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Nonlinear spacing policies for vehicle platoons: A geometric approach to decentralized control

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ABSTRACT

In this paper a decentralized approach to the platooning problem with nonlinear spacing policies is considered. A predecessor–follower control structure is presented in which a vehicle is responsible for tracking of a desired spacing policy with respect to its predecessor, regardless of the control action of the latter. From the perspective of geometric control theory, we state necessary and sufficient conditions for the existence of decentralized controllers that guarantee tracking and asymptotic stabilization of a general nonlinear spacing policy. Moreover, all nonlinear spacing policies for which there exists a decentralized state feedback controller that achieves asymptotic tracking are characterized. It is shown that string stability is a consequence of the choice of spacing policy and sufficient conditions for a spacing policy to imply string stability are given. As an example, we fully characterize all state feedback controllers that achieve the control goals for a given nonlinear spacing policy, guaranteeing asymptotic tracking for a heterogeneous platoon. The results are illustrated through simulations.

1. Introduction

Vehicle platooning amounts to the formation of closely-spaced groups of vehicles and has the potential to increase road safety, improve traffic flow, and reduce fuel consumption, see the overviews [1–4]. Consequently, the automatic control of vehicles for platooning has been studied extensively, *e.g.*, [5–9].

A key element in such control strategies for vehicle platooning is the specification of the desired distance between successive vehicles, known as the *spacing policy*. Well-known examples are the constant spacing policy [10], where a constant distance between vehicles is desired, and the constant headway spacing policy [11, 12], where the desired spacing is dependent on the velocity of the follower vehicle. These popular spacing policies have in common that they are *linear*, *i.e.*, the desired inter-vehicle distance is a linear function of the states of two successive vehicles. In this paper we focus on the design of vehicle controllers for *nonlinear* spacing policies to form a more general class of control strategies for platooning.

Although there is a very rich literature on nonlinear spacing policies as *models* of *human* driving behavior, see, *e.g.*, [13,14] for early results and [15] for an overview, their use as *desired* inter-vehicle distance for *automated* vehicles has received less attention. An exception is [16], where it is motivated that nonlinear spacing policies can reduce nominal inter-vehicle spacings as well as transient errors. Similarly, [17] proposes a nonlinear spacing policy aimed at increasing traffic flow capacity, see also [18]. Control for general nonlinear spacing policies is considered in [19]. For alternatives to linear spacing policies based on desired time delays between vehicles, see [20,21].

Motivated by these potential advantages of nonlinear spacing policies in vehicle platooning, this paper considers the decentralized platoon control for general nonlinear spacing policies. The main contributions are the following.

First, as a motivating example, we show that a spacing policy that is quadratic in the velocity of the following vehicle can be used to enforce a lower bound on the acceleration of the follower vehicle. When well-designed, such nonlinear spacing policy guarantees that no collisions occur even in case of limited braking capacity. This is of particular relevance in *heterogeneous* vehicle platoons, *e.g.*, for heavy-duty vehicles with distinct vehicle properties.

Second, we develop an approach for decentralized control of platoons using a predecessor–follower control structure. Following an idea from [22], the follower vehicle is responsible for maintaining the desired spacing policy with respect to its predecessor, *regardless* of the control actions of this predecessor. This robustness approach towards decentralized control allows for considering heterogeneous platoons.

Third, using tools from nonlinear geometric control theory, [23,24], we characterize *all* nonlinear spacing policies for which

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a decentralized controller can be synthesized. This is achieved by employing necessary and sufficient conditions for the existence of a controller for a given nonlinear spacing policy. Such a decentralized controller is given explicitly and is inherently nonlinear. Here, we note that the use of nonlinear controllers for platoon control has also been considered in, *e.g.*, [25,26], although popular linear spacing policies are generally considered in these works.

Fourth, for a class of nonlinear spacing policies, we show that string stability can be guaranteed. String stability, *e.g.*, [21,27–29], is a crucial performance criterion in vehicle platoons as it guarantees the attenuation of perturbations propagating through the platoon. In this paper it will be shown that, due to the decentralized control design, string stability is not a consequence of the choice of controller, but rather a property of the spacing policy.

The remainder of this paper is organized as follows. Section 2 provides a motivational example of a nonlinear spacing policy before Section 3 provides a detailed problem statement. The main results are stated in Section 4. Since the results are build on nonlinear geometric control theory, a brief review is given in Appendix A. Section 5 illustrates the results and conclusions are stated in Section 6.

2. A motivation for nonlinear spacing policies

Consider a platoon of $N+1$ vehicles with the dynamics

$$\begin{aligned} \dot{s}_i &= v_i, \\ \dot{v}_i &= a_i, \\ \tau_i \dot{a}_i &= -a_i + u_i, \end{aligned} \tag{1}$$

for $i \in \{0, 1, \ldots, N\}$. Here, s_i, v_i, and a_i (all in $\mathbb{R}$) are the longitudinal position, velocity and acceleration of vehicle i, respectively. The control input $u_i \in \mathbb{R}$ can be regarded as the desired acceleration. The time constant $\tau_i > 0$ represents the engine dynamics and are not necessarily identical for each vehicle. Hence a heterogeneous platoon is considered. This extends the model slightly compared to the models used in *e.g.*, [6,30]. We also note that (1) could be the result of applying feedback linearization to a nonlinear model, see [6]. For the remainder of this paper, the state of vehicle i is denoted by $\xi_i = \begin{bmatrix} s_i & v_i & a_i \end{bmatrix}^\top \in \mathbb{R}^3$ and the full state of the platoon is collected as $\xi = \begin{bmatrix} \xi_0^\top & \cdots & \xi_N^\top \end{bmatrix}^\top \in \mathbb{R}^{3(N+1)}$.

The distance between vehicle i and its predecessor with index $i-1$ is denoted as

$$\Delta_i = s_{i-1} - s_i, . \tag{2}$$

The *desired* or *reference* inter-vehicle distance is defined as a function $\Delta_i^{\mathrm{ref}} : \mathbb{R}^4 \to \mathbb{R}$ of $(v_{i-1}, a_{i-1}, v_i, a_i)$. This function is usually referred to as the *spacing policy*. The spacing error $z_i : \mathbb{R}^6 \to \mathbb{R}$ is then naturally defined by

$$z_i = \Delta_i - \Delta_i^{\mathrm{ref}}. \tag{3}$$

Common examples of *linear* spacing policies are the constant spacing policy, where $\Delta_i^{\mathrm{ref}} = d_0$, *e.g.*, [10], for some standstill distance $d_0 \in \mathbb{R}$, and the constant headway spacing policy given by

$$\Delta_i^{\mathrm{ref}} = d_0 + \lambda v_i, \tag{4}$$

with $\lambda > 0$, *e.g.*, [11,12].

In the case of a linear spacing policy, *e.g.*, (4), ensuring perfect tracking (*i.e.*, $\Delta_i = \Delta_i^{\mathrm{ref}}$ for all t) might require an acceleration or deceleration which is undesirable in practice. In the case of *e.g.*, an emergency stop, the acceleration $a_i(t)$ might need to subceed a certain bound *i.e.*, $a_i(t) < \bar{a}_i$ for some $\bar{a}_i < 0$ in order to maintain perfect tracking. However, it might be preferable that

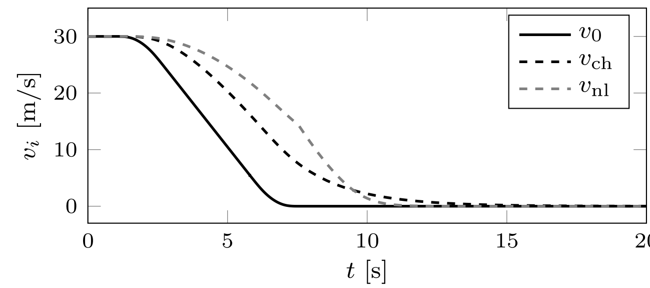

Fig. 1. Simulation of the velocity profile of a two-vehicle model (7) when perfect tracking is achieved for spacing policies (4) and (5), corresponding to the subscripts ch and nh respectively, time constants $\tau_1 = \tau_0 = 1$, time headway $\lambda = 2$, and, $\gamma = 0.1$.

