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Partial pole placement with time delay in structures using the receptance and the system matrices

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A B S T R A C T

Datta et al. solved the partial pole placement problem for the symmetric definite quadratic eigenvalue problem where part of the spectrum is relocated to predetermined locations and the rest of the spectrum remains unchanged. In this paper, the problem is solved by a hybrid combination of this result and the method of receptances. This allows for the partial assignment of desired poles with no spillover when there is time delay between the measured or estimated state and actuation of the control.

1. Introduction

Small vibrations about equilibrium position of viscously damped system are governed by the set of second order differential equations,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}, \quad \mathbf{M}, \mathbf{C}, \mathbf{K} \in \Re^{n\times n}, \quad \mathbf{M} = \mathbf{M}^T, \quad \mathbf{C} = \mathbf{C}^T, \quad \mathbf{K} = \mathbf{K}^T, \tag{1}$$

where $\mathbf{M}$ is positive definite, $\mathbf{C}$ and $\mathbf{K}$ are semi-positive definite, and dots denote derivatives with respect to time. In applications it is frequently required to rapidly reduce the level of oscillations in the system. This could be achieved by active vibration control implementing state feedback

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{b}u(t), \tag{2}$$

where

$$u(t) = \mathbf{f}^T\dot{\mathbf{y}} + \mathbf{g}^T\mathbf{y}. \tag{3}$$

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The constant vector $\mathbf{b}$ signifies the location of the various actuators implementing the control force and their setting of gains. The function $u(t)$ is the applied control. By (3) the control law is obtained from a linear combination of the state of the system, i.e., the position and the velocity of the various degrees of freedom. This explains the terminology *state feedback control*.

Substituting (3) in (2) gives

$$\mathbf{M}\ddot{\mathbf{y}} + (\mathbf{C} - \mathbf{b}\mathbf{f}^T)\dot{\mathbf{y}} + (\mathbf{K} - \mathbf{b}\mathbf{g}^T)\mathbf{y} = \mathbf{0}. \tag{4}$$

Separation of variables

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}, \tag{5}$$

and

$$\mathbf{y}(t) = \mathbf{w}e^{\mu t}, \tag{6}$$

applied to (1) and (4) gives the quadratic eigenvalue problems,

$$(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})\mathbf{v} = \mathbf{0}, \tag{7}$$

and

$$(\mu^2\mathbf{M} + \mu(\mathbf{C} - \mathbf{b}\mathbf{f}^T) + \mathbf{K} - \mathbf{b}\mathbf{g}^T)\mathbf{w} = \mathbf{0}, \tag{8}$$

corresponding to the *open-loop* system (1), and the *closed-loop* system (4), respectively. We remark in passing that in the quadratic eigenvalue problem (8), $\mathbf{C} - \mathbf{b}\mathbf{f}^T$ and $\mathbf{K} - \mathbf{b}\mathbf{g}^T$ are rank-one modifications to $\mathbf{C}$ and $\mathbf{K}$, featured in the quadratic eigenvalue problem (7).

The system (1) is a natural dissipative system that by the laws of the thermodynamics cannot generate energy perpetually. This means that the real parts of all its $2n$ eigenvalues λ_k, are non-positive. The objective of the control is to push the real part of these eigenvalues further in negativity, such that the response $\mathbf{y}(t)$ of the controlled system, with its eigenvalues μ_k, diminishes faster.

It thus follows that predetermining the eigenvalues $\{\mu_k\}_{k=1}^{2n}$, the problem of finding the control gains $\mathbf{f}$ and $\mathbf{g}$ such that (8) holds, is an inverse eigenvalue problem. Since the modification $\mathbf{b}\mathbf{f}^T$ and $\mathbf{b}\mathbf{g}^T$ are unsymmetric rank-one update to $\mathbf{C}$ and $\mathbf{K}$, arbitrary reassignment of eigenvalues from the set $\{\lambda_k\}_{k=1}^{2n}$ of simple eigenvalues to the set $\{\mu_k\}_{k=1}^{2n}$ of simple eigenvalues is possible, provided that $\mathbf{b}$ is not orthogonal to any of the eigenvectors $\mathbf{v}_k$. Otherwise, the system is said to be *not controllable*.

