and one meter is just* 1*, then you will get as boundary conditions… hmm… [writes u] hmmm… Well, we will call the number of particles 'N' [crosses u and changes it into N]. At point zero ehm, for all times, is zero. And at point one, one meter, also equal to zero [writes the following boundary conditions while talking:* $N(0, t) = 0$ *and* $N(1, t) = 0$*].*

This student defines a new variable N as the number of particles instead of using the physical quantity of concentration. He uses this new variable to define the (incorrect) BCs.

The second example shows the incorrect use of temperature where it should be heat:

> *Student G: The tube is closed so no particles can flow in or out of the tube, so that means that the flow should always be zero. So yeah, the tube starts at* $x = 0$ *and goes on until* $x = 1$. *[sketches a tube with boundaries 0 and 1] When x is smaller than or equal to zero or bigger than one, [writes* $x \leq 0$ *and* $x \geq 1$*] Then I think that mathematically... yeah... there is no temperature going out of the tube... hmmm yeah... Yes I would probably describe it as [writes* $\frac{\partial u}{\partial x} = 0$*], the way I think about it now, it should be zero.*

This student jumps from a particle diffusion problem to the statement "there is no temperature going out of the tube". He changes context on his own initiative. This could be interpreted positively because he clearly sees the similarities between heat flow (discussed in earlier questions in the interview) and particle diffusion. However, as can be seen in the quote, he does not correctly distinguish between heat and temperature. Difficulties with distinguishing between temperature and heat have been observed many times before (e.g. Clough & Driver, 1986; Goedhart & Kaper, 2003; Kesidou & Duit, 1993; Linn & Songer, 1991; Stavy & Berkovitz, 1980).

2.6.2 Reasoning in the mathematics space

We identified several steps of reasoning in the mathematics space that are hard for students. The observed difficulties can be grouped in two main categories: mathematical meaning of boundary conditions, and reasoning with functions of two variables.

The mathematical meaning of boundary conditions

The role of boundary conditions in solving a partial differential equation
When solving a partial differential equation, the general solution is found, which represents an infinite number of solutions. Next, the boundary conditions (and the initial condition) are needed to find the particular solution corresponding to a specific physical situation. Some students do not understand the order of these steps. Student J, for example, stated:

> *Student J: Then I think I should solve it [the partial differential equation] and just fill in zero...*

The student first wants to obtain the particular solution (without using any boundary conditions) and then fill in an x-value at the boundary to obtain the boundary conditions from this solution. Because the student persists this line of reasoning later on in the interview, the interviewer intervenes:

> *Student J: I would just solve it [the partial differential equation] and then see if I could get something from that, but...*
> *Interviewer: Yeah, but no, you actually cannot solve it. Because when you want to solve the equation, you need the boundary conditions at some point to be able to find the solution.*
> *Student J: Hmmm...*
> *Interviewer: So you cannot solve it [partial differential equation] yet, you first need to find the boundary conditions.*
> *Student J: Yeah alright that ehmm... I believe you [hesitantly].*

The student proposes once more to solve the partial differential equation and get the boundary conditions from this. After explicit intervention from the interviewer, saying that the boundary conditions are critical to find the particular solution, the student eventually agrees hesitantly, indicating that he does not really understand the point the interviewer is trying to make. This indicates that the student does not know that he needs boundary conditions to be able to find the particular solution for the problem.

Strong focus on boundary conditions of the form $u = 0$ Boundary conditions can take different forms. We generally distinguish between (in abbreviated notation): $u = g(t)$ (concentration at the boundary specified) and $\frac{\partial u}{\partial x} = g(t)$ (particle flow across the boundary specified) (Farlow, 1993). We observe that the form $u = 0$ is very appealing to students in the interview, five of the twelve participating students chose this as their final answer to the question in Figure 2.2, instead of the correct $\frac{\partial u}{\partial x} = 0$.

Below, we show an excerpt of a student who is also attracted to this '$u = 0$' solution, but he realizes that this cannot be correct.