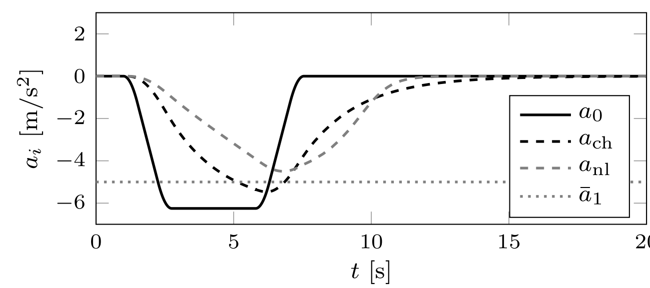

Fig. 2. Simulation of the acceleration profile of a two-vehicle model (7) when perfect tracking is achieved for spacing policies (4) and (5), for the parameters as in Fig. 1.

the acceleration of a following vehicle remains above $\bar{a}_i$, i.e. $\bar{a}_i \leqslant a_i(t)$ for all t.

The problem described above can be overcome by a nonlinear spacing policy, for example the *nonlinear headway spacing policy* defined for $\lambda > 0$ by

$$\Delta_i^{\mathrm{ref}} = d_0 + \lambda v_i + \gamma v_i^2. \tag{5}$$

Note that (4) is recovered from (5) in the case $\gamma = 0$. In the case of perfect tracking it follows that $\dot{\Delta}_i - \dot{\Delta}_i^{\mathrm{ref}} = 0$ for all t as well. Assuming that the platoon drives with a positive velocity, this then leads to

$$a_i(t) = \dot{v}_i(t) = \frac{-v_i(t) + v_{i-1}(t)}{(\lambda + 2\gamma v_i(t))} \geqslant \frac{-v_i(t)}{2\gamma v_i(t)} = \frac{1}{-2\gamma}.$$

Hence choosing $\gamma = -\frac{1}{2}\bar{a}_i^{-1}$ guarantees $a_i(t) \geqslant \bar{a}_i$ for all t. Furthermore, by choosing d_0 and λ appropriately such that $\Delta_i^{\mathrm{ref}} > 0$, collisions are guaranteed to be avoided in the case of perfect tracking.

Figs. 1 and 2 show a simulation comparison between the nonlinear headway and constant headway spacing policies in the case of perfect tracking (*i.e.*, $\Delta_i = \Delta_i^{\mathrm{ref}}$ for all t).

Fig. 2 illustrates that the acceleration $a_1(t)$ subceeds the value $\bar{a}_1$ when the constant headway spacing policy is tracked perfectly. However, Fig. 2 also shows that a nonlinear spacing policy, such as given by (5), ensures a safe emergency distance and enforces the acceleration to be greater than $\bar{a}_1$, even when the nominal spacing (*i.e.*, the inter-vehicle distance at a given nominal velocity) is the same as for the constant headway policy.

This example shows that nonlinear spacing policies potentially increase safety and it motivates the investigation of controller design for nonlinear spacing policies. In this paper the focus is on decentralized controller design. It will be shown that in the case of the nonlinear headway spacing policy it is possible to design a decentralized (nonlinear) controller that achieves asymptotic tracking. More generally, this paper characterizes *all* nonlinear spacing policies that can be asymptotically tracked by means of a decentralized controller.

3. Problem statement

As mentioned in the previous section, a *decentralized* approach towards controller design is pursued, in which a vehicle i is responsible for achieving the desired spacing with respect to its predecessor $i-1$ using only local measurements (*i.e.*, vehicle i has access to measurements on the state of its own and its predecessor). Moreover, the aim is to design the control for vehicle i such that it is robust with respect to the input of vehicle $i-1$. In that case, while vehicle i maintains the desired inter-vehicle distance with respect to its predecessor, the latter remains free to choose its control input u_{i-1} to achieve its own control objectives (e.g., tracking of a desired spacing policy with respect to vehicle $i-2$). Given this approach, which is similar to the control design in [22], it is sufficient to design a controller for two consecutive vehicles. Hence we consider a platoon consisting of only two vehicles (*i.e.*, $N=1$). The state of this platoon is given by

$$\begin{aligned} x &= [\xi_0^\top \quad \xi_1^\top \;]^\top \\ &= [s_0 \quad v_0 \quad a_0 \quad s_1 \quad v_1 \quad a_1]^\top \\ &= [x_1 \quad \cdots \quad x_6]^\top \end{aligned} \tag{6}$$

with the corresponding dynamics

$$\dot{x}(t) = Ax(t) + Bu_1(t) + Eu_0(t), \tag{7}$$

where

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}, \quad E = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}. \tag{8}$$

Here, A_i and B_i follow directly from (1) and are given by

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\tau_i^{-1} \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 \\ 0 \\ \tau_i^{-1} \end{bmatrix}. \tag{9}$$

Note that the spacing error, can be compactly written as a nonlinear output of (7) as

$$z(t) = h(x(t)). \tag{10}$$

Here we omit the index of z for the sake of notation, as will be done for Δ and Δ^{ref} as well.

As an example, using (2) and (5), it is readily observed that the nonlinear headway spacing policy leads to

$$h(x) = x_1 - x_4 - \lambda x_5 - \gamma x_5^2 - d_0. \tag{11}$$

For analytical purpose it is assumed in the remainder of the paper that $\Delta^{\mathrm{ref}}(0) = 0$. This assumption is without loss of generality, since Δ^{ref} can be regarded as an affine shift of a nominal spacing policy $\tilde{\Delta}^{\mathrm{ref}}$, *i.e.*, $\Delta^{\mathrm{ref}} = \tilde{\Delta}^{\mathrm{ref}} - \tilde{\Delta}^{\mathrm{ref}}(0)$. Simultaneously regarding s_1 as a deviation of a nominal position $\tilde{s}_1$, *i.e.*, $s_1 = \tilde{s}_1 + \tilde{\Delta}^{\mathrm{ref}}(0)$, and defining $s_0 = \tilde{s}_0$ and $\tilde{\Delta} = \tilde{s}_0 - \tilde{s}_1$ then yields

$$z(t) = \tilde{\Delta}(t) - \tilde{\Delta}^{\mathrm{ref}}(t) = \Delta(t) - \Delta^{\mathrm{ref}}(t).$$

This change of coordinates does not alter the model (7) due to the linearity of the dynamics, since $\tilde{\Delta}^{\mathrm{ref}}(0)$ is constant and hence $\dot{s}_1 = \dot{\tilde{s}}_1$. This shows that the consideration of Δ^{ref}, with $\Delta^{\mathrm{ref}}(0) = 0$ is indeed without loss of generality, given a proper change of coordinates. It is noted that in these coordinates $x(t) = 0$ and $z(t) = 0$ do not imply a collision of vehicles, but guarantee a safe standstill distance $\tilde{\Delta}^{\mathrm{ref}}(0)$ and perfect tracking of the nominal spacing policy $\tilde{\Delta}^{\mathrm{ref}}$ respectively.

The control objective of tracking and asymptotic stabilization of a given spacing policy can now be defined as follows.