However, if the control (2) and (3) is inappropriately applied, the stable system (1) may become unstable. This means that by applying the control (3), there is a risk for the vibration level to increase.

Inappropriate control may result from uncertainty and incomplete knowledge of the system parameters. In structural vibration the system (1) is a discrete model representing a continuous structure. Such a model is usually obtained by the finite element method. For the purpose of accuracy the model dimension n should be large. For a large model order calculation of all eigenvalues of (1) is not possible. Without a reference to the complete set of eigenvalues of the open loop system (7) it is not feasible to assign a reasonable set of eigenvalues for the closed-loop system (8); a set that could be implemented by a moderate control force that would not damage the structure, and at the same time would improve the dynamic response of the system.

Attempt to assign only part of the spectrum to given eigenvalues may result in *spillover*, a phenomenon that poles that are not intended to be changed are relocated to undesired locations that could increase the vibration level, or even destabilize the system. The problem of spillover was an obstacle in implementing state feedback control in vibrating structures, as described in [1].

Datta et al. [2] have solved the problem of assigning the desired poles without altering the rest of the spectrum. This problem is referred to as *partial pole placement*. Ram and Mottershead [4] have developed a method for complete assignment of poles by the method of *receptance*. The assignment of poles by receptances is restricted to the case where there are no common eigenvalues to the open and the closed loop systems. To overcome this limitation we first develop in this paper a hybrid pole assignment method that allows partial assignment of poles with measured receptances. Then we use the hybrid method in Section 5 to solve the partial pole placement with time delay. This problem cannot be solved by the methods provided in [2], [4] or [5].

2. Partial pole placement

Datta et al. have shown in [2] that partial pole placement, without spillover, could be made. We write (7) in the form

$$\mathbf{MV\Lambda}^2 + \mathbf{CV\Lambda} + \mathbf{KV} = \mathbf{0}, \tag{9}$$

and partition

$$\mathbf{\Lambda} = diag\{\mathbf{\Lambda}_1 \quad \mathbf{\Lambda}_2\}, \quad \mathbf{V} = [\mathbf{V}_1 \quad \mathbf{V}_2], \tag{10}$$

where

$$\mathbf{\Lambda}_1 = diag\{\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_m\}, \quad \mathbf{\Lambda}_2 = diag\{\lambda_{m+1} \quad \lambda_{m+2} \quad \cdots \quad \lambda_{2n}\}, \tag{11}$$

and

$$\mathbf{V}_1 = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_m], \quad \mathbf{V}_2 = [\mathbf{v}_{m+1} \quad \mathbf{v}_{m+2} \quad \cdots \quad \mathbf{v}_{2n}]. \tag{12}$$

Suppose that the set of eigenvalues in $\mathbf{\Lambda}_1$ is simple, and closed under conjugation. The problem solved in [2] is:

Problem 1

Given: $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, $\mathbf{b}$, $\mathbf{\Lambda}_1$, $\mathbf{V}_1$ and a self conjugate set of simple eigenvalues $\{\mu_k\}_{k=1}^m$
Find: $\mathbf{f}$ and $\mathbf{g}$ such that the eigenvalues of the closed loop system are included in the sets $\{\mu_k\}_{k=1}^m$, and $\{\mu_k = \lambda_k\}_{k=m+1}^{2n}$.

