Topics in inhomogeneous Bernoulli percolation

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1 Inhomogeneous Percolation on ladder graphs: Continuity of the critical curve

1.1 Overview of the chapter

In this chapter, we present an extension of the work of Szabó and Valesin [23], published in [10]. It regards the inhomogeneous Bernoulli bond percolation model on a graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, where the relevant edge set $\mathbb{E}$ can be written as a decomposition $\mathbb{E}' \cup \mathbb{E}''$, and parameters p and q, both in $[0,1]$, are assigned to the edges of $\mathbb{E}'$ and $\mathbb{E}''$, respectively. In [23], the authors considered $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ to be the graph induced by the cartesian product between an infinite and connected graph $G = (V, E)$ and the set of integers $\mathbb{Z}$; the set $\mathbb{E}''$ was chosen by selecting finite subsets $V' \subset V$, $E' \subset E$ and defining

$$\mathbb{E}'' = (\cup_{u \in V'} \{\{(u,n),(u,n+1)\} : n \in \mathbb{Z}\}) \cup (\cup_{\{u,v\} \in E'} \{\{(u,n),(v,n)\} : n \in \mathbb{Z}\}),$$

and $\mathbb{E}' = \mathbb{E} \setminus \mathbb{E}''$. They have proved the continuity of the critical curve $q \mapsto p_c(q)$ on the interval $(0,1)$, where $p_c(q)$ is the supremum of the values of p for which percolation with parameters p, q does not occur. In [10], we extend this result in the sense that the continuity of $p_c(q)$ still holds if V' and E' are infinite sets, provided that the set of vertices $V' \cup (\cup_{e \in E'} e)$ do not possess arbitrarily large connected components in G, and the graph-theoretic distance between any two such components is bigger than 2. This is achieved through the construction of a coupling (a combination of Lemmas 1.5 and 1.6), which allows us to understand how a small change in the parameters p and q of the model affects the percolation behavior.

Aspects of the critical curve have been explored by several authors in different models [8, 9, 18, 19, 23, 25]; some of these results are mentioned in the **Introduction**. With respect to our model, we shall define it rigorously and state the main result in Section 1.2. In Section 1.3, we develop some technical lemmas and prove the main result.

In the next sections, we use the following notation: for a graph $G = (V, E)$, vertices

$u, w \in V$ and subsets $U, W \subset V$, we denote by $\text{dist}_G(u, w)$ the graph-theoretic distance between $u, w \in V$, and $\text{dist}_G(U, W) := \min_{\substack{u \in U \\ w \in W}} \text{dist}_G(u, w)$. We also define $\mathsf{E}_U := \{e \in E : e \subset U\}$.

1.2 Definition of the model and main result

Let $G = (V, E)$ be an infinite and connected graph with bounded degree and define $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V} := V \times \mathbb{Z}$ and

$$\mathbb{E} := \big\{\{(u, n), (v, n)\} : \{u, v\} \in E, n \in \mathbb{Z}\big\} \cup \big\{\{(u, n), (u, n+1)\} : u \in V, n \in \mathbb{Z}\big\}.$$

Consider the following Bernoulli percolation process on $\mathbb{G}$. Every edge of $\mathbb{E}$ can be ***open*** or ***closed***, states represented by 1 and 0, respectively. Thus, a typical percolation configuration is an element of $\Omega = \{0, 1\}^{\mathbb{E}}$. As usual, the underlying σ-algebra $\mathcal{F}$ is the one generated by the finite-dimensional cylinder sets in Ω. Given $p \in [0, 1]$ and $q \in (0, 1)$, the governing probability $P_{p,q}$ of the process is the product measure on $(\Omega, \mathcal{F})$ with densities p and q on the edges of $\mathbb{E}$, specified as follows:

Fix a family of subgraphs of G, denoted by

$$\big\{G^{(r)} = \big(U^{(r)}, E^{(r)}\big)\big\}_{r \in \mathbb{N}}, \tag{1.1}$$

such that:

- $G^{(r)}$ is finite and connected for every $r \in \mathbb{N}$;
- $E^{(r)} = \mathsf{E}_{U^{(r)}}$;
- $\text{dist}_G(U^{(i)}, U^{(j)}) \geq 3$, for every $i \neq j$.

