Blending of mathematics and physics

van den Eynde, Sofie

DOI:
10.33612/diss.177955315

Publication date:
2021

Chapter 1

Introduction

> *"It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of a mathematical theory of great beauty and power, needing quite a high standard of mathematics for one to understand it. You may wonder: why is nature constructed along these lines? One can only answer that our present knowledge seems to show that nature is so constructed. We simply have to accept it. One could perhaps describe the situation by saying that God is a mathematician of a very high order, and He used very advanced mathematics in constructing the universe. Our feeble attempts at mathematics enable us to understand a bit of the universe, and as we proceed to develop higher and higher mathematics we can hope to understand the universe better."*

Paul Dirac (1963)

As this quote from Paul Dirac illustrates, physics and mathematics have been deeply related ever since their beginning. This mutual influence has played a crucial role in the development of both disciplines (Galili, 2018; Kjeldsen & Lützen, 2015). The way mathematical structures match physical phenomena and can be used to make predictions about the physical world appeals to the imagination. As Uhden et al. (2012) state, the role of mathematics in physics can be described in terms of multiple aspects: it serves as a tool (technical function), it acts as a language (communicative function), and it provides a way of logical deductive reasoning (structural function). Moreover, mathematics was not only crucial for the development of physics, a lot of mathematical concepts have been derived from the study of nature (Karam, 2015; Redish & Kuo, 2015). Mathematics is deeply interwoven with physics, and sometimes they are even inseparable. However, in an educational context, this mutual relationship is not always articulated that way. In physics education, mathematics is often seen as a tool to describe physical phenomena and do calculations. In mathematics education, physics is often only seen as a possible context to illustrate and apply abstract mathematical ideas. This dichotomy creates difficulties for students. They find it hard to integrate their knowledge of both topics. Despite plenty of research on the learning and instruction of mathematics and physics as such, the question of how to bring together mathematics and physics in a way that reaches most students remains unsolved (e.g. Karam, 2015; Pospiech, Michelini, & Eylon, 2019).

There exists a lot of research on student reasoning at the interface of mathematics and physics at secondary school level and the introductory undergraduate level (e.g. Billings & Klanderman, 2000; Brasell & Rowe, 1993; Carli, Lippiello, Pantano, Perona, & Tormen, 2020; Ceuppens, Bollen, Deprez, Dehaene, & De Cock, 2019; De Bock, Van Dooren, & Verschaffel, 2015; Goldberg & Anderson, 1989; McDermott & Redish, 1999; Tuminaro & Redish, 2004; Wemyss & van Kampen, 2013). The trend is often very similar: the use of mathematics in physics is difficult for students. One might expect these results at the secondary school level; not all students are as interested in mathematics and physics for example. However, in recent years, an increasing number of studies investigated student reasoning at the advanced undergraduate and graduate level, often with students who are majoring in physics (e.g. Bollen, van Kampen, Baily, & De Cock, 2016; Gupta, Redish, & Hammer, 2007; Modir, Thompson, & Sayre, 2019; Ryan, Wilcox, & Pollock, 2018; Wilcox, Caballero, Rehn, & Pollock, 2013; Wilcox & Pollock, 2015). Remarkably, the trends are very similar again. At this level, students are typically quite good at executing mathematical calculations, which refers to the technical role of mathematics in physics. However, understanding mathematical concepts and correctly applying them in a physics context is still causing difficulties, even at this advanced level. This indicates a deeper problem with understanding the structural role mathematics plays in physics.

In this dissertation, we aim to take a step forward in bringing mathematics and physics closer together for students. We study the interplay of mathematics and physics in student reasoning to develop a deeper understanding of reasoning processes when students combine mathematics and physics (theoretical aim). This knowledge should then form the basis for developing an instructional approach that scaffolds the blending of mathematics and physics in student reasoning (practice-oriented aim). The perspective that teaching should be research-based is central to the field of Physics Education Research (PER), which is founded upon the idea that teaching is also a science (McDermott, 2001).

In this first chapter, we provide context from literature to add some perspective to the choices we made in our own research project. First, we dig a bit deeper into the relation between mathematics and physics in section 1.1. This brings us to the specific context of our project in section 1.2: the heat equation, which is a partial differential equation that models the evolution of the temperature over time in a system. Next, we give an overview of some theoretical frameworks used in PER to study the interplay of physics and mathematics in student reasoning and problem solving (section 1.3). From this, we explain our choice for the conceptual blending framework as a lens in this project in section 1.4. To conclude this chapter, we give an overview of the research aims (section 1.5), the educational context (section 1.6) and the structure of the dissertation (section 1.7).

1.1 The role of mathematics in physics and physics education

As we introduced before, there is a deep entanglement of physics and mathematics (Galili, 2018; Kjeldsen & Lützen, 2015). If we look for example at some of the leading scientists of the seventeenth and eighteenth centuries, like Newton, Euler, Lagrange, and so on, it is hard to distinguish them into mathematicians and physicists. Mathematics has been very important for the development of physics. The notion of explaining a physical phenomenon by a physical mechanism was gradually complemented by the need to represent it by a mathematical formulation (Gingras, 2001, p.385). As Uhden et al. (2012) state, the role of mathematics in physics is twofold. There is the *technical* role, the algorithmic use of mathematics, where mathematics is seen as an external instrument, a tool without any physical content. Some examples are rote calculations, manipulations of variables and units. Second, there is the *structural* role, where mathematical structure penetrates into the construction of the physical concept itself. Mathematics partially defines what physics is because of the way of logical deductive reasoning it provides. It structures physical thought as it serves as a reasoning guide in the path to abstraction and thinking about new phenomena. Sometimes, different physical phenomena are described by similar or identical mathematical structures, e.g. the heat equation, which plays a central role in this dissertation, can be used to describe both heat transfer and particle diffusion. The equality in mathematical form reveals the analogy between both phenomena, while leaving space for their contextually different interpretations. Mathematics also provides predictive power in physics. By working within the inner structure of mathematics, it is possible to predict new physical phenomena (Galili, 2018). Considering the other direction, as we mentioned before, some mathematical domains were developed from a physical context (Redish & Kuo, 2015). The origin of calculus for example is almost inseparable from the description of motion, differential equations were developed in the context of classical mechanics problems, the development of vector calculus is related to the mathematization of electromagnetism, Fourier analysis was motivated by problems of waves in strings and propagation of heat, and so on (Doorman, 2005; Uhden et al., 2012). This all indicates that mathematics and physics are deeply interwoven at a structural level and that the role of mathematics in physics goes beyond the instrumental aspect.

From an epistemological point of view, mathematics and physics are of a different nature. Dirac (1939) stated in one of his lectures: “The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature.” The physicist is not primarily interested in the mathematical structures that underlie these generalizations, but the physical language uses mathematics. Physicists load physical meaning onto symbols and numbers in a way that mathematicians usually do not and that is an important difference between both disciplines (Redish & Kuo, 2015). The symbols in physics bring physical properties along with them, with physical units, and because of these units, physics equations have a particular structure (Redish, 2017). The description of advanced physical concepts is done in terms of mathematical structures. As such, proficiency in mathematics is required to understand physical phenomena, and being able to combine both fields is a prerequisite to become more proficient in physics. However, the mathematics in physics requires much more than the straightforward application of rules and mathematical procedures (technical role). It also requires being able to connect the mathematical operations in the equation to their physical meaning (structural role).

The interplay between mathematics and physics is important at all levels of physics *education*, but at the advanced level, the role of mathematics becomes even more important. As mentioned before, many studies have investigated student difficulties at the mathematics-physics interface, and research shows that students struggle with the mathematics used in physics at all levels of education (e.g. Billings & Klanderman, 2000; Bollen et al., 2016; Brasell & Rowe, 1993; Carli et al., 2020; Ceuppens et al., 2019; De Bock et al., 2015; Goldberg & Anderson, 1989; Gupta et al., 2007; McDermott & Redish, 1999; Modir et al., 2019; Ryan et al., 2018; Tuminaro & Redish, 2004; Wemyss & van Kampen, 2013; Wilcox et al., 2013; Wilcox & Pollock, 2015). At the advanced level, the mathematical structure is often very deeply interwoven with the physical concepts, and it becomes hard to separate the two disciplines. Therefore, students should be proficient in blending their mathematical and physical knowledge, form connections between mathematical and physical concepts and use these to get to new insights. The use of the word ‘blending’ is associated with the theoretical framework we use in this dissertation to investigate the students’ reasoning and will be introduced further on in this chapter.