The solution to Problem 1 is

$$\mathbf{f} = \mathbf{MV}_1\mathbf{\Lambda}_1\boldsymbol{\beta}, \tag{13}$$

$$\mathbf{g} = -\mathbf{KV}_1\boldsymbol{\beta}, \tag{14}$$

where the elements of $\boldsymbol{\beta}$ are given by

$$\beta_k = \frac{1}{\mathbf{b}^T\mathbf{v}_k}\frac{\mu_k - \lambda_k}{\lambda}\ {}_k \prod_{\substack{r=1\\ r\neq k}}^{m} \frac{\mu_r - \lambda_k}{\lambda_r - \lambda_k}, \quad k = 1, 2, \ldots, m. \tag{15}$$

It follows from (15) that Problem 1 is solvable provided that $\mathbf{b}^T\mathbf{v}_k \neq 0$, for $k = 1, 2, \ldots, m$. Note that the vectors $\mathbf{f}$ and $\mathbf{g}$ are real vectors, and that the partial assignment of eigenvalues is achieved with no spillover, based on only a small set of eigenvalues which are required to be changed. This is a practical solution to the practical engineering problem of pole placement.

From now on by *partial pole placement* we mean *partial pole placement with no spillover*, and without the assumption of knowing $\{\lambda_k\}_{k=m+1}^{2n}$ or $\{\mathbf{v}_k\}_{k=m+1}^{2n}$; as in Problem 1.

3. The idea behind the partial pole placement

For the purpose of the ensuing analysis it is important to understand the idea behind the partial pole placement (13) through (15), and in particular the choice of control vectors $\mathbf{f}$ and $\mathbf{g}$ according to (13) and (14).

We rewrite (8) in the form

$$(\mu^2\mathbf{M} + \mu\mathbf{C} + \mathbf{K} - \mathbf{bh}^T(\mu))\mathbf{w} = \mathbf{0}, \tag{16}$$

where

$$\mathbf{h}(\mu) = \mu\mathbf{f} + \mathbf{g}. \tag{17}$$

Lemma 1. *If* $\mathbf{h}^T(\lambda)\mathbf{v} = 0$ *then* (16) *has a non-trivial solution*

$$\{\mu = \lambda \quad \mathbf{w} = \mathbf{v}\}, \tag{18}$$

where $\{\lambda \quad \mathbf{v}\}$ *is an eigenpair of* (7).

Proof. The proof follows from reduction of the eigenvalue problem (16) to the eigenvalue problem (7) under the stipulation of Lemma 1. □

It follows from Lemma 1 that if we develop a method of choosing **f** and **g** such that $\mathbf{h}(\mu)$ in (17) is orthogonal to **v** then the control would not affect the eigenpair $\{\lambda \quad \mathbf{v}\}$. This is done by the selection of **f** and **g** according to (13) and (14), as shown in the following lemma.

Lemma 2. *If* **f** *and* **g** *are chosen according to (13) and (14), with an arbitrary vector* $\boldsymbol{\beta}$, *then*

$$\mathbf{h}^T(\lambda_r)\mathbf{v}_r = 0, \quad r \in \{k\}_{k=m+1}^{2n}. \tag{19}$$

Proof. From (7)

$$(\lambda_r^2\mathbf{M} + \lambda_r\mathbf{C} + \mathbf{K} - \mathbf{b}\mathbf{h}^T(\lambda_r))\mathbf{v}_r = -\mathbf{b}\mathbf{h}^T(\lambda_r)\mathbf{v}_r, \quad r \in \{k\}_{k=m+1}^{2n}. \tag{20}$$

Since **f** and **g** satisfy (13) and (14), respectively, we have

$$\mathbf{h}^T(\lambda_r)\mathbf{v}_r = \boldsymbol{\beta}^T(\lambda_r\boldsymbol{\Lambda}_1\mathbf{V}_1^T\mathbf{M} - \mathbf{V}_1^T\mathbf{K})\mathbf{v}_r = \mathbf{0}, \quad r \in \{k\}_{k=m+1}^{2n} \tag{21}$$

by virtue of the bi-orthogonal relation,

$$\boldsymbol{\Lambda}\mathbf{V}^T\mathbf{M}\mathbf{V}\boldsymbol{\Lambda} - \mathbf{V}^T\mathbf{K}\mathbf{V} = \mathbf{D}, \quad \mathbf{D} = diag\{d_1 \quad d_2 \quad \cdots \quad d_{2n}\}, \tag{22}$$

developed in [2]. □

It thus follows that by choosing **f** and **g** according to (13) and (14), with arbitrary $\boldsymbol{\beta}$, the eigenvalues $\{\lambda_k\}_{k=m+1}^{2n}$ become invariant under the control. This idea was seeded by Saad in [6].