Definition 1. Consider the platoon (7) and a spacing policy Δ^{ref} satisfying $\Delta^{\mathrm{ref}}(0,0,0,0) = 0$. Then, a controller $u_1 = \alpha(x) + \beta(x)\nu$ is said to

(i) *track* the spacing policy if for any $u_0(\cdot)$ and with $x(0) = 0$, it holds that $z(t) = 0$ for all $t \geqslant 0$;
(ii) *asymptotically stabilize* the spacing policy if for any $u_0(\cdot)$, $x(0) \in \mathbb{R}^6$, it holds that $\lim_{t\to\infty} z(t) = 0$;
(iii) achieve *string stability* if for any $u_0(\cdot)$ and with $x(0) = 0$, the following holds for all $T > 0$:

$$\int_0^T |v_1(t)|^2 \, dt \leqslant \int_0^T |v_0(t)|^2 \, dt. \tag{12}$$

The definition of string stability in (12) ensures disturbance attenuation in v_i when disturbances propagate through the platoon. This definition in terms of the velocities is in line with the classical literature on string stability *e.g.*, [27,31]. Although in [27] string stability is defined in terms of the positions s_i and s_{i-1}, this definition is equivalent to the definition in terms of the velocities given the model (1). Definitions in terms of the spacing error are also not uncommon, *e.g.*, [11,16], though it is recognized in *e.g.*, [29,32] that the definition in terms of the velocities is relevant as well.

Definition 1 leads to the following control problem.

Problem 1. Given (7) and spacing error (10) resulting from the spacing policy Δ^{ref}, find a (nonlinear) state feedback $u_1 = \alpha(x) + \beta(x)\nu$ such that the closed-loop system

$$\dot{x}(t) = Ax(t) + B\big(\alpha(x(t)) + \beta(x(t))\nu(t)\big) + Eu_0(t), \tag{13}$$

satisfies the following properties for any $u_0(\cdot)$:

(i) $x(0) = 0$ implies $z(t) = 0$ for all $t \geqslant 0$;
(ii) for all $x(0) \in \mathbb{R}^6$, it holds that $\lim_{t\to\infty} z(t) = 0$.

Clearly, properties (i) and (ii) in Problem 1 correspond to the objectives of tracking and asymptotic stabilization of the spacing policy as in items (i) and (ii) in Definition 1.

Even though there is no requirement on string stability in Problem 1, it will be shown that if property (i) is guaranteed, property (iii) is a consequence of the choice of Δ^{ref} rather than of the controller.

Before addressing Problem 1 in Section 4, it is noted that properties (i) and (ii) can be regarded as a disturbance decoupling problem with asymptotic stability. These are classical problems in nonlinear geometric control theory, of which a brief review is given in the Appendix.

Remark 2. As this problem fits within the framework of nonlinear geometric control theory, one could also assume a nonlinear vehicle model that is affine in the input, *i.e.*, of the form

$$\dot{x} = f(x) + g(x)u.$$

However, as the vehicle model (1) is most commonly used in the literature on platooning, it is adopted in this paper as well.

4. Decentralized nonlinear state feedback

4.1. General nonlinear spacing policies

Before designing a controller for tracking of the nonlinear headway spacing policy (11), we will first present a general result. We characterize all spacing policies for which part (i) of Problem 1 can be solved and then show that, for these spacing policies, solvability of the second part of Problem 1 is implied by the first part. To that extent we introduce the following concepts of nonlinear geometric control theory, which are crucial in the remainder of the section.

For a function $h : \mathbb{R}^6 \to \mathbb{R}$ and a smooth vector field $f : \mathbb{R}^6 \to \mathbb{R}^6$ the *Lie derivative* or *total derivative* of h along f is given by

$$L_f h(x) = \frac{\partial h}{\partial x}(x) f(x) \tag{14}$$

Note that $L_f h : \mathbb{R}^6 \to \mathbb{R}$. The *repeated Lie derivative* is then iteratively given by

$$L_f^k h(x) = L_f L_f^{k-1} h(x), \quad k = 1, 2, \ldots \tag{15}$$

where $L_f^0 h(x) = h(x)$. Given the definition of a Lie derivative, one can define the following relation between the input u_1 of (7) and the output (10).

Definition 3. The smallest integer ρ such that

$$L_B L_{Ax}^{\rho-1} h(x)$$

is different from the zero-function, is called the relative degree of the system (7) with output (11).

A more extended, yet brief review of nonlinear geometric control theory that is required for the proofs of the results is given in Appendix A.

In the remainder of the paper, we will omit the dependence on x of all functions and vector fields for the sake of notational convenience. Furthermore, all conditions are assumed to hold globally, *i.e.*, $f(x) = 0$ and $f(x) \neq 0$ means $f(x) = 0$ for all $x \in \mathbb{R}^6$ and $f(x)$ is not the zero-function, respectively, unless stated otherwise.

Theorem 4. *Consider the dynamics in (7) with the spacing error (3). Then, part (i) of Problem 1 has a solution if and only if*

$$\frac{\partial \Delta^{\mathrm{ref}}}{\partial a_0} = 0 \tag{16}$$

and the following implication holds

$$\frac{\partial \Delta^{\mathrm{ref}}}{\partial a_1} = 0 \implies \frac{\partial \Delta^{\mathrm{ref}}}{\partial v_0} = 0 \text{ and } \frac{\partial \Delta^{\mathrm{ref}}}{\partial v_1} \neq 0. \tag{17}$$

Proof. The proof is given in Appendix B. □

Remark 5. Condition (17) in Theorem 4 corresponds to two possible values of the relative degree ρ. Indeed, in the case that $\frac{\partial \Delta^{\mathrm{ref}}}{\partial a_1} \neq 0$, it follows from the proof of Theorem 4 that $L_B h(x) \neq 0$ and $\rho = 1$. Alternatively, if $\frac{\partial \Delta^{\mathrm{ref}}}{\partial a_1} = 0$, but $\frac{\partial \Delta^{\mathrm{ref}}}{\partial v_1} \neq 0$ then $L_B L_{Ax} h(x) \neq 0$ and hence $\rho = 2$.

Remark 6. In the case of $\rho = 1$ it holds that

$$\frac{\partial \Delta^{\mathrm{ref}}}{\partial v_0} = \frac{\partial \Delta^{\mathrm{ref}}}{\partial a_0} = \frac{\partial \Delta^{\mathrm{ref}}}{\partial a_1} = 0.$$

Consequently, $\Delta^{\mathrm{ref}} = \psi(v_1)$, *i.e.*, the spacing policy is a function depending on the velocity of the follower vehicle only.

The conditions (16)–(17) allow for an insightful interpretation. Namely, (16) states that the spacing policy should be independent of the acceleration of the predecessor, where we note that this is the state on which the control input u_0 acts. Similarly, (17) requires that the spacing policy is independent of the velocity of the predecessor if information on the acceleration of the follower is not included.

The following result states that the conditions of Theorem 4 are sufficient to also solve part (ii) of Problem 1.

Theorem 7. *Consider the dynamics in (7) and let Δ^{ref} be the spacing policy. If part (i) of Problem 1 can be solved, then part (i) and part (ii) can be solved simultaneously.*

Proof. The proof is given in Appendix C. □

As a consequence of feedback linearization, we can state the next result, which characterizes all inputs that achieve perfect tracking.

