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Strong asymptotic convergence of evolution equations governed by maximal monotone operators with Tikhonov regularization

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ABSTRACT

We consider the Tikhonov-like dynamics $-\dot{u}(t) \in A(u(t)) + \varepsilon(t)u(t)$ where A is a maximal monotone operator on a Hilbert space and the parameter function $\varepsilon(t)$ tends to 0 as $t \rightarrow \infty$ with $\int_0^\infty \varepsilon(t) dt = \infty$. When A is the subdifferential of a closed proper convex function f , we establish strong convergence of $u(t)$ towards the least-norm minimizer of f . In the general case we prove strong convergence towards the least-norm point in $A^{-1}(0)$ provided that the function $\varepsilon(t)$ has bounded variation, and provide a counterexample when this property fails.

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1. Introduction

We investigate the asymptotic behavior as $t \rightarrow \infty$ of solutions of the differential inclusion

$$-\dot{u}(t) \in A(u(t)) + \varepsilon(t)u(t); \quad u(0) = x_0, \quad (D)$$

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where $A : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is a maximal monotone operator on a Hilbert space \mathcal{H} , $\varepsilon(t) \geq 0$ is measurable, and $x_0 \in \text{dom}(A)$. Throughout this paper we assume that (D) admits a (necessarily unique) *strong solution*, namely, an absolutely continuous function $u : [0, \infty) \rightarrow \mathcal{H}$ such that (D) holds for almost every $t \geq 0$. Sufficient conditions for this existence may be found, among others, in [4,19,20], and [25].

The differential inclusion (D) is a perturbed version of

$$-\dot{u}(t) \in A(u(t)); \quad u(0) = x_0. \tag{I}$$

We denote by $S = \{x \in \mathcal{H} : 0 \in A(x)\}$ the set of rest points of the latter, and we assume that it is nonempty. The monotonicity of A implies that the dynamics (I) are dissipative, so one might expect that they converge to a point in S . This is not always the case as seen by considering a $\frac{\pi}{2}$ -rotation in \mathbb{R}^2 . However, if we perturb these dynamics as in (D) with a fixed $\varepsilon(t) \equiv \varepsilon > 0$, the operator $A + \varepsilon I$ is strongly monotone and we have strong convergence to the unique solution of $0 \in A(x) + \varepsilon x$. Hence, by introducing a vanishing parameter $\varepsilon(t) \rightarrow 0^+$ and under suitable conditions, one may hope to induce weak or even strong convergence of the solutions of (D) towards a point in S .

Several results are available for different classes of maximal monotone operators. In the unperturbed case $\varepsilon(t) \equiv 0$, while convergence does not hold in general, weak convergence was established in the classical paper [14] for the case of demi-positive operators. This class includes the subdifferentials of closed proper convex functions $A = \partial f$, as well as operators of the form $A = I - T$ with T a contraction having fixed points. As shown by the counterexample in [5], even in the case of subdifferential operators one may not expect this convergence to be strong.

Asymptotic results have also been proved for a variety of dynamics coupling a gradient flow with different approximation schemes. In the particular setting of (D) the convergence depends on whether $\varepsilon(t)$ is in $L^1(0, \infty)$ or not. When $\int_0^\infty \varepsilon(t) dt < \infty$ the results on asymptotic equivalence described in [32] (see also [2]) imply that the perturbation (D) preserves the qualitative convergence properties of (I). For the case $\int_0^\infty \varepsilon(t) dt = \infty$ the most general convergence result available goes back to [33] (based on previous work by [12]) and requires in addition $\varepsilon(t)$ to be non-increasing and convergent to 0 for $t \rightarrow \infty$. Under these conditions $u(t)$ converges strongly to x^* , the point of least norm in S . The main contributions in this paper are in the case $\int_0^\infty \varepsilon(t) dt = \infty$ with $\varepsilon(t) \rightarrow 0$. In Section 2 we consider the subdifferential case $A = \partial f$ and, with no extra assumptions, we prove in Theorem 2 the strong convergence of $u(t)$ towards x^* . For general maximal monotone operators we prove in Theorem 9 of Section 3 that the same result holds if in addition the function $\varepsilon(t)$ has bounded variation. Finally, in Section 4 we provide a counterexample showing that convergence may fail without this bounded variation property.

2. Tikhonov dynamics in convex minimization

Let $f : \mathcal{H} \rightarrow \mathbb{R} \cup \{\infty\}$ be closed, proper and convex, and consider the minimization problem

$$\min_{x \in \mathcal{H}} f(x) \tag{P}$$

whose optimal solution set $S = \{x \in \mathcal{H} : 0 \in \partial f(x)\}$ is assumed to be nonempty. The Tikhonov regularization scheme for (P) is the family of strongly convex problems

$$\min_{x \in \mathcal{H}} f_\varepsilon(x), \tag{P_\varepsilon}$$

where $f_\varepsilon(x) = f(x) + \frac{\varepsilon}{2}|x|^2$. It is well known (e.g. [37]) that the unique solution x_ε of (P_ε) converges strongly as $\varepsilon \rightarrow 0^+$ to the least-norm element of S , which we denote by x^* .