For each $r \in \mathbb{N}$, let

$$\begin{aligned} \mathbb{E}^{\text{in},(r)} := &\big\{\{(u, n), (v, n)\} : \{u, v\} \in E^{(r)}, n \in \mathbb{Z}\big\} \\ &\cup \big\{\{(u, n), (u, n+1)\} : u \in U^{(r)}, n \in \mathbb{Z}\big\}. \end{aligned} \tag{1.2}$$

We assign parameter q to each edge of $\mathbb{E}^{\text{in},(r)}$, for every $r \in \mathbb{N}$, and parameter p to each edge of $\mathbb{E} \setminus (\cup_{r \in \mathbb{N}} \mathbb{E}^{\text{in},(r)})$.

In what follows, we shall work with the notions of open paths, connectivity between vertices and percolation of vertices. The reader is referred to the **Introduction** of this thesis for an account of these definitions. First, given $p, q \in [0, 1]$, fix a vertex $v \in V$ and note that whether or not $P_{p,q}((v, 0) \leftrightarrow \infty) > 0$ depends on the values of p and q.

Our aim is to understand the shape of the surfaces determined by the sets of percolative and non-percolative parameters $(p,q) \in [0,1]^2$. Thus, our object of interest is the ***critical parameter function***, $p_c : [0,1] \to [0,1]$, defined by

$$p_c(q) := \sup\{p \in [0,1] : P_{p,q}((v,0) \leftrightarrow \infty) = 0\}.$$

Given $p, q \in (0,1)$ and $x, y \in \mathbb{V}$, the connectivity of $\mathbb{G}$ implies that $P_{p,q}(x \leftrightarrow y) > 0$. Thus, if $P_{p,q}(x \leftrightarrow \infty) > 0$, then the FKG Inequality (1) implies

$$P_{p,q}(y \leftrightarrow \infty) \geq P_{p,q}(y \leftrightarrow x, x \leftrightarrow \infty) \geq P_{p,q}(y \leftrightarrow x)P_{p,q}(x \leftrightarrow \infty) > 0,$$

since $\{y \leftrightarrow x\}$ and $\{x \leftrightarrow \infty\}$ are increasing events. Therefore, although the value of $P_{p,q}(x \leftrightarrow \infty)$ may depend on $x \in \mathbb{V}$, the value of $p_c(q)$ does not depend on the choice of $x \in \mathbb{V}$.

What we shall prove is a simple generalization of Theorem 1 of [23]. It states that the continuity of $p_c(q)$ still holds, provided that the cardinalities of the sets $U^{(r)}$ are uniformly bounded. In the context of Section 1.2, the model of [23] is the case where $G^{(r)} = (\emptyset, \emptyset)$ for every $r \geq 2$.

Theorem 1.1 (Continuity of the critical curve)**.** *If* $\mathrm{dist}_G(U^{(i)}, U^{(j)}) \geq 3$ *for every* $i \neq j$ *and* $\sup_{r\in\mathbb{N}}|U^{(r)}| < \infty$*, then* $p_c(q)$ *is continuous in* $(0,1)$.

Remark 1. Just as we have based our non-oriented percolation model upon the one of Szabó and Valesin [23], we can generalize the oriented model also present in [23] in an analogous manner. In this setting, the set of vertices $\mathbb{V}$ of the oriented graph is the same as the non-oriented case, and the set of oriented edges is $\vec{\mathbb{E}} = \{\langle (u,n),(v,n+1)\rangle : \{u,v\} \in E, n \in \mathbb{Z}\}$. The inhomogeneities are assigned to the set

$$\vec{\mathbb{E}}'' = \cup_{\{u,v\}\in E'}\big(\{\langle (u,n),(v,n+1)\rangle : n \in \mathbb{Z}\} \cup \{\langle (v,n),(u,n+1)\rangle : n \in \mathbb{Z}\}\big),$$

where E' is some finite subset of E. By a similar reasoning we shall present in the sequel, the continuity of the critical parameter for the oriented model also holds.

1.3 Proof of Theorem 1.1

Theorem 1.1 is a consequence of the following proposition:

Proposition 1.2. *Fix* $p, q \in (0,1)$ *and let* $\lambda = \min(p, 1-p)$. *If* $\sup_{r\in\mathbb{N}}|U^{(r)}| < \infty$ *and* $\mathrm{dist}_G\big(U^{(i)}, U^{(j)}\big) \geq 3$ *for every* $i \neq j$*, then for any* $\varepsilon \in (0,\lambda)$*, there exists* $\eta = \eta(p,q,\varepsilon) > 0$ *such that*

$$P_{p-\varepsilon,q+\delta}((v,0) \leftrightarrow \infty) \leq P_{p+\varepsilon,q-\delta}((v,0) \leftrightarrow \infty)$$

for every $\delta \in (0,\eta)$ *and* $v \in V \setminus (\cup_{r\in\mathbb{N}} U^{(r)})$.