> *Student J: So the boundary conditions, what would they be? Because it doesn't make sense to me that it [the value of the concentration at the boundary] stays at zero... but if it doesn't stay at zero, then we can't formulate boundary conditions because then... uhm it changes, unless of course you come up with a function for the boundary conditions but that is uhm a bit strange*

The student seems to think that $u = 0$ is the only possible form for a boundary condition. However, he realizes that $u = 0$ is not suitable in the case at hand. Next, he hesitantly comes up with the idea to construct a function that gives the values of the solution at the boundaries. He did not consider a boundary condition containing a derivative.

Later on in the interview, the student repeats his view on boundary conditions as follows:

> *Student J: I am looking at the solution at that point [points at $x = 0$], but that is ehm, that is how I think to understand boundary conditions.*

This illustrates how the student is focused on the idea that the boundary conditions give the solutions at the boundaries, which can explain his focus on the form $u = 0$. This idea is not fully correct. The boundary conditions provide a condition for the solution at the boundary. In the case of a boundary condition of the form $\frac{\partial u}{\partial x} = 0$, the flow or flux is fixed at zero, but it does not entail the complete solution at the boundary.

Reasoning with functions of two variables

A second category of mathematical difficulties relates to reasoning with functions of two variables. We give an overview of three different aspects we observed in the data and present examples for each aspect.

Not specifying the variable with respect to which the derivative is taken
Quite often, students talk about 'the' derivative of u. As u depends on both position and time, talking about 'the' derivative is not precise. In most cases, from the context in the transcripts it seems that students implicitly know which variable they are talking about. However, sometimes this being imprecise becomes problematic, because it might indicate that students do not realize that there are two different first derivatives for functions with two variables.

An example is observed in the following quote:

> *Student B: Hmm, the left and the right end of the tube are isolated, that doesn't mean that there are no particles, that means... Maybe... is it that the particles stay constant there? No. Or is it a derivative? Because nothing comes out.*

We see that this student correctly links derivative to change, but he does not specify the variable with respect to which the derivative is taken. Further on in the interview, the student continues as follows.

> *Student B: So if we look at this [draws a tube] we know nothing can go this direction, those particles bump in to boundary and can't go left [out of the tube], but they can go right [staying in the tube]. So that means that the derivative does not have to be zero.*

The student is talking about 'the derivative', but it is unclear which one he means or whether he realizes that there are actually two possible derivatives.

The difference between $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ To some students, it seems unclear that the different possible first and second derivatives all have a different meaning.

Student A chose boundary conditions of the form $\frac{\partial u}{\partial t} = 0$ as his final answer, which is incorrect. At the end of the interview, the interviewer showed him the correct mathematical description of the problem ($\frac{\partial u}{\partial x} = 0$).

> *Interviewer: Yeah, and here you have the solution to the problem from before and it was actually a derivative with respect to x...*
> *Student A: Yeah okay...*
> *Interviewer: Why would that be correct?*
> *Student A: Hmm... I don't know... Hmm I guess it is correct, but I could have made the same reasoning for t.*

This student acknowledges that he could not explain why it should be a derivative with respect to x instead of one with respect to t. The fact that he states that he could have reasoned the same way about a derivative with respect to t, indicates that he cannot explain the difference in meaning between the two different partial derivatives.

The following excerpt shows another example of this difficulty:

> *Student L: So I kind of want to write this down: delta t or delta x, that doesn't really matter [writes $\frac{\partial u}{\partial t} = 0$], but why would... x and this one? Second derivative?*
> *Interviewer: That is indeed the way you can write something like that down.*
> *Student L: [makes it a second derivative with respect to time]*
> *Interviewer: Hm wait, what have you done now?*
> *Student L: [crosses the t and turns it into an x en writes = $\frac{\partial u}{\partial t}$ behind it (which resulted in: $\frac{\partial^2 u}{\partial x^2} = 0 = \frac{\partial u}{\partial t}$] That, I would say the second derivative with respect to... Yeah I can also do it this way. This one is not really necessary [inaudible].*

This student seems to be gambling between different options. The student already decided before this excerpt that he needed something that expresses change, so he turned to derivatives. In this excerpt we see that he mentions derivatives with respect to t, and x, and also a second derivative with respect to x. In the first line he states that "it does not really matter" which derivative he chooses, which indicates he does not see a difference in meaning between the two partial derivatives. In the last sentence of the excerpt, the student writes a second derivative with respect to x and sets it equal to zero and also equal to the first derivative with respect to time. Again, the student does not distinguish in meaning between first and second partial derivatives with respect to different variables. This last sentence can also be interpreted in the light of the partial differential equation. The student writes $\frac{\partial^2 u}{\partial x^2} = 0 = \frac{\partial u}{\partial t}$, which has a form similar to that of the diffusion equation: $\frac{\partial u}{\partial t}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t)$. This could be part of the student's confusion.