In this setting, the dynamics (D) with $A = \partial f$ correspond to the coupling of the Tikhonov regularization scheme with a steepest descent dynamics, namely

$$-\dot{u}(t) \in \partial f_{\varepsilon(t)}(u(t)) = \partial f(u(t)) + \varepsilon(t)u(t); \quad u(0) = x_0. \tag{T}$$

Since (T) is a perturbed steepest descent method for $f(\cdot)$, we expect $u(t)$ to converge towards a point $x_\infty \in S$. The following slight variant of Gronwall's inequality will be used in the analysis.

Lemma 1. *Let $\theta : [0, \infty) \rightarrow \mathbb{R}$ be absolutely continuous with $\dot{\theta}(t) + \varepsilon(t)\theta(t) \leq \varepsilon(t)h(t)$ for almost all $t \geq 0$, where $h(t)$ is bounded and $\varepsilon(t) \geq 0$ with $\varepsilon \in L^1_{loc}(\mathbb{R}_+)$. Then the function $\theta(t)$ is bounded and if $\int_0^\infty \varepsilon(\tau) d\tau = \infty$ we have $\limsup_{t \rightarrow \infty} \theta(t) \leq \limsup_{t \rightarrow \infty} h(t)$.*

Proof. Let $\kappa_s = \sup\{h(t) : t \geq s\}$ so that $\dot{\theta}(t) + \varepsilon(t)[\theta(t) - \kappa_s] \leq 0$ for $t \geq s$. Multiplying by $\exp(\int_0^t \varepsilon(\tau) d\tau)$ and integrating over $[s, t]$ we get

$$[\theta(t) - \kappa_s] \leq [\theta(s) - \kappa_s] \exp\left(-\int_s^t \varepsilon(\tau) d\tau\right). \tag{1}$$

It follows that $\theta(t)$ is bounded and, if $\int_0^\infty \varepsilon(\tau) d\tau = \infty$, then letting $t \rightarrow \infty$ in (1) we get $\limsup_{t \rightarrow \infty} \theta(t) \leq \kappa_s$, so that $s \rightarrow \infty$ yields $\limsup_{t \rightarrow \infty} \theta(t) \leq \limsup_{t \rightarrow \infty} h(t)$. \square

In this section we improve the known results, showing that the asymptotic convergence of Tikhonov dynamics holds as soon as $\varepsilon(t) \rightarrow 0^+$ when $t \rightarrow \infty$, without any extra assumption (not even monotonicity of $\varepsilon(t)$).

Theorem 2. *Let $u : [0, \infty) \rightarrow \mathcal{H}$ be the strong solution of (T) with $\varepsilon(t) \rightarrow 0^+$ as $t \rightarrow \infty$.*

- (i) *If $\int_0^\infty \varepsilon(t) dt = \infty$ then $u(t) \rightarrow x^*$.*
- (ii) *If $\int_0^\infty \varepsilon(t) dt < \infty$ then $u(t) \rightarrow x_\infty$ for some $x_\infty \in S$.*

Proof. (i) Let $\theta(t) = \frac{1}{2}|u(t) - x^*|^2$ so that $\dot{\theta}(t) = \langle \dot{u}(t), u(t) - x^* \rangle$. Using (T) and the strong convexity of $f_\varepsilon(\cdot)$ we get

$$f_{\varepsilon(t)}(u(t)) + \langle -\dot{u}(t), x^* - u(t) \rangle + \frac{1}{2}\varepsilon(t)|u(t) - x^*|^2 \leq f_{\varepsilon(t)}(x^*)$$

which may be rewritten as

$$\dot{\theta}(t) + \varepsilon(t)\theta(t) \leq f_{\varepsilon(t)}(x^*) - f_{\varepsilon(t)}(u(t)).$$

Since $f_\varepsilon(x_\varepsilon) \leq f_\varepsilon(u(t))$ and $f(x^*) \leq f(x_\varepsilon)$ we deduce

$$\dot{\theta}(t) + \varepsilon(t)\theta(t) \leq \frac{1}{2}\varepsilon(t)[|x^*|^2 - |x_{\varepsilon(t)}|^2]$$

and since $x_\varepsilon \rightarrow x^*$ as $\varepsilon \rightarrow 0^+$ (see for instance [37]), we may use Lemma 1 with $h(t) = \frac{1}{2}[|x^*|^2 - |x_{\varepsilon(t)}|^2]$ to conclude $\limsup_{t \rightarrow \infty} \theta(t) \leq 0$, hence $u(t) \rightarrow x^*$.

