On approximations, complexity, and applications for copositive programming

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Chapter 2

Complexity of membership for the completely positive cone and its dual[1]

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As noted in the introduction, it has been proven by Murty and Kabadi [MK87] that the strong membership problem for the copositive cone is co-NP-complete. As a result, no polynomial time algorithms will exist for solving general copositive programs unless P = NP. Motivated by this result of Murty and Kabadi we consider the question of whether the strong membership problem for the completely positive cone is NP-hard. It is commonly believed that this is true and in fact this is to such an extent that many researchers in the field believe it to have already been proven. However the technical details concerning this question have never been considered in order to supply a proof for this result and confirm that the suspicion concerning the complexity is correct. It was conjectured by Jarre and Schmallowsky [JS09] that the strong membership problem for the completely positive cone is an NP-complete problem. In this chapter we provide a proof that the membership problem for the completely positive cone is indeed NP-hard and it is left as an open question as to whether this problem in NP as well. Note in particular the discussion concerning the CP-rank in the introduction of this thesis regarding this question. Moreover we will give a proof for the result that even weak membership for the completely positive cone is NP-hard. In particular, we prove this result in Section 2.3 by establishing a polynomial time Turing reduction from the stable set problem, a known NP-complete problem (see for example Garey and Johnson [GJ79, Section 3.1.3]) to the weak membership problem for the completely positive cone. Introductions to the stable set problem are provided by Bondy and Murty [BM76, Chapter 7] and Schrijver [Sch03, Chapter 64]. We recall from Grötschel, Lovász, and Schrijver [GLS88] that for two problems Π_1, Π_2, a *(polynomial time) Turing Reduction* from Π_1 to Π_2 is an algorithm A_1 which solves Π_1 using a hypothetical subroutine A_2 which solves Π_2 such that if A_2 is a polynomial time algorithm then so is A_1. A special case of Turing reductions is the many-one reduction where the algorithm A_2 is only called once. In these reductions it is vital that the encoding length of the input to the algorithm A_2 is polynomial in the input to the algorithm A_1, and providing such polynomial inputs will be the main work during this chapter.

In this chapter we will also provide an alternative proof for NP-hardness of the strong membership problem for the copositive cone in Section 2.2 together with a proof that even the weak membership problem for the copositive cone is an NP-hard problem.

We will however start off this chapter by providing some definitions and known results, as well as a technical lemma, in Section 2.1.

2.1 Problems for convex sets

We start off this section by recalling several definitions from Grötschel, Lovász, and Schrijver [GLS88], which are extended for the space of symmetric matrices. In this section we let $\mathbb{X}$ be equal to either $\mathbb{R}^n$ or $\mathbb{S}^n$ and correspondingly let $\mathcal{Q}$ be equal to either $\mathbb{Q}^n$ or $(\mathbb{Q}^{n\times n} \cap \mathbb{S}^n)$ respectively.

Definition 2.1. Let $K \subseteq \mathbb{X}$ and let $\varepsilon > 0$. Then we define,

$$\begin{aligned} \mathrm{S}\,(K,\varepsilon) &:= \{x \in \mathbb{X} \mid \|x - y\| \leq \varepsilon \text{ for some } y \in K\}, \\ \mathrm{S}\,(K,-\varepsilon) &:= \{x \in \mathbb{X} \mid \mathrm{S}\,(\{x\},\varepsilon) \subseteq K\}. \end{aligned}$$

When $K = \{a\}$ we shall write $\mathrm{S}\,(a,\varepsilon) := \mathrm{S}\,(\{a\},\varepsilon)$.

Note that we have the following relation, $\mathrm{S}\,(K,-\varepsilon) \subseteq K \subseteq \mathrm{S}\,(K,\varepsilon)$. Hence $\mathrm{S}\,(K,-\varepsilon)$ and $\mathrm{S}\,(K,\varepsilon)$ can be seen as inner and outer approximations of K respectively.