Traditionally, mathematics is seen as a prerequisite for the learning of physics. This is reflected in the way typical physics curricula at university level are structured, teaching students mathematics first in order to provide the tools for the subsequent physics courses. However, we know from research – and from teaching and learning experience – that proficiency in mathematics does not guarantee success in physics. On the other hand, we also know that the lack of mathematical skills is considered as one of the main reasons for students’ difficulties in physics courses (Karam, Uhden, & Höttecke, 2019). Redish and Kuo (2015), as teachers, were often surprised by how little mathematics students seem to know in their physics classes. They wondered why so many students seem unable to use mathematics in physics, despite their success in prerequisite math classes. Of course, sometimes the problem is that students still have difficulties with things they should have learned in their mathematics classes. However, from their research and teaching experience, Redish and Kuo suspected that the issue is more subtle and has something to do with the difficulties students have in translating the way symbols are used to make meaning differently in mathematics and science. This indicates that the classical solution to teach students more mathematics, hoping they take this with them when studying physics, is not sufficient (Eichenlaub & Redish, 2019; Greca & de Ataíde, 2019). Even if students have learned the relevant mathematics, they still need to be given the opportunity to learn a component of physics expertise not presented in math classes: tying those formal mathematical tools to physical meaning. Using mathematics in physics is not the same as doing mathematics. It has a different purpose. This interplay between mathematics and physics, which connects back to the structural role of mathematics in physics, is not often taught to students in an explicit way even though we know that this is not self-evident for them (Greca & de Ataíde, 2019). Therefore, it is important to address the differences between mathematics and physics explicitly with the students and help them in learning to blend physical and mathematical meaning (Karam et al., 2019; Redish & Kuo, 2015).

This illustrates the ongoing debate on the role of mathematics in a physics curriculum, which could be seen as a spectrum, with on one side a curriculum in which mathematics courses taught by mathematicians precede physics courses taught by physicists, which is the more traditional way, and on the other side a curriculum where mathematics and physics are taught in an integrated way by physicists. Most universities take a position in between these two extremes. Our research is conducted at KU Leuven (Belgium) and the University of Groningen (The Netherlands). We will discuss their curriculum in section 1.6.

1.2 The heat equation as a context where mathematics and physics are interwoven

Among the variety of mathematical structures being used to model the physical world, differential equations are of the most important ones. Many phenomena can be described in terms of a relation between a time-dependent variable and its rate of change. In mathematical terms, such a relation is an (ordinary) differential equation, i.e. an equation having a function as unknown and expressing a relation between the derivative of that unknown function and the function itself. For example, the function $x(t) = A\cos(\omega t + \phi)$ describing the motion of a harmonic oscillator is a solution of the ordinary differential equation $m\frac{d^2x}{dt^2} = -kx(t)$. Partial differential equations are differential equations that contain an unknown multi-variable function and its partial derivatives. In physics, they are used to model a wide variety of phenomena in different domains such as acoustics, heat transfer, fluid dynamics, electrodynamics, and quantum mechanics, because they are particularly suitable to describe multi-dimensional systems (both in spatial and time dimensions).

In order to understand a differential equation in its physical context, one needs good understanding of the physical and mathematical concepts involved and of the way they are related to each other (Pospiech, 2019). Hence, reasoning about differential equations provides good opportunities to investigate the blending of mathematical and physical concepts. In this section, we give a brief overview of the content matter that is relevant for the rest of this dissertation.

We focus on the heat equation, which is a mathematical model of how heat transfers in a medium over time. A mathematical model is a description of a system in terms of mathematical concepts and structures (Gilbert & Boulter, 2012; Malvern, 2000). This model can be used to study the effects of different components and to make predictions about the behavior of the system. In order to find a balance between a model that works and a model that is simple enough to work with, one needs to make simplifications and assumptions. Therefore, the modelled situation is often idealized. In this dissertation, we will focus on the one-dimensional heat equation, in which we assume that heat can only flow in one spatial direction in a homogeneous, isotropic solid (i.e., a material in which thermal conductivity is independent of location and direction).

If we consider a one-dimensional system of length L in which the temperature at position x and time t is described by $T(x,t)$, the evolution of the temperature in the system is modelled by the following partial differential equation:

$$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial x^2}(x,t)$$

for $0 < x < L$ and $0 < t < \infty$. In this equation, α is the thermal diffusivity [m^2/s], a measure of the rate at which heat can spread in a specific medium.

This equation is also referred to as the diffusion equation, often in the form

$$\frac{\partial u}{\partial t}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t).$$

In this case, the equation models particle diffusion in one dimension, with $u(x,t)$ expressing the concentration in terms of position and time and α a measure of the rate at which particles can spread in a specific medium. In what follows, we will continue the description in the context of heat transfer, but everything can be analogously formulated in terms of particle diffusion.

The equation relates the quantities $\frac{\partial T}{\partial t}$, the rate of change in temperature with respect to time at a certain position, and $\frac{\partial^2 T}{\partial x^2}$, the concavity of the temperature distribution at that position. The concavity essentially compares slopes of the tangent lines in nearby points, or in other words compares temperature gradients $\frac{\partial T}{\partial x}$ in nearby points. To understand why these two terms are related, we first have to understand how the temperature gradient $\frac{\partial T}{\partial x}$ is related to the heat flow through that point, which is defined as the amount of heat passing through a point per unit of time, expressed in Watts (Farlow, 1993; Folland, 1976). Depending on the textbook, heat flow can also be referred to as heat rate (e.g. Hahn & Özışık, 2012).

Fourier's law, also known as the law of thermal conduction, relates the temperature gradient to heat flow. It states that the heat Q [J] that flows through a unit area per unit of time is proportional to the negative temperature gradient. In the one-dimensional differential form, this law can be written as:

$$\frac{\partial Q}{\partial t}(x,t) = -kA_x\frac{\partial T}{\partial x}(x,t)$$

where $\frac{\partial Q}{\partial t}(x,t)$ is the heat flow, i.e. the amount of heat passing through a point per unit of time [W]. k is the thermal conductivity [$\mathrm{Wm^{-1}K^{-1}}$], which is a measure of how well the material conducts heat, and is related to the thermal diffusivity α as $\alpha = \frac{k}{\rho c_p}$ with ρ the density [$\mathrm{kgm^{-3}}$] and c_p the specific heat capacity [$\mathrm{Jkg^{-1}K^{-1}}$] (Farlow, 1993). In this dissertation, we consider k to be constant. A_x [$\mathrm{m^2}$] is the cross-sectional area perpendicular to the x-direction in which the heat can flow, and is also considered constant in this dissertation.

Fourier's law shows that the temperature gradient, which essentially expresses the local temperature difference, is a measure of the heat flow. Therefore, if the temperature gradient increases, which means that the temperature difference between nearby points increases, more heat will flow through that point per unit of time. The minus sign is introduced to make the heat flow a positive quantity in the positive coordinate direction (i.e., opposite to the temperature gradient) (Hahn & Özışık, 2012). This is represented graphically in Figure 1.1 by the slopes of $T(x)$ in different points of the system. The slope is negative, which means that the heat flow will be to the right (visualized by the arrows beneath the x-axis). The steeper the tangent line, the stronger the heat flow will be (visualized by the length of the arrows).