Proof of Theorem 1.1. Since $q \mapsto p_c(q)$ is non-increasing, any discontinuity, if it exists, must be a jump. Suppose p_c is discontinuous at some point $q_0 \in (0,1)$, let $a = \lim_{q\downarrow q_0} p_c(q)$ and $b = \lim_{q\uparrow q_0} p_c(q)$. Then, for any $p \in (a,b)$, we can find an $\varepsilon > 0$ such that

$$P_{p+\varepsilon,q_0-\delta}((v,0) \leftrightarrow \infty) = 0 < P_{p-\varepsilon,q_0+\delta}((v,0) \leftrightarrow \infty)$$

for every $\delta > 0$ and $v \in V$, a contradiction to Proposition 1.2. □

The proof of Proposition 1.2 is based on the construction of a coupling which allows us to understand how a small change in the parameters p and q of the model affects the percolation behavior. This construction is done in several steps. First, we split $\mathbb{G} = (\mathbb{V},\mathbb{E})$ into an appropriate family of connected subgraphs $(\mathbb{V}_\alpha,\mathbb{E}_\alpha)$, $\alpha \in I$, such that $\{\mathbb{E}_\alpha\}_{\alpha\in I}$ constitutes a decomposition of $\mathbb{E}$. Second, we define coupling measures on each $\mathbb{E}_\alpha$, in such a way that the increase of parameter p compensates the decrease, by some small amount, of parameter q, in the sense of preserving the connections between boundary vertices of $\mathbb{V}_\alpha$. This property will play an important role when we consider percolation on the graph $\mathbb{G}$ as a whole. Third, we verify that we can set the same parameters for each coupling measure, provided that $\sup_{r\in\mathbb{N}}|U^{(r)}| < \infty$. Finally, we merge these couplings altogether by considering the product measure of each one. The first and second steps consist of ideas developed in [9] and [23]. The third step, which allows extending the result of [23], is the main result of [10]. To introduce them rigorously, we begin with some definitions.

For $r \in \mathbb{N}$, $n \in \mathbb{Z}$, let $L_r := |U^{(r)}|$ and

$$\begin{aligned}
\mathbb{V}_n^{(r)} &:= \big\{(v,m) \in \mathbb{V} : \mathrm{dist}_G\big(v,U^{(r)}\big) \le 1, (2L_r+2)n \le m \le (2L_r+2)(n+1)\big\};\\
\mathbb{E}_n^{(r)} &:= \mathsf{E}_{\mathbb{V}_n^{(r)}} \setminus \mathsf{E}_{V\times\{2L_r(n+1)\}};\\
\mathbb{E}^{(r)} &:= \cup_{n\in\mathbb{Z}}\mathbb{E}_n^{(r)}.
\end{aligned} \tag{1.3}$$

Based on these definitions, note that:

- Since $G = (V,E)$ has bounded degree and $|U^{(r)}| < \infty$, it follows that the graph $(\mathbb{V}_n^{(r)},\mathbb{E}_n^{(r)})$ is finite;
- $\mathbb{E}_n^{(r)} \cap \mathbb{E}_{n'}^{(r)} = \emptyset$, for every $n \ne n'$;
- Since we are assuming $\mathrm{dist}_G(U^{(r)},U^{(r')}) \ge 3$, for every $r \ne r'$, it follows that $\mathrm{dist}_{\mathbb{G}}(\mathbb{V}_n^{(r)},\mathbb{V}_{n'}^{(r')}) \ge 1$ for every $n,n' \in \mathbb{Z}$. Therefore, $\mathbb{E}_n^{(r)} \cap \mathbb{E}_{n'}^{(r)} = \emptyset, \forall n,n' \in \mathbb{Z}$ and $r \ne r'$.

