The impact of individual differences on network relations

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Appendix A

Mathematical Proofs Chapter 2

A.1 Proof Proposition 1: Best responses in Γ

Proposition 1 presents the best response functions in the productive exchange game. The proof is the solution to the optimization problem of the payoff function in Equation 2.1:

$$\max_{x_{ii}} \quad u_i(\delta_i, \delta_j, x_i, x_{N_i}(g)) = \rho x_{ii}^{\delta_i} + \sum_{j\neq i}^{n} x_{ij}^{\delta_i} x_{ji}^{\delta_j} \tag{A.1}$$

$$\text{s.t.} \quad x_{ii} + \sum_{j\neq i}^{n} x_{ij} \leq \Omega$$

The First Order Conditions show that:

$$\frac{\partial u_i}{\partial x_{ii}} = \rho\delta_i x_{ii}^{(\delta_i - 1)} - \lambda = 0,$$

$$\rho\delta_i x_{ii}^{\delta_i} = \lambda x_{ii} \tag{A.2}$$

$$\frac{\partial u_i}{\partial x_{ij}} = \delta_i x_{ij}^{(\delta_i - 1)} x_{ji}^{\delta_j} - \lambda = 0,$$

$$\delta_i x_{ij}^{\delta_i} x_{ji}^{\delta_j} = \lambda x_{ij} \tag{A.3}$$

$$\frac{\partial u_i}{\partial \lambda} = x_{ii} + \sum_{j \neq i}^{n} x_{ij} - \Omega = 0,$$

$$x_{ii} + \sum_{j \neq i}^{n} x_{ij} = \Omega \tag{A.4}$$

Summing Equation A.3 in j:

$$\delta_i \sum_{j \neq i}^{n} x_{ij}^{\delta_i} x_{ji}^{\delta_j} = (\Omega - x_{ii})\lambda \tag{A.5}$$

Adding Equation A.2 and A.5:

$$\delta_i(\rho x_{ii}^{\delta_i} + \sum_{j \neq i}^{n} x_{ij}^{\delta_i} x_{ji}^{\delta_j}) = \lambda \Omega \tag{A.6}$$

Dividing Equation A.2 by Equation A.6, we obtain the best response of player i on her allocation to an individual project, x_{ii}^*:

$$x_{ii}^* = \frac{\rho x_{ii}^{*\delta_i}}{\rho x_{ii}^{*\delta_i} + \sum_{j \neq i}^{n} x_{ij}^{*\delta_i} x_{ji}^{*\delta_j}} \Omega \tag{A.7}$$

Dividing Equation A.3 by Equation A.6, we obtain the best response of player i on her allocation to a combined project with j, x_{ij}^*:

$$x_{ij}^* = \frac{x_{ij}^{*\delta_i} x_{ji}^{*\delta_j}}{\rho x_{ii}^{*\delta_i} + \sum_{j \neq i}^{n} x_{ij}^{*\delta_i} x_{ji}^{*\delta_j}} \Omega \tag{A.8}$$

A.2 Proof Lemma 1: Optimal allocations in a dyad

Lemma 1 presents the optimal allocations for the interaction between two players in the productive exchange game Γ. For this proof, we denote the set of resources a player i has as $\hat{\Omega}$, where $\hat{\Omega} \leq \Omega$. This means that we can generalize the proof for any proportion of resources considered from the entire endowment Ω. This is a useful consideration for the extension of the results to networks of any size $n \geq 2$. Consider the optimization problem below, where a player i decides on the optimal way of allocating her resources between an individual and a combined project:

$$\max_{x_{ii}} \quad u_i = \rho x_{ii}^{\delta_i} + (\hat{\Omega} - x_{ii})^{\delta_i} x_{ji}^{\delta_j}$$

The First Order Conditions show that:

$$\frac{\partial u_i}{\partial x_{ii}} = \rho \delta_i x_{ii}^{(\delta_i - 1)} - \delta_i (\hat{\Omega} - x_{ii})^{(\delta_i - 1)} x_{ji}^{\delta_j} = 0$$

and the Second Order Conditions show that:

$$\frac{\partial^2 u_i}{\partial x_{ii}^2} = \rho \delta_i (\delta_i - 1) x_{ii}^{(\delta_i - 2)} + \delta_i (\delta_i - 1)(\hat{\Omega} - x_{ii})^{(\delta_i - 2)} x_{ji}^{\delta_j} \gtreqless 0$$

so that:

$$\begin{cases} u_i^{''} > 0 & \text{if } \delta_i > 1 : \nexists \text{ internal maximum} \\ u_i^{''} = 0 & \text{if } \delta_i = 1 : u_i' = \rho - x_{ji}^{\delta_j} \gtreqless 0 \\ u_i^{''} < 0 & \text{if } \delta_i < 1 : \text{internal maximum is feasible} \end{cases}$$