Fourier's law can be related back to the heat equation to further explain how $\frac{\partial T}{\partial t}$ and $\frac{\partial^2 T}{\partial x^2}$ are related. By calculating the concavity $\frac{\partial^2 T}{\partial x^2}$ at a point, one compares how much heat is flowing in and out of that point. This can also be seen in Figure 1.1. The concavity at point x_1 is positive. On the left of x_1, the slope of $T(x)$ is steeper than than it is on the right of that point. This means that more heat is flowing in than out. In the figure, the heat flow left and right of x_1 is visualized by arrows that indicate direction and strength. As a result, the temperature at x_1 will increase over time. This process continues until an equilibrium state is reached, which means that the temperature at every point will not change over time anymore. This can only happen when the concavity is zero, which

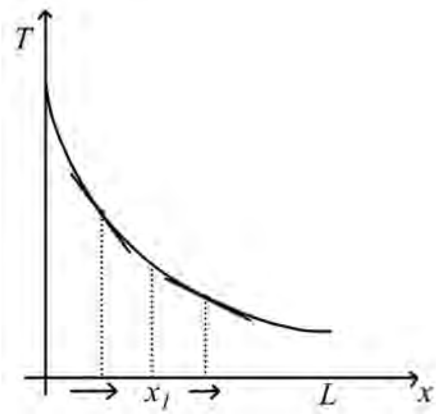

Figure 1.1: Graphical representation of a temperature distribution $T(x)$ at a certain time t. The arrows visualize the heat flow through points nearby x_1 (direction and strength).

means that the temperature distribution will have the form of a linear function. If the slope $\frac{\partial T}{\partial x}$ is constant everywhere, there is as much heat flowing in as out of every point and the temperature stays constant over time.

To fully describe the evolution of the temperature in a physical system, the initial temperature distribution $T(x, t = 0)$ and the conditions describing what happens at the boundaries of the system are needed. Boundary conditions refer to the conditions physical quantities must satisfy at the boundaries of the system during the whole process. There are different types of boundary conditions, but we will limit ourselves to Dirichlet and Neumann boundary conditions, as these are the types that are typically introduced in a course on partial differential equations at the undergraduate level. A Dirichlet boundary condition has the form $T(x_b, t) = c$, with c being a constant. In the context of the heat equation, it means that the boundary $x = x_b$ has a constant temperature c at all times, which is often explained as the boundary being in contact with a heat reservoir of a constant temperature c. The contact with the heat reservoir can be considered as an idealized situation in which the reservoir is infinitely large and keeps a constant temperature regardless of how much heat flows in or out of it through the boundary with the system. A Neumann boundary condition has the form $\frac{\partial T}{\partial x}(x_b, t) = c$ with c being a constant, and is related to the heat flowing through the boundary $x = x_b$ over time. If c is equal to zero, this indicates that no heat can flow through the boundary, and thus it is thermally isolated.

1.3 Theoretical frameworks to study the role of mathematics in physics

The interplay of mathematics and physics in student reasoning has been studied using different frameworks, each with a different focus. A theoretical framework can guide our thinking and alert us to things we might otherwise miss (Redish, 2014). It provides a handhold for systematic observation and analysis. In this section, we give a brief overview of three groups of frameworks used in literature to study the interplay of mathematics and physics in student reasoning. We first give a brief overview of the ideas of *transfer*. Second, we introduce the *mathematical modelling cycle* and third we discuss some frameworks from PER focusing on *mathematical modelling in physics*. It is important to note that theoretical frameworks are not necessarily mutually exclusive and might sometimes even overlap or complement each other. One framework might be more suitable than the other, and it is important to make an informed choice depending on the research aims, data, context, and so on. We explained for each presented framework why it did not match well enough with our research aims. In the next section, we introduce the framework that will guide the rest of this dissertation: the conceptual blending framework (Fauconnier & Turner, 2003b).

1.3.1 Transfer and resources

Generally, transfer can be defined as the ability to extend what has been learned in one context to new contexts (Byrnes, 1996). The word 'context' is defined in a broad way: this can range from one problem to another, or from one topic to another, to even from one discipline to another. Without an adequate level of initial learning, transfer cannot be expected (Bransford, 2000).

In a more traditional view, transfer brings up the idea of knowledge as a thing that is acquired in one context and imported (or not) to another. Hammer, Elby, Scherr, and Redish (2005) built their own view on transfer, in terms of *activation* as the central construct. They view students' thinking as involving cognitive resources, and they frame their questions in terms of when and how students activate those resources. In PER, this view is often referred to as the resources framework. It builds on the idea of transfer but emphasizes the active construction of knowledge rather than 'carrying it' as a finished entity from one context to another.

PER often focuses on how students transfer or activate knowledge that they should have learned in their mathematics classes to or in their physics classes. However, this approach does not account for the idea that learning a concept in a physical context could improve the mathematical understanding of that concept, or that physics and mathematics could be learned in an integrated way. Making the bridge to our own

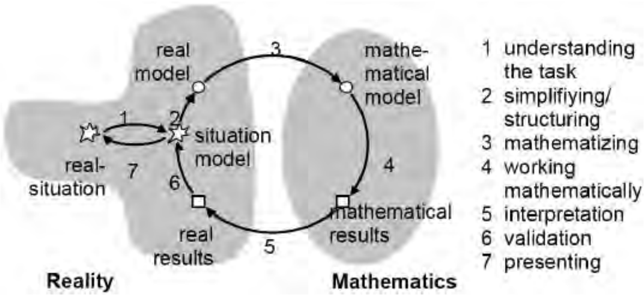

Figure 1.2: Visualization of the mathematical modelling cycle (Blum & Leiß, 2007).

research topic, we are interested in investigating how students combine mathematical and physical knowledge in their reasoning. Doing this from a transfer or resources perspective would entail a direction from mathematics to physics and would not account for a back and forth movement between these two domains, or a symbiosis between them, and thus does not match well enough with our research aims.

1.3.2 The mathematical modelling cycle

Mathematical modelling is generally understood as 'the process of translating between the real world and mathematics in both directions' and has always been an important topic in mathematics education research (MER) (Blum & Borromeo Ferri, 2009). Modelling tasks are usually difficult for students and one of the reasons for that is that many competencies have to be combined. We give a brief introduction to the modelling cycle of Blum and Leiß (2005).

As shown in Figure 1.2, the modelling cycle consists of seven steps. First, the problem situation has to be understood by the problem solver, who in this process constructs a situation model. Next, the situation has to be simplified, structured and idealized, leading to a real model of the situation. The real model is then mathematized into a mathematical model. Working mathematically (calculating, solving the equations, etc.) yields mathematical results, which are next interpreted in the real world as real results. A validation of these results may show that it is necessary to go around the loop a second time. Blum and Borromeo Ferri (2009) state that the advantage of this particular model is that all the seven steps are essential stages in modelling processes, as well as potential cognitive barriers for students, which makes it interesting to study both the process of modelling and the difficulties with it.

In this dissertation, we focus on how students combine mathematical and physical knowledge in their reasoning. Our focus would therefore be mostly on only two

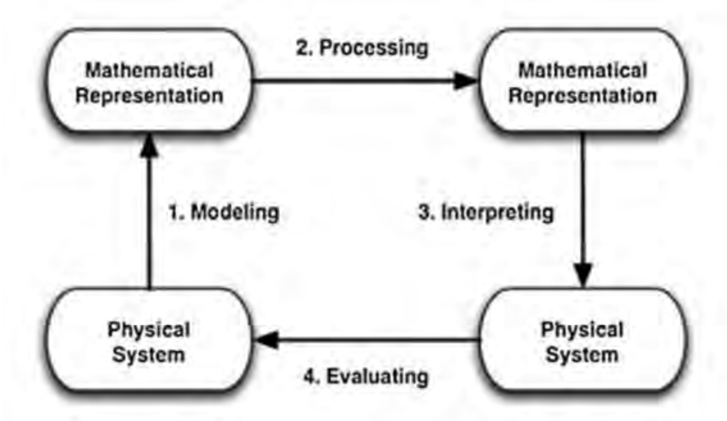

Figure 1.3: Representation of physical modelling according to Redish (2006).

steps in the modelling cycle, namely steps 3 (mathematizing) and 5 (interpretation). An important part of the modelling cycle focuses on translating the real world to an idealized situation that can be modelled mathematically (steps 1, 2 and 7). In our research, however, we will not focus on this aspect, which is already contained in the construction of the heat equation itself. We will also not focus on the technical role of mathematics, represented in step 4 (working mathematically) in Figure 1.2. Therefore, we conclude that there is not a good enough match between our research aims and the mathematical modelling cycle.

1.3.3 Mathematical modelling in physics

Another set of theoretical frameworks focuses more specifically on mathematization and mathematical modelling *in physics*. We briefly discuss two frameworks.