We now consider the following problems for a set $K \subseteq \mathbb{X}$.

Definition 2.2. The Strong Membership Problem (MEM). Let $K \subseteq \mathbb{X}$. Given an instance $y \in \mathcal{Q}$, decide that either

1. $y \in K$, or
2. $y \notin K$.

Definition 2.3. The Weak Membership Problem (WMEM). Let $K \subseteq \mathbb{X}$. Given an instance $(y, \delta) \in \mathcal{Q} \times \mathbb{Q}_{++}$, decide that either

1. $y \in \mathrm{S}\,(K,\delta)$, or
2. $y \notin \mathrm{S}\,(K,-\delta)$.

Note that for $K \neq \emptyset, \mathbb{X}$ and $\delta > 0$, we have that $\mathrm{S}\,(K,\delta) \setminus \mathrm{S}\,(K,-\delta) \neq \emptyset$. Therefore, in general, for some instances either answer would be valid.

Definition 2.4. The Weak Validity Problem (WVAL). Let $K \subseteq \mathbb{X}$. Given an instance $(c, \gamma, \varepsilon) \in \mathcal{Q} \times \mathbb{Q} \times \mathbb{Q}_{++}$, either

1. decide that $\langle c, x\rangle \leq \gamma + \varepsilon$ for all $x \in \mathrm{S}\,(K,-\varepsilon)$, or

2. decide that $\exists y \in \mathrm{S}(K, \varepsilon)$ for which $\langle c, y\rangle \geq \gamma - \varepsilon$.

Note that again, in general, for some instances either answer would be valid.

We now consider what WVAL tells us about the values of $\langle c, x\rangle$ for $x \in K$, rather than just for x in some approximation of K. However, before we do this we first define a special type of convex body.

Definition 2.5. Consider a convex set $K \subseteq \mathbb{X}$ with the following properties,

1. $N = \dim \mathbb{X}$,
2. $\exists R \in \mathbb{Q}_{++}$ such that $K \subseteq \mathrm{S}(0, R)$, and
3. $\exists r \in \mathbb{Q}_{++}$, $a_0 \in \mathcal{Q}$ such that $\mathrm{S}(a_0, r) \subseteq K$.

Then K is called an a_0-centered convex body which is denoted as the quintuple $(K; N, R, r, a_0)$.

Lemma 2.6. *Consider WVAL with K being a convex body $(K; N, R, r, a_0)$ as defined in Definition 2.5. If we assume that $\varepsilon < r$ then we have the following,*

1. *$\langle c, x\rangle \leq \gamma + \varepsilon$ for all $x \in \mathrm{S}(K, -\varepsilon)$ implies that*

$$\langle c, z\rangle \leq \left(\gamma + \varepsilon - \frac{\varepsilon}{r}\langle c, a_0\rangle\right) \Big/ \left(1 - \frac{\varepsilon}{r}\right) \qquad \text{for all } z \in K.$$

2. *$\exists y \in \mathrm{S}(K, \varepsilon)$ for which $\langle c, y\rangle \geq \gamma - \varepsilon$ implies that*

$$\exists z \in K \text{ such that } \langle c, z\rangle \geq \gamma - (1 + \|c\|)\varepsilon.$$

Proof. We shall prove both points of the theorem separately.

1. Let $z_0 \in K$, then

$$\operatorname{conv}\big(\{z_0\} \cup \mathrm{S}(a_0, r)\big) \subseteq K,$$

Therefore, if we let $z_\theta = (1-\theta)z_0 + \theta a_0$, then we have that $\mathrm{S}(z_\theta, \theta r) \subseteq K$ for all $0 \leq \theta \leq 1$. Hence in particular $z_{\varepsilon/r} \in \mathrm{S}(K, -\varepsilon)$. We now get that

$$\begin{aligned}
\langle c, z_0\rangle &= \big(\langle c, z_\theta\rangle - \theta\langle c, a_0\rangle\big) \big/ (1 - \theta) \qquad \text{for all } 0 \leq \theta < 1 \\
&= \left(\langle c, z_{\varepsilon/r}\rangle - \frac{\varepsilon}{r}\langle c, a_0\rangle\right) \Big/ \left(1 - \frac{\varepsilon}{r}\right) \\
&\leq \left(\gamma + \varepsilon - \frac{\varepsilon}{r}\langle c, a_0\rangle\right) \Big/ \left(1 - \frac{\varepsilon}{r}\right).
\end{aligned}$$