Corollary 8. *Consider the dynamics in (7). Let Δ^{ref} be a spacing policy and ρ be the relative degree. Perfect tracking is achieved if and only if for some function $\phi(\cdot)$ with the property that $z(t) = 0$ for all $t \geqslant 0$ implies $\phi(t) = 0$ for all $t \geqslant 0$, the control $u_1(t)$ is of the form*

$$u_1(t) = (L_B L_{Ax}^{\rho-1} h(x))^{-1} (\phi(t) - L_{Ax}^{\rho} h(x(t))). \tag{18}$$

Proof. Suppose perfect tracking is achieved by $u_1(\cdot)$. Then $z(t) = 0$ for all $t \geqslant 0$ implies $\dot{z}(t) = 0$ and $\ddot{z}(t) = 0$ for all $t \geqslant 0$. Omitting the time dependence, this means that in the case $\rho = 1$, *i.e.*, $L_B h(x) \neq 0$, we have that

$$\begin{aligned} \dot{z}(t) &= \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}(Ax + Bu_1 + Gu_0) \\ &= L_{Ax} h(x) + L_B h(x) u_1 \\ &= 0. \end{aligned}$$

Now consider the case $\rho = 2$, *i.e.*, $L_B L_{Ax} h(x) \neq 0$. Then we obtain

$$\ddot{z}(t) = L_{Ax}^2 h(x) + L_B L_{Ax} h(x) u_1 = 0.$$

Consequently $u_1(t) = (L_B L_{Ax}^{\rho-1} h(x))^{-1}(\phi(t) - L_{Ax}^{\rho} h(x(t)))$ where $z(t) = \dot{z}(t) = 0$ implies $\phi(t) = 0$.

Conversely, it is readily verified that if $u_1(\cdot)$ is of the form (18) perfect tracking is achieved. □

Corollary 8 allows for a straightforward characterization of a general class of controllers that achieve perfect tracking and stabilization of the spacing error. However, as Problem 1 requires, we will confine ourselves to state-feedback controller of the form $u_1 = \alpha(x) + \beta(x)\nu$.

Corollary 9. *Consider the dynamics in (7) and let the spacing error (10) satisfy the conditions of Theorem 4. Then in the case $\rho = 1$, Problem 1 is solved by*

$$u_1 = (L_B h)^{-1} \left(\phi_1(z) - L_{Ax} h\right), \tag{19}$$

where $\phi_1(z)$ is such that the origin is a globally asymptotically stable equilibrium of the differential equation $\dot{z} = \phi_1(z)$. In the case $\rho = 2$, Problem 1 is solved by

$$u_1 = (L_B L_{Ax} h)^{-1} \left(\phi_2(z, \dot{z}) - L_{Ax}^2 h\right), \tag{20}$$

where $\phi_2(z, \dot{z})$ is such that the origin is a globally asymptotically stable equilibrium of $\ddot{z} = \phi_2(z, \dot{z})$.

Example 10. A simple (linear) function which globally asymptotically stabilizes the origin of $\dot{z} = \phi_1(z)$ is the function $\phi_1(z) = -\theta z$ for some $\theta > 0$. Similarly $\phi_2(z, \dot{z}) = -\theta_1 z - \theta_2 \dot{z}$ is a typical example of a function that globally asymptotically stabilizes the origin of $\ddot{z} = \phi_2(z, \dot{z})$ if $\theta_1, \theta_2 > 0$.

Remark 11. The convergence (to zero) of the spacing error $z = h(x)$ is determined by the choice of ϕ_ρ, as follows from (C.2). Hence, the function ϕ_ρ can be designed to achieve desired behavior. For example, if an additional vehicle is attached to the platoon, but it is far behind, the (relatively large) spacing error can be reduced with mild control action, *e.g.*, a controller ensuring a bounded acceleration $a_i(t)$ for all t. On the other hand, if the spacing error is very small, due to a disturbance in Δ_i, more aggressive control action is desirable due to safety reasons.

4.2. Internal dynamics and zero dynamics

By construction, the controllers of Corollary 8 achieve input–output linearization with respect to the output $z = h(x)$ and a virtual input v that is chosen as $v = \phi_2(z, \dot{z})$ in case $\rho = 2$. In this section we study the remaining internal dynamics.

To identify the relevant remaining internal dynamics, we note that the lead vehicle cannot be controlled and we are only interested in the behavior of the follower vehicle *with respect to* this lead vehicle. To make this more explicit, consider the case $\rho = 2$ and the change of coordinates

$$\begin{bmatrix} s_0 \\ v_0 \\ a_0 \\ z \\ \dot{z} \\ \zeta \end{bmatrix} = S(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ h(x) \\ L_{Ax}h(x) \\ \lambda(x) \end{bmatrix} = \begin{bmatrix} s_0 \\ v_0 \\ a_0 \\ s_0 - s_1 - \psi(v_1) \\ v_0 - v_1 - \frac{\partial \psi}{\partial v_1}(v_1)a_1 \\ \lambda(x) \end{bmatrix},$$

where we have used that necessarily $\Delta^{\text{ref}} = \psi(v_0)$ for some ψ, see Remark 6. Now, the remaining dynamics with state ζ should be chosen such that (1) the function S is invertible (*i.e.,* S is a coordinate transformation) and (2) it is relevant for describing vehicle-following behavior.

Clearly, the choice

$$\zeta = \lambda(x) = v_0 - v_1, \tag{21}$$

satisfies these two criteria. Its dynamics is readily obtained from (7) and (10) as

$$\dot{\zeta} = a_0 - a_1 = -\left(\frac{\partial \psi}{\partial v_1}(v_1)\right)^{-1} (\zeta - \dot{z}) + a_0, \tag{22}$$

leading to the zero dynamics

$$\dot{\zeta} = a_0 - a_1 = -\left(\frac{\partial \psi}{\partial v_1}(v_1)\right)^{-1} \zeta + a_0. \tag{23}$$

Now, we immediately obtain the following result.

Lemma 12. *Consider the spacing policy $\Delta^{\text{ref}} = \psi(v_1)$ and assume that there exists $\delta > 0$ such that*

$$\frac{\partial \psi}{\partial v_1}(v_1) \geqslant \delta \tag{24}$$

for all v_1. Then, the zero dynamics (23) is input-to-state stable with respect to the input a_0.

Hence, under the condition of Lemma 12, the state $\zeta = v_0 - v_1$ remains bounded if the predecessor vehicle acceleration a_0 is bounded, thus leading to desirable vehicle-following behavior.

A similar, but more involved analysis can be done for the case $\rho = 1$. It can be shown that

$$\zeta_1 = \lambda_1(x) = v_0 - v_1, \qquad \zeta_2 = \lambda_2(x) = a_0 - a_1, \tag{25}$$

yield valid and relevant internal states. The controller as in (19) leads to the following internal dynamics

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \frac{\partial \Delta^{\text{ref}}}{\partial a_1}\dot{\zeta}_2 &= \phi_0(z) - \zeta_1 - \frac{\partial \Delta^{\text{ref}}}{\partial v_1}\zeta_2 + \frac{1}{\tau_0}\frac{\partial \Delta^{\text{ref}}}{\partial a_1}u_0 \\ &\quad + \left(\frac{\partial \Delta^{\text{ref}}}{\partial v_0} + \frac{\partial \Delta^{\text{ref}}}{\partial v_1} - \frac{1}{\tau_0}\frac{\partial \Delta^{\text{ref}}}{\partial a_1}\right) a_0 \end{aligned} \tag{26}$$

This yields a second order nonlinear differential equation, of which the stability properties are heavily dependent on the specific choice of spacing policy Δ^{ref}. Hence we restrict attention to the special case that the partial derivatives of Δ^{ref} are constant, *i.e.,*

$$\frac{\partial \Delta^{\text{ref}}}{\partial v_0} = h_{v_0}, \quad \frac{\partial \Delta^{\text{ref}}}{\partial v_1} = h_{v_1}, \quad \frac{\partial \Delta^{\text{ref}}}{\partial v_{a_1}} = h_{a_1},$$

for some constants h_{v_0}, h_{v_1} and $h_{a_0} \neq 0$. For this case, and for $z = 0$, the zero dynamics are given by

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ h_{a_1}\dot{\zeta}_2 &= -\zeta_1 - h_{v_1}\zeta_2 + \frac{1}{\tau_0}h_{a_1}u_0 \\ &\quad + \left(h_{v_0} + h_{v_1} - \frac{1}{\tau_0}h_{a_1}\right) a_0. \end{aligned} \tag{27}$$

This leads to the following result.

