Formation control in the port-Hamiltonian framework
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Chapter 1

Introduction
Chapter 1

Introduction

Formation control is a specific control problem within the broad class of coordination control. The main challenge in coordination control is to achieve a prescribed group behavior for a group of agents employing local feedback rules, rather than centralized controllers [3]. For formation control the prescribed group behavior is to achieve a geometrical shape for the network of agents (i.e., to achieve a formation) [58, 83, 115]. Two other well known types of coordination control which are outside the scope of this thesis are consensus (or agreement) [55, 84, 93, 95] and synchronization [23, 80, 81, 84].

All systems in this thesis are modeled as port-Hamiltonian systems. Port-Hamiltonian systems theory is an energy-based modeling framework [86] which provides powerful tools for the analysis and design of controllers by exploiting the physical structure of the system. The port-Hamiltonian framework considers systems as the interconnection of energy storing and energy dissipating components which exchange energy through power ports. External power ports enable the system to interact with external systems, such as controllers and the environment [39, 100]. Furthermore, the framework provides insight into the energy supplied by the controllers.

This chapter starts with a theoretical background on port-Hamiltonian systems theory and formation control (Section 1.1). Section 1.2 continues with the two main application areas for the results in this thesis. The main contributions, thesis outline, and publications are given in respectively Sections 1.3, 1.4 and 1.5. Finally Section 1.6 introduces the (mathematical) notation for the subsequent chapters.

1.1 Theoretical background

Formation control has received a wide interest from different perspectives in recent years. Two major topics include formation stability (i.e., to achieve and maintain a formation) [30, 55, 72, 83], and changing formation patterns (i.e., to change from one formation to another) [30, 67]. Another distinction can be made according to the systems under investigation (e.g. nonholonomic robots [30, 34, 40, 67, 72] and satellites [2, 23, 94, 102]). The passivity-based design tool for group coordination in [3] provides a starting point for the formation control algorithms in this thesis.
In this thesis the physical structure of the network is exploited for the analysis and design of formation control algorithms. The physical structure not only refers to the real physical structure of the agents in the network, but also to the virtual structure, which is part of the controller design. Here, formation control is achieved by assigning virtual couplings between real agents (i.e., robots and satellites) in the network [43, 70]. Each virtual coupling consists of a spring and an optional damper in parallel. Virtual springs achieve formation control by shaping the energy function of the network, while virtual dampers guarantee stability by injecting damping.

The integration of real physical systems with virtual couplings is a perfect example of a cyber-physical system. Cyber-physical systems deal with the integration of physical processes and computation and have become a hot topic in different disciplines in recent years [7, 69, 103]. The ability of cyber-physical systems to interact with and expand the capabilities of the physical world through computation, communication and control is a key enabler for future technology developments [7].

All cyber-physical networks in this thesis are modeled as port-Hamiltonian systems, which is an energy-based modeling framework. The remainder of this section provides the background on port-Hamiltonian systems (Section 1.1.1), followed by the formation control (Section 1.1.2). Section 1.1.2 also provides the intuition behind the algorithms in Chapters 3, 4, and 5.

1.1.1 Port-Hamiltonian systems theory

Port-Hamiltonian systems theory was introduced in the early 90s by Maschke and Van der Schaft [73] as an energy-based modeling framework for the modeling of a large class of multi-domain physical systems. The theory brings together the traditions of port-based modeling [12, 88], geometric mechanics [14, 71], and systems and control theory [99]. Energy concepts are well-known to practitioners and port-Hamiltonian models may therefore serve as a lingua franca amongst engineers, by interpreting the models and control actions using physical intuitions [86]. The Euler-Lagrange and Brayton-Moser modeling frameworks provide similar advantages, but are outside the scope of this thesis. The interested reader is referred to e.g. [31, 61, 85, 86] for more information on these frameworks.

Not only does port-Hamiltonian systems theory enable systematic and intuitive modeling and analysis of multi-domain systems, it also provides a starting point for control design. It is widely recognized that physical properties of systems should be exploited in the design of robust and physically interpretable control systems. Exploiting the physical structure for design and control inherits robustness and fault-tolerance for the design from the intrinsic physical robustness. Furthermore, port-Hamiltonian systems theory offers powerful tools and concepts for doing this
1.1. Theoretical background

(e.g. [19, 38, 45, 62, 87, 98, 99]) and has proven successful in many applications (e.g. [39, 41, 86, 106, 107] and references therein).