(ii) The proof is based on a result by [10]. Let $\bar{x} \in S$ and set $\theta(t) = \frac{1}{2}|u(t) - \bar{x}|^2$. Proceeding as in part (i) we get

$$\dot{\theta}(t) + \varepsilon(t)\theta(t) \leq f(\bar{x}) - f(u(t)) + \frac{1}{2}\varepsilon(t)[|\bar{x}|^2 - |u(t)|^2] \tag{2}$$

from which it follows that $\dot{\theta}(t) \leq \frac{1}{2}|\bar{x}|^2\varepsilon(t)$. Thus $\theta(t) - \frac{1}{2}|\bar{x}|^2 \int_0^t \varepsilon(\tau) d\tau$ is decreasing and hence convergent so that $\theta(t)$ has a limit for $t \rightarrow \infty$. Invoking Opial's Lemma [30] the proof will follow if

we show that every weak accumulation point of $u(t)$ belongs to S , for which it suffices to establish that $f(u(t)) \rightarrow \alpha := \inf_{x \in \mathcal{H}} f(x)$. To prove the latter we note that (T) may be written as $-\dot{u}(t) \in \partial f(u(t)) + v(t)$ with $v(t) = \varepsilon(t)u(t) \in L^1(0, \infty; \mathcal{H})$, so that [10, Lemma 3.3] implies that $f(u(t))$ is absolutely continuous with

$$\frac{d}{dt}[f(u(t))] = -\langle \dot{u}(t) + \varepsilon(t)u(t), \dot{u}(t) \rangle \quad \text{a.e. } t \geq 0.$$

The latter may be bounded from above by $\delta(t) = \frac{1}{4}\varepsilon(t)^2|u(t)|^2 \in L^1(0, \infty; \mathbb{R})$, so that $\frac{d}{dt}[f(u(t)) - \int_0^t \delta(\tau) d\tau] \leq 0$ implying that $f(u(t)) - \int_0^t \delta(\tau) d\tau$ is decreasing and hence convergent. It follows that $f(u(t))$ converges as well. Now, using (2) we get $0 \leq f(u(t)) - f(\bar{x}) \leq -\dot{\theta}(t) + \frac{1}{2}|\bar{x}|^2\varepsilon(t)$ so that

$$\int_0^T [f(u(t)) - \alpha] dt \leq \theta(0) - \theta(T) + \frac{1}{2}|\bar{x}|^2 \int_0^T \varepsilon(t) dt \leq \theta(0) + \frac{1}{2}|\bar{x}|^2 \int_0^\infty \varepsilon(t) dt < \infty$$

which allows to conclude that the limit of $f(u(t))$ is indeed α as claimed. \square

Remark. As mentioned in the introduction, when $\varepsilon(t)$ is non-increasing, part (i) was proved in [33]. This result went unnoticed and several special cases of it were rediscovered in [3,7,15] as examples of couplings of the steepest descent method with general approximation schemes. Particular cases of (ii) were described in [15,17], though we note that this may be deduced from the general results in [20] or, alternatively, from the results on asymptotic equivalence presented in [32].

Theorem 2 still holds, with essentially the same proof, when the regularizing kernel $\frac{1}{2}|x|^2$ is replaced by any strongly convex term. Moreover, part (i) admits the following straightforward generalization.

Proposition 3. *Let $f_\varepsilon(\cdot)$ be strongly convex with parameter $\beta(\varepsilon) > 0$, namely, for each $x \in \mathcal{H}$ and $y \in \partial f_\varepsilon(x)$*

$$f_\varepsilon(x) + \langle y, z - x \rangle + \frac{1}{2}\beta(\varepsilon)|z - x|^2 \leq f_\varepsilon(z), \quad \forall z \in \mathcal{H}.$$

Assume that the minimum x_ε of $f_\varepsilon(\cdot)$ has a strong limit x^ as $\varepsilon \rightarrow 0^+$. Suppose further that there is $y_\varepsilon \in \partial f_\varepsilon(x^*)$ with $|y_\varepsilon| \leq M\beta(\varepsilon)$ for some $M \geq 0$. If $\int_0^\infty \beta(\varepsilon(t)) dt = \infty$ then any solution of $-\dot{u}(t) \in \partial f_{\varepsilon(t)}(u(t))$ satisfies $u(t) \rightarrow x^*$ for $t \rightarrow \infty$.*

Proof. Proceeding as in the previous proof we get

$$\begin{aligned} \dot{\theta}(t) + \beta(\varepsilon(t))\theta(t) &\leq f_{\varepsilon(t)}(x^*) - f_{\varepsilon(t)}(x_{\varepsilon(t)}) \\ &\leq \langle y_{\varepsilon(t)}, x^* - x_{\varepsilon(t)} \rangle \\ &\leq M\beta(\varepsilon(t))|x^* - x_{\varepsilon(t)}| \end{aligned}$$

so the conclusion follows again from Lemma 1 since $h(t) := M|x^* - x_{\varepsilon(t)}| \rightarrow 0$. \square

3. Tikhonov dynamics for maximal monotone maps

Let us consider now the case of a maximal monotone operator $A : \mathcal{H} \rightarrow 2^{\mathcal{H}}$, and let $S = A^{-1}(0)$ denote the solution set of the inclusion $0 \in A(x)$. We suppose that S is nonempty and, as before, we denote x^* its least-norm element (recall that S is closed and convex). In contrast with the subdifferential case, the strong solution of (I) need not converge when $t \rightarrow \infty$ towards a point in S , unless

some further restriction is imposed on the operator A . On the other hand, for any fixed $\varepsilon > 0$, the perturbed operator $A_\varepsilon = A + \varepsilon I$ is strongly monotone and the solution of the differential inclusion

$$-\dot{u}(t) \in A(u(t)) + \varepsilon u(t)$$

converges strongly to $x_\varepsilon = A_\varepsilon^{-1}(0)$.