The FOC and SOC give different outcomes depending on player i's expertise, even when playing with the same partner j. For the case specialist, $\delta_i > 1$, no interior point can be a local maximum, thus neither a global one. There are two candidates ($x_{ii} = 0; x_{ii} = \hat{\Omega}$). The payoff functions for each are $u_i(N_E) = \rho \hat{\Omega}^{\delta_i}$ and $u_i(F_E) = \hat{\Omega}^{\delta_i} x_{ji}^{\delta_j}$, respectively. Thus:

$$BR = \begin{cases} x_{ii}^* = 0 & \text{iff } x_{ji}^{*\delta_j} > \rho \\ x_{ii}^* = \hat{\Omega} & \text{iff } x_{ji}^{*\delta_j} \leq \rho \end{cases}$$

If a player is a semi-specialist, $\delta_i = 1$:

$$BR = \begin{cases} x_{ii}^* = 0 & \text{iff } x_{ji}^{*\delta_j} > \rho \\ x_{ii}^* \in [0, \hat{\Omega}] & \text{iff } x_{ji}^{*\delta_j} = \rho \\ x_{ii}^* = \hat{\Omega} & \text{iff } x_{ji}^{*\delta_j} < \rho \end{cases}$$

If a player is a generalist, $\delta_i < 1$, from the FOC we know that $\rho \delta_i x_{ii}^{\delta_i - 1} = \delta_i (\hat{\Omega} - x_{ii})^{\delta_i - 1} x_{ji}^{\delta_j}$, where $\rho x_{ii}^{\delta_i - 1} = (\hat{\Omega} - x_{ii})^{\delta_i - 1} x_{ji}^{\delta_j}$, so that $\hat{\Omega} = x_{ii}[1 + (\frac{1}{\rho})^{(\frac{1}{1-\delta_i})} x_{ji}^{\frac{\delta_j}{1-\delta_i}}]$:

$$BR = \left\{ \; x_{ii}^* = \hat{\Omega}[1 + (\tfrac{1}{\rho})^{(\frac{1}{1-\delta_i})} x_{ji}^{*\frac{\delta_j}{1-\delta_i}}]^{-1} \quad \text{if } x_{ji}^{*\delta_j} \gtreqless \rho \right. \tag{A.9}$$

Note that this equation is the best response for local internal maximum. In order to check wether this is a global best reply, we compare it to the two corner candidates $x_{ii} = \{0, \Omega\}$. Recall that:

$$u_i = \rho\Omega^{\delta_i} \geq \Omega^{\delta_i} x_{ji}^{\delta_j} = u_i \ \text{ iff } x_{ji}^{\delta_j} \leq \rho : x_{ji} \leq \rho^{\frac{1}{\delta_j}}$$

Case 1: $\delta_i < 1$ and $x_{ji}^{\delta_j} \leq \rho$.

So that we compare x_{ii}^* only to $u_i(x_{ii} = \Omega)$. First, we replace Eq. A.9, in u_i:

$$u_i(x_{ii}^*) = \rho(\Omega[1 + (\tfrac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}]^{-1})^{\delta_i} + [\Omega - (\Omega[1 + (\tfrac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}]^{-1})]^{\delta_i} x_{ji}^{\delta_j}$$

$$u_i(x_{ii}^*) = \frac{\rho\Omega^{\delta_i}}{[1+(\frac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}]^{\delta_i}} + [\Omega - \frac{\Omega}{1+(\frac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}}]^{\delta_i} x_{ji}^{\delta_j}$$

$$u_i(x_{ii}^*) = \frac{\rho\Omega^{\delta_i} + \Omega^{\delta_i}[(\frac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}]^{\delta_i} x_{ji}^{\delta_j}}{[1+(\frac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}]^{\delta_i}} = \frac{\Omega^{\delta_i}(\rho + \rho^{\frac{\delta_i}{\delta_i - 1}} x_{ji}^{\frac{\delta_j\delta_i}{1-\delta_i} + \delta_j})}{[1+(\frac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}]^{\delta_i}} = \frac{\Omega^{\delta_i}\rho(1+\rho^{\frac{\delta_i}{\delta_i-1}-1} x_{ji}^{\frac{\delta_j}{1-\delta_i}})}{[1+(\frac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}]^{\delta_i}}$$

$$u_i(x_{ii}^*) = \rho\Omega^{\delta_i}[1 + (\frac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}]^{1-\delta_i} \tag{A.10}$$

Now, the question is when is $u_i(x_{ii}^*) \geq u_i(x_{ii} = \Omega)$. We say this condition is satisfied when:

$$\rho\Omega^{\delta_i}[1 + (\tfrac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}]^{1-\delta_i} \geq \rho\Omega^{\delta_i}$$

$$[1 + (\tfrac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}]^{1-\delta_i} \geq 1$$

which is always.