We start with the simplest one, shown in Figure 1.3. Redish (2006) describes the modelling process as moving in and between two domains: the physical system and the mathematical representation. It can be considered as a simplification of the mathematical modelling cycle from Blum and Leiß, e.g. steps 1, 2 and 3 from the mathematical modelling cycle are reduced to step 1 in Redish's model. In this view, 'modelling' is the step where students construct a mathematical representation for a physical system (e.g. setting up an integral). The mathematical processing (e.g. calculating the integral) happens in the mathematical domain, away from the physical context and gives a possible result (Redish & Kuo, 2015). The obtained mathematical results then need to be interpreted within the physical model. The physical results are finally validated either according to the problem statement or to the physical situation.

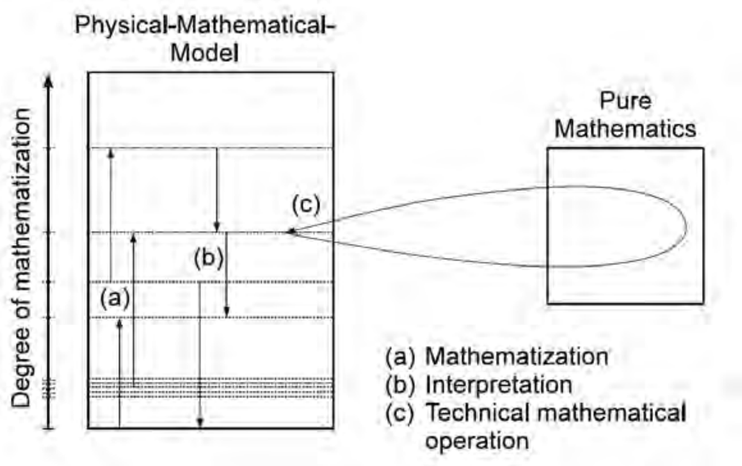

Figure 1.4: Model of the use of mathematics in physics by Uhden et al. (2012).

In instruction, the process step tends to receive a lot of attention while students are rarely asked explicitly to interpret their results and even less often to evaluate whether the initial model is adequate (Redfors, Hansson, Hansson, & Juter, 2014; Redish, 2014).

Uhden et al. (2012) developed a more sophisticated framework, based on the mathematical modelling cycle discussed in the previous subsection (Blum & Leiß, 2005). Figure 1.4 shows their model, which incorporates the entanglement of mathematics and physics and also enables to distinguish between the technical ("pure mathematics" in the framework) and structural role ("physical-mathematical-model" in the framework) of mathematics in physics. Each level in the physical-mathematical-model corresponds to a degree of mathematization. Moving upwards means to make an abstraction from the physical system. As students more formally describe the physical system, the level of mathematization ((a) in Figure 1.4) increases. We can take the concept of velocity as an example: the idea of "distance over time" is rather physical, but when described in terms of a ratio of differences $v = \frac{\Delta x}{\Delta t}$, it increases in the level of mathematization. The step towards the vector differential representation of $\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ illustrates how one can further increase the level of mathematization. Separately, there is a part of the framework representing the "purely mathematical" part, where calculations can be done before returning to the physical-mathematical-model ((c) in Figure 1.4). Interpreting these results ((b) in Figure 1.4) corresponds to moving downwards, towards a lower degree of mathematization.

In the presented frameworks on mathematical modelling in physics, the emphasis is always on the process of constructing a model, which is not the focus in this dissertation. We are interested in how students understand an existing model, i.e. the heat equation.

We study the role of mathematical and physical knowledge when students reason about (aspects of) this mathematical model. Another reason is that these frameworks rather emphasize the way mathematics can be used to "serve" physics, while we are interested in how mathematical and physical understanding can enhance each other by combining them.

1.4 Conceptual blending

Conceptual blending, or sometimes called mental space integration, was originally introduced by Fauconnier and Turner (1998) to model how people create new meaning in linguistic contexts by selectively combining information from previous experiences. In recent years, the framework has been proven useful to investigate the combination of physics and mathematics in student reasoning. In this section, we will first give a brief overview of the original theory and review how the theory has been implemented in science education research. Last, we explain why we opted for the conceptual blending framework as a lens in this dissertation.

1.4.1 The original theory of conceptual blending

A general schematic representation of the conceptual blending framework is shown in Figure 1.5. In its basic form, a conceptual blending network consists of four connected mental spaces: two partially matched input spaces, a generic space, and the blended space. Generally, a mental space is comprised of conceptual packets or knowledge elements that tend to be activated together, and has an organizing frame that specifies the relationships, or connections between the elements (Bollen et al., 2016). Input spaces are small self-contained regions of conceptual ideas. The generic space describes the underlying structure of the input spaces, identifying commonalities in content and structure (Fauconnier & Turner, 2003b). A blended space is constructed through selective projection from the inputs. The projection is selective, because the elements and organizing frames of the input spaces can be blended in a variety of combinations. This implies that the construction of a blended space may strongly depend on cues and contexts, and that not all elements in the input spaces are activated in the blending process. Consequently, the blended space may partially inherit the structure of the input spaces, but also has its own emergent structure. Fauconnier and Turner distinguish three mechanisms to develop emergent structure. The first one is composition, which says that blending can compose elements from the input spaces to provide relations that do not exist in the separate input spaces. The second one is completion, in which the person adds new elements based on background models that are brought into the

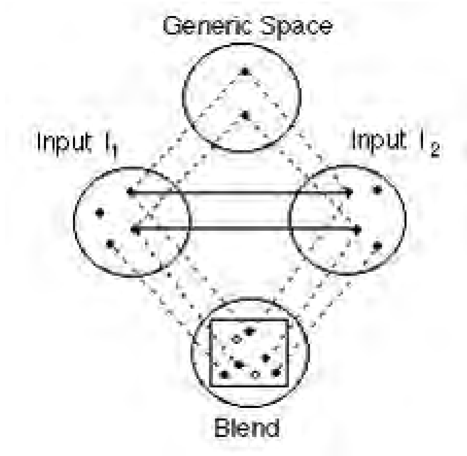

Figure 1.5: Schematic representation of a blending diagram from the original work of Fauconnier and Turner (2003a).

blend unconsciously. Third, there is elaboration, which refers to treating the blend as a simulation and running it imaginatively, which creates new insights.

‘The boat race’ is an accessible example to introduce the different constructs used in the blending framework (Fauconnier & Turner, 2003a, p.2). A modern catamaran is sailing from San Francisco to Boston in 1993, trying to go faster than a clipper that sailed the same course in 1853. A sailing magazine reports:

> *As we went to press, Rich Wilson and Bill Biewenga were barely maintaining a 4.5 day lead over the ghost of the clipper Northern Light, whose record run from San Francisco to Boston they’re trying to beat. In 1853, the clipper made the passage in 76 days, 8 hours.*

Actually, there are two distinct events in this story, the run by the clipper in 1853 and the run by the catamaran in 1993 on (approximately) the same course. In the magazine quote, the two runs are merged into a single event, a race between the catamaran and the clipper’s "ghost". Coming from a blending perspective, the two distinct events are identified as two input spaces. Each space contains relevant elements for the respective event: the voyage, the departure and arrival points, the period and time of travel, the boat, its positions at various times. The two events share the structure of sailing from San Francisco to Boston; this corresponds to the generic space. In the blended space, the two runs are merged and we have two boats on the same course that left San Francisco at the same time. Completion allows us to interpret this situation as a race and therefore to import the familiar background frame of racing and the emotions that go with it. This is an example of emergent structure constructed in the blending process. It is important to note that the blended space remains connected to the input spaces. This way, inferences can be computed in the inputs from the imaginary situation in the blended space. For example, we can deduce that the catamaran is going faster overall in 1993 than the clipper did in 1853, and more precisely, we have some idea ("four and a half days") of their relative performances. The ‘boat race’ example is a simple case of blending. Two inputs share structure. They are linked by cross-space mapping and elements are projected selectively to a blended space. The projection allows emergent structure to develop based on composition, completion and elaboration.