Before analyzing the conditions for convergence in the non-autonomous case $\varepsilon(t)$ as in (D), we recall the following asymptotic property for the trajectory $\varepsilon \mapsto x_\varepsilon$. This corresponds to Lemma 1 in [13] and can be traced back to [29]. See also [16] for a recent extension with the identity operator replaced by a c -uniformly maximal monotone operator V . For the reader's convenience we include a short proof.

Lemma 4. *If $S \neq \emptyset$ then $x_\varepsilon \rightarrow x^*$ as $\varepsilon \rightarrow 0^+$.*

Proof. Monotonicity of A gives $\langle -\varepsilon x_\varepsilon, x_\varepsilon - x^* \rangle \geq 0$ so that $|x_\varepsilon| \leq |x^*|$ and x_ε remains bounded as $\varepsilon \rightarrow 0^+$. Thus $\varepsilon x_\varepsilon \rightarrow 0$ and since $\text{gph}(A)$ is weak-strong sequentially closed, it follows that every weak cluster point $x_\infty = w - \lim x_{\varepsilon_k}$ with $\varepsilon_k \rightarrow 0$ belongs to S . The inequality $|x_{\varepsilon_k}| \leq |x^*|$ then gives $|x_\infty| \leq |x^*|$ by weak lower-semicontinuity of the norm, and then $x_\infty = x^*$ so that $x_\varepsilon \rightarrow x^*$. Since we also have $|x_\varepsilon| \rightarrow |x^*|$, the convergence is strong. \square

Let us go back to the Tikhonov dynamics (D) with $\varepsilon(t) \rightarrow 0^+$ as $t \rightarrow \infty$. The case when $\int_0^\infty \varepsilon(t) dt < \infty$ may be completely analyzed by combining [32, Proposition 7.9] and [32, Proposition 8.5]: the trajectories of (D) converge (either weakly or strongly) to a point in S if and only if the corresponding property holds for the unperturbed dynamics (I). Let us then address the question whether $\int_0^\infty \varepsilon(t) dt = \infty$ is enough to ensure the convergence of the trajectories. We shall see that the answer is negative in general, but under some additional assumptions one can establish strong convergence to x^* . For instance, adapting the arguments in [3], we can easily prove the following:

Proposition 5. *Suppose $\varepsilon(t)$ is decreasing to 0 and let $u(t)$ be the strong solution of (D). Assume $\int_0^\infty \varepsilon(t) dt = \infty$ and also that either the path $\varepsilon \mapsto x_\varepsilon$ has finite length or the parameter function satisfies $\dot{\varepsilon}(t)/\varepsilon(t)^2 \rightarrow 0$ as $t \rightarrow \infty$. Then $u(t) \rightarrow x^*$ strongly.*

Proof. The proof consists in showing that $\theta(t) = \frac{1}{2}|u(t) - x_{\varepsilon(t)}|^2$ tends to 0. We recall that $x_\varepsilon = (A + \varepsilon I)^{-1}(0)$ is absolutely continuous on $(0, \infty)$ (see e.g. [3, p. 530]). Differentiating we get

$$\dot{\theta}(t) = \left\langle \dot{u}(t) - \dot{\varepsilon}(t) \frac{d}{d\varepsilon} x_{\varepsilon(t)}, u(t) - x_{\varepsilon(t)} \right\rangle$$

for almost all $t \geq 0$, and then using the strong monotonicity of $A + \varepsilon I$ we deduce

$$\dot{\theta}(t) \leq -2\varepsilon(t)\theta(t) - \dot{\varepsilon}(t) \left| \frac{d}{d\varepsilon} x_{\varepsilon(t)} \right| \sqrt{2\theta(t)}$$

which is the same inequality obtained in [3] so that the arguments in that paper yield $\theta(t) \rightarrow 0$ as required. \square

This extension, included here for completeness, was suggested in [28] and it appeared in the recent thesis [22]. Now, the case $\dot{\varepsilon}(t)/\varepsilon(t)^2 \rightarrow 0$ was already studied in [24] and, as a matter of fact, it may be obtained as a particular case of a more general statement [33, Theorem 1.4] which can be itself traced back to [12, Theorem 10.12] for a special class of operators (see also [34,35]). These more general results do not require finite length of $\varepsilon \mapsto x_\varepsilon$ nor $\dot{\varepsilon}(t)/\varepsilon(t)^2 \rightarrow 0$, but only $\varepsilon(t)$ to be decreasing. We shall prove that even this monotonicity condition can be relaxed. We begin by characterizing the strong convergence of the solutions of (D).