Seeing a function of two variables as a function of one variable When one of the variables is kept constant, some students think exclusively in terms of the resulting function of one variable and forget about the original function of two variables. This results in thinking that only one of the partial derivatives makes sense. We identified two examples of this difficulty in our sample.

In the first one, the student looks at $u(x,t)$ at different values of t. By setting the time variable equal to a constant value, the student apparently thinks that u is a function of x only, and no longer of t and uses this as an argument to not take the derivative with respect to t:

> *Interviewer: Can you explain why it should not be a derivative with respect to t using the three sketches you made? [Sketches are u(x) graphs at several times t]*
> *Student A: Well, I'm thinking now, you have graphs in function of x, so it makes much more sense to take the derivative with respect to x, but yeah.*

In the second example, the student plugs $x = 0$ into $u(x,t)$, which then becomes $u(0,t)$. From this, the student thinks that x is not a variable anymore, so one cannot take the derivative with respect to x anymore:

> *Interviewer: And can you explain to me why it is a derivative with respect to time?*
> *Student C: Ehm, yeah, it would be a bit stupid to take the derivative with respect to x because there is no x anymore.*

From the context of the transcript, we interpret this as the student saying 'in $u(0,t)$ there is no x'. She does not seem to realize that boundary conditions, more generally, give a condition for $u(x, t)$ at position x and that this condition can be stated in terms of the derivative with respect to x in $x = 0$. Even if we fill in a numerical value for x (or t), we can still discuss the derivative with respect to that variable.

2.6.3 Reasoning in the blended space

In this section, we give an overview of difficulties related to making the connection between mathematical and physical reasoning. We identified four ways in which blending failed.

An incorrect combination of elements from the input spaces

Some students start reasoning correctly about the closed tube in physical terms, but when asked about the mathematical description, they connect this to a derivative with respect to time. This indicates that the student has an incorrect idea about the physical meaning of the time derivative. We discuss two examples.

In the first excerpt, we observe a student trying to make connections between the physical situation and a mathematical description:

Student A: No, it is different because the derivative will be zero now ehm [writes $\frac{\partial u}{\partial t} = 0$*]. Yeah it's going to be the derivative which is equal to zero, sorry, ehm... I think it should be like this... and it will be the same for* $x = 1$.
Interviewer: Yes, and you say the derivative with respect to time, why to time?
Student A: Ehm... because the flow does not change... yeah no... There is no drain of particles and no particles can flow in, but indeed, maybe that... I think it should be a derivative because ehm yes there is no change. It's not something constant or so, there is just no change.

Later on in the transcript, he concludes the following:

Interviewer: Yes, but is it a derivative with respect to t or to x? That is the question.
Student A: Ehm... yes... yeah no, it should definitely be t, my initial thought was correct. [crosses the x in the denominator and changes it to t]. Yeah, cause ehm, over time, no new particles are flowing in.

We observe that this student shows some understanding of the physical and the mathematical aspect of his reasoning, but makes an incorrect combination of P and M elements. Concerning the physics space, the student shows some doubt in formulating what a closed system means (lines 7 and 8). Concerning the mathematics space, the student connects ‘change’ to a derivative in general (lines 9 and 10). However, in this case the derivative is referring to a derivative with respect to time and the student has no clear concept of what he means by ‘no change’. This could be referring to no change in concentration at the boundaries, or to no particles flowing through the boundaries. The first option is incorrect in this situation, and the second option is incompatible with the derivative with respect to time.

We observe a similar reasoning pattern in excerpts from the interview with student E. At the start of his reasoning, he states:

> *Student E: And then, the left and right end of the tube are isolated, so the derivative with respect to time is zero there because no particles can flow in or out of the tube.*

A bit further on in the interview he concludes:

> *Interviewer: So why is it a derivative with respect to time? Conclusion?*
> *Student E: Well, my conclusion is that you want to describe how the situation evolves here over time and that you don’t want any change in that. Because you don’t want particles to be able to flow out.*

Again, the student seems to understand the physical meaning of a closed system (lines 2, 3 and 7). Mathematically, the student knows that the derivative with respect to time expresses change over time (lines 5 and 6). However, in blending he connects these two elements which are incompatible.