Control of port-Hamiltonian systems is achieved by interconnecting the system to be controlled to a controller using power ports. A power port has two corresponding port variables, whose product has the dimension of power. Usually the port variables are the input $u$ and output $y$ of the system, implying that the product of the input and output $y^T u$ equals the external power supplied through the port.

In addition the framework provides insight into the energy consumption of the controller. Consider a control input $u = -k(z - z^*) - d\dot{z}$ for simple set-point regulation problem, with controller gains $k, d$, position $z$, set-point $z^*$. This simplified control law is closely related to the formation control algorithms in this thesis. The two terms of the control input $u$ are interpretable as respectively a virtual spring and damping force. The energy supplied by the controller equals the integral of the power supplied through the power port and is given by

$$\int_0^T y^T u \, dt = \left. \frac{1}{2} k(z - z^*)^2 \right|_0^T - \left. \frac{1}{2} k(z - z^*)^2 \right|_0^T - \int_0^T d \dot{z}^2 \, dt. \quad (1.1)$$

Note that this energy is the real energy supplied by the controller, which differs from the usually considered measure for energy consumption $\int u^2 \, dt$. Analyzing (1.1) for different values of $k$ and $d$ facilitates energy-efficient controller designs.

The Hamiltonian $H$ is the total energy stored in the system (i.e., $H$ equals the sum of the kinetic and potential energy). For the mechanical systems in this thesis, the physical structure ensures that $\dot{H} \leq y^T u$ (see Section 2.3). In other words the energy flow in the system $\dot{H}$ is always equal or smaller than the external power supplied $y^T u$ through the power port $(u, y)$. This implies that port-Hamiltonian systems are passive from the input $u$ to the output $y$. Section 2.3 provides more technical details on port-Hamiltonian systems.

Port-Hamiltonian systems theory can also be used to model complex networks of dynamical agents [100]. The theory of port-Hamiltonian systems on graphs is used for the modeling, analysis and design of formation control algorithms in the remainder. In this approach both the agents and the virtual couplings are modeled as port-Hamiltonian systems and a power-continuous interconnection structure is imposed on the network. This interconnection structure is derived from the graph structure modeling the network. In this thesis, all networks are modeled as undirected graphs, where agent dynamics are assigned to the nodes and virtual coupling dynamics are assigned to the edges of the graph [100]. The next section continues with a more detailed background on the formation control problem.
1.1.2 Formation control of networks

Coordination control is about achieving prescribed group behaviors for a group of agents (e.g. robots, satellites) with local feedback rules, rather than with centralized controllers [3]. Formation control is a specific type of coordination problem where the prescribed group behavior is to achieve prescribed displacements between the agents in the network. (see Figure 1.1). The approach to formation control in this thesis consists of three components: the agents $A$ in the network, the virtual couplings $V$ to achieve formation control, and the interconnection structure $D$ to interconnected the first two components (Figure 1.2). For example, consider Figure 1.1, where $A$, $V$, and $D$ correspond to respectively the seven robots, six green lines, and the way in which robots are interconnected by lines.

Starting point for the formation control algorithms in this thesis is the passivity-based design tool for group coordination in [3, 9]. The approach consists of an internal and an external feedback. The internal feedback is a local feedback which renders agents in the network passive with respect to an error output, to track reference velocities. The external feedback achieves group coordination by rendering target sets invariant and asymptotically stable by interconnecting agents in the network. The target sets correspond to group coordination tasks such as consensus, synchronization, and formation control. The symmetry in the interconnection structure presented in [3] is also exploited in this thesis.

The dynamics of the agents $A$ in Figure 1.2 play an important role in the formation control problem. Various classes of dynamic agents have been considered e.g. [9, 81, 93, 100]. The dynamics considered here include fully actuated agents (Chapter 3), nonholonomic wheeled robots (Chapter 4), and satellites (Chapter 5), which are all modeled as (mechanical) port-Hamiltonian systems. Dissipation due to friction and damping plays an important role in the stability analysis of networks.
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Figure 1.2: General setup for formation control with agents $A$, virtual couplings $V$, and interconnection structure $D$.

of dynamical agents (e.g. [9, 57]). Different types of dissipation are considered here, ranging from discontinuous Coulomb friction to virtual dampers in between the agents.