Proposition 6. *The strong solution $u(t)$ of (D) is bounded and if $\int_0^\infty \varepsilon(\tau) d\tau = \infty$ then the following properties are equivalent:*

- (a) *all weak cluster points of $u(t)$ for $t \rightarrow \infty$ belong to S ,*
- (b) *$\liminf_{t \rightarrow \infty} |u(t)| \geq |x^*|$,*
- (c) *$u(t) \rightarrow x^*$ strongly.*

Proof. Let $\theta(t) = \frac{1}{2}|u(t) - x^*|^2$. Differentiating and using the monotonicity of A we get

$$\begin{aligned} \dot{\theta}(t) &= \langle \dot{u}(t), u(t) - x^* \rangle \\ &= \langle \dot{u}(t) + \varepsilon(t)u(t), u(t) - x^* \rangle + \varepsilon(t)\langle u(t), x^* - u(t) \rangle \\ &\leq \varepsilon(t)\langle u(t), x^* - u(t) \rangle \\ &= \frac{\varepsilon(t)}{2} [|x^*|^2 - |u(t)|^2 - |x^* - u(t)|^2] \end{aligned}$$

so that setting $h(t) = \frac{1}{2}[|x^*|^2 - |u(t)|^2]$ we obtain

$$\dot{\theta}(t) + \varepsilon(t)\theta(t) \leq \varepsilon(t)h(t).$$

Applying Lemma 1 we deduce that $\theta(t)$ is bounded and therefore so is $u(t)$. On the other hand, (a) \Rightarrow (b) follows from the weak lower-semicontinuity of the norm, while (c) \Rightarrow (a) is straightforward (both implications hold no matter what the value of $\int_0^\infty \varepsilon(\tau) d\tau$ is). Finally, (b) \Rightarrow (c) follows from Lemma 1 provided that $\int_0^\infty \varepsilon(\tau) d\tau = \infty$ since then $\limsup_{t \rightarrow \infty} \theta(t) \leq \limsup_{t \rightarrow \infty} h(t) \leq 0$ so that $\theta(t) \rightarrow 0$. \square

Remark. The implication (b) \Rightarrow (c) may fail if $\int_0^\infty \varepsilon(\tau) d\tau < \infty$. To see this, take $A = \partial f$ given by Baillon’s counterexample for strong convergence in [5]: the solutions of (D) converge weakly but not strongly to some element of S , thus they satisfy (a) and (b), but not (c). To see the latter we invoke the equivalence result in [32] to deduce that the systems with or without $\varepsilon(t)$ have the same asymptotic behavior.

The next lemmas provide tools to check that condition (a) in Proposition 6 holds. From now on we exploit the fact that the function $\varepsilon(t)$ has bounded variation.

Lemma 7. *Suppose $\varepsilon(t) \rightarrow 0^+$ for $t \rightarrow \infty$ and $\dot{u}(t) \rightarrow 0$ when $t \rightarrow \infty$, $t \in D$, where D is a dense subset of $[0, \infty)$. Then all weak cluster points of $u(t)$ for $t \rightarrow \infty$ are in S .*

Proof. Let \bar{x} be a weak cluster point of $u(t)$ and choose $t_k \rightarrow \infty$ with $u(t_k) \rightarrow \bar{x}$. Since $u(\cdot)$ is continuous we may find $\tilde{t}_k \in D$ close enough to t_k so that $|u(\tilde{t}_k) - u(t_k)| \leq \frac{1}{k}$ and therefore $u(\tilde{t}_k) \rightarrow \bar{x}$. Then $\dot{u}(\tilde{t}_k) \rightarrow 0$ and since $\varepsilon(t) \rightarrow 0$ and $u(t)$ is bounded it follows that $v_k := -\dot{u}(\tilde{t}_k) - \varepsilon(\tilde{t}_k)u(\tilde{t}_k) \rightarrow 0$ with $v_k \in A(u(\tilde{t}_k))$, from which we conclude $0 \in A(\bar{x})$ as required. \square

Lemma 8. *If $\int_0^\infty \varepsilon(t) dt = \infty$ and $\int_0^\infty |\dot{\varepsilon}(t)| dt < \infty$ then there exists $D \subset [0, \infty)$ with full measure such that $\dot{u}(t) \rightarrow 0$ when $t \rightarrow \infty$, $t \in D$.*

Proof. Let $\theta(t) = \frac{1}{2}|u(t + \delta) - u(t)|^2$ with $\delta > 0$ so that

$$\begin{aligned} \dot{\theta}(t) &= \langle \dot{u}(t + \delta) - \dot{u}(t), u(t + \delta) - u(t) \rangle \\ &\leq \varepsilon(t + \delta)\langle u(t + \delta), u(t) - u(t + \delta) \rangle + \varepsilon(t)\langle u(t), u(t + \delta) - u(t) \rangle \end{aligned}$$

$$= -[\varepsilon(t + \delta) + \varepsilon(t)]\theta(t) + \frac{1}{2}[\varepsilon(t) - \varepsilon(t + \delta)][|u(t + \delta)|^2 - |u(t)|^2].$$