Case 2: $\delta_i < 1$ and $x_{ji}^{\delta_j} > \rho$.

So that we compare x_{ii}^* only to $u_i(x_{ii} = 0)$. Now, the question is when is $u_i(x_{ii}^*) \geq u_i(x_{ii} = 0)$. We say this condition is satisfied when:

$$\rho\Omega^{\delta_i}[1 + (\tfrac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}]^{1-\delta_i} \geq \Omega^{\delta_i} x_{ji}^{\delta_j}$$

$$\rho^{\frac{1}{1-\delta_i}}[1 + (\tfrac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{\frac{\delta_j}{1-\delta_i}}] \geq x_{ji}^{\frac{\delta_j}{1-\delta_i}}$$

$$\rho^{\frac{1}{1-\delta_i}} + x_{ji}^{\frac{\delta_j}{1-\delta_i}} \geq x_{ji}^{\frac{\delta_j}{1-\delta_i}}$$

$$\rho^{\frac{1}{1-\delta_i}} \geq 0 \to \rho > 0$$

which is always. Then, x_{ii}^* in Eq. A.9, is the best response for $\delta_i < 1$ in the 2-person game:

$$BR = \begin{cases} x_{ii}^* = \Omega[1 + (\tfrac{1}{\rho})^{\frac{1}{1-\delta_i}} x_{ji}^{*\frac{\delta_j}{1-\delta_i}}]^{-1} & \text{if } x_{ji}^{*\delta_j} \gtreqless \rho \end{cases}$$

A.3 Proof Lemma 2: Optimal connectivity in equilibrium

Lemma 2 shows the optimal level of connectivity a player reciprocates to, given her type of expertise. Consider the set of players $N = \{N_i^+ \cup N_i^-\}$, where $N_i^+ = \{j : x_{ji} > 0\}$ and $N_i^- = \{j : x_{ji} = 0\}$, for all $j \in N$. It is straightforward to check that given no output can be obtained in a combined productive project when only one partner is making positive allocations, no *zero* allocations from j will be reciprocated by i. Thus:

$$x_{ij} = 0 \ \text{ if } j \in N_i^- \tag{A.11}$$

In addition, from Proposition 1 and Lemma 1 we know that not all positive allocations received from a $j \in N_i^+$ will be reciprocated by i. The question is then, which of the potential projects in the set N_i^+ will be formed? Recall that an allocation vector x_i is part of a Nash allocation profile if player i has no incentives to make any unilateral reallocation of her resources from one project to another.

We can express this by a sequence of *improving paths* (Jackson and Watts, 2000) reached *unilaterally* by a reallocation of resources from $x_i = \{x_{i1}, \ldots, x_{ij}, \ldots, x_{ik}, \ldots, x_{in}\}$

to $x'_i = \{x_{i1}, \ldots, x'_{ij}, \ldots, x'_{ik}, \ldots, x_{in}\}$. That is, players can refuse to form links and thus can severe formed connections. Assume that $j, k \in N_i^+$, and $x_{ij}, x_{ik} > 0$. We say there is an improving path from the allocation vector that includes both projects x_i to that in which only project with j is formed x'_i, where $x'_{ij} = x_{ij} + x_{ik}$ and $x'_{ik} = 0$, if $u'_i > u_i$. This analysis is straightforward for cases where players also form an individual productive project, simply replace $\hat{\Omega}$ for $(\hat{\Omega} - x_{ii})$, so that the resources used for exchange between j and k are those not invested in individual production. Then, for every pair of projects, either one or both are reciprocated to if:

$$u_i(\Gamma_{\{ij\}}) = \hat{\Omega}^{\delta_i} x_{ji}^{\delta_j} > (\alpha\hat{\Omega})^{\delta_i} x_{ji}^{\delta_j} + [(1-\alpha)\hat{\Omega}]^{\delta_i} x_{ki}^{\delta_k} = u_i(\Gamma_{\{ij,ik\}})$$

$$(1-\alpha^{\delta_i})\hat{\Omega}^{\delta_i} x_{ji}^{\delta_j} > (1-\alpha)^{\delta_i} \hat{\Omega}^{\delta_i} x_{ki}^{\delta_k}$$

$$\frac{x_{ji}^{\delta_j}}{x_{ki}^{\delta_k}} - \frac{(1-\alpha)^{\delta_i}}{(1-\alpha^{\delta_i})} > 0. \tag{A.12}$$

where $\alpha \in (0,1)$, so that resources need not be symmetrically allocated. Notice that, independently of $\delta_i \gtreqless 1$ there are two scenarios: either (i) $x_{ji}^{\delta_j} = x_{ki}^{\delta_k}$, or (ii) $x_{ji}^{\delta_j} > x_{ki}^{\delta_k}$. Depending on δ_i we have three cases.