Lemma 13. *Consider the platoon (7) with the linear spacing policy $\Delta^{\text{ref}}(v_0, a_0, v_1, a_1) = h_{v_0}v_0 + h_{v_1}v_1 + h_{a_1}a_1$. Then the zero dynamics (27) is input-to-state stable with respect to the inputs u_0 and a_0 if and only if $h_{v_1} > 0$ and $h_{a_1} > 0$.*

Proof. As the internal dynamics (27) is linear, input-to-state stability is equivalent to asymptotic stability of the autonomous system

$$\begin{bmatrix} \dot{\zeta}_1 \\ h_{a_1}\dot{\zeta}_2 \end{bmatrix} = \begin{bmatrix} \zeta_2 \\ -\zeta_1 - h_{v_1}\zeta_2 \end{bmatrix} \tag{28}$$

This is the case if and only if $h_{v_1} > 0$ and $h_{a_1} > 0$. □

Remark 14. It can be shown that $\tilde{\zeta}_1 = s_0 - s_1$ and $\tilde{\zeta}_2 = v_0 - v_1$ are valid internal states as well for the case $\rho = 1$. These alternative internal states have dynamics given by

$$\begin{aligned} \dot{\tilde{\zeta}}_1 &= \tilde{\zeta}_2 \\ \dot{\tilde{\zeta}}_2 &= a_0 - a_1. \end{aligned} \tag{29}$$

As $\frac{\partial \Delta^{\text{ref}}}{\partial a_1} \neq 0$, it follows from the implicit function theorem and the relation $z = s_0 - s_1 - \psi(v_0, v_1, a_1)$ that there exists a function $\Gamma(\cdot)$ such that $a_1 = \Gamma(v_0, z, \zeta_1, \zeta_2)$. Input-to-state stability of (29) can be studied further in the case that $\Gamma(\cdot)$ can be written down explicitly or is known to have a specific structure for a given spacing policy. However, further investigation of spacing policies that result in ISS of (29) is beyond the scope of this paper.

4.3. Controller design and string stability

In the following we will show that for a properly chosen Δ^{ref}, string stability is induced via property (i) of Problem 1 and therefore is not a property of the controller. This is formalized in the next proposition.

Proposition 15. *Consider the platoon dynamics (7) with spacing policy Δ^{ref}. If there exists a controller $u_1(\cdot)$ that achieves perfect tracking and results in string stable behavior, then any controller that achieves perfect tracking results in string stable behavior.*

Proof. If the controllers $u_1(\cdot)$ and $\tilde{u}_1(\cdot)$ achieve perfect tracking, then $z(t) = 0$ for all $t \geqslant 0$. By Corollary 8 this implies $\phi(t) = \tilde{\phi}(t) = 0$ and hence $u_1(t) = \tilde{u}_1(t)$. Consequently, in the case of perfect tracking, the dynamics of vehicle i is independent of the choice of input. Hence, if $u_1(\cdot)$ results in string stable behavior, $v_1(t)$ establishes this as well. Since $u_1(\cdot)$ and $\tilde{u}_1(\cdot)$ were chosen arbitrarily, it follows that if there exists one controller achieving perfect tracking and results in string stable behavior, for any controller that achieves perfect tracking string stability is implied. □

Thus, in the case that a controller solves Problem 1, string stability is induced by properties of the spacing policy. This means that all spacing policies for which there exists a decentralized controller $u(\cdot)$ satisfying the control objective as in Definition 1, are those spacing polices that satisfy the conditions of Theorem 4 and induce string stability.

Determining which spacing policies, regardless whether they satisfy the conditions of Theorem 4, will result in string stability in the case of perfect tracking is in general difficult. However, for some special cases it is possible to state sufficient conditions. Consider for example spacing policies of the form

$$\Delta^{\mathrm{ref}} = \psi(v_1), \tag{30}$$

for some differentiable function $\psi(\cdot)$, *i.e.*, the spacing policy is a function of v_1 only. Such spacing policies satisfy the conditions of Theorem 4 as long as $\frac{d\psi}{dv_1} \neq 0$ for all v_1, *e.g.*, when $\psi(v_1)$ is a strictly monotone function of v_1. In that case Problem 1 is solvable for Δ^{ref} in (30) and we can state the following result.

Theorem 16. *Consider the platoon (7) with the nonlinear spacing policy (30). Let the controller achieve perfect tracking. If $\frac{d\psi}{dv_1}(v_1) \geqslant \varepsilon > 0$ for all v_1, we have for $x(0) = 0$ that*

$$\int_0^T |v_1(t)|^2 \, dt \leqslant \int_0^T |v_0(t)|^2 \, dt,$$

for all $T \geqslant 0$.

Proof. Since $x(0) = 0$ we have that $z(t) = 0$ for all $t \geqslant 0$ by invariance of D^*, as the controller solves Problem 1. Therefore it follows that $\dot{z}(t) = -v_1 + v_0 - \frac{d\psi}{dv_1}\dot{v}_1 = 0$ for all $t \geqslant 0$, which means that

$$\dot{v}_1 = \frac{-v_1 + v_0}{\frac{d\psi}{dv_1}}.$$

Considering the storage function $V(v_1) = \varepsilon v_1^2$, we obtain

$$\dot{V}(v_1) = \varepsilon \frac{-2v_1^2 + 2v_0 v_1}{\frac{d\psi}{dv_1}} \leqslant \varepsilon \frac{-v_1^2 + v_0^2}{\frac{d\psi}{dv_1}} \leqslant -v_1^2 + v_0^2,$$

where the final inequality follows from the fact that $\frac{d\psi}{dv_1} \geqslant \varepsilon > 0$. Thus, we obtain

$$V(v_1(T)) - V(v_1(0)) \leqslant -\int_0^T v_1^2(t) \, dt + \int_0^T v_0^2(t) \, dt.$$

Then, by noting that $V(v_1(T)) \geqslant 0$ by definition of V and $v_1(0) = 0$ due to $x(0) = 0$, the result (12) follows after rearranging terms. □

In the case that the spacing policy is also dependent on the acceleration a_1 of the follower vehicle, one could try to obtain similar results. Following the lines of the proof of Theorem 16 we see that if $\Delta^{\mathrm{ref}}(v_1, a_1)$ is perfectly tracked, the spacing error $z(t) = 0$ for all $t \geqslant 0$ and hence

$$\dot{z}(t) = -v_1 + v_0 - \frac{\partial \Delta^{\mathrm{ref}}}{\partial v_1}\dot{v}_1 - \frac{\partial \Delta^{\mathrm{ref}}}{\partial a_1}\ddot{v}_1 = 0, \tag{31}$$

for all $t \geqslant 0$. In Eq. (31) a time-varying second order differential equation can be recognized. In the case that the spacing policy is linear in v_1 and a_1, Eq. (31) yields a time invariant second order transfer function and we obtain the following result of which the proof can be found in [22].