1.4.2 Conceptual blending to investigate the role of mathematics in science education

The conceptual blending framework has been adopted in science education research in many different contexts (e.g. Close & Scherr, 2015; Dreyfus, Gupta, & Redish, 2015; Edwards, 2009; Gregorcic & Haglund, 2021; Hutchins, 2005; Podolefsky & Finkelstein, 2007; Wittmann, 2010; Yoon, Thomas, & Dreyfus, 2011; Zandieh, Roh, & Knapp, 2014). In this section, we limit ourselves to the way conceptual blending has been used in physics, mathematics and chemistry education research (PER, MER and CER) studies focusing on the role of mathematics in student reasoning. We observe that, generally, the blending framework is used as an analytical framework. It has not yet served to design instructional materials. We distinguish between two main methodological approaches. Some studies use blending diagrams in their analysis and in other studies a coding scheme is developed based on conceptual blending to analyze student reasoning.

A first way to use conceptual blending as an analytical lens is to make blending diagrams that visualize student reasoning. Bing and Redish (2007) were the first to introduce the language of conceptual blending as a way to analyze problem solving in physics at the introductory level. They identified two input spaces, which they called ‘mathematical machinery’ and ‘physical world’ (see Figure 1.6). In their adaptation, the role of mathematics is limited to its technical role. Hu and Rebello (2013), and Bollen et al. (2016) extended the adaptation of the blending framework to also consider the conceptual aspect of mathematics, looking at how students blended concepts from electrodynamics with the mathematical concepts of integration and vector differential operators respectively. They identified three input spaces: the symbolic space (abstract mathematical symbols and notations), the math notion space (knowledge about mathematical concepts and notations) and the physics space (physical quantities associated with the object) (see Figure 1.7). The addition of the math notion space accounts for the structural role of mathematics.

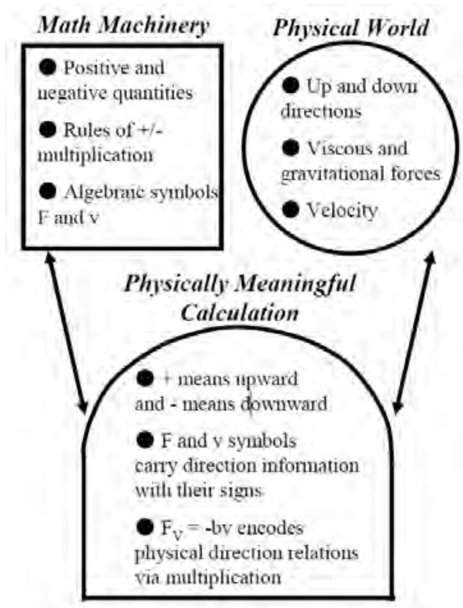

Figure 1.6: Example of a blending diagram from the work of Bing and Redish (2007).

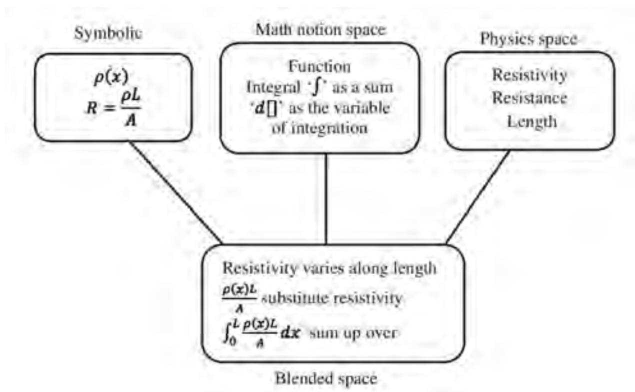

Figure 1.7: Example of a blending diagram from the work of Hu and Rebello (2013).

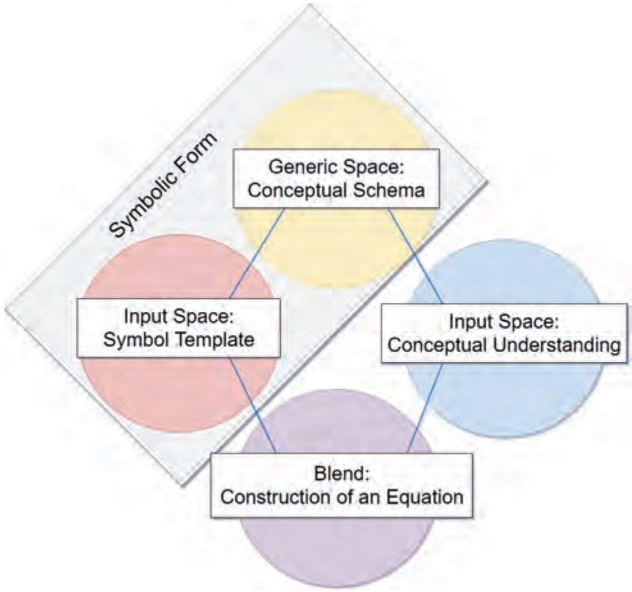

Figure 1.8: Diagram of conceptual blending incorporating symbolic forms to investigate equation construction Schermerhorn (2018).

Schermerhorn (2018) combined the conceptual blending framework with the notion of symbolic forms to describe students' construction of differential length vectors. Sherin (2001) developed symbolic forms to show how conceptual content and equations can be connected. A symbolic form consists of a symbol template and a conceptual schema. For example, the symbol template "□ = □" is a template for an equation in which two expressions are set equal. The □ can be replaced by any expression. The conceptual schema gives the idea expressed in the equation and the symbol template specifies how that idea is written in symbolic form. Schermerhorn states that a symbolic form reveals students' structural understanding and associations related to the mathematics context, but it does little to account for the students' conceptual understanding that dictates the need for the specific form. As an answer, they propose a combination of the symbolic forms and the conceptual blending framework (Figure 1.8). The symbol template and conceptual schema are both considered mental spaces. The second input space is called 'conceptual understanding'. This refers to students' conceptual understanding regarding a specific topic, in this case the students' understanding of a differential length vector. Blending this conceptual understanding with the symbol template results in the construction of an equation, which is situated in the blended space. The conceptual schema plays a special role here, as it can be identified as the common structure of all the other mental spaces. Therefore, it takes on the role of generic space. The generic space has been absent in other studies in PER and MER.

In the context of MER, Gerson and Walter (2008) used conceptual blending diagrams to investigate students' understanding of Calculus concepts in a teaching experiment in the setting of university honors Calculus. This study is not focusing on the role of mathematics in another discipline, but we include it here because the method of constructing the diagram differs from the other studies in an interesting way. Gerson and Walter developed multi-response tasks to elicit calculus content. The content was new for the students and they were encouraged to develop multiple solution strategies. In their paper, Gerson and Walter analyzed episodes of student reasoning in response to a problem about the flow of water in and out of a reservoir. They suspected that important calculus content such as interpreting rates, the anti-derivative, concavity, extrema, points of inflection, area between curves, and average rate of change would emerge from the discussion of this task. Figure 1.9 shows an example of one of the conceptual blending diagrams presented in their paper. The researchers identified two input spaces: the ideas concerning in- and out-flow, and one about the context of water in a reservoir. These two input spaces are combined into a blended space where the students discuss the quantity of water in the reservoir. The diagrams presented in this paper are more detailed than the ones commonly used in PER (Bing & Redish, 2007; Bollen et al., 2016; Hu & Rebello, 2013). The authors give a more fine-grained visualization of the reasoning by identifying individual connections between elements in the mental spaces instead of connecting the entire spaces. Gerson and Walter state that analyzing both the content and the connections students make among content, context and previous knowledge, gives a richer picture of the emergent meaning students are creating as they explore meaningful mathematics tasks.

A second way of using conceptual blending as an analytical lens is to code data from a blending perspective. In the context of PER, Taylor and Loverude (2018) used the conceptual blending framework to investigate student understanding of the function concept in kinematics. Specifically, they probed the ability of introductory physics students to interpret graphical representations of position vs. time functions and their corresponding derivatives and to translate the graphical representation into a meaningful symbolic representation. In their analysis, they consider three input spaces: a mathematical formalism space, a graphical space, and a physical space. They created an "ideal expert blend" and compared this with the student responses. The input spaces of the expert blend are used as a codebook to categorize student statements. Extended sequences of statements with the same code indicate reasoning within a single mental space, while multiple different codes grouped together in a part of the reasoning suggests blending.