The practical advantage of the method is that Eqs. (13) and (14) are not expressed in terms of $\{\lambda_k \quad \mathbf{v}_k\}_{k=m+1}^{2n}$. If, in addition, the vector $\boldsymbol{\beta}$ chosen as in (15), the m poles $\{\lambda_k\}_{k=1}^{m}$ are relocated to the desired set $\{\mu_k\}_{k=1}^{m}$.

4. Partial pole placement by the method of receptances

For a complex variable s the *receptance* matrix corresponding to the open loop system (7) is defined as follows,

$$\mathbf{H}(s) = (s^2\mathbf{M} + s\mathbf{C} + \mathbf{K})^{-1}. \tag{23}$$

The importance of the receptance in applications is by virtue of it measurability. It is possible to evaluate the point receptance $H_{kr}(s)$, the $k - r$ element of $\mathbf{H}(s)$, by applying a dynamic force to the kth degree of freedom and measuring its dynamic influence on the rth degree of freedom. Such a test is called a *modal test* [3]. Hence, we may not know the system model, **M**, **C** and **K**, but yet $\mathbf{H}(s)$ may be available from physical tests.

Ram and Mottershead [4] have shown that pole placement can be done by using receptances. Let $\widehat{\mathbf{H}}(s)$ be the matrix receptance associated with the closed loop system (8). Then the Sherman–Morrison formula gives

$$\widehat{\mathbf{H}}(s) = (s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} - \mathbf{b}(\mathbf{g}^T + s\mathbf{f}^T))^{-1} = \mathbf{H}(s) + \frac{\mathbf{H}(s)\mathbf{b}(\mathbf{g}^T + s\mathbf{f}^T)\mathbf{H}(s)}{1 - (\mathbf{g}^T + s\mathbf{f}^T)\mathbf{H}(s)\mathbf{b}}. \tag{24}$$

For μ_k the receptance $|\widehat{\mathbf{H}}(\mu_k)| \to \infty$. It thus follows from (24) that

$$(\mathbf{g}^T + \mu_k\mathbf{f}^T)\mathbf{H}(\mu_k)\mathbf{b} = 1, \quad k = 1, 2, \ldots, 2n. \tag{25}$$

The equations in (25) may be written in matrix form,

$$\begin{bmatrix} \mu_1 \mathbf{b}^T \mathbf{H}(\mu_1) & \mathbf{b}^T \mathbf{H}(\mu_1) \\ \mu_2 \mathbf{b}^T \mathbf{H}(\mu_2) & \mathbf{b}^T \mathbf{H}(\mu_2) \\ \vdots & \vdots \\ \mu_{2n} \mathbf{b}^T \mathbf{H}(\mu_{2n}) & \mathbf{b}^T \mathbf{H}(\mu_{2n}) \end{bmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{26}$$

or

$$\mathbf{G} \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \mathbf{p}, \tag{27}$$

with the obvious definition of **G** and **p**. Note that for (26) to be solvable the sets $\{\lambda_k\}_{k=1}^{2n}$ and $\{\mu_k\}_{k=1}^{2n}$ of simple eigenvalues have to be distinct in the sense that there is no common eigenvalue $\lambda = \mu$. Otherwise, some entries in the matrix in (26) are unbounded.

Then by knowing $\mathbf{H}(s)$, $\mathbf{b}$ and a complete set of assigned poles, $\{\mu_k\}_{k=1}^{2n}$, the control vectors **f** and **g** may be found from (27).