Next, recall the definition of $\mathbb{E}^{\text{in},(r)}$ in (1.2) and let

$$\mathbb{E}_n^{\partial,(r)} := \mathbb{E}_n^{(r)} \setminus \mathbb{E}^{\text{in},(r)}, \quad \mathbb{E}_n^{\text{in},(r)} := \mathbb{E}_n^{(r)} \cap \mathbb{E}^{\text{in},(r)}, \quad \mathbb{E}_O := \mathbb{E} \setminus \big(\cup_{r\in\mathbb{N}}\mathbb{E}^{(r)}\big).$$

One should also observe that $\mathbb{E}$ is a disjoint union of the sets above:

$$\mathbb{E} = \mathbb{E}_O \cup \big[\cup_{r\in\mathbb{N}}\mathbb{E}^{(r)}\big] = \mathbb{E}_O \cup \Big[\cup_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}}\mathbb{E}_n^{(r)}\Big] = \mathbb{E}_O \cup \Big[\cup_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}}\big(\mathbb{E}_n^{\partial,(r)} \cup \mathbb{E}_n^{\text{in},(r)}\big)\Big].$$

Thus, letting

$$\Omega_O := \{0,1\}^{\mathbb{E}_O}, \quad \Omega_n^{(r)} := \{0,1\}^{\mathbb{E}_n^{(r)}}, \quad \Omega_n^{\partial,(r)} := \{0,1\}^{\mathbb{E}_n^{\partial,(r)}}, \quad \Omega_n^{\text{in},(r)} := \{0,1\}^{\mathbb{E}_n^{\text{in},(r)}},$$

we can write

$$\Omega = \Omega_O \times \prod_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}} \Omega_n^{(r)} = \Omega_O \times \prod_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}} \big(\Omega_n^{\partial,(r)} \times \Omega_n^{\text{in},(r)}\big).$$

Finally, let

$$\begin{aligned} \partial\mathbb{V}_n^{(r)} := \big\{(v,m) \in \mathbb{V}_n^{(r)} : \text{dist}_G(v,U^{(r)}) = 1\big\} \\ \cup \big[U^{(r)} \times \{(2L_r+2)\}\big] \cup \big[U^{(r)} \times \{(2L_r+2)\}\big], \end{aligned} \tag{1.4}$$

and, for $A \subset \partial\mathbb{V}_n^{(r)}$ and $\omega_n^{(r)} \in \Omega_n^{(r)}$, define the random set

$$C_n^{(r)}\big(A,\omega_n^{(r)}\big) := \Big\{(v,m) \in \partial\mathbb{V}_n^{(r)} : \exists(v_0,m_0) \in A, (v,m) \xleftrightarrow{(\mathbb{V}_n^{(r)},\mathbb{E}_n^{(r)})} (v_0,m_0) \text{ in } \omega_n^{(r)}\Big\}, \tag{1.5}$$

where $a \xleftrightarrow{G'} b$ indicates that a and b are connected by a path entirely contained in the graph G'.

For $p,q \in [0,1]$ and $E' \subset \mathbb{E}$, let $P_{p,q}|_{E'}$ be the measure $P_{p,q}$ restricted to $\{0,1\}^{E'}$. It is clear that

$$P_{p,q} = P_{p,q}|_{\mathbb{E}_O} \times \prod_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}} P_{p,q}|_{\mathbb{E}_n^{(r)}}.$$

With these definitions in hand, we are ready to establish the facts necessary for the proof of Proposition 1.2.

Lemma 1.3. *Let $p,q \in (0,1)$ and $\lambda = \min(p,1-p)$. For any $\varepsilon \in (0,\lambda)$ and $\delta \in (0,1)$ such that $(q-\delta,q+\delta) \subset [0,1]$, there exists a coupling $\mu_O = (\omega_O,\omega'_O)$ on Ω_O^2 such that*

- $\omega_O \overset{(d)}{=} P_{p-\varepsilon,q+\delta}|_{\mathbb{E}_O}$;
- $\omega'_O \overset{(d)}{=} P_{p+\varepsilon,q-\delta}|_{\mathbb{E}_O}$;

- *$\omega_O \le \omega'_O$ for every $(\omega_O, \omega'_O) \in \Omega_O^2$.*

Proof. This construction is standard. Let $Z = (Z_1, Z_2) \in \Omega_O^2$ be a pair of random elements defined in some probability space, such that the marginals Z_1 and Z_2 are independent on every edge of $\mathbb{E}_O$ and assign each edge to be open with probabilities $p - \varepsilon$ and $2\varepsilon/(1 - p + \varepsilon)$, respectively. Taking $\omega_O = Z_1$ and $\omega'_O = Z_1 \vee Z_2$, define μ_O to be the distribution of (ω_O, ω'_O) and the claim readily follows. □

To properly compare percolation configurations in $(\mathbb{V}_n^{(r)}, \mathbb{E}_n^{(r)})$ at different parameter values, we make use of the following result, proved in [9] and also used in [23]. It is based on Doeblin's maximal coupling lemma (see [24], Chapter 1.4).