Conflicts between correct mathematical reasoning and incorrect physical intuition

Student H solved the problem incorrectly. His final answer consisted of boundary conditions of the form $N(0, t) = 0$ and $N(1, t) = 0$ (Note the choice for N being number of particles, as discussed in section 2.6.1).

In a follow-up question in the interview, we show the student the analytic solution and we ask him to reflect on its physical meaning. This question is not the focus of our investigation, but while answering this question, the student gets back to the topic of boundary conditions and reflects on what he answered before. At the point of the excerpt, the student has drawn a graph of $u(x)$ as time approaches infinity, which is the horizontal line with vertical ends at the boundaries in Figure 2.4. He added the vertical lines because his choice of boundary conditions determined that the value

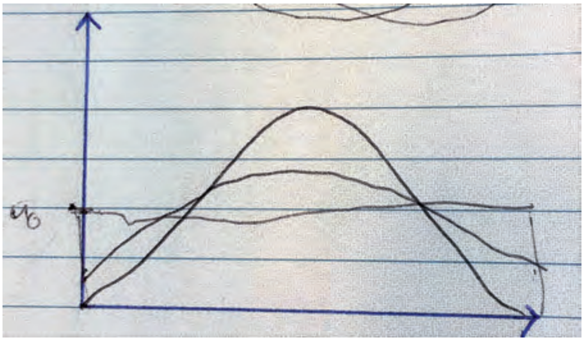

Figure 2.4: Graphs sketched by student H.

should always be kept at zero at the boundaries. However, while reasoning about the limit behavior of the analytic solution when time goes to infinity, he adds the following:

> *Interviewer: What are you thinking?*
> *Student H: Ehm yeah, what the value at the boundaries is, so if it is zero or if it is u_0 when you fill in $x = 0$ here [in the limit $t \to \infty$ of the analytic solution]. It doesn't matter then, because you just get u_0. That does not correspond to the boundary conditions I chose before.*

In the last sentence of this excerpt, we observe a conflict between his answer from before (of the form $u(0, t) = 0$) and the mathematical limit behavior of the analytic solution (of the form $u(0, \infty) = u_0$). To stimulate further reasoning, the interviewer asked the student to draw graphs at intermediate times t, to which the student added the other two graphs in Figure 2.4. Afterwards, the student concluded the following:

> *Interviewer: And you mentioned before that the boundaries should stay at zero. That does not follow from the analytic solution, but does it makes sense physically that they stay at zero?*
> *Student H: Yeah the boundary conditions... but... Yeah that is because for all times there is...*
> *Interviewer: So what are you saying?*
> *Student H: ... The boundary conditions are not correct I think.*
> *Interviewer: Why not?*
> *Student H: Well, ehm, no particles can flow out, so then I thought it should just be zero at the boundaries for all ehm for all times t. But yeah, maybe that nothing can flow out when it is non-zero, but I would find that illogical.*

In this last excerpt, the student acknowledges the conflict between his physics intuition and the mathematical limit. He thinks that when a tube is closed and no particles can flow out, that ‘it’ should be equal to zero at the boundaries for all times, which contradicts with his mathematical line of reasoning which says it should be u_0 at the boundaries. Note that he does not specify the ‘it’ as being concentration or amount of particles. This student used correct mathematical reasoning about the limiting behavior of the analytic solution when $t \to \infty$, but was unable to connect this to a correct and complete physical interpretation, which resulted in not finding the correct boundary conditions for the problem.

Correct physical and correct mathematical understanding, but difficulties connecting both

It is possible to have a good understanding of both the physical and mathematical aspects used in the reasoning, but to have difficulties connecting both. An example of this is observed in the reasoning of student J.

Because the student failed to formulate an answer, the interviewer decided to write down two suggestions: $\frac{\partial u}{\partial t} = 0$ and $\frac{\partial u}{\partial x} = 0$. He asked the student which option is correct and which is not and why. The following excerpt shows the reasoning of the student about option $\frac{\partial u}{\partial x} = 0$.