For the application areas in this thesis (see Section 1.2.2), formation control alone is not enough. The whole network also has to move by tracking a prescribed reference velocity (i.e., it has to sweep [22, 66] or cover [25, 82] a surface). Velocity tracking is achieved by rendering the agent dynamics passive from the input to the velocity error output [3]. Generalized canonical transformations enable velocity tracking for port-Hamiltonian systems [45], by deriving and stabilizing the error dynamics preserving the port-Hamiltonian structure.

Another, more practical, challenge is the presence of disturbances, which distort the formation shape or even render the system unstable. Proportional-integral control with quantized information and time-varying topologies has been proposed for a network of single-integrator robots [120], while adaptive internal model control was studied for a single port-Hamiltonian system [47, 48]. Section 4.5 provides new insights using an internal model controller for disturbance rejection in a network of nonholonomic wheeled robots.

In this thesis, formation control is achieved by assigning virtual couplings ($V$ in Figure 1.2) between the agents. Different types of virtual couplings have been considered [43, 70]. Here virtual couplings are interpreted as virtual springs, with an optional virtual damper in parallel. The springs shape the energy function of the whole network, while the dampers inject damping for stabilization. The virtual damper is an extension to the external feedback in [3], which enables only virtual springs. Moreover, the setup fits within the IDA-PBC paradigm [87] and builds upon the theory of port-Hamiltonian systems on graphs [100].

The interconnection structure $D$ in Figure 1.2 models the way in which agents are interconnected by virtual couplings. Graph theory [11, 50] provides powerful tools for modeling the interconnection structure of complex networks. Nodes and edges of the graph corresponds to respectively agents and virtual couplings in the network [100]. The topology of the network consists of two layers. The communication topology models which agents exchange information with other agents in the network. The interconnection topology models the (virtual) physical structure of the network of agents and virtual couplings. Due to the underlying physical structure the interconnection topology is modeled as an undirected graph.
[3, 9, 93]. The communication topology on the other hand is often assumed to be directed [81, 100]. In this work the communication topology is assumed to be equal to the interconnection topology and therefore only undirected graphs are considered.

All undirected graphs in this thesis are assumed to be connected and acyclic, except for the cycle graphs in Chapter 5. The graph being connected prevents agents in the network to drift away from others [3, 9, 81, 93, 100]. Considering acyclic graphs has pros and cons. An advantage of acyclic graphs is that they impose the communication network to be loop-free, which prevents error accumulation during parallel distributed signal processing at the robots. On the other hand, cyclic graphs provide robustness to link failures. Excluding cyclic graphs in the approach yields less general results, while on the other hand cyclic graphs require an additional (restrictive) condition on the desired relative displacement of the formation [3, 100] (see also Remark 4.3). Furthermore, for some types of virtual springs undesired equilibria may arise [3, 8, 9] when the graph has cycles.

In the remainder, the graph topology is a design freedom and considering only acyclic graphs is therefore not restrictive. More technical details on graph theory are provided in Section 2.2.

1.2 Application areas

The formation control algorithms developed in this thesis concern a broad class of applications (see e.g. [2, 23, 30, 34, 40, 67, 72, 94, 102, 104, 105] and references therein). The two main application areas for this thesis both originate from the ROSE project, which is introduced in Section 1.2.1. Section 1.2.2 continues with the first application area, being the inspection of dikes using robotic sensor networks. The second application area concerns the formation flying of satellite constellations, which is elaborated in Section 1.2.3.

1.2.1 ROSE project

ROSE is the acronym for “Energy-efficient design and control of mobile RObotic SEnsor networks”. Funded by Dutch Technology Foundation STW, ROSE is a collaboration between scientists (academia) and users (industry) (see Appendix B for an overview of the partners). ROSE started in 2010 and is part of the Autonomous Sensor Systems (ASSYS) program of STW under project 10550.

ROSE can be divided into two parts. The first part concerns the energy-efficient design of autonomous and mobile sensor-integrated robotic devices for data acquisition on dikes. This part is carried out by the University of Twente, where the focus is on the design of a mobile robotic sensor which acquires data
1.2. Application areas

1.2.1 Understanding dike composition, consistency and condition in an autonomous way [36]. The novelty of the design is a new concept for the locomotion of the robot, using Continuously Variable Transmissions [37].