Multiplying this inequality by $\exp(E_t^\delta)$ where $E_t^\delta = \int_0^t [\varepsilon(\tau + \delta) + \varepsilon(\tau)] d\tau$, we may integrate over $[s, t]$ in order to obtain

$$\exp(E_t^\delta)\theta(t) \leq \exp(E_s^\delta)\theta(s) + \frac{1}{2} \int_s^t \exp(E_\tau^\delta)[\varepsilon(\tau) - \varepsilon(\tau + \delta)][|u(\tau + \delta)|^2 - |u(\tau)|^2] d\tau.$$

Now $u(\cdot)$ is differentiable on a set $D \subseteq [0, \infty)$ of full measure, so that multiplying the previous inequality by $2/\delta^2$ and letting $\delta \rightarrow 0^+$ it follows that for all $s, t \in D$ with $s \leq t$ we have

$$\begin{aligned} \exp(E_t^0)|\dot{u}(t)|^2 &\leq \exp(E_s^0)|\dot{u}(s)|^2 - 2 \int_s^t \exp(E_\tau^0)\dot{\varepsilon}(\tau)\langle \dot{u}(\tau), u(\tau) \rangle d\tau \\ &\leq \exp(E_s^0)|\dot{u}(s)|^2 + \int_s^t \exp(E_\tau^0)|\dot{\varepsilon}(\tau)|[|\dot{u}(\tau)|^2 + |u(\tau)|^2] d\tau. \end{aligned}$$

Denoting $\phi(t) = \exp(E_t^0)|\dot{u}(t)|^2$ and $R = \sup_{\tau \geq 0} |u(\tau)|$ we get

$$\phi(t) \leq \phi(s) + R^2 \int_s^t \exp(E_\tau^0)|\dot{\varepsilon}(\tau)| d\tau + \int_s^t |\dot{\varepsilon}(\tau)|\phi(\tau) d\tau$$

and since the quantity $\kappa(s, t) = \phi(s) + R^2 \int_s^t \exp(E_\tau^0)|\dot{\varepsilon}(\tau)| d\tau$ is non-decreasing in t , we may use Gronwall's inequality to deduce

$$\phi(z) \leq \kappa(s, t) \exp\left(\int_s^z |\dot{\varepsilon}(\tau)| d\tau\right), \quad \forall z \in [s, t].$$

In particular, for $z = t$ this gives

$$\begin{aligned} |\dot{u}(t)|^2 &\leq \left[\phi(s) \exp(-E_t^0) + R^2 \int_s^t \exp(E_\tau^0 - E_t^0)|\dot{\varepsilon}(\tau)| d\tau \right] \exp\left(\int_s^t |\dot{\varepsilon}(\tau)| d\tau\right) \\ &\leq \left[\phi(s) \exp(-E_t^0) + R^2 \int_s^t |\dot{\varepsilon}(\tau)| d\tau \right] \exp\left(\int_s^t |\dot{\varepsilon}(\tau)| d\tau\right) \end{aligned}$$

and letting $t \rightarrow \infty$ with $t \in D$ we obtain

$$\limsup_{t \rightarrow \infty, t \in D} |\dot{u}(t)|^2 \leq R^2 \exp\left(\int_s^\infty |\dot{\varepsilon}(\tau)| d\tau\right) \int_s^\infty |\dot{\varepsilon}(\tau)| d\tau.$$

Since the right-hand side expression tends to 0 for $s \rightarrow \infty$, we conclude that $\dot{u}(t) \rightarrow 0$ for $t \rightarrow \infty$, $t \in D$. \square

Combining Proposition 6 with Lemmas 7 and 8 we obtain the announced extension of [33, Theorem 1.4].

Theorem 9. *Let $u(t)$ be the strong solution of (D) and assume that $\varepsilon(t) \rightarrow 0$ as $t \rightarrow \infty$ with $\int_0^\infty \varepsilon(t) dt = \infty$ and $\int_0^\infty |\dot{\varepsilon}(t)| dt < \infty$. Then $u(t) \rightarrow x^*$ strongly.*

4. Counterexamples

4.1. A non-convergent Tikhonov-like trajectory

In this subsection we give a counterexample showing that Theorem 9 may fail if $\varepsilon(t)$ is not of bounded variation. The idea is as follows. Consider $A(x) = (1 - x_2, x_1 - 1)$ the $\frac{\pi}{2}$ -rotation around the unique rest point $p = (1, 1)$. The Tikhonov trajectory is $x_\varepsilon = \frac{1}{1+\varepsilon^2}(1 - \varepsilon, 1 + \varepsilon)$ and describes a half-circle with center at $(\frac{1}{2}, \frac{1}{2})$ and radius $\frac{1}{\sqrt{2}}$ (see dotted line in Fig. 1). For the dynamics, let us start from a point x_0 on the other half of this circle and let d be its distance to p . Fix $\varepsilon > 0$ and follow the trajectory of $-\dot{u}(t) = Au(t) + \varepsilon u(t)$ which spirals towards x_ε . On a first phase $u(t)$ increases its distance to p and afterwards it comes closer again (see Fig. 1). Stop exactly when the distance is again d and shift to $\varepsilon = 0$ in such a way that the trajectory now turns around p until it comes back to the initial point x_0 , from where we restart a new cycle with a smaller ε . To make this idea more precise and to simplify the computations we use complex numbers, identifying \mathbb{R}^2 with \mathbb{C} .

