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On approximations, complexity, and applications for copositive programming

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On Approximations, Complexity, and Applications for Copositive Programming

Luuk Gijben



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Furthermore I have had the opportunity to meet and work with a number of people in the past four years. In particular I would like to thank Dr. Peter Dickinson for a number of fruitful discussions resulting in several successful publications. By the same token I would also like to express my gratitude to Dr. Roland Hildebrand. I would furthermore like to thank Dr. Juan Vera for a series of very interesting and eye opening conversation that have given me a better understanding of how to approach doing research.

During my time in Groningen I have also received a lot of support from my friends and family. For this I am grateful, particularly I would like to thank my parents and my brother who are always there for me.

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Contents

Personal contributions	iii
1 Introduction to copositive programming	1
1.1 Notation	2
1.2 The copositive cone and its dual	4
1.3 Approximation hierarchies	12
1.3.1 Approximations via simplicial partitioning	13
1.3.2 The polyhedral approximation cones \mathcal{C}_n^r	14
1.3.3 The Parrilo r -cones	15
1.3.4 The SOS approximation cones \mathcal{Q}_n^r	16
1.3.5 Approximating cones via DSOS and SDSOS	18
1.3.6 Other approximations for \mathcal{COP}^n and \mathcal{CP}^n	20
1.4 Copositive (Completely Positive) Programming	22
2 Complexity of membership for the completely positive cone and its dual	29
2.1 Problems for convex sets	31
2.2 The Copositive Cone	34
2.3 The Completely Positive Cone	36
3 Irreducible elements of the copositive cone	41
3.1 Notation	43
3.2 Irreducible copositive matrices	44
3.3 S -matrices	49
3.4 Auxiliary results	53
3.5 Classification of 5×5 copositive matrices	58
3.5.1 Property 3.4	59
3.5.2 Property 3.10 but not Property 3.4	62
3.5.3 Irreducible matrices of \mathcal{COP}^5	68

4	Scaling relationship between the copositive cone and Parrilo's first level approximation	69
4.1	Scaling a matrix out of \mathcal{K}_n^r	71
4.2	Non-decreasing scalings	75
4.3	Scaling a matrix into \mathcal{K}_5^1	80
4.4	Conjectures and open problems	85
5	Graph isomorphism and copositive programming	89
5.1	Notation	92
5.2	Graph Isomorphism as a Copositive Program	92
5.2.1	Properties of the matrix D	96
5.3	The Graph Isomorphism Problem as an LP	99
5.4	Solving the \mathcal{GIP} via approximation hierarchies	100
5.5	Reformulating the copositive formulation	104
	Summary	107
	Samenvatting	109
	Bibliography	111
	Nomenclature	121
	Index	125

Personal contributions

During my time as a PhD-student at the University of Groningen I have produced along with my co-authors the following papers.

Published articles:

[DG14] Peter J.C. Dickinson and Luuk Gijben. On the computational complexity of membership problems for the completely positive cone and its dual. *Computational Optimization and Applications*, 57(2):403-415, 2014.

[DDGH13a] Peter J.C. Dickinson, Mirjam Dür, Luuk Gijben, and Roland Hildebrand. Irreducible elements of the copositive cone. *Linear Algebra and its Applications*, 439(6):1605-1626, 2013.

[DDGH13b] Peter J.C. Dickinson, Mirjam Dür, Luuk Gijben and Roland Hildebrand. Scaling relationship between the copositive cone and Parrilo's first level approximation. *Optimization Letters*, 7(8):1669-1679, 2013. (*won the 2013 OPTL Best Paper Award*)

Articles in construction:

Mirjam Dür and Luuk Gijben. Graph Isomorphism and Copositive Programming.

Besides these papers I have obtained several separate currently unpublished results. What follows is a more detailed explanation of my personal contributions to the field of copositive programming as presented in this thesis.

It has been shown by Murty and Kabadi [MK87] that the strong membership problem for the copositive cone, that is deciding whether or not a given matrix is in the copositive cone, is a co-NP-complete problem. From this it has long been assumed that this implies that the question of whether or not the strong membership problem for the dual of the copositive cone, the completely positive cone, is also an NP-hard problem. However, the technical details for this have not previously been looked at to confirm that this is indeed the case. In *Chapter 2*, which is based on [DG14], I prove together with Dr. Peter J.C. Dickinson that the strong membership problem for the

completely positive cone is indeed NP-hard. Furthermore, we show that even the so called weak membership problems for both of these cones are NP-hard. An alternative proof of the NP-hardness of the strong membership problem for the copositive cone is also provided.