Lemma 1.4. *Let $\{P_\theta\}_{\theta \in (0,1)}$ be probability measures on a finite set S, such that $\theta \mapsto P_\theta(z)$ is continuous in $(0,1)$ for every $z \in S$. If $P_\tau(\bar{x}) > 0$ for some $\tau \in (0,1)$ and $\bar{x} \in S$, then, for every $\alpha, \beta \in (0,1)$ close enough to τ, there exists, on a larger probability space $(S^2, \mathbb{P})$, a coupling $X, Y \in S$, such that $X \overset{(d)}{=} P_\alpha$, $Y \overset{(d)}{=} P_\beta$ and*

$$\mathbb{P}(\{X = Y\} \cup \{X = \bar{x}\} \cup \{Y = \bar{x}\}) = 1.$$

Proof. Since $\theta \mapsto P_\theta(z)$ is continuous in $(0,1)$ for every $z \in S$, then the function

$$h(\theta, \gamma) := 1 - \sum_{z \neq \bar{x}} P_\theta(z) \vee P_\gamma(z)$$

is also continuous. By hypothesis, we have $h(\tau, \tau) = 1 - \sum_{z \neq \bar{x}} P_\tau(z) = P_\tau(\bar{x}) > 0$, so that $h(\alpha, \beta) > 0$ for every (α, β) close enough to (τ, τ).

Now, let $\mathbb{P}$ be the probability measure on S^2 defined by

$$\mathbb{P}(z_1, z_2) = \begin{cases} 1 - \sum_{z \neq \bar{x}} P_\alpha(z) \vee P_\beta(z), & \text{if } z_1 = z_2 = \bar{x}; \\ P_\alpha(z) \wedge P_\beta(z), & \text{if } z_1 = z_2 = z \neq \bar{x}; \\ [P_\alpha(z) - P_\beta(z)]^+, & \text{if } z_1 = z \neq \bar{x} \text{ and } z_2 = \bar{x}; \\ [P_\beta(z) - P_\alpha(z)]^+, & \text{if } z_1 = \bar{x} \text{ and } z_2 = z \neq \bar{x}; \\ 0 & \text{if } z_1 \neq z_2 \text{ and } z_1, z_2 \neq \bar{x}. \end{cases}$$

Thus, if $X, Y : S^2 \to S$ are defined by $X(x, y) = x$ and $Y(x, y) = y$, the result readily follows. □

The next lemma is one of the fundamental facts established in [23]. It is motivated by the observation that if a vertex $v \in \mathbb{V} \setminus \mathbb{V}_n^{(r)}$ percolates, then closing some edges within $\mathbb{V}_n^{(r)} \setminus \partial \mathbb{V}_n^{(r)}$ does not change the percolative behavior of v, as long as these closed edges do not interfere in the connectivity between the vertices of $\partial\mathbb{V}_n^{(r)}$. To make this assertion precise, we make use of the set $C_n^{(r)}(A, \omega_n^{(r)})$, defined by (1.5).

Lemma 1.5 (Coupling two configurations inside a finite cylinder)**.** *Let $r \in \mathbb{N}$, $n \in \mathbb{Z}$, $p, q \in (0,1)$ and $\lambda = \min(p, 1-p)$. For any $\varepsilon \in (0,\lambda)$, there exists $\eta^{(r)} > 0$, such that if $\delta \in (0, \eta^{(r)})$, there is a coupling $\mu_n^{(r)} = (\omega_n^{(r)}, {\omega'}_n^{(r)})$ on $\Omega_n^{(r)} \times \Omega_n^{(r)}$ with the following properties:*

- $\omega_n^{(r)} \overset{(d)}{=} P_{p-\varepsilon,q+\delta}|_{\mathbb{E}_n^{(r)}}$;
- ${\omega'}_n^{(r)} \overset{(d)}{=} P_{p+\varepsilon,q-\delta}|_{\mathbb{E}_n^{(r)}}$;
- $C_n^{(r)}(A, \omega_n^{(r)}) \subset C_n^{(r)}(A, {\omega'}_n^{(r)})$ *for every* $A \in \partial\mathbb{V}_n^{(r)}$ *almost surely.*

Moreover, the value of $\eta^{(r)} > 0$ depends only on the choice of q, p, ε and the graph $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$.