Case 1 ($\delta_i > 1$): For any of the two scenarios, the condition in Equation A.12 is always satisfied for a specialist player. That is, a player i with a specialist type of expertise $\delta_i > 1$ is better off allocating all her resources to a single project, independently of the size of the set of players. Being that project with the partner that allocates the most to the combined project.

Case 2 ($\delta_i = 1$): Semi-specialist players are indifferent between the two options in case (i) and they are better off in a single combined exchange instead of two in case (ii).

Case 3 ($\delta_i < 1$): The condition is never satisfied in (i) for generalists, but can be satisfied in (ii), depending on the parameters of the model Ω and ρ. It is possible to observe that the lower δ_i is for generalists, the more likely the condition is satisfied.

A.4 Proof Proposition 2: Pairwise stable Nash equilibria

Proposition 2 shows the set of pairwise stable Nash equilibria. It follows from the conditions in the previous Proposition 1 and Lemmas 1 and 2. Simply take into account that if players are specialists , they reciprocate to a single project with the most profitable partner available, and if no partner is available, they produce individually, as shown in Equation 2.4. For the case of the generalists, from Equation 2.6 and invoking the condition of proportionality from Proposition 1, players have incentives to reciprocate to productive exchanges with multiple partners and always find it optimal to create simultaneously an individual productive project. The semi-specialists, given Equation 2.5, follow the behavior of either a specialists or a generalist, depending on the size of the endowment. Finally, the cases of heterogeneous distributions of types of expertise vary depending on the size of the network and the actors in it.

Consider the set of players $N = N^{>} \cup N^{=} \cup N^{<}$ where $N^{>} = \{i : \delta_i > 1\}$ is the set of specialists, with cardinality $n^{>}$, $N^{=} = \{i : \delta_i = 1\}$ is the set of semi-specialists, with cardinality $n^{=}$, and $N^{<} = \{i : \delta_i < 1\}$ is the set of generalists, with cardinality $n^{<}$. From this, we divide the distribution of types of expertise in the network into seven cases: [1] $n^{>} = n$, [2] $n^{=} = n$, [3] $n^{<} = n$, [4] $n^{>} + n^{=} = n(n^{<} = 0)$, [5] $n^{>} + n^{<} = n(n^{=} = 0)$, [6] $n^{=} + n^{<} = n(n^{>} = 0)$, and [7] $n^{>} + n^{=} + n^{<} = n$. The first three cases represent homogeneous distributions of types of expertise and the last four represent heterogeneous distributions.

Case 1: If $\Omega > \rho$ and n is even, the only stable configuration is F_E where all pairs of players i and j create a single combined project such that $x_{ij} = x_{ji} = \Omega$. Pairwise stability rules out the N_E. When n is an odd number, then $n - 1$ behave as mentioned and the remaining player is *excluded* and produces alone. If $\Omega \leq \rho$, for any n, the only PNE is N_E, where all players produce alone.

Case 2: As shown in Equation 2.5, if $\Omega \geq \rho$, players behave as if their $\delta_i > 1$ forming F_E configurations as in Case 1. If $\Omega < \rho$ they all produce alone forming a N_E configuration.

Case 3: For any size of Ω and n players show link monotonicity, as shown in Equation

A.12, so they always find optimal to create new links. Thus the PNE is H_E where all players are connected.

Case 4: It is straightforward that if $\Omega > \rho$ a F_E configuration emerges as in Cases 1 and 2, where the specialists connect between each other in pairs. If $n^>$ is odd, the remaining player connects with j who is a semi-specialist ($\delta_i = 1$), and the other $n^= - 1$ connect also in pairs. If $n^= - 1$ is odd, there will be a player producing alone. If $\Omega = \rho$ the $n^>$ produce alone and the semi-specialists ($n^=$) full exchange in pairs. If $\Omega < \rho$, all players produce alone.

Case 5: If $\Omega > \rho$ and $n^>$ is odd, $n^> - 1$ form dyads as in Case 1 and the remaining player connects with a $j \in N^<$, as long as $x_{ji}^{\delta_j} > \rho$. Thus, j has $\Omega - x_{ji}$ to allocate in projects with the remaining $n^< - 1$. In the same way, all $n^< - 1$ players connect between each other.

Case 6: This case follows as in Case 5 but allows for $x_{ij} \geq \rho$ for the combined project between the two players with different types of expertise.

Case 7: Follows from Cases 1 to 6, where the specialists ($n^>$) connect between them in pairs, the semi-specialists ($n^=$) connect between them in pairs, and the generalists ($n^<$) connect between them in a hybrid exchange network. If both $n^>$ and $n^=$ are odd, there will be a dyad formed by one player from each set. If only one but not both is odd, the remaining player connects with a $j \in N^<$ as in Case 5. If the size of the first two sets is even, there will be no mixing between types of expertise.