Blending has also found its way to CER, where it has been used to study the blending of chemistry and mathematics as students solve problems in the context of chemical kinetics. Bain, Rodriguez, Moon, and Towns (2018) interviewed undergraduate students majoring in chemistry when solving chemical kinetics problems. They

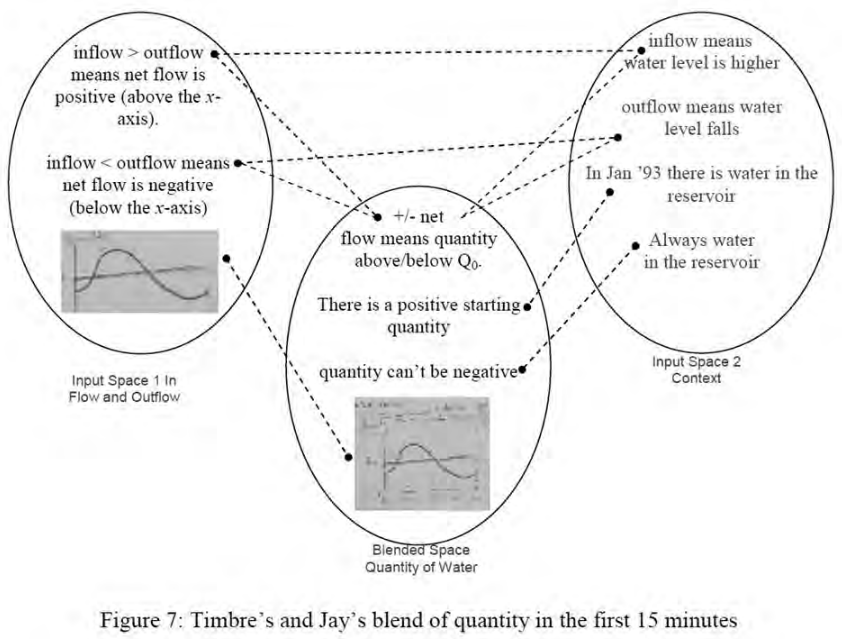

Figure 1.9: Blending diagram from the work of Gerson and Walter (2008).

identified problem solving "steps" in student reasoning and used a coding approach to categorize each step in terms of their understanding: as part of one of the two input spaces representing mathematical or chemical knowledge or as blended when students explicitly integrated mathematical and chemical knowledge. In the blended category, Bain and colleagues developed some subcodes to represent the variety in the data: they characterized the blending process in terms of the specific content category (context), movement from one mental space to another (directionality) and spectrum, in terms of the caliber of discussion (quality). Each sub code allowed them to investigate an aspect of the blending process. In terms of context, they were able to list topics that were commonly discussed while blending (e.g. molecularity and order, concentration). Concerning the directionality, they more frequently observed mathematics to chemistry blending, where students attribute chemical meaning to mathematical expressions. In terms of the quality of blending, they found that students that exhibited more instances of blending generally also exhibit a higher level of blending. Last, they identified a relation between blending ability and problem solving. By comparing students that exhibited more instances of blending (high-frequency blenders) with students that did not engage in blending (non-blenders), they concluded that the type of problem solving observed in high-frequency blenders is reminiscent of expert-like problem solving, where integration of conceptual and mathematical reasoning is a prominent characteristic. In comparison, the non-blenders exhibited more variation in problems solving, with students proportionally showing less ability or even attempts to approach the problems conceptually. This indicates that blending is necessary for expert-like understanding.

In their search of how students blend chemistry and mathematics, this research group published a second paper in which they investigated the same data set for the role of graphs and equations in the blending of chemistry and mathematics. In this study, Rodriguez, Santos-Diaz, Bain, and Towns (2018) combine the resources framework with symbolic forms (Sherin, 2001) and the conceptual blending framework (Fauconnier & Turner, 2003b). The transcripts were coded with the codebook from the previous paper (Bain et al., 2018) but an additional layer of analysis was added based on symbolic and graphical forms to investigate the role of graphs and equations in the blending process. The idea behind symbolic forms was explained before. Rodriguez, Bain, Towns, Elmgren, and Ho (2019) developed an analogical construct for graphs: graphical forms. Graphical forms are defined as intuitive mathematical ideas the students associate with graphs. An example of a graphical form is 'steepness as rate', which states that steepness of a graph gives a measure of the rate of change. In graphical forms, the symbol template is now a graphical template, which is a distilled version of a graph with a specific shape depending on the graphical form. The findings of Rodriguez et al. (2018) suggest that the ability to engage in reasoning characterized by symbolic and graphical forms supported students' engagement in their blending process. They observed that often when students used these symbolic or graphical forms, it aided in their understanding of the chemical phenomena being described, leading the students to discuss the mathematical narrative represented in the graph, which indicates blending.

In the context of electromagnetism, Huynh and Sayre (2019) used conceptual blending to demonstrate that different physical meanings, such as directionality and location, could be associated with positive and negative signs. They analyzed the struggles of upper-division students as they worked on an introductory level problem where they must employ multiple signs with different meanings in one mathematical expression. In the course of solving the problem, the students should attribute directionality to the sign of the physical quantity at some point, which can happen in multiple ways. The authors constructed several small blending diagrams that each represent a possible way in which one can blend information about directionality and location with sign, and thus each represent a different 'meaning' of the signs. These different blends formed a codebook to guide the analysis of student reasoning. Figure 1.10 shows three examples of blends that Huynh and Sayre proposed in their codebook. While solving the problem, students switch between the different blends to solve different aspects of the problem. In the data analysis, the authors coded the transcripts for the presence of the different blends throughout the students' reasoning. Students continuously switched among the blends to use, and combined those blends. The authors concluded that the problem

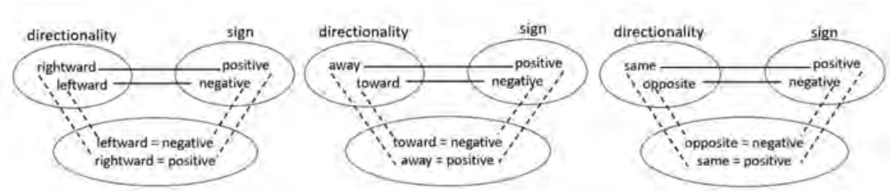

FIG. 3. From left to right: space-fixed blend (□), body-fixed blend (△), and comparative blend (○).

Figure 1.10: Huynh and Sayre (2019) distinguish three different blends between directionality and sign.

given to the students was challenging for them because there are not only multiple signs coming within a single expression, but also multiple meanings with which these signs could be associated, where students need to carefully choose what blends to use.

Some constructs from the original theory of Fauconnier and Turner (2003b) have not found their way yet to the incorporation in science education. The generic space is absent in most of the work presented here. The studies in science education also do not distinguish between the three mechanisms of blending (composition, completion and elaboration). These constructs appear to be difficult to adapt in the context of student reasoning in science education. It remains a challenge to better understand these constructs and discover how they match the science education context.

The theory of conceptual blending shows some overlap with the resources perspective. Taylor and Loverude (2018, p. 1) described conceptual blending as "the theory that all human knowledge is pervaded by metaphors constructed from bodily experiences; knowledge is grouped into resources that are activated together at appropriate moments (Fauconnier & Turner, 2003a). In the blending framework, these resource groups are called mental spaces. When confronted with a new concept, the mind inputs mental spaces into a blend to make sense of the notion at hand (Fauconnier & Turner, 1998)." It is remarkable how we can identify the language and ideas of the resources framework in this description of the blending framework. Wittmann (2010) discussed the overlap between both frameworks by comparing blending diagrams to resource graphs, i.e. simple graphical representations of the resources that are activated by an individual in a particular setting. He adopted both frameworks to describe how students combine elements from different experiences when answering a question about wave pulses propagating along a long taut spring. He concluded that the conceptual blending approach provides a more complete description of student thinking because it makes fewer assumptions about the nature of pre-existing ideas and has greater explanatory power. He finds that in the resources description, he must make more assumptions about how an idea is applied, while the conceptual blending diagram directly shows how an idea is used in the specific setting. Furthermore, Bollen (2017) states that one does not necessarily have to choose between the two, because both frameworks can be considered compatible. Resources can be of a blended nature, and the conceptual blending framework can therefore be applied within the boundaries of the resources perspective. He argues that the "conceptual packets or knowledge elements" (the elements in a mental space) as described by Fauconnier and Turner are very similar, if not identical, to the fine-grained cognitive structures, or resources, as defined by Hammer et al. (2005).