[interviewer writes $\frac{\partial u}{\partial x} = 0$]
Student J: Let’s see... so then it stays flat. Yes. So. Wait, what does this mean? What is the physical meaning of this possible boundary condition? Well, it stays flat... hey... If I look at this [the graph he sketched before] I think it should be possible because hmm... but that is, does it... Yes! It is possible I think, because it is definitely flat there yes yes, okay, that is it.
Interviewer: Yes, and do you know this just because it is flat?
Student J: Well, because we have such a cosine ehm... then it has to be flat at the ends... Yes okay and if I say now... So there is stated that the tube is closed at the boundaries, so what does that mean? No particles can pass the boundary, yes.

When the interviewer suggests $\frac{\partial u}{\partial x} = 0$, student J immediately says "then it stays flat", which indicated a connection to the mathematical meaning of a derivative being equal to zero: a horizontal tangent line (line 2). Further on, he also reflects on the physical situation stated in the problem statement (lines 9 and 10). In this excerpt, both the mathematical and physical insights he mentions are correct, but he does not make the connection between them. Later on in the interview, the student elaborates on this:

Student J: But ehm, I did not think that this [$\frac{\partial u}{\partial x} = 0$] would always apply if you know nothing can ehm escape. And I can't really... well yeah probably after reasoning for a long time I would be able to find out why this always applies, but it's not something that is immediately clear to me.

Here, the student admits not seeing the connection between his mathematical and physical understanding from before, which shows that the blending did not happen.

Unproductive blending

Sometimes, students are actively blending mathematical and physical elements in their reasoning, but it does not help them in their search for the answer. This can be called unproductive blending.

Student J, for example, is intensively looking for ways to describe the physical system mathematically in his search for the boundary conditions.

Student J: So the question is what happens with those boundary conditions... Oh yeah if you take the integral, it should always be u_0...

This student recognizes that the number of particles in the system (u_0) is conserved and links this to the use of an integral. Further on in the interview he continues with this idea:

Student J: Oh! Maybe you can... Maybe you can show this [the tube being closed] by giving that integral... But it does not really say something about the boundaries... No yeah yeah okay, that is maybe how I would show that the tube is closed, by taking the integral from zero to one... and you want that to say something about the boundaries...

The student is clearly blending throughout this reasoning: he connects the closed tube, and so conservation of particles, to a constant integral of $u(x,t)$ over x. However, the blending is not productive to find the answer to the problem.

2.7 Discussion

The blending of mathematics and physics is a general aim of physics education. We want students to be able to think mathematically about a physical situation and go back and forth between those worlds. Boundary conditions play a key role when describing a physical system mathematically in terms of a partial differential equation. Therefore, they are a powerful topic to investigate the blending of mathematics and physics in student reasoning. In this paper, we report on findings from an interview study in which we asked students to mathematically formulate boundary conditions for a given physical situation. The results show that this is a hard task for them. We identified a set of difficulties that can be used as a starting point to design student activities. We demonstrated the power of the conceptual blending framework to categorize these difficulties in the mathematics, the physics and the blended space.

In the physics space (see section 2.6.1), some students had difficulties formulating the correct physical quantity that is described by the diffusion equation. We identified confusion between concentration and number of particles. First, it is known from literature that density or concentration is a difficult concept in general (C. Smith, Maclin, Grosslight, & Davis, 1997; Xu & Clarke, 2012). Second, the fact that we are considering diffusion along only one dimension of the rod might make it harder for students. Third, the problem statement itself may also have played a role (see Figure 2.2). We called the initial number of particles in the tube u_0, which might have led students to believe that the physical quantity u must stand for number of particles instead of concentration. Another factor in the problem statement is that we chose the length of the tube to be one meter, which means that u_0 represents both the number of particles and the value of the overall concentration in the tube. Choosing a different length in the problem statement may help students to explicitly think about the difference between concentration and amount of particles. Classic textbooks often use 'easy' values in their problems to make the mathematical calculation easier, but we observe here that this may not always be the best for the learning process.