We develop now a hybrid method that combines the results in [2] with the method of receptances which allows partial assignment of poles and hence overcomes the limitation that prevents the open and the closed loop system from having common poles. In particular we wish to assign only $m < 2n$ poles and leave the others unaltered. Then it is required that

$$\mathbf{G}_1 \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \mathbf{p}_1, \tag{28}$$

where

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix}, \quad \mathbf{G}_1 \in C^{m \times 2n}, \quad \mathbf{p} = \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix}, \quad \mathbf{p}_1 \in \Re^m. \tag{29}$$

By Lemmas 1 and 2, for an arbitrary vector $\boldsymbol{\beta}$ the control gains,

$$\begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \begin{bmatrix} \mathbf{M}\mathbf{V}_1 \boldsymbol{\Lambda}_1 \\ -\mathbf{K}\mathbf{V}_1 \end{bmatrix} \boldsymbol{\beta}, \tag{30}$$

leave the eigenvalues $\{\mu_k = \lambda_k\}_{k=m+1}^{2n}$ unchanged.

Lemma 3. *An alternative formula for* $\boldsymbol{\beta}$ *is given by*

$$\boldsymbol{\beta} = \left(\mathbf{G}_1 \begin{bmatrix} \mathbf{M}\mathbf{V}_1 \boldsymbol{\Lambda}_1 \\ -\mathbf{K}\mathbf{V}_1 \end{bmatrix} \right)^{-1} \mathbf{p}_1. \tag{31}$$

Proof. The proof is obtained by substitution of (30) in (28). □

Example 1. Consider the open loop system (7) with

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}. \tag{32}$$

The eigenvalues and eigenvectors of this system are:

$$\lambda_{1,2} = \pm i, \quad \mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_{3,4} = -1 \pm 2i, \quad \mathbf{v}_{3,4} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{33}$$

In the ensuing developments we are not assuming the knowledge of $\lambda_{3,4}$ or $\mathbf{v}_{3,4}$.

Let $\mathbf{b} = (1 \quad 0)^T$. We wish to assign $\lambda_{1,2} = \pm \mathrm{i}$ to $\mu_{1,2} = -1 \pm \mathrm{i}$ and leave $\lambda_{3,4}$ unchanged. Then by (15)

$$\boldsymbol{\beta} = \begin{pmatrix} 0.5 + \mathrm{i} \\ 0.5 - \mathrm{i} \end{pmatrix}. \tag{34}$$

By (29)

$$\mathbf{G}_1 = \frac{1}{15}\begin{bmatrix} -7+\mathrm{i} & -2-4\mathrm{i} & 4+3\mathrm{i} & -1+3\mathrm{i} \\ -7-\mathrm{i} & -2+4\mathrm{i} & 4-3\mathrm{i} & -1-3\mathrm{i} \end{bmatrix}, \tag{35}$$

and

$$\begin{bmatrix} \mathbf{MV}_1\mathbf{\Lambda}_1 \\ -\mathbf{KV}_1 \end{bmatrix} = \begin{bmatrix} \mathrm{i} & -\mathrm{i} \\ \mathrm{i} & -\mathrm{i} \\ -1 & -1 \\ -1 & -1 \end{bmatrix}. \tag{36}$$

Therefore,

$$\mathbf{G}_1 \begin{bmatrix} \mathbf{MV}_1\mathbf{\Lambda}_1 \\ -\mathbf{KV}_1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -5\mathrm{i} & -2+\mathrm{i} \\ -2-\mathrm{i} & 5\mathrm{i} \end{bmatrix}, \tag{37}$$

and by (31)

$$\boldsymbol{\beta} = \left(\mathbf{G}_1 \begin{bmatrix} \mathbf{MV}_1\mathbf{\Lambda}_1 \\ -\mathbf{KV}_1 \end{bmatrix}\right)^{-1}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{4}\begin{bmatrix} 5\mathrm{i} & 2-\mathrm{i} \\ 2+\mathrm{i} & -5\mathrm{i} \end{bmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0.5+\mathrm{i}\\0.5-\mathrm{i}\end{pmatrix}, \tag{38}$$

which is the same as (34). From (30) we have

$$\mathbf{f} = \begin{pmatrix}-2\\-2\end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix}-1\\-1\end{pmatrix}, \tag{39}$$

so that (8) gives

$$\begin{bmatrix} \mu^2+3\mu+4 & \mu-1 \\ -\mu-2 & \mu^2+\mu+3 \end{bmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{40}$$

The characteristic polynomial of (40) is

$$\mu^4 + 4\mu^3 + 11\mu^2 + 14\mu + 10 = 0 \tag{41}$$

with roots, as requested,

$$\mu_{1,2} = -1 \pm \mathrm{i}, \quad \mu_{3,4} = -1 \pm 2\mathrm{i}. \tag{42}$$

The method described in Example 1 for partial pole placement is useful in the case where the measured receptances from the real structure are combined with an analytical model with uncertainties. The method presented reconciles the data coming from these two sources.