Proof. Let $r \in \mathbb{N}$ and $n \in \mathbb{Z}$. The measures $\mu_n^{(r)}$ will be translations of $\mu_0^{(r)}$, hence we shall construct only the latter. Our aim is to use Lemma 1.4 to properly compare percolation configurations in $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$ at different parameter values. To do so, we must first point out the relevant objects in the setting of the referred lemma.

For the finite set, we consider $S = \Omega_0^{\partial,(r)} \times \Omega_0^{\partial,(r)} \times \Omega_0^{\text{in},(r)}$.

Next, for $p \in (0,1)$ and $\lambda = \min(p, 1-p)$, fix $\varepsilon \in (0,\lambda)$ and let $\{P_{p,\varepsilon,t}\}_{t\in(0,1)}$ be the family of probability measures on $S = \Omega_0^{\partial,(r)} \times \Omega_0^{\partial,(r)} \times \Omega_0^{\text{in},(r)}$ such that, independently, each edge in the first copy of $\mathbb{E}_0^{\partial,(r)}$ is open with probability $p-\varepsilon$, each edge in the second copy of $\mathbb{E}_0^{\partial,(r)}$ is open with probability $2\varepsilon/(1-p+\varepsilon)$, and each edge in $\mathbb{E}_0^{\text{in},(r)}$ is open with probability t. For every $z \in S$, the application $t \mapsto P_{p,\varepsilon,t}(z)$ is a polynomial, therefore it is continuous on $(0,1)$.

In this context, we consider $\bar{x} = (\bar{x}^{\partial,(r),1}, \bar{x}^{\partial,(r),2}, \bar{x}^{\text{in},(r)}) \in S$, where $\bar{x}^{\partial,(r),1}(e) = 0$ for every edge e in the first copy of $\mathbb{E}_0^{\partial,(r)}$, $\bar{x}^{\partial,(r),2}(e) = 1$ for every edge e in the second copy of $\mathbb{E}_0^{\partial,(r)}$, and $\bar{x}^{\text{in},(r)}$ is defined according to the following rule:

Let $G^{(r)} = (U^{(r)}, E^{(r)})$ be the subgraph of G specified in (1.1) and recall that $L_r = |U^{(r)}|$. Define $\Delta U^{(r)} := \{v \in V : \text{dist}_G(v, U^{(r)}) = 1\}$ and assume that the vertices $w_1, \dots, w_{L_r} \in U^{(r)}$ are enumerated so that

$$\text{dist}_G\big(w_j, \Delta U^{(r)}\big) \le \text{dist}_G\big(w_{j+1}, \Delta U^{(r)}\big) \quad \forall j = 1, \dots, L_r - 1.$$

For a fixed $j = 1, \dots, L_r - 1$, choose a vertex $w'_j \in \Delta U^{(r)}$ such that $\text{dist}_G(w_j, \Delta U^{(r)}) = \text{dist}_G(w_j, w'_j)$, and a shortest path $\gamma_j = \{w_j = x_1, x_2, \dots, x_k = w'_j\}$ from w_j to w'_j, both of them specified according to some predefined order. Let

$$\Gamma_j := \gamma_j \cap U^{(r)} = \gamma_j \setminus \{w_j'\},$$

and, for $m, m' \in \mathbb{N}$, $m < m'$, denote

$$W_m^{m'}(j) := \{(w_j, m), (w_j, m+1), \dots, (w_j, m')\}.$$

We set $\bar{x}^{\mathrm{in},(r)}(e) = 1$ if and only if

$$e \subset \big(U^{(r)} \times \{L_r + 1\}\big), \tag{1.6}$$

or, for some $j \in \{1, \dots, L_r\}$, we have

$$e \subset \big[W_0^j(j) \cup (\Gamma_j \times \{j\})\big] \cup \big[W_{2L_r+2-j}^{2L_r+2}(j) \cup (\Gamma_j \times \{2L_r + 2 - j\})\big]. \tag{1.7}$$