Theorem 17. *Consider the platoon (7) with a spacing policy $\Delta^{\mathrm{ref}}(v_1, a_1)$, i.e., depending on v_1 and a_1 only. Let $\frac{\partial \Delta^{\mathrm{ref}}}{\partial v_1} = h_v > 0$ and $\frac{\partial \Delta^{\mathrm{ref}}}{\partial a_1} = h_a > 0$, for some constants h_v, h_a. Let the controller be of the form (20). Then, the platoon is string stable if and only if $h_v \geqslant \sqrt{2h_a}$.*

4.4. Example: nonlinear headway spacing policy

Returning to the nonlinear headway spacing policy (11), we have that this spacing policy satisfies the conditions of Theorem 4. This leads to the following corollary.

Corollary 18. *Given the dynamics in (7) with the nonlinear headway spacing policy (5). Then Problem 1 is solvable.*

Proof. The nonlinear headway spacing policy leads to the spacing error (11). As $\frac{\partial h}{\partial x_6} \neq 0$ in this case, we have $\rho = 2$ and obtain

$$\begin{aligned} D^* &= \ker(\mathrm{span}\{dh, dL_{Ax}h\}) \\ &= \ker \begin{bmatrix} -1 & 0 & 0 & 1 & -(h+2\gamma x_5) & 0 \\ 0 & -1 & 0 & 0 & 1+2\gamma x_6 & -(h+2\gamma x_5) \end{bmatrix}. \end{aligned}$$

It is clear that $E \in D^*$, such that, by Problem 1, there does indeed exist a feedback that solves the disturbance decoupling problem in Problem 1. It is easily verified that a controller of the form (20) with, for example $\phi_2(z, \dot{z}) = -\theta_1 z - \theta_2 \dot{z}$, where $\theta_1, \theta_2 > 0$ satisfies the conditions of Corollary 9 and hence solves Problem 1. □

The next result shows that any controller that renders D^* invariant guarantees the string stability property as in the third item in Definition 1.

Corollary 19. *Consider the platoon (7) with the nonlinear spacing policy (5). Let the controller be of the form (20). Then for all trajectories $x(\cdot)$ for which $\lambda + 2\gamma v_1(t) \geqslant \varepsilon > 0$ for all $t \geqslant 0$, we have for $x(0) = 0$ that for all $T \geqslant 0$,*

$$\int_0^T |v_1(t)|^2 \, dt \leqslant \int_0^T |v_0(t)|^2 \, dt.$$

Proof. The proof follows immediately by noting that the conditions of Theorem 16 are satisfied. □

Combining the results of Corollaries 18 and 19 yields that we can track the nonlinear headway spacing policy asymptotically with a decentralized controller while guaranteeing string stability. Moreover, Corollary 9 states that this controller is not unique. As a consequence, suitable controllers can be designed, depending on additional requirements or control objectives.

5. Illustrative simulation results

To illustrate the results obtained in the previous section, a nonlinear state feedback controller (20) is designed for the nonlinear headway spacing policy, with the function $\phi_2(z, \dot{z}) = -z - 2\dot{z}$. A simulation of the two vehicle model (7), with this controller is given in Fig. 3. As parameters, the time constants τ_i are chosen randomly from the interval $[0.6, 1.4]$, the time headway $h = 1.5$, $\gamma_1 = 0.1$ and $x_0^\top = [\,-100\ 2\ -0\ 17\ 0\,]$. Furthermore, the input of the leading vehicle was chosen as $u_0(t) = 1$ for $t \in [25, 28]$ and $u(t) = 0$ otherwise. It is observed that asymptotic tracking of the spacing policy is achieved. Moreover, the spacing error is unaffected by the acceleration of the predecessor vehicle and the conditions of Problem 1 are satisfied.

Fig. 4 shows a comparison of two choices of functions $\phi_2, \tilde{\phi}_2$. The spacing error z is stabilized using the function $\phi_2(z, \dot{z}) = -\tanh(z) - 2\sinh(\dot{z})$, whereas $\tilde{z}$ is stabilized using $\tilde{\phi}_2(z, \dot{z}) = -z - 2\dot{z}$. We see that for larger spacing errors, the convergence of z is slower, whereas for small errors the convergence is almost identical. This shows exactly what is claimed in Remark 11, *i.e.*, depending on the desired behavior in the case of relatively large or small spacing errors a controller can be designed.

Fig. 5 depicts the behavior of a platoon of six vehicles using the same nonlinear state feedback controllers and illustrates the

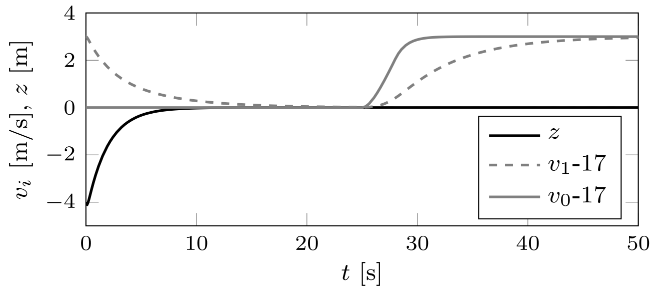

Fig. 3. Simulation of the two-vehicle model (7) for $i = 1$, time constants τ_i, $i \in \{0, 1\}$ chosen randomly from the interval [0.6, 1.4], time headway $h = 1.5$, $\gamma_1 = 0.1$, and initial conditions $x^\top(0) = [\,-100\ 20\ 0\ 0\ 17\ 0\,]$. The input to the lead vehicle is given as $u_0(t) = 1$ when $t \in [25, 28]$ and $u_0(t) = 0$ otherwise.

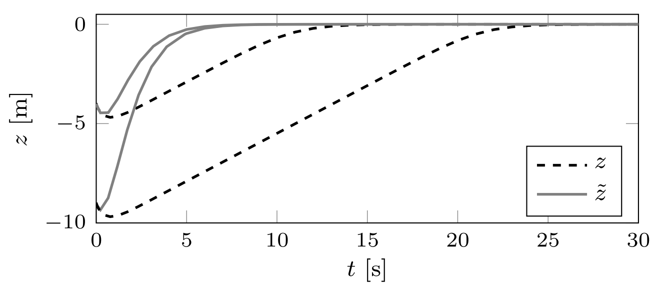

Fig. 4. A simulation of the spacing error z and $\tilde{z}$ with different initial conditions. The error z is stabilized with the function $\phi_2(z, \dot{z}) = -\tanh(z) - 2\sinh(\dot{z})$, whereas $\tilde{z}$ is stabilized with $\tilde{\phi}_2(z, \dot{z}) = -z - 2\dot{z}$.

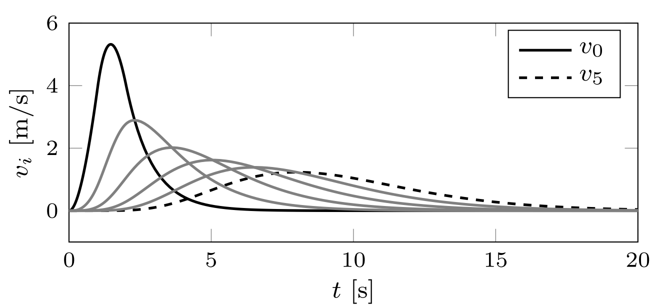

Fig. 5. Simulation of a platoon of six vehicles with time constants τ_i, $i \in \{0, \ldots, 5\}$ chosen randomly from the interval [0.6, 1.4], time headway $h = 1.5$, γ_i chosen randomly from $[-0.1, 0.1]$, and equilibrium initial conditions with $x_i(0) = 0$, $i \in \{1, \ldots, 5\}$. The input to the lead vehicle is given as $u_0(t) = 10$ for $t \in [0, 1]$, $u_0(t) = -10$ for $t \in [1, 2]$, and $u_0(t) = 0$ otherwise.

string stability property, as guaranteed by Theorem 16. For this platoon the τ_i, $i \in \{0, 5\}$ were again chosen arbitrarily from the interval [0.6, 1.4], γ_i was similarly chosen randomly from $[-0.1, 0.1]$ and the time headway remained $h = 1.5$. Furthermore, the input to the lead vehicle was chosen as $u_0(t) = 10$ for $t \in [0, 1]$ and $u_0(t) = -10$ for $t \in [1, 2]$ and $u_0(t) = 0$ otherwise.

