Expressivity of Logics of Knowledge and Action

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Chapter 7

Conclusion

7.1 Summary

In this conclusion we mainly want to mention a few areas that might be interesting for further research. But before doing so it is convenient to refresh our memories about the results presented in this thesis.

In this thesis we looked at the concept of expressivity, which can be defined in the following way:

> A logic $\mathcal{L}_2$ is *at least as expressive* as a logic $\mathcal{L}_1$ if for every formula φ_1 of $\mathcal{L}_1$ there is an equivalent formula φ_2 of $\mathcal{L}_2$.

Or, equivalently, as follows.

> A logic $\mathcal{L}_2$ is *at least as expressive* as a logic $\mathcal{L}_1$ if there is a function t from the formulas Φ_1 of $\mathcal{L}_1$ to the formulas Φ_2 of $\mathcal{L}_2$ such that $t(\varphi) \equiv \varphi$ for all $\varphi \in \Phi_1$.

If a logic $\mathcal{L}_2$ is at least as expressive as a logic $\mathcal{L}_1$ we denote this by $\mathcal{L}_1 \preceq \mathcal{L}_2$. If $\mathcal{L}_1 \preceq \mathcal{L}_2$ and $\mathcal{L}_2 \preceq \mathcal{L}_1$ we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are equally expressive, denoted $\mathcal{L}_1 \equiv \mathcal{L}_2$. If on the other hand $\mathcal{L}_1 \preceq \mathcal{L}_2$ and $\mathcal{L}_2 \not\preceq \mathcal{L}_1$ we say that $\mathcal{L}_2$ is (strictly) more expressive than $\mathcal{L}_1$, denoted $\mathcal{L}_1 \prec \mathcal{L}_2$.

In Chapters 2 and 3 we proved a number of expressivity results. Specifically, in Chapter 2 we showed that $\mathcal{L}_{\text{CP}} \prec \mathcal{L}_{\text{CPS}}$ over the classes **KD45**, **S4** and **S5** of models, where $\mathcal{L}_{\text{CP}}$ is a logic with operators for common knowledge and public announcements, while $\mathcal{L}_{\text{CPS}}$ is a logic with operators for common knowledge, public announcements and public substitutions. It was already known that $\mathcal{L}_{\text{CPS}} \equiv \mathcal{L}_{\text{R}}$ over all these classes of models, where $\mathcal{L}_{\text{R}}$ is a logic with an operator for relativized common knowledge. The results presented in Chapter 2 therefore also show that $\mathcal{L}_{\text{CPS}} \prec \mathcal{L}_{\text{R}}$ over **KD45**, **S4** and **S5**. This extends the result from [van Benthem et al., 2006], which showed that $\mathcal{L}_{\text{CPS}} \prec \mathcal{L}_{\text{R}}$ over **K**.

There is one limitation to the results that $\mathcal{L}_{\text{CP}} \prec \mathcal{L}_{\text{CPS}}$ over **KD45**, **S4** and **S5** though. This restriction is that the set of agents needs to be large enough. For **KD45** and **S4** we need $|\mathcal{A}| \geq 2$ and for **S5** we need $|\mathcal{A}| \geq 3$. If $|agents| = 1$ then it can easily be seen that $\mathcal{L}_{\text{CP}} \equiv \mathcal{L}_{\text{CPS}}$ over **KD45**, **S4** and **S5**. But this leaves one case open: we don't know whether $\mathcal{L}_{\text{CPS}}$ is more expressive than $\mathcal{L}_{\text{CP}}$ over **S5** if $|\mathcal{A}| = 2$.

In Chapter 3 we proved two other expressivity results. These results are that $\mathcal{L}_{\mathrm{R}} \prec \mathcal{L}_{\mathrm{U}^*}$ and that $\mathcal{L}_{\mathrm{UC}} \equiv \mathcal{L}_{\mathrm{U}^*}$, where $\mathcal{L}_{\mathrm{U}^*}$ is a logic with an arrow common knowledge operator and $\mathcal{L}_{\mathrm{UC}}$ is a logic with arrow update and (normal) common knowledge operators. Together with previously known results, this fully determines the expressivity landscape of all logics using any combination of public announcements, arrow updates, common knowledge, relativized common knowledge and arrow common knowledge.

In order to show that $\mathcal{L}_{\mathrm{R}} \prec \mathcal{L}_{\mathrm{U}^*}$ we had to prove two things: that $\mathcal{L}_{\mathrm{R}} \preceq \mathcal{L}_{\mathrm{U}^*}$ and that $\mathcal{L}_{\mathrm{U}^*} \not\preceq \mathcal{L}_{\mathrm{R}}$. Usually $\preceq$ proofs are easy and $\not\preceq$ proofs are relatively hard, because $\preceq$ is an existence proof (of a truth preserving translation) while $\not\preceq$ is a non-existence proof. The $\mathcal{L}_{\mathrm{R}} \preceq \mathcal{L}_{\mathrm{U}^*}$ and $\mathcal{L}_{\mathrm{U}^*} \not\preceq \mathcal{L}_{\mathrm{R}}$ proofs are no exceptions to this rule; the proof that $\mathcal{L}_{\mathrm{R}} \preceq \mathcal{L}_{\mathrm{U}^*}$ is so trivial that we left it to the reader whereas the proof that $\mathcal{L}_{\mathrm{U}^*} \not\preceq \mathcal{L}_{\mathrm{R}}$ takes several pages.