The operator: Since A is the $\frac{\pi}{2}$ clockwise rotation in the plane around the point $p = 1 + i$, Eq. (D) may be rewritten as

$$\dot{u}(t) = -i(u(t) - p) - \varepsilon(t)u(t). \tag{3}$$

The parameter function: Let ε_n be a sequence of positive real numbers with $\varepsilon_n \rightarrow 0$ and $\sum \varepsilon_n = \infty$. Take $a_0 = 0$ and let $b_n = a_n + \tau_n$, $a_{n+1} = b_n + \sigma_n$ with $\tau_n > 0$, $\sigma_n > 0$ to be fixed later on, and consider the step function

$$\varepsilon(t) = \begin{cases} \varepsilon_n & \text{if } a_n \leq t < b_n, \\ 0 & \text{if } b_n \leq t < a_{n+1}. \end{cases}$$

Clearly $\varepsilon(t) \rightarrow 0^+$ and we get $\int_0^\infty \varepsilon(t) dt = \infty$ provided τ_n is bounded away from zero.

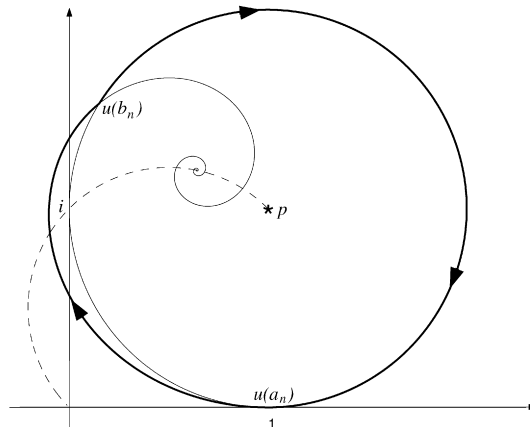


Fig. 1. The trajectory $u(t)$ on the interval $[a_n, a_{n+1}]$, starting from 1 and back.

The dynamics: Let $u(a_n) = 1 \in \mathbb{C}$. On the interval $[a_n, b_n)$ the solution of (3) is

$$u(t) = \frac{1}{\varepsilon_n + i} [i - 1 + (1 + \varepsilon_n)e^{-(\varepsilon_n + i)(t - a_n)}]. \tag{4}$$

Let $t = b_n$ be the first time after a_n with $|u(t) - p| = 1$, so that $\tau_n = b_n - a_n$ may be characterized as the first positive zero of the function

$$\psi_n(s) = (1 + \varepsilon_n)e^{-2\varepsilon_n s} + 2\varepsilon_n e^{-\varepsilon_n s} [\sin(s) - \cos(s)] + \varepsilon_n - 1.$$

We claim that if $\varepsilon_n \leq \frac{1}{2}$ then $\tau_n \in [\frac{1}{4}, \frac{3}{2}\pi]$. For the lower bound, since $\psi_n(0) = 0$ it suffices to show that $\psi'_n(s) > 0$ for all $s \in (0, \frac{1}{4})$. Now, $\psi'_n(s) = 2\varepsilon_n e^{-\varepsilon_n s} \phi_n(s)$ with $\phi_n(s) = (1 + \varepsilon_n) \cos(s) + (1 - \varepsilon_n) \sin(s) - (1 + \varepsilon_n)e^{-\varepsilon_n s}$, and since $\phi_n(0) = 0$ it suffices to check $\phi'_n(s) > 0$ for $s \in (0, \frac{1}{4})$, which follows from

$$\phi'_n(s) = (1 - \varepsilon_n) \cos(s) - (1 + \varepsilon_n) \sin(s) + \varepsilon_n(1 + \varepsilon_n)e^{-\varepsilon_n s} > \frac{1}{2} [\cos(s) - 3 \sin(s)] > 0.$$

For the upper bound we just prove that $\psi_n(\frac{3}{2}\pi) < 0$. To this end we set $\rho = e^{-\frac{3}{2}\pi\varepsilon_n}$ so that $\rho \in (0, 1)$ and therefore

$$\psi_n\left(\frac{3}{2}\pi\right) = (\rho - 1)[1 + \rho + \varepsilon_n(\rho - 1)] = (\rho - 1)[2\rho + (1 - \varepsilon_n)(1 - \rho)] < 0.$$

On the interval $[b_n, a_{n+1})$ the solution is $u(t) = p + (u(b_n) - p)e^{-i(t - b_n)}$, and we may pick σ_n such that $u(a_{n+1}) = 1$ in order for the solution to cycle indefinitely. More precisely, let σ_n be the first positive solution of $e^{is} = i(u(b_n) - p)$. Such a positive solution exists because $|u(b_n) - p| = 1$. On the interval $[b_n, a_{n+1})$, the trajectory $u(t)$ travels from $u(b_n)$ to 1 along the circle $|z - p| = 1$. Now, Eq. (4) implies that the real part of $u(b_n)$ is strictly less than 1. Therefore, the trajectory covers at least the arc joining (clockwise) the points $1 + 2i$ and 1 on the circle $|z - p| = 1$ as t goes from b_n to a_{n+1} , so it cannot converge as $t \rightarrow \infty$.