With respect to reasoning in the mathematics space (see section 2.6.2), we identified several aspects of the reasoning process that were difficult for students. We clustered these aspects in two groups.
The first group contained findings related to the mathematical meaning of boundary conditions. Some students did not correctly understand the role of boundary conditions in the process of solving a partial differential equation. They were convinced that the boundary conditions could be found by evaluating the solution of the partial differential equation for the coordinates of the spatial boundaries of the system. Concerning the form of the boundary conditions, we opted for $\frac{\partial u}{\partial x} = 0$ in the interview question, which expresses a closed tube where nothing can pass through the boundaries. However, almost half of the students were focused on boundary conditions of the form '$u = 0$'. The focus on '$u =$' might originate from thinking that boundary conditions provide the solution at the boundary, which is not fully correct. Boundary conditions provide a condition satisfied by the system at the boundary. The persistence of the '$= 0$' might originate from limited and typical examples in class. In general, many examples in textbooks or classes have boundary conditions of the form '$= 0$' because it simplifies

the mathematical complexity of the problem. From this, we may expect difficulties with more general boundary conditions.
The second group contained findings related to different aspects of reasoning with functions of two variables. First, we observed that some students did not seem to be aware of the two different possible derivatives. These students would typically talk about 'the' derivative, without specifying the variable. This continues in a second difficulty: some students could not distinguish between the meaning of different partial derivatives. Third, we observed that when one of the variables is kept constant, some students think exclusively in terms of the resulting function of one variable and forget about the original function of two variables. The two discussed examples can be interpreted as the graphical and symbolic version of the same difficulty. Graphically, one student struggled with combining multiple graphs of $u(x, t = c)$ as a representation of a function of two variables. Symbolically, a student struggled with the order of operations: the student first plugged in a fixed value for the variable and then tried to take the derivative with respect to that variable, while it should be the reversed way. These examples confirm the findings of Thompson, Manogue, Roundy, and Mountcastle (2012). They developed and assessed curricular materials designed to help address student understanding of the mathematics used in thermodynamics, particularly the mathematics of partial derivatives and differentials. They found that students have difficulties with what it means to keep one or more variables fixed while taking the derivative with respect to a different variable.

To our knowledge, little research has been done on student understanding of functions of two variables. Martinez-Planell, Gaisman, and McGee (2015, 2017) used APOS theory to investigate undergraduate students' understanding of concepts related to directional derivatives, partial derivatives, tangent planes, and their interrelationship. APOS, which stands for Action, Process, Object and Schema, is a framework for research and curriculum development in mathematics education (Arnon et al., 2014; Dubinsky & Mcdonald, 2002). Among other findings, Martinez-Planell and colleagues observed that some students did not show flexibility in the use of variables, and they had difficulties deciding which of the variables are dependent and which are independent.

We categorized reasoning as blended when it explicitly connected a mathematical and a physical idea or concept. We identified four ways in which blending failed (see section 2.6.3).
A first way was when students selected elements in the input spaces but connected them in the wrong way, which leads to an incorrect blend. In our sample, students often connected the closed tube (physical system) to a partial derivative with respect to time equal to zero (mathematical description), which is incorrect. In these examples, they fail to explicitly translate the derivative with respect to time to its physical meaning. Students generally made quite correct statements about the physical situation (e.g. the tube is closed, particles cannot flow in or out), but then they connected these to $\frac{\partial u}{\partial t} = 0$ with limited explanation. Two factors play a role here. First, the physics space of these

students is typically incomplete. They are mostly repeating what is already stated in the problem. Deeper physical understanding, e.g. being aware that the concentration can change at the boundaries, is missing. Second, the mathematics space is incomplete. They use a partial derivative with respect to time in their answer and reasoning, but they do not explicitly show conceptual understanding, e.g. a derivative as change of a variable with respect to another variable, or a derivative as slope. In conclusion, stimulating more elaborate reasoning may help students in completing their input spaces, which may result in a better selection of elements for the blending process.

Second, we observed that sometimes students express a correct line of reasoning in the mathematics space, but experience a cognitive conflict with their incorrect physical intuition. In the discussed example (see section 2.6.3, it is interesting that even though the mathematics clearly showed the incorrectness of his physical answer, the student held on to his own idea and expressed doubt about the physical consequences of his mathematical findings.