The second part concerns the design of algorithms for coordinating a network of such robotic sensors, using ideas from passive systems [3] and port-Hamiltonian system theory [39, 99]. Exploiting the physical structure enables the generalization of the dike inspection application to other application areas such as the formation flying of satellites. The results in this thesis encompass this part of the ROSE project. The following two sections elaborate on the two application areas.

1.2.2 Dike inspection using robotic sensor networks

The first application area aligns with the algorithms developed in Chapters 3 and 4 and concerns the inspection of dikes. Recent dike breaches show that current inspection methods are not sufficient to guarantee dike safety (see Section A.1 for more background information on dike inspection in The Netherlands). Sensor technology has shown great potential for improving dike safety during several ground breaking experiments of the Stichting FloodControl IJkdijk (see Section A.2 for more background information on dike inspection using sensor technology). Figure 1.3 shows the experimental setup of one of these experiments, which was carried out by Stichting IJkdijk in 2012.

The results in this thesis enable the next step in the use of sensor technology for dike inspection, by developing formation control algorithms for robotic sensor networks. A robotic sensor network is a group (or network) of mobile sensor-
equipped robots, which gathers measurements in a coordinated fashion. Recent advances in sensor-equipped autonomous mobile robots enable the use of robotic sensor networks for a wide range of applications [21]. In addition, the experiments executed at the IJkdijk test facility show that sensor technology provides crucial information not attainable by traditional visual inspection methods.

Formation control is a crucial aspect for the use of robotic sensor networks in the dike inspection application. Acquiring high resolution measurements on the dike interior imposes a prescribed acquisition geometry to the sensor network [109], which is corresponds to achieving prescribed relative displacements between the robotic sensors. Furthermore, the robots need to cover the whole dike surface which requires coverage or deployment control. At this moment there is no definition yet on the optimal formation shape for dike monitoring. Therefore the algorithms developed here are able to deal with different types of formations, based on different interconnection topologies.

Three application areas are foreseen for the use of robotic sensor networks within the dike inspection application. First of all, for existing dikes it is infeasible to install static sensors everywhere. The number of kilometers of dike and the costs for positioning static sensors is simply too high. A mobile robotic sensor network is more flexible and often less costly for the coordinated monitoring of dikes. Second, the profile and substance (interior) of dikes is in many cases unknown. Mobile robotic sensors can be used for the exploration of both. Finally, robotic sensors can be used for the guidance of drilling during the installation of existing static sensors.
1.2. Application areas

Figure 1.5: Galileo satellite constellation (Source: ESA - J. Huart, 2002).

1.2.3 Formation flying of satellite constellations

The second application area concerns the results of Chapter 5 and deals with formation flying of satellites. Formation flying of satellites is usually divided according to the dynamic environment in which the satellites operate. For deep space formation flying the satellite translational dynamics are approximated as (double) integrators, while for planetary orbital formation flying the satellites are subject to significant gravitational dynamics and other environmental disturbances [101, 102]. Along the same line, two application areas are identified.

For deep space applications formation flying aims to coordinate several (or many) small satellites, such that the network acts as one big instrument. The dimensions of such a setup enable specifications which outperform single satellite setups. Other advantages of using a network of satellites are the ability to reconfigure formations, adapt baselines and acquire targets. Accurate control of the relative displacements amongst the satellites is important to achieve sufficiently high resolutions. Two missions which involve formation flying in deep space are the Far-InfraRed Interferometry (FIRI) mission of SRON Netherlands Institute for Space Research and the Proba-3 mission of ESA. FIRI aims to provide very large baselines and thus a high spatial resolution for the measuring of wavelengths between 25 and 300 microns. Proba-3 (see Figure 1.4) is a two satellite setup, which forms a 150 m long solar coronagraph to study the Sun’s faint corona closer to the solar rim than has ever before been achieved (source: sci.esa.int/sre-ft/37936-formation-flying).

The second application area considers formation flying on planetary orbits, which is related to coverage. Coverage here refers to the fact that each point on the planetary surface is always covered by one (or more) satellite(s). Well-known
examples are Global Navigation Satellite Systems (GNSS) like the Global Positioning System (GPS) and Galileo (Figure 1.5). For GNSSs it is of the utmost importance to phase satellites on the orbits, to ensure that there are always at least four satellites covering the object to be localized. The algorithms in Chapter 5 are generalizations of the algorithms in Chapters 3 and 4, dealing with the highly complex satellite dynamics.