Thus, note that, for any $q \in (0,1)$, we have $P_{p,\varepsilon,q}(\bar{x}) > 0$. Hence, Lemma 1.4 implies the existence of $\eta^{(r)} = \eta(p, \varepsilon, q, \mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)}) > 0$, such that if $\delta \in (0, \eta^{(r)})$, then there exists a coupling

$$X = \big(X_0^{\partial,(r),1}, X_0^{\partial,(r),2}, X_0^{\mathrm{in},(r)}\big), \quad Y = \big(Y_0^{\partial,(r),1}, Y_0^{\partial,(r),2}, Y_0^{\mathrm{in},(r)}\big),$$

where $X, Y \in S$ possess the following properties:

- The values of $X_0^{\partial,(r),1}, X_0^{\partial,(r),2}, X_0^{\mathrm{in},(r)}$ are independent on all edges, and the same is true for $Y_0^{\partial,(r),1}, Y_0^{\partial,(r),2}, Y_0^{\mathrm{in},(r)}$;
- $X_0^{\partial,(r),1}$ and $Y_0^{\partial,(r),1}$ assign each edge of the first copy of $\mathbb{E}_0^{\partial,(r)}$ to be open with probability $p - \varepsilon$;
- $X_0^{\partial,(r),2}$ and $Y_0^{\partial,(r),2}$ assign each edge of the second copy of $\mathbb{E}_0^{\partial,(r)}$ to be open with probability $2\varepsilon/(1 - p + \varepsilon)$;
- $X_0^{\mathrm{in},(r)}$ and $Y_0^{\mathrm{in},(r)}$ assign each edge of $\mathbb{E}_0^{\mathrm{in},(r)}$ to be open with probabilities $q + \delta$ and $q - \delta$, respectively;
- $\mathbb{P}(\{X = Y\} \cup \{X = \bar{x}\} \cup \{Y = \bar{x}\}) = 1$.

Now, let $\omega_0^{(r)}, {\omega'}_0^{(r)} \in (\Omega_0^{\partial,(r)} \times \Omega_0^{\mathrm{in},(r)}) = \Omega_0^{(r)}$ be given by

$$\omega_0^{(r)} = \big(X_0^{\partial,(r),1}, X_0^{\mathrm{in},(r)}\big), \quad {\omega'}_0^{(r)} = \big(Y_0^{\partial,(r),1} \vee Y_0^{\partial,(r),2}, Y_0^{\mathrm{in},(r)}\big),$$

and define $\mu_0^{(r)}$ to be the distribution of the pair $(\omega_0^{(r)}, {\omega'}_0^{(r)})$. The first four properties of X and Y listed above imply that $\omega_0^{(r)} \overset{(d)}{=} P_{p-\varepsilon,q+\delta}|_{\mathbb{E}_0^{(r)}}$ and ${\omega'}_0^{(r)} \overset{(d)}{=} P_{p+\varepsilon,q-\delta}|_{\mathbb{E}_0^{(r)}}$, so that the first two properties listed in the statement of Lemma 1.5 are satisfied. To show that $C_0^{(r)}(A, \omega_0^{(r)}) \subset C_0^{(r)}(A, \omega'^{(r)}_0)$ for every $A \in \partial\mathbb{V}_0^{(r)}$ almost surely, it suffices to check that this property holds in the event $\{X = Y\} \cup \{X = \bar{x}\} \cup \{Y = \bar{x}\}$ of probability one. As a matter of fact,

- If $X = Y$, then $\omega_0^{(r)}(e) \leq \omega'^{(r)}_0(e)$ for every $e \in \mathbb{E}_0^{(r)}$, so that the property immediately follows.
- If $X = \bar{x}$, then $\omega_0^{(r)} = (\mathbf{0}, \bar{x}^{\text{in},(r)}) \in (\Omega_0^{\partial,(r)} \times \Omega_0^{\text{in},(r)})$. The only open edges in this configuration are those indicated in (1.6) and (1.7), which are not capable of connecting any two vertices of $\partial\mathbb{V}_0^{(r)}$. Therefore, $C_0^{(r)}(A, \omega_0^{(r)}) = A \subset C_0^{(r)}(A, \omega'^{(r)}_0)$ for every $A \subset \partial\mathbb{V}_0^{(r)}$.
- If $Y = \bar{x}$, then $\omega'^{(r)}_0 = (\mathbf{1}, \bar{x}^{\text{in},(r)}) \in (\Omega_0^{\partial,(r)} \times \Omega_0^{\text{in},(r)})$. Since in this configuration every edge of $\mathbb{E}_0^{\partial,(r)}$ and every edge indicated in (1.7) is open, any vertex of $\partial\mathbb{V}_0^{(r)}$ is connected to $U^{(r)} \times \{L_r + 1\}$. By (1.6), every edge inside this set is also open, so that $C_0^{(r)}(A, \omega'^{(r)}_0) = \partial\mathbb{V}_0^{(r)} \supset C_0^{(r)}(A, \omega_0^{(r)})$ for every non-empty subset $A \subset \partial\mathbb{V}_0^{(r)}$.