2. Let $y \in \mathrm{S}(K, \varepsilon)$ such that $\langle c, y\rangle \geq \gamma - \varepsilon$ and let $z \in \mathrm{S}(y, \varepsilon) \cap K$ $(\neq \emptyset)$, then

$$\begin{aligned}\langle c, z\rangle &\geq \min\{\langle c, u\rangle \mid u \in \mathrm{S}(y, \varepsilon)\}\\ &= \min\{\langle c, y\rangle + \varepsilon\langle c, v\rangle \mid v \in \mathrm{S}(0,1)\}\\ &= \langle c, y\rangle - \|c\|\varepsilon\\ &\geq \gamma - (1 + \|c\|)\varepsilon. \qquad \square\end{aligned}$$

We will now consider how the problems in this section are related to each other. It is immediately apparent that an oracle for MEM would provide us with an oracle for WMEM. For $\mathbb{X} = \mathbb{R}^n$, we now connect WMEM and WVAL using the following lemma by Yudin and Nemirovski [YN76].

Lemma 2.7. *For $\mathbb{X} = \mathbb{R}^n$, there exists an algorithm that solves WVAL for every quintuple $(K; n, R, r, a_0)$ given by a weak membership oracle. This algorithm is oracle-polynomial time with respect to the encoding lengths of the quintuple and the instance for WVAL.*

It is relatively easy to show that this theorem also holds for $\mathbb{X} = \mathbb{S}^n$. However, when doing this, care must be taken to maintain rationality. One way to do this would be to define the one-to-one mapping "svec" which maps $\mathbb{S}^n$ to $\mathbb{R}^N$, where $N = \dim(\mathbb{S}^n) = \frac{1}{2}n(n+1)$, by stacking the elements of the upper triangle, i.e.

$$\operatorname{svec}(X) = (x_{11},\ x_{12},\ x_{22},\ x_{13}, \ldots,\ x_{1k},\ x_{2k}, \ldots,\ x_{kk}, \ldots,\ x_{nn})^{\mathsf{T}}.$$

For any two matrices $C, X \in \mathbb{S}^n$, we then get that

$$\begin{aligned}&\langle C, X\rangle = \langle \operatorname{svec}((2E - I) \circ C), \operatorname{svec}(X)\rangle,\\ &\tfrac{2}{3}\|X\| \leq \|\operatorname{svec}(X)\| \leq \|X\|.\end{aligned}$$

The important things about this mapping are that it is linear, rationality is maintained and the norms are linearly related. (An alternative definition for the svec mapping which often appears in the literature has the off-diagonal elements multiplied by $\sqrt{2}$. Although this has the advantage of maintaining the inner product and norms, we would not maintain rationality, as required.) Using this, along with Lemma 2.7, it is now trivial to prove the following theorem.

Theorem 2.8. *For $\mathbb{X}$ equal to $\mathbb{R}^n$ or $\mathbb{S}^n$, there exists an algorithm that solves WVAL for every quintuple $(K; N, R, r, a_0)$ given by a weak membership oracle. This algorithm is oracle-polynomial time with respect to the encoding lengths of the quintuple and the instance for WVAL.*