1.3 Contributions

The contributions of this thesis are summarized as follows:

- This thesis provides an extension of the passivity-based design tool for network coordination in [3] to port-Hamiltonian systems. Coordination of the network is achieved using virtual couplings [58–60, 110–116], which consist of a virtual springs and dampers in parallel. The springs shape the energy function of the network, while the dampers inject damping for stabilization. In addition to standard formation control, several related control problems are considered (see next bullet).

- In addition to standard formation control, this thesis addresses formation control in the presence of Coulomb friction, deployment, velocity tracking, disturbance rejection for nonholonomic systems, and orbital phasing for satellites. Coulomb friction renders the agent dynamics non-smooth and rigorous stability proofs using tools from non-smooth analysis are given for both continuous virtual springs and their discontinuous counterpart [58, 59]. Deployment is achieved using a combination of virtual couplings and virtual walls, which are positioned at prescribed reference points [110]. Velocity tracking is achieved by stabilizing the error dynamics with respect to the reference velocity, which are obtained using generalized canonical transformations [113, 114]. Internal model control is able to counteract the effect of harmonic matched input disturbances for networks of nonholonomic wheeled robots [60, 116], while for constant disturbances only stability is proven [60]. Finally, the versatility of virtual couplings is shown, by their application to orbital phasing of satellites [114].

- The concept of virtual couplings is used for the formation control of three different classes of systems. For each system the virtual couplings are tailored to the system under consideration. Chapter 3 starts with continuous springs and develops a discontinuous counterpart to achieve formation control in the presence of Coulomb friction [58, 59]. For the nonholonomic wheeled robots in Chapter 4, the virtual couplings are assigned to the front ends of
1.4 Thesis outline

The outline of the remainder of this thesis is as follows. Chapter 2 starts with the preliminaries on stability theory, graph theory, port-Hamiltonian systems, and non-smooth analysis. Subsequent Chapters 3, 4, and 5 deal with formation control of three types of systems.

Chapter 3 starts with formation control and deployment of fully actuated systems. Special attention is devoted to agents in the presence of Coulomb friction, which renders the agent dynamics non-smooth and requires tools from non-smooth systems theory for the stability analysis. Chapter 4 starts with the derivation of a dynamical model of the wheeled robot in the port-Hamiltonian framework. Then algorithms are designed for the problems of formation control, velocity tracking and matched input disturbance rejection. Chapter 5 deals with orbital phasing of the robots rather than the center of mass [60, 113, 115, 116]. This change of point of action is required to deal with the nonholonomic constraint on the wheel axle of the robot. For the satellites in Chapter 5, both translational and rotational couplings are considered [111, 112, 114]. The translational couplings guarantee convergence to the desired altitude, while the rotational couplings achieve orbital phasing and convergence to the desired angular velocity.

• Virtual couplings provides a clear physical interpretation to practitioners thereby facilitating implementation of the algorithms [86]. The physical structure of the network of agents interconnected by virtual couplings is exploited for the analysis and design of the control algorithms. Since the algorithms only exchange local information amongst agents in the network all algorithms are distributed and therefore easily scalable.

• The rigorous stability proofs for formation control in the presence of Coulomb friction require several tools for non-smooth dynamics. The use of the Krasovskii notion of solution, generalized Clarke gradient, and a non-smooth version of LaSalle’s invariance principle fits completely into port-Hamiltonian systems theory. These powerful tools enable the generalization of existing non-smooth theory to port-Hamiltonian systems.

• The algorithms and theory in this thesis are illustrated and validated using simulations and experiments. Using a testbed of e-puck wheeled robots experimental results are provided for formation control and deployment of fully actuated systems (Sections 3.3 and 3.5), and for formation control and velocity tracking of wheeled robots [113, 115]. For all other systems, extensive simulation results are provided [58–60, 110–112, 114, 116].
1. Introduction

A network of satellites. A local controller keeps each satellite on the orbit, while a distributed controller equally distributes the satellites on the orbit. Concluding remarks and recommendations for future research are presented in Chapter 6.

The appendices provide more background information on dike inspection in The Netherlands (Appendix A), partners of the ROSE project (Appendix B), the experimental setup used in Chapters 3 and 4 (Appendix C), and complementary simulation and experimental results (Appendix D).