Here we use this result as a stepping stone for solving the problem of partial pole placement with time delay.

5. Partial pole placement with time delay

In this section, we extend the hybrid method to solve the partial pole placement with time delay. This problem cannot be solved by the results in [2], [4] or [5].

In the model of feedback control described by (2) and (3) the time delay between the measurements of the state and the implementation of control was ignored. In practice, time delay between measurements and actuation, which is depended on the measured data, is unavoidable. With time delay the model (2) and (3) is changed to,

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}\dot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) = \mathbf{b}u(t-\tau), \tag{43}$$

where

$$u(t-\tau) = \mathbf{f}^T\dot{\mathbf{y}}(t-\tau) + \mathbf{g}^T\mathbf{y}(t-\tau), \tag{44}$$

and where τ is the known time delay. Separation of variables,

$$\mathbf{y}(t) = \mathbf{w}e^{\mu t}, \tag{45}$$

gives in this case

$$(\mu^2\mathbf{M} + \mu\mathbf{C} + \mathbf{K})\mathbf{w}e^{\mu t} = \mathbf{b}(\mu\mathbf{f}^T + \mathbf{g}^T)\mathbf{w}e^{\mu(t-\tau)}, \tag{46}$$

which leads to the transcendental eigenvalue problem

$$(\mu^2\mathbf{M} + \mu(\mathbf{C} - e^{-\mu\tau}\mathbf{b}\mathbf{f}^T) + \mathbf{K} - e^{-\mu\tau}\mathbf{b}\mathbf{g}^T)\mathbf{w} = \mathbf{0}. \tag{47}$$

Ram et al. [5] have shown that $2n$ poles of the time delayed system may be assigned as follows. Let $\widetilde{\mathbf{H}}(s)$ be the receptance matrix associated with the closed-loop delayed eigenvalue problem (47). Then, the Sherman–Morrison formula gives

$$\widetilde{\mathbf{H}}(s) = (s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} - e^{-s\tau}\mathbf{b}(\mathbf{g}^T + s\mathbf{f}^T))^{-1} = \mathbf{H}(s) + \frac{e^{-s\tau}\mathbf{H}(s)\mathbf{b}(\mathbf{g}^T + s\mathbf{f}^T)\mathbf{H}(s)}{1 - e^{-s\tau}(\mathbf{g}^T + s\mathbf{f}^T)\mathbf{H}(s)\mathbf{b}} \tag{48}$$

From the condition $|\widetilde{\mathbf{H}}(\mu_k)| \to \infty$ we have

$$(\mathbf{g}^T + \mu_k\mathbf{f}^T)\mathbf{H}(\mu_k)\mathbf{b} = e^{\mu_k\tau}, \quad k = 1, 2, \ldots, 2n. \tag{49}$$

So that assignment of $2n$ poles is achieved by solving the linear set of equations

$$\begin{bmatrix} \mu_1\mathbf{b}^T\mathbf{H}(\mu_1) & \mathbf{b}^T\mathbf{H}(\mu_1) \\ \mu_2\mathbf{b}^T\mathbf{H}(\mu_2) & \mathbf{b}^T\mathbf{H}(\mu_2) \\ \vdots & \vdots \\ \mu_{2n}\mathbf{b}^T\mathbf{H}(\mu_{2n}) & \mathbf{b}^T\mathbf{H}(\mu_{2n}) \end{bmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} e^{\mu_1\tau} \\ e^{\mu_2\tau} \\ \vdots \\ e^{\mu_{2n}\tau} \end{pmatrix}, \tag{50}$$

or

$$\mathbf{G}\begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \mathbf{q}, \tag{51}$$

with the obvious definition of $\mathbf{q}$.