2.2 The Copositive Cone

If a graph G contains a stable set of size λ then it is easy to see that it contains a stable set of size $t \in \mathbb{Z}_{++}$ for all $t \leq \lambda$. Recall that we denote by $\alpha_G \in \mathbb{Z}_{++}$ the stability number of a graph G. Note that G contains a stable set of size $t \in \mathbb{Z}_{++}$ if and only if $\alpha_G \geq t$. Combining this observation with (1.30) and (1.31) we get the following alternative formulations of the stability number of a graph G

$$\begin{aligned} \alpha_G &= \min\left\{\lambda \;\middle|\; \big((I + A_G)\lambda - E\big) \in \mathcal{COP}\right\}, && (2.1)\\ &= \max\left\{\langle E, X\rangle \mid \langle I + A_G, X\rangle = 1,\ X \in \mathcal{CP}\right\}, && (2.2)\\ &= \max\left\{\langle E, X\rangle \mid \langle I + A_G, X\rangle \leq 1,\ X \in \mathcal{CP}\right\}, && (2.3) \end{aligned}$$

It can now be seen that the stable set problem is Turing reducible to MEM for the copositive cone.

Lemma 2.9. *The graph G contains a stable set of size $t \in \mathbb{Z}_{++}$ if and only if*

$$\big((I + A_G)\big(t - \tfrac{1}{2}\big) - E\big) \notin \mathcal{COP}.$$

Proof. For any $\lambda \geq \alpha_G$ we have that $\big((I + A_G)\lambda - E\big) \in \mathcal{COP}$. This can be seen from the fact that $\big((I + A_G)\alpha_G - E\big) \in \mathcal{COP}$ and $(I + A_G)$ is nonnegative (so also copositive). Therefore

$$\begin{aligned} \big((I + A_G)\big(t - \tfrac{1}{2}\big) - E\big) \notin \mathcal{COP} &\Leftrightarrow \alpha_G > t - \tfrac{1}{2}\\ &\Leftrightarrow \alpha_G \geq \left\lceil t - \tfrac{1}{2}\right\rceil = t\\ &\Leftrightarrow \text{There is a stable set of size } t. \end{aligned}$$

□

This immediately allows to state the following theorem, confirming again the complexity result by Murty and Kabadi for the copositive cone.

Theorem 2.10. *The stable set problem is Turing reducible to the strong membership problem for the copositive cone with a many-one reduction, and thus the strong membership problem for the copositive cone is NP-hard.*

Proof. This comes from Lemma 2.9, and from noting that the encoding length of $\big((I + A_G)\big(t - \tfrac{1}{2}\big) - E\big)$ is polynomial in the encoding length of the stable set problem. □

In order to extend this to WMEM for the copositive cone we provide the following lemma.

Lemma 2.11. *Let $\varepsilon, \lambda \in \mathbb{R}_+$ such that $\varepsilon \leq 1$ and $\lambda \geq (1-\varepsilon)\alpha_G + \varepsilon(n+1)$. Now define $Z_\lambda := (I + A_G)\lambda - E$. Then we have that $Z_\lambda \in \mathrm{S}(\mathcal{COP}, -\varepsilon)$, or equivalently $\mathrm{S}(Z_\lambda, \varepsilon) \subseteq \mathcal{COP}$.*

Proof. We consider an arbitrary $Y \in \mathrm{S}(Z_\lambda, \varepsilon)$. There exists $V \in \mathbb{S}^n$ such that $\|V\| = 1$ and $Y = Z_\lambda + \varepsilon V$. Now, for all $\mathbf{x} \geq 0$ such that $\|\mathbf{x}\| = 1$ we have

$$\begin{aligned}
\mathbf{x}^\mathsf{T} Y \mathbf{x} &= \mathbf{x}^\mathsf{T}\big((I + A_G)\lambda - E + \varepsilon V\big)\mathbf{x} \\
&= (1-\varepsilon)\mathbf{x}^\mathsf{T}\big((I + A_G)\alpha_G - E\big)\mathbf{x} - \varepsilon \mathbf{x}^\mathsf{T} E \mathbf{x} + \varepsilon \mathbf{x}^\mathsf{T} V \mathbf{x} \\
&\qquad + (\lambda - (1-\varepsilon)\alpha_G)\big(\mathbf{x}^\mathsf{T} I \mathbf{x} + \mathbf{x}^\mathsf{T} A_G \mathbf{x}\big) \\
&\geq 0 - \varepsilon n - \varepsilon + \varepsilon(n+1)(1+0) = 0.
\end{aligned}$$

Therefore $Y \in \mathcal{COP}$, completing the proof. □

We can now state the following lemma and theorem concerning WMEM for the copositive cone.