1.5 Publications

All publications contributing to this thesis are enlisted below, divided into journal papers, conference papers, conference abstracts, posters, and graduation projects.

Journal papers


Conference papers


1.5. Publications

*the International Federation of Automatic Control, pages 6662–6667, Cape Town, South Africa, 2014*


### Conference abstracts


### Posters


1. Introduction


Graduation projects

The following graduation projects to obtain the Master or Bachelor degree in Industrial Engineering and Management contributed directly or indirectly to this thesis:


1.6 Notation

Throughout the remainder of this thesis, the following notion is adopted.
1.6. Notation

- Let $n$ denote the dimension of the state space under consideration and let $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n \times n}$ denote respectively the set of real numbers, the set of $n$-dimensional column vectors of real numbers, and the set of $n \times n$ matrices of real numbers. Moreover, $\mathbb{I}_n$ denotes the $n$-dimensional column vector of ones, given by $\mathbb{I}_n = (1, \ldots, 1)^T$. Unless otherwise stated $0$ is always assumed to be appropriately dimensioned.

- For two vectors $a, b \in \mathbb{R}^n$, their duality product is denoted by $\langle a \mid b \rangle$.

- For a scalar function $H(x) : \mathbb{R}^n \to \mathbb{R}$ with $x \in \mathbb{R}^n$ the column vector of partial derivatives $\frac{\partial H}{\partial x}(x)$ and the $n \times n$ Hessian matrix $\frac{\partial^2 H}{\partial x^2}(x)$ are defined respectively as

\[
\frac{\partial H}{\partial x}(x) = \begin{pmatrix}
\frac{\partial H}{\partial x_1}(x) \\
\vdots \\
\frac{\partial H}{\partial x_n}(x)
\end{pmatrix}, \quad \frac{\partial^2 H}{\partial x^2}(x) = \begin{pmatrix}
\frac{\partial^2 H}{\partial x_1^2}(x) & \cdots & \frac{\partial^2 H}{\partial x_1 \partial x_n}(x) \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 H}{\partial x_n \partial x_1}(x) & \cdots & \frac{\partial^2 H}{\partial x_n^2}(x)
\end{pmatrix}.
\]

For a vector function $v(x) = (v_1(x), \ldots, v_m(x))^T$, with scalar functions $v_i(x) : \mathbb{R}^n \to \mathbb{R}$, the $m \times n$ Jacobian matrix $\frac{\partial v}{\partial x}(x)$ is defined as

\[
\frac{\partial v}{\partial x}(x) = \begin{pmatrix}
\frac{\partial v_1}{\partial x_1}(x) & \cdots & \frac{\partial v_1}{\partial x_n}(x) \\
\vdots & \ddots & \vdots \\
\frac{\partial v_m}{\partial x_1}(x) & \cdots & \frac{\partial v_m}{\partial x_n}(x)
\end{pmatrix}.
\]

- Consider two column vectors $S_i(x), S_j(x) \in \mathbb{R}^n$ with $x \in \mathbb{R}^n$. Then, the Lie-bracket denoted by $[S_i, S_j](x)$ is defined as

\[
[S_i, S_j](x) = \frac{\partial S_j}{\partial x}(x)S_i(x) - \frac{\partial S_i}{\partial x}(x)S_j(x),
\]

with $\frac{\partial S_i}{\partial x}(x)$ the Jacobian matrix.

- For graph $G(V, E)$ let $N = |V|$ denote the number of nodes and let $E = |E|$ denote the number of edges. Moreover, let $N_i = |V_i|$ denote the number of internal nodes and let $N_b = |V_b|$ denote the number of boundary nodes (see Section 2.3.4).

- For two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ the Kronecker product $A \otimes B \in \mathbb{R}^{mp \times nq}$ is defined as

\[
A \otimes B = \begin{pmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{pmatrix}.
\]
Furthermore, let \( \ker A \) denote the kernel of matrix \( A \) and \( \im A \) the image.

- The Krasovskii operator acting on some (possibly discontinuous) function \( f(x) \) is denoted by \( Kf(x) \) (see Section 2.4).

- The convex hull is denoted by \( \co \), while the convex closure is denoted by \( \overline{\co} \). A disc of radius \( r \) centered at \( x \in \mathbb{R}^n \) is denoted by \( B(x, r) \) (see Section 3.4).