Lemma 4. *With the control vectors* $\mathbf{f}$ *and* $\mathbf{g}$ *given by (13) and (14), where*

$$\boldsymbol{\beta} = \left(\mathbf{G}_1\begin{bmatrix} \mathbf{M}\mathbf{V}_1\boldsymbol{\Lambda}_1 \\ -\mathbf{K}\mathbf{V}_1 \end{bmatrix}\right)^{-1}\mathbf{q}_1, \quad \mathbf{q}_1 = (e^{\mu_1\tau} \quad e^{\mu_2\tau} \quad \cdots \quad e^{\mu_m\tau})^T, \tag{52}$$

the sets $\{\mu_k\}_{k=1}^m$ *and* $\{\lambda_k\}_{k=m+1}^{2n}$ *are included in the spectrum of the transcendental eigenvalue problem (47).*

Proof. We first write (47) in the form

$$(\mu^2\mathbf{M} + \mu\mathbf{C} + \mathbf{K} - e^{-\mu\tau}\mathbf{b}\mathbf{h}^T(\mu))\mathbf{w} = \mathbf{0}, \tag{53}$$

where $\mathbf{h}(\mu)$ is given by (17). Then

$$(\lambda_r^2\mathbf{M} + \lambda_r\mathbf{C} + \mathbf{K} - e^{-\lambda_r\tau}\mathbf{b}\mathbf{h}^T(\lambda_r))\mathbf{v}_r = -e^{-\lambda_r\tau}\mathbf{b}\mathbf{h}^T(\lambda_r)\mathbf{v}_r, \quad r \in \{k\}_{k=m+1}^{2n} \tag{54}$$

by virtue of (7). From the bi-orthogonal relations (22) we have $\mathbf{h}^T(\lambda_r)\mathbf{v}_r = 0, r \in \{k\}_{k=m+1}^{2n}$, as in (21). It thus follows that the set $\{\lambda_k\}_{k=m+1}^{2n}$ is included in the spectrum of (46).

By the assignment (13), (14) and (52), stipulated in the lemma, the set $\{\mu_k\}_{k=1}^m$ is included in the spectrum of (46) as well. □

Example 2. Consider the system studied in Example 1 with time delay $\tau = 0.1$. As before we wish to assign $\lambda_{1,2} = \pm\mathrm{i}$ to $\mu_{1,2} = -1 \pm \mathrm{i}$ and leave $\lambda_{3,4}$ unchanged, where $\mathbf{b} = (1 \quad 0)^T$.

Here (52) gives,

$$\boldsymbol{\beta} = \frac{1}{4}\begin{bmatrix} 5\mathrm{i} & 2 - \mathrm{i} \\ 2 + \mathrm{i} & -5\mathrm{i} \end{bmatrix}\begin{pmatrix} e^{0.1(-1+\mathrm{i})} \\ e^{0.1(-1-\mathrm{i})} \end{pmatrix} \cong \begin{pmatrix} 0.314659 + 0.855150\mathrm{i} \\ 0.314659 - 0.855150\mathrm{i} \end{pmatrix}, \tag{55}$$

by virtue of (38). The control gain vectors are then determined by (13) and (14),

$$\mathbf{f} \cong \begin{pmatrix} -1.710301 \\ -1.710301 \end{pmatrix}, \quad \mathbf{g} \cong \begin{pmatrix} -0.629318 \\ -0.629318 \end{pmatrix}. \tag{56}$$

For confirmation

$$|\det(\mu_k^2\mathbf{M} + \mu_k(\mathbf{C} - e^{-0.1\mu_k}\mathbf{b}\mathbf{f}^T) + \mathbf{K} - e^{-0.1\mu_k}\mathbf{b}\mathbf{g}^T)| < 10^{-13}, \quad k = 1, 2, 3, 4. \tag{57}$$

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