Third, we showed that it is possible to have a good understanding of both the physical and the mathematical side of the reasoning separately, but having difficulties connecting both. This is in line with earlier findings of Bollen et al. (2016). They established that correct information in the input spaces does not automatically result in the intended new meaning after blending, which shows that student difficulties may sometimes not be due to a lack of prior knowledge, but may stem from improper blending. In this case, we observe input spaces containing the necessary elements to theoretically form the desired blend. However, the blended space is empty because students fail to connect their understanding.

Fourth, we observed that it is impossible to formulate an answer if students fail to select the necessary elements in the input spaces for their blend. We showed examples where students focused on the global aspect of the closed tube (conservation of number of particles), instead of the local aspect (particles cannot flow through the boundary). In general, these are promising examples of blended reasoning; the student is actively combining both his physical and mathematical understanding in search of the solution to the problem. However, the blending is unproductive because it does not lead to an answer to the problem.

2.8 Implications for teaching

In summary, it seems that students are not always prepared to blend physical and mathematical knowledge when reasoning about boundary conditions of physical systems. So, the next question is ‘How can we prepare them in a better way?’

In general, we found that both the physical and mathematical components of the reasoning are required for a successful blending in the process of formulating boundary conditions for a described physical situation. Both the physics and mathematics space separately have to be sufficiently developed in order to overcome difficulties in the separate input spaces. It is important that students are aware of the physical quantities represented in the mathematical model. Explicitly defining variables can resolve this problem and make students aware of the physical process connected to the mathematical model (Rowland, 2006). In the process of mathematically modelling a physical situation, instructors should give explicit attention to the assumptions and abstractions that are made, e.g. the abstraction of the tube to a one-dimensional system, as these are not always trivial to students (e.g. Crouch & Haines, 2004). Concerning the mathematics space, especially the difficulties with functions of two variables are recurring in our sample. As an instructor, it is important to be aware that the step from functions depending on one variable to functions depending on two variables is not trivial for students and probably needs more explicit attention.

Moreover, there should be explicit attention to the connections between the physics and mathematics spaces in order to form a blend. Formulating questions that cue those connections may help. We recommend to use a variety of boundary conditions in class, both of the forms $u(x,t) = c$ and $\frac{\partial u}{\partial x} = c$ and with c not only equal to zero, in order to broaden students' experience. It is important to give explicit attention to the physical interpretations of those boundary conditions and to the mathematical conceptual understanding of the partial derivative. We observed in the interviews that graphs often helped students in understanding and connecting both the physical and mathematical side of the boundary conditions. Stimulating this graphical reasoning, and using graphs in teaching this topic may help students in building their understanding.

2.9 Future research and limitations of the study

This paper is part of a larger project in which we investigate student reasoning with boundary conditions in the description of processes of diffusion and heat flow, and we intend to design an intervention targeting this. In this study, we performed a first exploration of student difficulties with the topic. It is important to acknowledge the limitations in the study design and implementation described in this article.

The formulation of the interview question could have had an impact on student answers, which is a limitation that we have to keep in mind while interpreting the results. First, the question forced students to go from a physical description to a mathematical description. The physical situation was described in the problem statement, which could partly explain the low number of difficulties identified in the physics space. Moreover, it is possible that the direction implied in the question influenced the elements that were activated by the students, and which aspects of blending were observable. Therefore, we will explore different directions in interview problems in follow-up research: from physics to mathematics, but also vice versa. Second, the problem statement itself might have fostered some of the difficulties. It was not the best choice to use u_0 for a number of particles, while u stands for concentration in the differential equation. Moreover, because of the choice of length of the tube (one meter), the 'u_0' in the problem statement could be interpreted both as number of particles and as overall concentration in the tube. As mentioned in the discussion, these aspects might have led to more confusion in distinguishing between number of particles and concentration.

Because of the small sample size, the difficulties listed here are only a first step in investigating student understanding of boundary conditions. More research should be conducted to come to a more complete overview. Moreover, because of the small sample size, we could not connect any conclusions to differences in students' performance or the university they study at.

In this study, we used the blending framework as an analytical lens to categorize difficulties in the reasoning process of students. However, conceptual blending also has potential that exceeds categorization. In future research, we aim to extend the use of the blending framework to visualize the entire reasoning and not only the difficulties in student reasoning.