6. Conclusion

In this paper we considered nonlinear spacing policies for vehicle platoons, motivated by their potential in improving safety. All spacing policies that can be asymptotically tracked with a decentralized nonlinear state feedback controller are characterized. It was observed that given a spacing policy which can be asymptotically tracked, the controller that achieves this tracking is not unique. Furthermore, it was shown that in this framework for decentralized controller synthesis, string stability is a consequence of the choice of spacing policy. The results are illustrated through simulations and two different controllers are compared.

Future work will focus on characterizing all spacing policies which guarantee string stability in the case of perfect tracking. Another direction of research will be the investigation of the existence of dynamic *output* feedback controllers that achieve asymptotic tracking for nonlinear spacing policies.

CRediT authorship contribution statement

Paul Wijnbergen: Conceptualization, Formal analysis, Investigation, Methodology, Visualization, Writing - original draft, Writing - review & editing. **Mark Jeeninga:** Formal analysis, Investigation, Methodology, Writing - Original Draft, Writing - review & editing. **Bart Besselink:** Conceptualization, Methodology, Supervision, Writing - original draft, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Review of geometric control theory

In the following, we consider the non-linear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + e(x)w, \\ z &= h(x), \end{aligned} \tag{A.1}$$

where $x = (x_1, \ldots, x_n)$ are local coordinates for a smooth manifold M, u is the input, f, g and e are smooth vector fields on M, z is the output, h is a function on M and w is an unknown disturbance. Throughout this paper, we have that $M = \mathbb{R}^6$ and $h : M \to \mathbb{R}$, u is a scalar input and w a scalar disturbance. Recall the following definitions of nonlinear control theory from [23,24,33].

Recall the definition of the Lie derivative of a function $h(x)$ along the vector field $f(x)$ and the repeated Lie derivative in (14) and (15) respectively. Furthermore, recall the definition of the relative degree given in Definition 3.

The differential of h at $x \in M$ is given by the one-form

$$dh(x) = \left[\frac{\partial f}{\partial x_1}(x) \ldots \quad \frac{\partial f}{\partial x_n}(x)\right].$$

For any two smooth vector fields f and g on M we can define a new vector field denoted as $[f, g]$ and called the *Lie bracket*, which is defined as

$$[f, g](x) = \frac{\partial g}{\partial x}(x)f(x) - \frac{\partial f}{\partial x}(x)g(x).$$

The following definition formalizes when an output is invariant under some disturbance, *i.e.*, not affected by it. This concept is also known as *output invariance*.

Definition 20. Consider the system (A.1). The output z is called invariant under w if for all x_0, u and w,

$$z(t, x_0, u, w) = z(t, x_0, u, 0), \quad \forall t \geqslant 0,$$

where $z(\cdot, x_0, u, w)$ is the solution to (A.1) for initial condition x_0, input u and disturbance w.

In order to state results on output invariance the notion of *(involutive) distributions* is introduced.

Definition 21. Let $d_1, \ldots, d_k$ be smooth vector fields on a smooth manifold M. Let $D(x) = \text{span}\{d_1(x), \ldots, d_k(x)\}$ be a vector space at $x \in M$. Then the collection of all vector spaces $D(x)$ for $x \in M$ is called a *distribution* D, generated by $D(x)$. A distribution D is called *involutive*, if $[d_i, d_j] \in D$ whenever d_i and d_j are vector fields in D.

The dual object of a distribution is a codistribution, of which the definition equals Definition 21 if the smooth vector fields $d_1, \dots, d_k$ are replaced by *smooth one-forms* $\sigma_1, \dots, \sigma_k$. For a codistribution we can define the following.

Definition 22. The kernel of a smooth codistribution P, denoted $\ker P$ is defined as the smooth distribution generated by

$$(\ker P)(x) = \operatorname{span}\left\{ d(x) \,\middle|\, \begin{array}{l} d \text{ is a vector field s.t.} \\ \sigma(d) = 0 \text{ for all } \sigma \in P \end{array} \right\}.$$

The concept of invariant subspaces in linear control theory can be extended to the nonlinear case.

Definition 23. A smooth distribution D on M is invariant for the nonlinear system (A.1) if

$$[f, D] \subset D, \quad [g, D] \subset D,$$

where $[\,\cdot\,, D] \subset D$ if $[\,\cdot\,, d] \in D$ for all vector fields $d \in D$.

Similarly, we introduce the concept of controlled invariant distributions, which is the nonlinear generalization of a controlled invariant subspace.

Definition 24. A smooth distribution D on M is called *controlled invariant* if there exists a feedback $u = \alpha(x) + \beta(x)v$ such that for $\tilde{f}(x) = f(x) + g(x)\alpha(x)$ and $\tilde{g}(x) = g(x)\beta(x)$

$$[\tilde{f}, D] \subset D, \quad [\tilde{g}, D] \subset D.$$

Lemma 25. *Let G be the distribution generated by $G(x) = \operatorname{span}\{g(x)\}$. A smooth distribution D on M is controlled invariant if and only if*

$$[f, D] \subset D + G, \quad [g, D] \subset D + G.$$

The concepts introduced so far contribute to solving the following problem [23, Problem 7.7].

Problem 2 (*Disturbance Decoupling Problem (DDP)*)**.** Consider the nonlinear system (A.1). Find a nonlinear state feedback $u = \alpha(x) + \beta(x)v$ such that in the feedback modified dynamics

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v + e(x)w$$

is invariant under w.

Under certain conditions this problem can be solved, which are stated by the following proposition, where $\ker dh$ is the codistribution, generated by $(\ker dh)(x)$.

Proposition 26 (*Proposition 7.8 [23]*)**.** *The DDP is solvable for the system (A.1) if and only if there exists a constant dimensional involutive distribution D, which is controlled invariant and satisfies*

$$e \in D \subset \ker dh. \tag{A.2}$$

Proposition 26 gives conditions for global solvability of the DDP. To see this, note that in the case that the distribution D in Proposition 26 is generated by $D(x)$, the condition in (A.2) could be rewritten as

$$e(x) \in D(x) \subset \ker dh(x), \quad \forall x \in M.$$

The next result tells us that we only need to find the maximal distribution that satisfies (A.2).

Lemma 27 (*Corollary 7.12 [23]*)**.** *There exists a unique involutive distribution $D^* \subset \ker dh$ that is controlled invariant and which contains all controlled invariant distributions in $\ker dh$.*

In the case of a single-input single-output system the distribution $D^* \subset \ker dh$ can be computed straightforwardly.

Theorem 28 (*Theorem 7.21 [23]*)**.** *Consider (A.1) with $e(x) = 0$. Let $\rho < \infty$ be the relative degree. Then the distribution $D^* \subset \ker dh$ is given by*

$$D^* = \ker(\operatorname{span}\{dh, dL_f h, \dots, dL_f^{\rho-1} h\}). \tag{A.3}$$

Furthermore, the feedback $u = \alpha(x) + \beta(x)v$ with $\alpha(x) = -(L_g L_f^{\rho-1} h)^{-1} L_f^{\rho} h(x)$ and $\beta(x) = (L_g L_f^{\rho-1} h(x))^{-1}$ renders D^* invariant.