Lemma 2.12. *Let $G = (V, E)$ be a graph with $|V| = n \in \mathbb{Z}_{++}$ and let*

$$\begin{aligned}
Y &= (I + A_G)\left(t - \tfrac{1}{2}\right) - E, \\
\delta &= 1/(2n+1), \\
K &= \mathcal{COP},
\end{aligned}$$

where $n \geq t \in \mathbb{Z}_{++}$. Then considering the WMEM for these parameters we have that

1. *$Y \in \mathrm{S}(K, \delta)$ would imply that the graph G does not contain a stable set of size t.*
2. *$Y \notin \mathrm{S}(K, -\delta)$ would imply that the graph G does contain a stable set of size t.*

Proof. We shall prove these results separately.

1. Suppose that $Y \in \mathrm{S}(\mathcal{COP}, \delta)$. There must exist $Z \in \mathrm{S}(0, 1)$ such that $(Y + \delta Z) \in \mathcal{COP}$.

   From Lemma 2.11 (setting $\varepsilon = 1$ and $\lambda = n + 1$) we have that

   $$((I + A_G)(n+1) - E - Z) \in \mathcal{COP}.$$

   As the copositive cone is convex the following matrix must again be copositive,

   $$\begin{aligned}
   &\tfrac{\delta}{1+\delta}\Big((I + A_G)(n+1) - E - Z\Big) + \left(1 - \tfrac{\delta}{1+\delta}\right)\Big((I + A_G)\big(t - \tfrac{1}{2}\big) - E + \delta Z\Big) \\
   &\qquad = (I + A_G)\left(t - \frac{2t-1}{4(n+1)}\right) - E.
   \end{aligned}$$

From this we see that $\alpha_G \leq \left\lfloor t - \frac{2t-1}{4(n+1)} \right\rfloor = t - 1$.

Therefore the graph does not contain a stable set of size t.

2. Suppose that $Y \notin \mathrm{S}(\mathcal{COP}, -\delta)$.

   From Lemma 2.11 we have that $t - \frac{1}{2} < (1-\delta)\alpha_G + \delta(n+1) \leq \alpha_G + \delta n$.

   Therefore $\alpha_G \geq \left\lceil t - \frac{1}{2} - \frac{n}{2n+1} \right\rceil = \left\lceil t - \frac{4n+1}{4n+2} \right\rceil = t$, and so the graph does contain a stable set of size t.

□

Theorem 2.13. *The stable set problem is Turing reducible to the weak membership problem for the copositive cone with a many-one reduction, and thus the weak membership problem for the copositive cone is NP-hard.*

Proof. This comes from Lemma 2.9 and noting that the encoding lengths of Y and δ from this lemma are polynomial in the encoding length of the stable set problem. □

2.3 The Completely Positive Cone

In this section we consider the weak membership problem for the completely positive cone. In order to do this, rather than reformulating to a WMEM problem as we did for the copositive case, this time we reformulate to a WVAL problem, using the following quintuple.

Lemma 2.14. *Consider a graph G with n vertices, let $K = \{X \in \mathcal{CP} \mid \langle I + A_G, X\rangle \leq 1\}$ and furthermore set*

$$\begin{aligned} N &= \tfrac{1}{2}n(n+1), \\ R &= 1, \\ r &= \frac{1}{4n^2}, \\ A_0 &= \tfrac{1}{2n}I + \tfrac{1}{4n^2}E, \end{aligned}$$

then the quintuple $(K; N, R, r, A_0)$ is an A_0-centered convex body as defined in Definition 2.5.

