On approximations, complexity, and applications for copositive programming

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On Approximations, Complexity, and Applications for Copositive Programming

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Contents

Personal contributions

During my time as a PhD-student at the University of Groningen I have produced along with my co-authors the following papers.

Published articles:

[DG14] Peter J.C. Dickinson and Luuk Gijben. On the computational complexity of membership problems for the completely positive cone and its dual. *Computational Optimization and Applications*, 57(2):403-415, 2014.

[DDGH13a] Peter J.C. Dickinson, Mirjam Dür, Luuk Gijben, and Roland Hildebrand. Irreducible elements of the copositive cone. *Linear Algebra and its Applications*, 439(6):1605-1626, 2013.

[DDGH13b] Peter J.C. Dickinson, Mirjam Dür, Luuk Gijben and Roland Hildebrand. Scaling relationship between the copositive cone and Parrilo's first level approximation. *Optimization Letters*, 7(8):1669-1679, 2013. *(won the 2013 OPTL Best Paper Award)*

Articles in construction:

Mirjam Dür and Luuk Gijben. Graph Isomorphism and Copositive Programming.

Besides these papers I have obtained several separate currently unpublished results. What follows is a more detailed explanation of my personal contributions to the field of copositive programming as presented in this thesis.

It has been shown by Murty and Kabadi [MK87] that the strong membership problem for the copositive cone, that is deciding whether or not a given matrix is in the copositive cone, is a co-NP-complete problem. From this it has long been assumed that this implies that the question of whether or not the strong membership problem for the dual of the copositive cone, the completely positive cone, is also an NP-hard problem. However, the technical details for this have not previously been looked at to confirm that this is indeed the case. In *Chapter* 2, which is based on [DG14], I prove together with Dr. Peter J.C. Dickinson that the strong membership problem for the completely positive cone is indeed NP-hard. Furthermore, we show that even the so called weak membership problems for both of these cones are NP-hard. An alternative proof of the NP-hardness of the strong membership problem for the copositive cone is also provided.

Chapter 3 comes from the paper [DDGH13a] I published together with Dr. Peter J.C. Dickinson, Prof. Dr. Mirjam Dür and Dr. Roland Hildebrand. An element A of the $n \times n$ copositive cone $\mathcal{COP}^n$ is defined to be *irreducible* with respect to the nonnegative cone if it cannot be written as a nontrivial sum $A = C + N$ of a copositive matrix C and an element-wise nonnegative matrix N (note that this concept of irreducibility differs from the standard one normally studied in matrix theory). This property was studied by Baumert [Bau65] who gave a characterization of irreducible matrices. It is demonstrated in this chapter that Baumert's characterization is incorrect and a correct version of his theorem is given instead. This establishes a necessary and sufficient condition for a copositive matrix to be irreducible. For the case of 5×5 copositive matrices a complete characterization of all irreducible matrices is given. It is shown that those irreducible matrices in $\mathcal{COP}^5$ which are not positive semidefinite can be parameterized in a semi-trigonometric way. Finally, a proof is given for the result that every 5×5 copositive matrix which is not the sum of a nonnegative and a positive semidefinite matrix can be expressed as the sum of a nonnegative and a single irreducible matrix.

This result is then used in *Chapter* 4, the content of which is from the paper [DDGH13b] (which received the 2013 OPTL Best Paper Award) that I also published with Dr. Peter J.C. Dickinson, Prof. Dr. Mirjam Dür and Dr. Roland Hildebrand. In particular we investigate the relation between the cone $\mathcal{COP}^n$ of $n \times n$ copositive matrices and the approximating cone $\mathcal{K}_n^1$ introduced by Parrilo [Par00]. These cones are known to be equal for $n \leq 4$, and for $n \geq 5$ it is shown that they are not equal. Our result is based on the fact that $\mathcal{K}_n^1$ is not invariant under diagonal scaling while $\mathcal{COP}^n$ is. In particular it is shown that for any copositive matrix which is not the sum of a nonnegative and a positive semidefinite matrix, we can find a scaling which is not in $\mathcal{K}_n^1$. In fact, it can be shown that if all scaled versions of a matrix are contained in $\mathcal{K}_n^r$ for some fixed r, then the matrix must be in $\mathcal{K}_n^0$. For the specific 5×5 case, the more surprising result that we can scale any copositive matrix X into $\mathcal{K}_5^1$ is given. In particular it is shown that any scaling matrix D such that $(DXD)_{ii} \in \{0, 1\}$ for all i yields $DXD \in \mathcal{K}_5^1$. This provides a way to use the cone $\mathcal{K}_5^1$ to check if any order 5 matrix is copositive. Another consequence of this is a complete characterization of $\mathcal{COP}^5$ in terms of $\mathcal{K}_5^1$. Moreover, during Chapter 4 I provide an explicit way to construct a scaling matrix D for $A \in \mathcal{K}_n^1 \backslash \mathcal{K}_n^0$ such that $DAD \notin \mathcal{K}_n^1$, a result that was not included in [DDGH13b]. Another result given in Chapter 4 not included in this paper, is a negative result regarding the complexity of the problem of scaling arbitrary matrices in the opposite direction of the hierarchy of Parrilo cones. Finally I introduce the concept of non-decreasing scalings which also had not been discussed before in [DDGH13b]. At the end of Chapter 4 several conjectures are provided concerning scalings.

In *Chapter* 5 I suggest a copositive formulation for the graph isomorphism problem, that is, the problem of deciding whether or not two graphs are the same after a (possible) relabeling of the vertices. This problem is particular in the sense that its complexity is currently unknown. The hierarchies $\mathcal{C}_n^1$ and $\mathcal{K}_n^r$ are then applied to the copositive formulation and it is shown that an optimal solution always exists in one of these approximating cones for a finite r. This is particularly important for this problem because the graph isomorphism problem only has 'yes' and 'no' as answers. Finally, I rewrite the initial copositive formulation several times obtaining an LP formulation, as well as a possible method to construct a certificate for a polynomial time solution to the graph isomorphism problem.