Thus, we conclude the proof of Lemma 1.5. □

The key fact that allows us to extend the results in [23] to the model defined in Section 1.2 is our main contribution to this study and the last ingredient used in the proof of Proposition 1.2.

Lemma 1.6 (Same coupling parameter for all cylinders)**.** *If* $\sup_{r\in\mathbb{N}}|U^{(r)}| < \infty$, *then for any* $\varepsilon > 0$ *fixed, the sequence* $\{\eta^{(r)}\}_{n\in\mathbb{N}}$ *in Lemma 1.5 may be chosen bounded away from* 0.

Proof. From Lemma 1.5, it follows that, for every $r \in \mathbb{N}$, the value of $\eta^{(r)} > 0$ depends on the choice of q, p, ε and the graph $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$. Note that while the values of q, p and ε are the same for every $r \in \mathbb{N}$, the graphs $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$ may differ. However, there are only a finite number of graphs that $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$ can assume. As a matter of fact, recalling that $\Delta U^{(r)} = \{v \in V : \text{dist}_G(v, U^{(r)}) = 1\}$, one may observe by definition (1.3) that $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$ is constructed from the vertex set $U^{(r)} \cup \Delta U^{(r)}$ and from the edges with both endpoints within $U^{(r)} \cup \Delta U^{(r)}$. Since $\sup_{r\in\mathbb{N}}|U^{(r)}| < \infty$ and G has bounded degree, we have $M = \sup_{r\in\mathbb{N}}|U^{(r)} \cup \partial U^{(r)}| < \infty$. Since there are only a finite number of graphs of bounded degree with at most M vertices, the claim regarding $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$ follows, that is, $\eta := \inf_{r\in\mathbb{N}} \eta^{(r)} > 0$. □

Proof of Proposition 1.2. Lemmas 1.5 and 1.6 imply the following result: let $p, q \in (0, 1)$ and $\lambda = \min(p, 1-p)$. For any $\varepsilon \in (0, \lambda)$, there exists $\eta > 0$ such that if $\delta \in (0, \eta)$, there is a family of couplings $\{\mu_n^{(r)}\}_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}}$, with each $\mu_n^{(r)} = (\omega_n^{(r)}, \omega'^{(r)}_n)$ defined on $\Omega_n^{(r)} \times \Omega_n^{(r)}$ and having the following property:

- $\omega_n^{(r)} \overset{(d)}{=} P_{p-\varepsilon,q+\delta}|_{\mathbb{E}_n^{(r)}}$;
- ${\omega'}_n^{(r)} \overset{(d)}{=} P_{p+\varepsilon,q-\delta}|_{\mathbb{E}_n^{(r)}}$;
- $C_n^{(r)}(A,\omega_n^{(r)}) \subset C_n^{(r)}(A,{\omega'}_n^{(r)})$ for every $A \in \partial\mathbb{V}_n^{(r)}$ almost surely.

Let μ_O be the coupling of Lemma 1.3 and define the coupling $\mu = (\omega, \omega')$ on Ω^2 by

$$\mu = \mu_O \times \prod_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}} \mu_n^{(r)}.$$

Thus, it is clear that $\omega \overset{(d)}{=} P_{p-\varepsilon,q+\delta}$, $\omega' \overset{(d)}{=} P_{p+\varepsilon,q-\delta}$ and, almost surely, for every $v \in V \setminus (\cup_{r\in\mathbb{N}} U^{(r)})$, if $(v,0) \leftrightarrow \infty$ in ω, then $(v,0) \leftrightarrow \infty$ in ω'. $\square$