1.4.3 Conceptual blending as a lens in this project

In this dissertation, we adhere to a perspective where mathematics and physics are treated equally in hierarchy, none of the two solely serving the other one. Thompson, Bucy, and Mountcastle (2006) state that failure to make connections between mathematics and physics in both directions may prevent a full understanding of the relevant physical phenomena. This point of view, where mathematics and physics are considered equally important and connections have to be made in both directions, suits very well with the conceptual blending perspective. The concepts and structures of mathematics have to be combined with physical knowledge and intuition, which results in an enhanced understanding of both. New ideas and inferences emerge after this combination (Bing & Redish, 2007).

Brahmia (2017, p. 4) formulated beautifully why the blending perspective has the potential to be used to study the combination of mathematical and physical knowledge in reasoning:

> *"Seen through the lens of conceptual blending, we suggest that the math-physics blending may be tighter than has been previously discussed in theoretical models proposed in PER. Rather than a back and forth between the math world and the physics world, we find it productive to think in terms of symbiotic cognition in which a homogeneous blended cognitive space, at a subconscious level, can be cultivated and can catalyze cognitive flexibility; the physics informs the mathematical thinking which informs physics reasoning."*

This quote can be related to the discussion of the different frameworks in Section 1.3. The frameworks of mathematization eminently focus on this back and forth movement between the physical and mathematical world. We agree with Brahmia on the mutual influence of mathematics and physics on each other and on the learning process and believe that the conceptual blending framework is well suited to express this view.

1.5 Research aims

In this chapter, we have introduced all elements central to this research project. It appears that combining mathematical and physical knowledge is difficult for students at all levels of education. We contribute to the increasing number of studies investigating student reasoning at the advanced undergraduate and graduate level, specifically with mathematics and physics majors. We are interested in understanding the process of reasoning when students combine mathematical and physical knowledge, identifying possible pitfalls and opportunities. Furthermore, we want to study the dynamics of this reasoning process in detail (theoretical aim). This knowledge will then form the basis for investigating how the process of combining mathematics and physics can be fostered in instruction (practice-oriented aim). This way, the theoretical findings lead to research-based instructional methods and materials.

As explained earlier, (partial) differential equations are among the most often used mathematical concepts to describe physical phenomena. We choose to conduct this research in the context of the heat equation. As one needs good understanding of both the physical and mathematical concepts involved and of the way they are related to each other, we consider the heat equation as a promising topic to study student reasoning on the mathematics-physics interplay. We approach this from a conceptual blending perspective, which acknowledges the mutual influence of mathematics and physics in the learning process. It is important to note that all aims are focused on studying students' reasoning processes in detail. Therefore, we take on qualitative methodologies: we conduct task-based interviews with small numbers of students. Qualitative methods are eminently suitable to do a fine-grained analysis of very detailed and rich data. In section 1.7, we present the research design and the aims of each separate study, but first we give some more background information about the educational context.

1.6 Educational context

Most of the data collected in the context of this dissertation are collected with students from the Bachelor of Physics and the Bachelor of Mathematics at KU Leuven (Belgium). In the first study, we also conducted interviews with students from the Bachelor of Physics from the University of Groningen (The Netherlands). In order to interpret results and understand some of the choices we made in the design of the studies, it is important to discuss the respective degree programmes and background knowledge of the students.

1.6.1 KU Leuven

In the bachelor programme of physics at KU Leuven, there is a tradition of first building a strong mathematical foundation before moving on to the more advanced physics courses. The mathematics courses in the first year (Calculus I and II, linear algebra, mathematical proving, and probability theory) are taught by mathematicians. The physics component of the first year focuses on general physics (based on typical textbooks like Giancoli (2008) or Jewett and Serway (2008)) and physics labs, taught by physicists. The students follow one more advanced physics course in their first year: statistical thermodynamics. Other advanced physics courses, like electricity and magnetism, classical, and quantum mechanics, start in the second year.

The first year of the Bachelor of Mathematics is almost identical to the Bachelor of Physics. Mathematics students take the same mathematics courses and general physics courses together with the physics students.

Our research focus fits within the frame of a course called ‘Differential equations’, which is situated in the first semester of the second year of the Physics and Mathematics Bachelor programmes and is compulsory for both. As was shown by the description of the first year of the programme, the students have had very similar preparation in spite of their different majors (mathematics or physics). The course is divided in two main parts: ordinary differential equations, currently taught by a mathematician, and partial differential equations, currently taught by a plasma physicist. The course notes are written by the respective professors and are not explicitly based on a specific textbook.

The part of the course on partial differential equations is centered around the typical physical partial differential equations like the heat, wave and Schrödinger equations. In the chapter on the heat equation (also called the diffusion equation), the one-dimensional heat equation is derived from Newton’s law of cooling combined with conservation of energy. The rest of the chapter focuses on analytically solving the heat equation in several situations with different boundary conditions using the technique of separation of variables. At the end of the chapter, there is also some attention for inhomogeneous problems and problems with the multi-dimensional heat equation, but the focus lies mostly on homogeneous problems in one dimension.

The course notes are written mostly from a mathematical perspective. The derivation is compact, and relevant physical concepts like the definition of heat, heat flow or temperature gradient are not explicitly introduced. The physics majors might have prior knowledge on these concepts from their thermodynamics course. In secondary school, heat and temperature are also discussed in the physics curriculum. This could be a source of prior knowledge for all students. In the rest of the chapter, where the focus is on solving the equation analytically, the role of physics is mostly reduced to being a context for a mathematics question. There is not much attention for reflection on the physical meaning of a problem statement or solution.

1.6.2 University of Groningen

In our first study, we also conducted interviews with students from the Bachelor programmes of Physics at the University of Groningen. The rest of the studies focuses on data collected with KU Leuven students. The physics curricula at KU Leuven and the University of Groningen differ slightly. At the University of Groningen, there are less ‘pure mathematics’ courses and they start earlier with more advanced physics courses.

Similarly to the programme at KU Leuven, the first year contains compulsory mathematics courses taught by mathematicians (Calculus I and II and linear algebra). Apart from these ’pure mathematics’ courses, the students also take a mathematical physics course, taught by an experimental physicist, which differs from the KU Leuven approach. Mathematics physics focuses on advanced calculus topics, but specifically directed towards physics students. The content ranges from sequences and series, limits, power series, Taylor expansions, Fourier series, Fourier integrals, to second order ordinary differential equations and partial differential equations.

The physics component of the first year also differs from the KU Leuven approach. There are no general physics courses. Instead the first year contains more advanced topics with the compulsory courses ‘Mechanics and Relativity’, and ‘Electricity and Magnetism’, complemented with elective subjects on different physics topics (e.g. astronomy, nanophysics, energy and environment,...). Similarly to KU Leuven the programme contains two physics lab courses.

The different structure of the curricula can be observed in the position of the course ‘Electricity and Magnetism’. At both universities, this is taught using the textbook ‘Introduction to electrodynamics’ from Griffiths (1999). However, at the University of Groningen it is placed in the second semester of the first year and at KU Leuven in the second semester of the second year.

This difference in structure between the curricula is also noticeable in the placement of the topic of partial differential equations. At the University of Groningen, ordinary and partial differential equations are part of the course on mathematical physics. The students take this course in the second half of the second semester of their first year. It is taught by a physicist and the objective is to teach the students the necessary mathematical concepts for their advanced physics courses. The heat equation is solely discussed as an example of a partial differential equation and the students are expected to be able to solve it using separation of variables. There is little to no attention to the relevant physical concepts behind this equation like heat flow or the physical meaning of boundary conditions.

1.7 Research overview

In this dissertation, we aim to investigate blending of mathematics and physics in students' reasoning in the context of one-dimensional systems modeled by the heat equation. Figure 1.11 shows the research overview in a schematic way. In this section, we briefly discuss the three studies presented in this dissertation. We report on these studies in Chapters 2 to 5. Finally, in Chapter 6, we summarize the most important findings, reflect on the research project and formulate some relevant implications for instruction and further research.