Proof. First we will show that $K \subseteq \mathrm{S}\,(0, R)$. Note that $(I + A_G)$ is in the interior of the copositive cone, therefore,

$$\begin{aligned}
&\max\{\|X\| \mid X \in K\}\\
&\quad= \max\left\{\|X\| \;\middle|\; X \in \operatorname{conv}\left(\{0\} \cup \{\mathbf{b}\mathbf{b}^{\mathsf{T}} \mid \mathbf{b} \in \mathbb{R}^n_+,\ \mathbf{b}^{\mathsf{T}}(I + A_G)\mathbf{b} = 1\}\right)\right\}\\
&\quad= \max\left\{\|\mathbf{b}\mathbf{b}^{\mathsf{T}}\| \;\middle|\; \mathbf{b} \in \mathbb{R}^n_+,\ \mathbf{b}^{\mathsf{T}}(I + A_G)\mathbf{b} = 1\right\}\\
&\quad\leq 1 = R.
\end{aligned}$$

Next we will show that $S(A_0, r) \subseteq K$. Let $X \in \mathrm{S}\,(A_0, r)$, then we get the following entrywise inequalities for X,

$$0 \leq \frac{1}{2n} I = A_0 - Er \leq X \leq A_0 + Er = \frac{1}{2n} I + \frac{1}{2n^2} E. \tag{2.4}$$

Hence it now follows that

$$\langle I + A_G, X\rangle \leq \langle E, X\rangle \leq \frac{1}{2n}\langle E, I\rangle + \frac{1}{2n^2}\langle E, E\rangle = \frac{n}{2n} + \frac{n^2}{2n^2} = 1.$$

What is left to show now is that $X \in \mathcal{CP}$. To do this we use a result from Kaykobad [Kay87] that says that if we have an entrywise nonnegative matrix $Y \in \mathbb{S}^n$, which is diagonally dominant, i.e. $(Y)_{ii} \geq \sum_{j \neq i}(Y)_{ij}$ for all $i = 1, \ldots, n$, then $Y \in \mathcal{CP}$.

From (2.4), we note that for all i, j such that $i \neq j$, we have $0 \leq X_{ij} \leq \frac{1}{2n^2}$ and $\frac{1}{2n} \leq X_{ii}$. The result then immediately follows. □

We can now state the following theorem from which the NP-hardness of MEM for the completely positive cone will follow.

Theorem 2.15. *Consider a graph G with n vertices, let $t \in \mathbb{Z}_{++}$, and let $(K; N, R, r, A_0)$ be the A_0-centered convex body as described in Lemma 2.14. Then set*

$$\begin{aligned}
C &= E,\\
\gamma &= t - \tfrac{1}{2},\\
\varepsilon &= \frac{1}{16n^2 t}.
\end{aligned}$$

Considering the WVAL for these parameters we have that

1. *$\langle C, X\rangle \leq \gamma + \varepsilon$ for all $X \in \mathrm{S}\,(K, -\varepsilon)$ would imply that the graph does not contain a stable set of size t.*

2. *$\exists Y \in \mathrm{S}(K,\varepsilon)$ such that $\langle C, Y\rangle \geq \gamma - \varepsilon$ would imply that the graph does contain a stable set of size t.*

From this we then have that the stable set problem is Turing reducible to WMEM for K.