Appendix B. Proof of Theorem 4

Proof. Necessity Observe that the system (7) is of the form (A.1) with $f(x) = Ax$, $g(x) = B$, and $e(x) = E$, such that Problem 1, part (i), is equivalent to the DDP in Problem 2. Consequently, by Proposition 26 and Theorem 28, we have that $E \in D^* \subset \ker dh$, where D^* is given by (A.3).

A direct computation of dh gives

$$dh = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} & \frac{\partial h}{\partial x_4} & \frac{\partial h}{\partial x_5} & \frac{\partial h}{\partial x_6} \end{bmatrix}.$$

Since $E \in \ker dh$, it is necessary that $\frac{\partial h}{\partial x_3} = 0$. Note that this implies

$$\frac{\partial h}{\partial x_3} = \frac{\partial \Delta}{\partial x_3} - \frac{\partial \Delta^{\text{ref}}}{\partial x_3} = -\frac{\partial \Delta^{\text{ref}}}{\partial a_0} = 0,$$

which proves (16).

Next, assume that $\frac{\partial h}{\partial x_6} = 0$ in addition to $\frac{\partial h}{\partial x_3} = 0$. Then, $L_B h = 0$ and ρ in Theorem 28 satisfies $\rho \geqslant 2$, such that $D^* \subset \ker(\operatorname{span}\{dh, dL_{Ax} h\})$. To evaluate this, we compute

$$L_{Ax} h = \frac{\partial h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_2} x_3 + \frac{\partial h}{\partial x_4} x_5 + \frac{\partial h}{\partial x_5} x_6,$$

and, consequently,

$$dL_{Ax} h = \begin{bmatrix} \frac{\partial L_{Ax} h}{\partial x_1} & \frac{\partial L_{Ax} h}{\partial x_2} & \frac{\partial h}{\partial x_2} & \frac{\partial L_{Ax} h}{\partial x_4} & \frac{\partial L_{Ax} h}{\partial x_5} & \frac{\partial h}{\partial x_5} \end{bmatrix}.$$

As $E \in D^*$, this leads to $\frac{\partial h}{\partial x_2} = 0$, proving

$$\frac{\partial h}{\partial x_2} = \frac{\partial \Delta}{\partial x_2} - \frac{\partial \Delta^{\text{ref}}}{\partial x_2} = -\frac{\partial \Delta^{\text{ref}}}{\partial v_0} = 0.$$

To finalize the proof of (17), let $\frac{\partial h}{\partial x_5} = 0$, or equivalently $\frac{\partial \Delta^{\text{ref}}}{\partial v_1} = 0$, with the aim of establishing a contradiction. Then, following a similar reasoning as before, we have $\rho \geqslant 3$ and $D \subset \ker(\operatorname{span}\{dh, dL_{Ax} h, dL_{Ax}^2 h\})$. We obtain

$$L_{Ax}^2 h = \frac{\partial L_{Ax} h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_1} x_3 + \frac{\partial L_{Ax} h}{\partial x_4} x_5 + \frac{\partial h}{\partial x_4} x_6.$$

and

$$dL_{Ax}^2 h = \begin{bmatrix} \frac{\partial L_{Ax}^2 h}{\partial x_1} & \frac{\partial L_{Ax} h}{\partial x_1} & \frac{\partial h}{\partial x_1} & \frac{\partial L_{Ax}^2 h}{\partial x_4} & \frac{\partial L_{Ax} h}{\partial x_4} & \frac{\partial h}{\partial x_6} \end{bmatrix},$$

such that $E \in D^*$ implies $\frac{\partial h}{\partial x_1} = 0$. However,

$$\frac{\partial h}{\partial x_1} = \frac{\partial \Delta}{\partial s_0} = 1,$$

which leads to a contradiction.

Sufficiency. It can be verified by direct computation that spacing errors satisfying the conditions are such that $E \in D^* \subset \ker dh$, where D^* satisfied (A.3). The result then follows from Proposition 26. More explicitly, we note that $\frac{\partial h}{\partial x_6} \neq 0$ implies that $\rho = 1$, in which case it can be verified that (16) is sufficient to prove $E \in D^*$. Similarly, $\frac{\partial h}{\partial x_6} = 0$ and $\frac{\partial h}{\partial x_5} \neq 0$ give $\rho = 2$ and the implications (16) and (17) guarantee $E \in D^*$. □

Appendix C. Proof of Theorem 7

Proof. Building on the proof of Theorem 4, we have that part (i) of Problem 1 implies that $E \in D^*$, with D^* the maximum controlled invariant distribution in $\ker dh$ as in (A.3). By Theorem 28, the controller $u_1 = \alpha(x)+\beta(x)\nu$ renders D^* invariant, where $\alpha(x) = -(L_B L_{Ax}^{\rho-1} h)^{-1} L_{Ax}^{\rho} h$, $\beta(x) = (L_B L_{Ax}^{\rho-1} h)^{-1}$ and ρ is the smallest integer such that $L_B L_{Ax}^{\rho-1} h \neq 0$. Furthermore, α and β are such that $[Ax + B\alpha(x), D^*] \subseteq D^*$ and $[B\beta(x), D^*] \subset D^*$.

We will proceed by considering various cases separately.

First, consider the case $\rho = 1$. It follows from input–output linearization that

$$\dot{z} = \nu. \tag{C.1}$$

Substituting $\nu = \phi_1(z)$, where $\phi_1(z)$ globally asymptotically stabilizes the origin of the dynamics $\dot{z} = \phi_1(z)$ yields property (ii) of Problem 1. It remains to show that property (i) is not violated by $\phi_1(z)$.

To that extent, we consider $d_j \in D^*$ and compute

$$\begin{aligned}[B\beta(x)\nu, d_j] &= \nu[B\beta(x), d_j] + B\beta(x)\frac{\partial \phi_1}{\partial z}\frac{\partial h}{\partial x} d_j \\ &= \nu[B\beta(x), d_j] \in D^*,\end{aligned}$$

where we have used that $\frac{\partial h}{\partial x} d_j = 0$ as $d_j \in D^* \subset \ker dh$. Hence, $u_1 = \alpha(x) + \beta(x)\phi_1(z)$ renders D^* invariant and both properties (i) and (ii) are guaranteed.

Next, consider the case $\rho = 2$. Following a similar reasoning as before, substituting $\nu = \phi_2(z, \dot{z})$ where $\phi_1(z, \dot{z})$ globally asymptotically stabilizes the origin of

$$\frac{d}{dt}\begin{bmatrix} z \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \dot{z} \\ L_{Ax}^2 h(x) + (L_B L_{Ax} h)u \end{bmatrix} = \begin{bmatrix} \dot{z} \\ \phi_2(z, \dot{z}) \end{bmatrix}. \tag{C.2}$$

Property (i) is again proven by noting that

$$\begin{aligned}&[B\beta(x)\nu, d_j] \\ &= \nu[B\beta(x), d_j] - B\beta(x)\left(\frac{\partial \phi_2}{\partial z}\frac{\partial h}{\partial x} + \frac{\partial \phi_2}{\partial \dot{z}}\frac{\partial \dot{h}}{\partial x}\right) d_j \\ &= \nu[B\beta(x), d_j] \in D^*.\end{aligned}$$

To obtain the latter, we have used $\dot{h} = L_{Ax} h$ and $D^* = \ker(\mathrm{span}\{dh, dL_{Ax} h\})$. □

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