1.7.1 Study 1: Difficulties with boundary conditions for the diffusion equation

As there is little research on student understanding of partial differential equations in PER and MER, we start with an exploratory study. We developed broad task-based interviews that probe student reasoning with a variety of different questions related to the diffusion/heat equation and its solution. The aim was to explore instances where blending of mathematics and physics might play a role in student reasoning in this context, e.g. physically interpreting a mathematical solution, or constructing mathematical boundary conditions based on a description of a physical system. We conducted the interviews with six students from the Bachelor programmes of Physics and Mathematics at KU Leuven and six students from the Bachelor programme of Physics at the University of Groningen.

Based on preliminary data analysis, we observed that mathematically formulating boundary conditions based on a description of a physical system was a topic where the blending of mathematics and physics proved to be difficult, which made it particularly interesting for further analysis. This narrowed our focus and resulted in a paper, presented in Chapter 2, in which we aim to identify reasoning difficulties and characterize them in terms of their position in the blending framework, i.e. difficulties of physical or mathematical nature, or difficulties that are related to the blending of both.

Important note: We conducted the exploratory interviews in the context of both particle diffusion and heat transfer. The analysis presented in Chapter 2 focused on boundary conditions. The relevant interview question happened to be posed in the context of diffusion. However, in the design of the subsequent studies we decided that heat transfer in a system offers a more interesting physical context for our investigation because we observed that reasoning about heat transfer was more challenging for the students than reasoning about particle diffusion. As a result of this, Chapter 2 focuses on diffusion, while the other chapters focus on heat transfer.

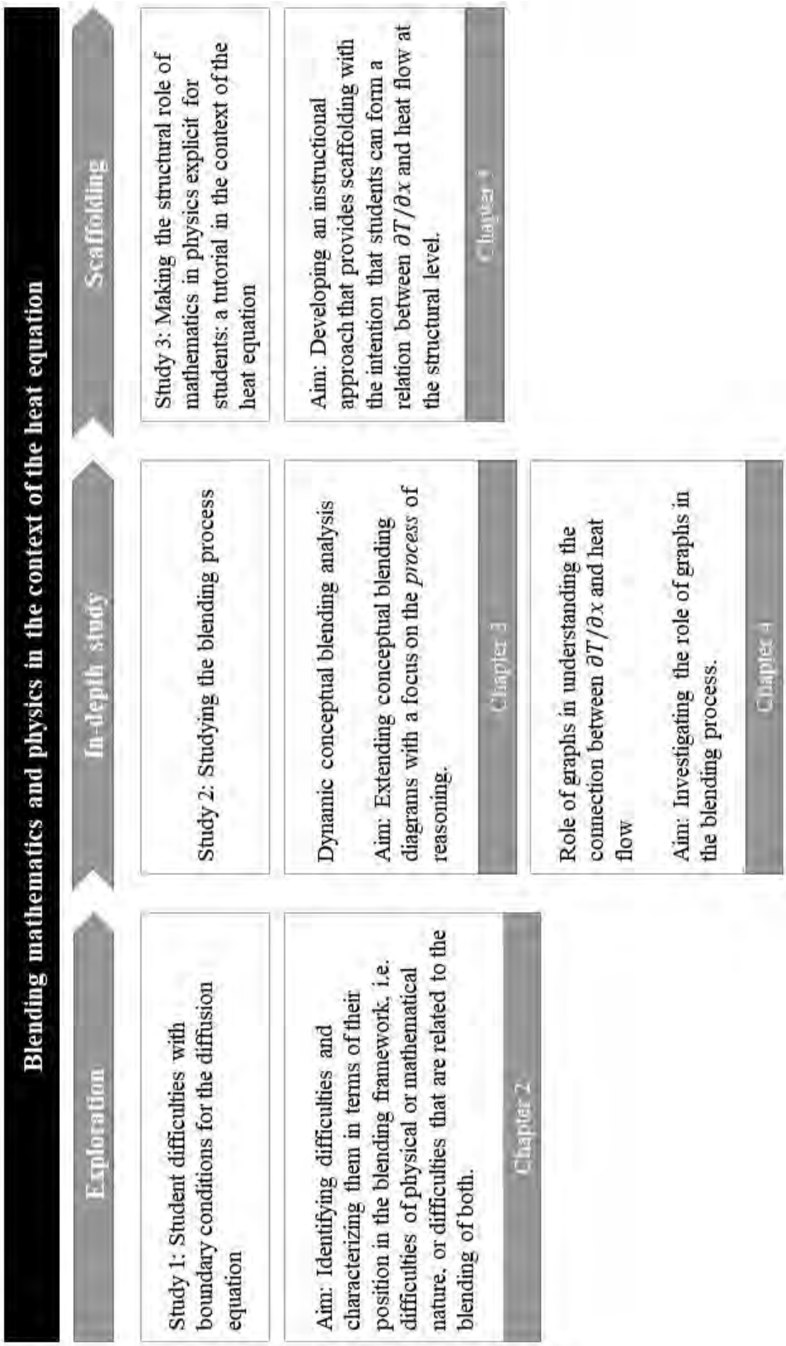

Figure 1.11: Research overview.

1.7.2 Study 2: Studying the blending process

In order to scaffold meaningful blending of mathematics and physics, which is the ultimate aim of the project, insight into the blending process itself is needed. Therefore, after investigating students' difficulties, we focused on the blending process, i.e. the way students combine mathematics and physics in their reasoning, in more detail. We developed a new task-based interview, which solely focused on boundary conditions in their mathematical and physical form and the translation between them in both directions. We conducted this second round of interviews with four pairs of physics and mathematics majors from KU Leuven. This study resulted in two papers, presented in Chapters 3 and 4. We give a brief overview of each paper below.

Dynamic conceptual blending analysis to model student reasoning processes while integrating mathematics and physics

As explained in section 1.4, conceptual blending diagrams offer a way to visualize and analyze student reasoning. However, we felt that the current way the framework is implemented in PER (e.g. Bing & Redish, 2007; Bollen et al., 2016; Hu & Rebello, 2013) mostly focuses on visualizing the result of student reasoning instead of the reasoning process leading to that result. As such, the diagram gives a compact overview of the used concepts in each mental space, but the reasoning process, i.e. how and in what order these concepts were used by the students and to what extend these concepts were connected and combined, is not captured in this visualization. Therefore, we extent the way blending diagrams have been used in PER to give a more detailed representation of the dynamics of the reasoning process, which we will call *dynamic blending diagrams*. Chapter 3 presents the methodology of constructing a dynamic blending diagram using two cases from our second interview round.

Role of graphs in understanding the connection between $\frac{\partial T}{\partial x}$ and heat flow from a dynamic conceptual blending perspective

The interview questions focused on boundary conditions of the form $T(x_b, t) = c$ and $\frac{\partial T}{\partial x}(x_b, t) = c$ (as discussed in section 1.2). Especially boundary conditions of the latter form were challenging for students. We observed in the data that students often spontaneously constructed graphs when they trying to give physical meaning to the partial derivative $\frac{\partial T}{\partial x}$. We therefore studied the role of graphs in the blending process, using dynamic blending diagrams, with a focus on the relation between $\frac{\partial T}{\partial x}$ and heat flow. In Chapter 4, we report on the analysis and findings.

1.7.3 Study 3: Making the structural role of mathematics in physics explicit for students: a tutorial in the context of the heat equation

In the last study, we brought everything together in the development of a tutorial with the aim to foster the blending of mathematics and physics in student reasoning. Specifically, we aimed to help students in formulating and interpreting boundary conditions for systems that can be described by the heat equation and to give physical meaning to the partial derivatives $\frac{\partial T}{\partial t}$ and $\frac{\partial T}{\partial x}$ in the context of heat transfer. The design of the tutorial was based on different design principles, all aiming to promote the blending of mathematics and physics in their own way.

We report on the development of the tutorial and we focus specifically on the part where we guide students in exploring the relation between $\frac{\partial T}{\partial x}$ and heat flow. We already found in Chapter 4 that graphs could foster the blending in the context of this relation. However, we also found that more scaffolding is necessary in order for students to be able to formulate this relation. In Chapter 5, we discuss the development of an instructional approach that provides that necessary scaffolding, specifically with the intention to make the structural role of mathematics in physics explicit for students. We conducted teaching-learning interviews with small groups of students to evaluate the instructional approach.