Proof. Using Lemma 2.6 we look at what the results of WVAL for our choice of parameters would mean. A major point in the implications is that $\alpha_G \in \mathbb{Z}_{++}$ and we recall from (2.3) that

$$\alpha_G = \max\left\{\langle C, X\rangle \mid X \in K\right\}.$$

1. Let $\langle C, X\rangle \leq \gamma + \varepsilon$, for all $X \in \mathrm{S}(K, -\varepsilon)$. Then for all $Z \in K$ we have that

$$\begin{aligned}\langle C, Z\rangle &\leq \left(\gamma + \varepsilon - \frac{\varepsilon}{r}\langle C, A_0\rangle\right)\Big/\left(1 - \frac{\varepsilon}{r}\right)\\ &= \left(t - \frac{1}{2} + \frac{1}{16n^2t} - \frac{4n^2}{16n^2t}\left\langle E, \frac{1}{2n}I + \frac{1}{4n^2}E\right\rangle\right)\Big/\left(1 - \frac{4n^2}{16n^2t}\right)\\ &= \left((t - \tfrac{1}{4})(16n^2t - 4n^2) - 4n^2 + 1\right)\big/\left(16n^2t - 4n^2\right)\\ &= t - \tfrac{1}{4} - \frac{4n^2 - 1}{4n^2(4t-1)} \leq t - \tfrac{1}{4}.\end{aligned}$$

   Therefore $\alpha_G \leq \left\lfloor t - \frac{1}{4}\right\rfloor = t - 1$ and so the graph does not contain a stable set of size t.

2. Assume $\exists Y \in \mathrm{S}(K,\varepsilon)$ such that $\langle C, Y\rangle \geq \gamma - \varepsilon$. Then again by Lemma 2.6, $\exists Z \in K$ such that

$$\begin{aligned}\langle C, Z\rangle &\geq \gamma - (1 + \|C\|)\varepsilon\\ &= t - \frac{5}{8} + \frac{2n^2t - n - 1}{16n^2t}\\ &\geq t - \frac{5}{8}.\end{aligned}$$

   Therefore $\alpha_G \geq \left\lceil t - \frac{5}{8}\right\rceil = t$ and so the graph does contain a stable set of size t.

It can now be seen that the stable set problem is Turing reducible to WMEM for K by using Theorem 2.8 and noting that the encoding lengths of $(K; N, R, r, A_0)$, C, γ and ε are polynomial in the encoding length of the stable set problem. □

We now get the following result on MEM for the completely positive cone.

Theorem 2.16. *The strong membership problem for the completely positive cone is NP-hard.*

Proof. From Theorem 2.15, we have that the stable set problem is Turing reducible to WMEM for K. This is in turn many-one reducible to MEM for the completely positive cone, as the encoding length of $(I + A_G)$ is polynomial in the encoding length of a matrix in $\mathbb{S}^n$. □

In order to extend this to WMEM for the completely positive cone we need a way of solving WMEM for K from Lemma 2.14 given a weak membership oracle for the completely positive cone.

Lemma 2.17. *We consider K from Lemma 2.14, define $\mathcal{H} := \{X \in \mathbb{S} \mid \langle I + A_G, X\rangle \leq 1\}$ and let $\delta \in \mathbb{Q}_{++}$. Then we have that*

1. $\mathrm{S}\,(K, -\delta) = \mathrm{S}\,(\mathcal{H}, -\delta) \cap \mathrm{S}\,(\mathcal{CP}, -\delta) \subseteq \mathcal{H} \cap \mathrm{S}\left(\mathcal{CP}, -\delta/(1 + n^2)\right)$,
2. $\mathrm{S}\,(K, \delta) \supseteq \mathcal{H} \cap \mathrm{S}\left(\mathcal{CP}, \delta/(1 + n^2)\right)$.

From this we then have that WMEM for K is Turing reducible to WMEM for the completely positive cone.

Proof. We consider each of these parts separately.