In order to show that $\mathcal{L}_{\mathrm{UC}} \equiv \mathcal{L}_{\mathrm{U}^*}$ we had to prove that $\mathcal{L}_{\mathrm{UC}} \preceq \mathcal{L}_{\mathrm{U}^*}$ and that $\mathcal{L}_{\mathrm{U}^*} \preceq \mathcal{L}_{\mathrm{UC}}$. The proof that $\mathcal{L}_{\mathrm{UC}} \preceq \mathcal{L}_{\mathrm{U}^*}$ is very simple[1] so it is a typical $\preceq$ result. The proof that $\mathcal{L}_{\mathrm{U}^*} \preceq \mathcal{L}_{\mathrm{UC}}$ on the other hand is highly a-typical because it is very difficult, complex and long.[2]

The results presented in the remaining chapters are not, strictly speaking, expressivity results. The results are *related* to expressivity though. In Chapter 4 we looked at the *arbitrary public announcement* operator $\Diamond$ from Arbitrary Public Announcement Logic (APAL). The intuition behind $\Diamond$ is that $\Diamond\varphi$ holds if and only if there is some public announcement $\langle\psi\rangle$ such that $\langle\psi\rangle\,\varphi$ holds. Unfortunately, in order to avoid circularity, the semantics of $\Diamond$ are defined not as

$$\mathcal{M}, w \models \Diamond\varphi \Leftrightarrow \exists\psi \in \Phi_{APAL} : \mathcal{M}, w \models \langle\psi\rangle\,\varphi$$

but as

$$\mathcal{M}, w \models \Diamond\varphi \Leftrightarrow \exists\psi \in \Phi_{PAL} : \mathcal{M}, w \models \langle\psi\rangle\,\varphi.$$

This means that there could be some $\psi \in \Phi_{APAL} \setminus \Phi_{PAL}$, $\varphi \in \Phi_{APAL}$ and some $\mathcal{M}, w$ such that $\mathcal{M}, w \models \langle\psi\rangle\,\varphi \wedge \neg\Diamond\varphi$. But note that it is not immediately obvious that such ψ, φ and $\mathcal{M}, w$ exist. Maybe for every $\psi \in \Phi_{APAL}$ there is some $\psi' \in \Phi_{PAL}$ such that $\models \langle\psi\rangle\,\varphi \leftrightarrow \langle\psi'\rangle\,\varphi$.[3]

In Chapter 4 we showed that such ψ' do not always exist by finding $\psi \in \Phi_{APAL} \setminus \Phi_{PAL}$, $\varphi \in \Phi_{APAL}$ and $\mathcal{M}, w$ such that $\mathcal{M}, w \models \langle\psi\rangle\,\varphi \wedge \neg\Diamond\varphi$. The methods used in order to find these ψ, φ and $\mathcal{M}, w$ are very similar to the methods used in [Balbiani et al., 2007] to show that APAL is more expressive than PAL.

In Chapter 5 we considered a result that was not, like Chapter 4, related to expressivity in a technical way. Instead we used very different methods to investigate a question very similar to that of expressivity: what can be represented in a logic, and what cannot?

Specifically, we investigated whether deontic logics with semantics based on the idea of a single "sanction" representing wrongdoing can represent so-called Contrary-to-Duty Obligations (CTDOs). We discovered that they cannot.

Finally, in Chapter 6 we considered a concept expressivity$_g$ that is related to expressivity because it is a generalization of expressivity. We used expressivity$_g$ to provide a

[1] It was also already known from [Kooi and Renne, 2011].

[2] In addition to 11 pages in Chapter 3 the proof takes all 26 pages of the Appendix.

[3] Note that if APAL and PAL would have had the same expressivity it would have followed immediately that such ψ' exist. However, [Balbiani et al., 2007] showed that APAL is strictly more expressive than PAL.

possible explanation of why results from [Thomason, 1974, Gasquet and Herzig, 1996, Goranko and Jamroga, 2004, Broersen et al., 2006a,b] are interesting.

7.2 Further Research

Of course there are many possible directions for further research. For every two logics that share a class of models, we can ask whether one of them is more expressive than the other. And for many more logics we could ask whether one is more expressive$_g$ than the other. But there are a few questions that are particularly salient subjects for further research.

The first of these questions is whether $\mathcal{L}_{\text{CPS}}$ is more expressive than $\mathcal{L}_{\text{CP}}$ over **S5** if $|\mathcal{A}| = 2$. We already know that $\mathcal{L}_{\text{CP}} \prec \mathcal{L}_{\text{CPS}}$ over **K**, over **KD45** and **S4** if $|\mathcal{A}| \geq 2$ and over **S5** if $|\mathcal{A}| \geq 3$. Furthermore, we know that $\mathcal{L}_{\text{CP}} \equiv \mathcal{L}_{\text{CPS}}$ over **KD45**, **S4** and **S5** if $|\mathcal{A}| = 1$. But in this one case, where $|\mathcal{A}| = 2$ and we consider the class **S5** of models, we do not know whether $\mathcal{L}_{\text{CP}} \prec \mathcal{L}_{\text{CPS}}$ or $\mathcal{L}_{\text{CP}} \equiv \mathcal{L}_{\text{CPS}}$.

The second salient question is whether we can find alternative semantics for APAL such that $\Diamond$ represents a truly arbitrary public announcement. That is, can we find semantics such that for all $\varphi \in \Phi_{APAL}$ and all $\mathcal{M}, w \in \mathfrak{M}_{APAL}$ we have $\mathcal{M}, w \models \Diamond\varphi$ if and only if there is some $\psi \in \Phi_{APAL}$ such that $\mathcal{M}, w \models \langle\psi\rangle\, \varphi$.

The third salient question is whether we can find interesting results that follow a pattern that is even more general than expressivity$_g$. In particular, we could ask whether there are any interesting translations that are not finitely generated but that are finitely generated$_g$ in one of the two ways discussed in Chapter 6.