On approximations, complexity, and applications for copositive programming

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Summary

In this thesis we investigated a number of properties regarding the copositive and completely positive cone, motivated by results obtained in copositive optimization. These two cones are respectively defined as:

$$\mathcal{COP}^n := \{A \in \mathbb{S}^n \mid \mathbf{x}^\mathsf{T} A \mathbf{x} \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n_+\},$$
$$\mathcal{CP}^n := \{A \in \mathbb{S}^n \mid \; A = \sum_{i=1}^{k} \mathbf{b}_i \mathbf{b}_i^\mathsf{T},\, \mathbf{b}_i \geq 0 \text{ for all } i\},$$

and are mutually dual.

We first studied the complexity of the membership problem for both of these cones. It is a known result that verifying copositivity is co-NP-complete. On the other hand, the complexity of verifying that a matrix is completely positive was still unknown. In this thesis we confirm the expected result that the membership problem for the completely positive cone is NP-hard. In fact, we show that both the weak and the strong membership problem for the copositive as well as the completely positive cone belong to the class NP-hard.

We also investigated the property of irreducibility with respect to the cone of nonnegative matrices, a weaker version of extremality. In particular, we provide a necessary and sufficient condition for a copositive matrix to be irreducible. For the 5×5 copositive cone we give a complete characterization of all irreducible matrices. Then, we show that every 5×5 copositive matrix that is not the sum of a nonnegative and a positive semidefinite matrix can be expressed as the sum of a nonnegative and a single irreducible matrix. This latter result is used to show that the 5×5 copositive cone reduces to the level one Parrilo cone under specific scalings.

We furthermore proved the result that we can scale any matrix, that is not the sum of a nonnegative and a positive semidefinite matrix, out of any level r of the Parrilo cone for $r \geq 1$ and $n \geq 5$. For the level one Parrilo cone an explicit way to construct such scalings is provided. We then investigated scalings in the opposite direction and find that an algorithm that constructs such scalings cannot have a polynomial running time unless P= NP. We introduce the concept of non-decreasing scalings, which are scalings that cannot scale a matrix out of any level of a hierarchy it is in. We moreover give an example of a non-decreasing scaling that is capable of scaling matrices downward in (amongst others) the Parrilo hierarchy.

Finally, we provide an application in the form of a copositive formulation of the graph isomorphism problem and show that we can in fact decide isomorphism using finite levels of some approximation hierarchies of the copositive cone. Then, several alternative formulations are suggested, one of which implies a potential method to construct a certificate that the graph isomorphism problem is in P.