Remark. The lack of continuity of the function $\varepsilon(t)$ is not the problem, nor is it the fact that $\varepsilon(t)$ vanishes in some intervals. In fact, one can find $\eta \in C^\infty(\mathbb{R}_+; \mathbb{R}_{++})$ such that $\eta \notin L^1(0, \infty)$ while $\varepsilon - \eta \in L^1(0, \infty)$. Obviously this η will not be of bounded variation. The arguments in [32] show that Eq. (4) with $\eta(t)$ instead of the previous $\varepsilon(t)$ has the same asymptotic behavior and therefore it will not converge.

4.2. A non-convergent discrete trajectory

Given the close connection between evolution equations and the proximal point method [18,19,26, 27,31,32,35], a natural question is whether one may find sequences $\{\lambda_n\}$ and $\{\theta_n\}$ with $\sum \lambda_n \theta_n = \infty$ and such that the discrete trajectory generated by the (perturbed) proximal point algorithm

$$\frac{x_{n-1} - x_n}{\lambda_n} \in Ax_n + \theta_n x_n$$

does not converge. This is strongly related to [34]. Observe that in the unperturbed case ($\theta_n \equiv 0$) the sequence x_n converges weakly in average [6]. For $A = \partial f$ the sequence converges weakly [11], but the counterexample in [21] (based on that of [5]) shows that this convergence need not be strong; answering a question posed earlier in [36]. More examples of this kind have appeared recently in [8,9], based on results of [23].

Let $\varepsilon(t)$ be the function defined in Section 4.1. One can select a non-increasing sequence $\{\lambda_n\}$ in such a way that the function ε is constant on each interval of the form $[\Lambda_n, \Lambda_{n+1})$, where $\Lambda_n = \sum_{k=1}^n \lambda_k \rightarrow \infty$. Define $\theta_n = \varepsilon(\Lambda_n)$ and observe that

$$\sum_{n=1}^{\infty} \lambda_n \theta_n = \int_0^{\infty} \varepsilon(t) dt = \infty.$$

With these conditions, a corollary of Kobayashi’s inequality (see [26] as well as [21], [1] or [32]) states that

$$|u(t) - x_n| \leq |u(s) - x_k| + |Bx_k| \sqrt{[(\Lambda_n - \Lambda_k) - (t - s)]^2 + \sum_{j=k+1}^n \lambda_j^2}, \tag{5}$$

where B is any maximal monotone operator, $x_n = \prod_{j=1}^n (I + \lambda_j B)^{-1} x$ is a corresponding proximal sequence, and u satisfies $-\dot{u}(t) \in Bu(t)$.

Consider now the indices J_n such that the discontinuities of the function $\varepsilon(t)$ lie precisely on the set $\{\Lambda_{J_n}\}$. We have

$$\sum_{k=J_n+1}^{J_{n+1}} \lambda_k^2 \leq \lambda_{J_{n+1}} (\Lambda_{J_{n+1}} - \Lambda_{J_n}) \leq 2M\lambda_{J_n},$$

where M is an upper bound for the τ_n ’s and the σ_n ’s.

Let $U(t, s)x = u(t)$, where $-\dot{u}(t) = Au(t) + \varepsilon(t)u(t)$ and $u(s) = x$. Define also $V(t, s)x = \prod_{k=v(s)+1}^{v(t)} [I + \lambda_k(A + \theta_k I)]^{-1} x$, where $v(t) = \max\{k \in \mathbb{N} \mid \Lambda_k \leq t\}$. Applying inequality (5) repeatedly for $B_n = A + \theta_n I$ in the appropriate subintervals one gets

$$|U(t, s)x - V(t, s)x| \leq K \sum_{n=v(s)+1}^{v(t)} \sqrt{\lambda_{J_n}}$$

for some constant K , which depends on a bound for the sequence $\{Ax_n + \varepsilon(\Lambda_n)x_n\}$. If $\sum_{k=1}^{\infty} \sqrt{\lambda_{J_k}}$ is finite, this implies that the trajectories $t \mapsto U(t, s)x$ converge if and only if the same holds for $t \mapsto V(t, s)x$. Therefore the proximal point algorithm cannot always converge.

Sequences satisfying $\sum_{k=1}^{\infty} \sqrt{\lambda_{J_k}} < \infty$ and not being in ℓ^1 are difficult to characterize. However we can provide a very simple example. First, let m be a positive lower bound for the τ_n ’s and the σ_n ’s. Define $\{\lambda_n\}$ as follows: for $4^{k-1} < n \leq 4^k$ set $\lambda_n = 4^{-k}m$. We then have $\sum_{n \geq 0} \lambda_n = \infty$, while $\sum_{n \geq 1} \sqrt{\lambda_{J_n}} \leq m \sum_{n \geq 0} 2^{-n} < \infty$.

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