Chapter 3 comes from the paper [DDGH13a] I published together with Dr. Peter J.C. Dickinson, Prof. Dr. Mirjam Dür and Dr. Roland Hildebrand. An element A of the $n \times n$ copositive cone \mathcal{COP}^n is defined to be *irreducible* with respect to the nonnegative cone if it cannot be written as a nontrivial sum $A = C + N$ of a copositive matrix C and an element-wise nonnegative matrix N (note that this concept of irreducibility differs from the standard one normally studied in matrix theory). This property was studied by Baumert [Bau65] who gave a characterization of irreducible matrices. It is demonstrated in this chapter that Baumert's characterization is incorrect and a correct version of his theorem is given instead. This establishes a necessary and sufficient condition for a copositive matrix to be irreducible. For the case of 5×5 copositive matrices a complete characterization of all irreducible matrices is given. It is shown that those irreducible matrices in \mathcal{COP}^5 which are not positive semidefinite can be parameterized in a semi-trigonometric way. Finally, a proof is given for the result that every 5×5 copositive matrix which is not the sum of a nonnegative and a positive semidefinite matrix can be expressed as the sum of a nonnegative and a single irreducible matrix.

This result is then used in *Chapter 4*, the content of which is from the paper [DDGH13b] (which received the 2013 OPTL Best Paper Award) that I also published with Dr. Peter J.C. Dickinson, Prof. Dr. Mirjam Dür and Dr. Roland Hildebrand. In particular we investigate the relation between the cone \mathcal{COP}^n of $n \times n$ copositive matrices and the approximating cone \mathcal{K}_n^1 introduced by Parrilo [Par00]. These cones are known to be equal for $n \leq 4$, and for $n \geq 5$ it is shown that they are not equal. Our result is based on the fact that \mathcal{K}_n^1 is not invariant under diagonal scaling while \mathcal{COP}^n is. In particular it is shown that for any copositive matrix which is not the sum of a nonnegative and a positive semidefinite matrix, we can find a scaling which is not in \mathcal{K}_n^1 . In fact, it can be shown that if all scaled versions of a matrix are contained in \mathcal{K}_n^r for some fixed r , then the matrix must be in \mathcal{K}_n^0 . For the specific 5×5 case, the more surprising result that we can scale any copositive matrix X into \mathcal{K}_5^1 is given. In particular it is shown that any scaling matrix D such that $(DXD)_{ii} \in \{0, 1\}$ for all i yields $DXD \in \mathcal{K}_5^1$. This provides a way to use the cone \mathcal{K}_5^1 to check if any order 5 matrix is copositive. Another consequence of this is a complete characterization of \mathcal{COP}^5 in terms of \mathcal{K}_5^1 . Moreover, during Chapter 4 I provide an explicit way to construct a scaling matrix D for $A \in \mathcal{K}_n^1 \setminus \mathcal{K}_n^0$ such that $DAD \notin \mathcal{K}_n^1$, a result that was not included in [DDGH13b]. Another result given in Chapter 4 not included in this paper, is

a negative result regarding the complexity of the problem of scaling arbitrary matrices in the opposite direction of the hierarchy of Parrilo cones. Finally I introduce the concept of non-decreasing scalings which also had not been discussed before in [DDGH13b]. At the end of Chapter 4 several conjectures are provided concerning scalings.

In *Chapter 5* I suggest a copositive formulation for the graph isomorphism problem, that is, the problem of deciding whether or not two graphs are the same after a (possible) relabeling of the vertices. This problem is particular in the sense that its complexity is currently unknown. The hierarchies \mathcal{C}_n^1 and \mathcal{K}_n^r are then applied to the copositive formulation and it is shown that an optimal solution always exists in one of these approximating cones for a finite r . This is particularly important for this problem because the graph isomorphism problem only has 'yes' and 'no' as answers. Finally, I rewrite the initial copositive formulation several times obtaining an LP formulation, as well as a possible method to construct a certificate for a polynomial time solution to the graph isomorphism problem.