1. This comes from noting that

$$\begin{aligned}
\mathrm{S}\,(K, -\delta) &= \{X \in \mathbb{S} \mid \mathrm{S}\,(X, \delta) \subseteq \mathcal{H} \cap \mathcal{CP}\} \\
&= \{X \in \mathbb{S} \mid \mathrm{S}\,(X, \delta) \subseteq \mathcal{H}\} \cap \{X \in \mathbb{S} \mid \mathrm{S}\,(X, \delta) \subseteq \mathcal{CP}\} \\
&= \mathrm{S}\,(\mathcal{H}, -\delta) \cap \mathrm{S}\,(\mathcal{CP}, -\delta)\,, \\
\mathrm{S}\,(\mathcal{H}, -\delta) &\subseteq \mathcal{H}, \\
\mathrm{S}\,(\mathcal{CP}, -\delta) &\subseteq \mathrm{S}\left(\mathcal{CP}, -\delta/(1 + n^2)\right).
\end{aligned}$$

2. Consider an arbitrary $X \in \mathcal{H} \cap \mathrm{S}\left(\mathcal{CP}, \delta/(1 + n^2)\right)$ and let $\varepsilon = \delta/(1 + n^2)$. Then there exists $Y \in \mathcal{CP}$ such that $\|X - Y\| \leq \varepsilon$ and we have that

$$\langle I + A_G, Y\rangle \leq \langle I + A_G, X\rangle + \varepsilon\|I + A_G\| \leq 1 + \varepsilon n^2,$$

$$\begin{aligned}
&\|Y\| \leq \max\left\{\|U\| \;\middle|\; U \in \mathcal{CP},\ \langle I + A_G, U\rangle \leq 1 + \varepsilon n^2\right\} \\
&= \max\left\{\|U\| \;\middle|\; U \in \mathrm{conv}\left(\{0\} \cup \{\mathbf{b}\mathbf{b}^\mathsf{T} \mid \mathbf{b} \in \mathbb{R}^n_+,\ \mathbf{b}^\mathsf{T}(I + A_G)\mathbf{b} = 1 + \varepsilon n^2\}\right)\right\} \\
&\leq 1 + \varepsilon n^2.
\end{aligned}$$

We now let $Z = \frac{1}{1+\varepsilon n^2} Y$. We have that $Z \in \mathcal{CP}$ and $\langle I + A_G, Z\rangle \leq 1$. Therefore $Z \in \mathcal{H} \cap \mathcal{CP}$. We finish the proof by noting that

$$\begin{aligned} \|X - Z\| &= \left\| X - \frac{1}{1+\varepsilon n^2} Y \right\| \\ &= \left\| X - Y + \frac{\varepsilon n^2}{1+\varepsilon n^2} Y \right\| \\ &\leq \|X - Y\| + \frac{\varepsilon n^2}{1+\varepsilon n^2} \|Y\| \\ &\leq \varepsilon(1+n^2) \\ &= \delta, \end{aligned}$$

and hence $X \in \mathrm{S}\left(\mathcal{H} \cap \mathcal{CP}, \delta\right) = \mathrm{S}\left(K, \delta\right)$.

We then have that WMEM for K is Turing reducible to WMEM for the completely positive cone by noting that the encoding lengths of $(I + A_G)$ and $\delta/(1+n^2)$ are polynomial in the encoding lengths of the input δ and a matrix in $\mathbb{S}^n$. □

We now get the following result, which is the main result of this paper,

Theorem 2.18. *Both the weak and strong membership problems for the completely positive cone are NP-hard.*

Proof. From Theorem 2.15, and Lemma 2.17, it follows that the stable set problem is Turing reducible to WMEM for the completely positive cone, and thus this problem is NP-hard. To show that MEM for the completely positive cone is also NP-hard we can either use Theorem 2.16 or a many-one reduction to WMEM for the completely positive cone. □

This finally establishes that the strong and weak membership problems for the copositive as well as the completely positive cone are in NP-hard. As we mentioned in the Introduction of this thesis whether or not membership for the completely positive cone is in NP is still an open question at the moment of writing. Another interesting question that can be asked considering the results of this chapter is whether or not this type of complexity result holds in general. More specifically we define the following open problem.

Open Problem 2.19. *Let $K \in \mathbb{R}^{n\times n}$ and suppose that the (weak) membership problem for K is NP-hard. Does this then imply that the (weak) membership problem for K^* is also NP-hard.*