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Note

Some remarks on the analysis of large strain torsion-like problems

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Summary. These notes are concerned with some fundamental aspects in the analysis of torsion-like problems involving large inelastic deformations. Accepting a spatially fixed cylindrical coordinate system as a preferred frame of reference, the significance of the material rate of change of the associated basis vectors during such deformation processes is emphasized. So-called shifters and their rates are shown to provide a convenient tool to account for this. Explicit expressions are given and applied in coordinate expressions of rate equations of the type occurring in large strain plasticity theories.

1 Introduction

To date, analyses of inelastic solids involving large strains, like the simulation of metal forming processes, are carried out on an almost routine basis. Most of them are two-dimensional plane or axisymmetric analyses. Recently, the analysis of torsion-like problems such as torsion of tubes or solid circular bars under conditions of fixed ends (e.g. [1], [2], [3]) or axially free ends (e.g. [4], [3]), and torsion of a cylinder between rigid casings [5] has attracted special attention in the large strain plasticity community. The main reason for this is that it was found that such problems are well-suited to study the adequacy of constitutive models for deformation-induced anisotropy.

The use of convected (or material) coordinates embedded in the material has been very popular in the analysis of large strain plasticity problems (see, e.g., [6], [7]). Such coordinates have been used in the analysis of fixed-end torsion in [1]. However, for torsion-like problems, with full cylindrical symmetry, a cylindrical coordinate system that is spatially fixed seems a more natural candidate and has been applied in e.g. [2], [3], [4], [5]. However there is a difficulty associated with such a frame in the case of torsion-like problems which is easily overlooked. As we will point out in detail here, this difficulty relates to the fact that during such deformation processes, the local basis for a material point continually rotates although the coordinate system is spatially fixed. This observation is essential for the application of rate equations as they appear for instance in constitutive theories for large strain plasticity. In fact, omission of these effects — as in [5] — leads to erroneous results.

Although these aspects are concerned with basic kinematical aspects of continuum mechanics, they have not yet been included in textbooks. Therefore, it is felt worthwhile to present here a brief but complete account.

Standard tensor notation is used, with tensors denoted by bold-face characters. The tensor product is denoted by $\otimes$, a single dot denotes the usual scalar product of vectors and a: denotes the double-dot product ($\boldsymbol{a}:\boldsymbol{b} = a^{ij}b_{ij}$ in terms of components of tensors $\boldsymbol{a}$ and $\boldsymbol{b}$ on a common basis). A superscript T denotes the transpose of a second-order tensor and a superposed dot denotes the material time derivative.

2 Problem definition

Let X^α and x^i denote the coordinates of a material point X at the initial state ($t = 0$) and the current deformed state, respectively. Here we will take the coordinates in a spatially fixed cylindrical coordinate system such that $x^i = (r, \theta, z)$ (see Fig. 1). The associated (covariant) base vectors at x^i are

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{x}}{\partial x^i},$$

with $\boldsymbol{x}$ the position vector, and their reciprocals in the usual sense being $\boldsymbol{g}^i$. The components of the metric tensor are

$$g_{22} = \frac{1}{g^{22}} = r^2, \qquad g_{ij} = g^{ij} = \delta_{ij} \quad \text{otherwise}, \tag{2.1}$$

with δ_{ij} the Kronecker delta, and the nonvanishing Christoffel symbols of the second kind are

$$\Gamma^1_{22} = -r, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}. \tag{2.2}$$

For future reference we note that in this orthogonal coordinate system the physical components $a_{\langle ij \rangle}$ of any second-order tensor $\boldsymbol{a}$ may be obtained from its covariant, contravariant or mixed tensor components in the basis $\{\boldsymbol{g}_i\}$ by

$$a_{\langle ij \rangle} = \sqrt{g_{ii}g_{jj}}\, a^{ij} = \sqrt{g^{ii}g^{jj}}\, a_{ij} = \sqrt{g_{ii}g^{jj}}\, a^i_{.j} = \sqrt{g^{ii}g_{jj}}\, a_i^{.j} \tag{2.3}$$

where i, j are not summed.

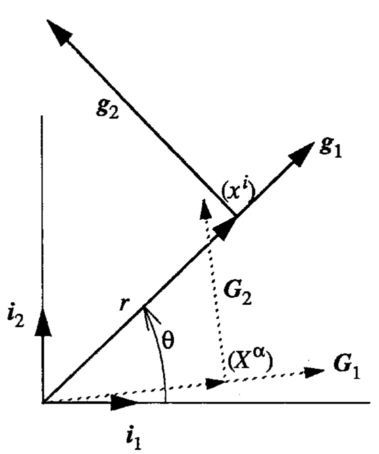

Fig. 1. Spatially fixed cylindrical coordinate system and covariant base vectors at current position x^i (solid) of material point X and at initial position X^α (dashed)

Consider now a class of torsion-like problems defined by the following instantaneous, cylindrically symmetric velocity field $\boldsymbol{v} = v^i \boldsymbol{g}_i$:

$$v^1 = \dot{r}(r; t), \qquad v^2 = \dot{\theta}(r, z; t), \qquad v^3 = \dot{z}(z; t). \tag{2.4}$$

Torsion of a circular bar with fixed ends is included as a special case with $\dot{z} = 0$ and $\dot{\theta} = \dot{\theta}(z; t) = z\dot{\psi}(t)$, ψ denoting the twist per unit length of the bar; torsion of an infinite cylinder bounded by two rigid casings is obtained by taking $\dot{\theta} = \dot{\theta}(r; t)$ instead. The physical components of the velocity gradient tensor

$$\boldsymbol{L} = \operatorname{grad} \boldsymbol{v} = L^i{}_j \boldsymbol{g}_i \otimes \boldsymbol{g}^j, \qquad L^i{}_j = v^i|_j = \frac{\partial v^i}{\partial x^j} + \Gamma^i_{jk} v^k, \tag{2.5}$$

are then found in a standard way to be given by (cf., e.g. [8])

$$[L_{\langle ij \rangle}] = \begin{bmatrix} \dfrac{\partial \dot{r}}{\partial r} & -\dot{\theta} & 0 \\[2ex] \dot{\theta} + r\dfrac{\partial \dot{\theta}}{\partial r} & \dfrac{\dot{r}}{r} & r\dfrac{\partial \dot{\theta}}{\partial z} \\[2ex] 0 & 0 & \dfrac{\partial \dot{z}}{\partial z} \end{bmatrix}, \tag{2.6}$$

so that the components of the strain-rate tensor $\boldsymbol{D} = \dfrac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T)$ and the spin tensor $\boldsymbol{W} = \dfrac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T)$ are

$$[D_{\langle ij \rangle}] = \begin{bmatrix} \dfrac{\partial \dot{r}}{\partial r} & \dfrac{1}{2} r \dfrac{\partial \dot{\theta}}{\partial r} & 0 \\[2ex] \dfrac{1}{2} r \dfrac{\partial \dot{\theta}}{\partial r} & \dfrac{\dot{r}}{r} & \dfrac{1}{2} r \dfrac{\partial \dot{\theta}}{\partial z} \\[2ex] 0 & \dfrac{1}{2} r \dfrac{\partial \dot{\theta}}{\partial z} & \dfrac{\partial \dot{z}}{\partial z} \end{bmatrix}, \tag{2.7.1}$$

$$[W_{\langle ij \rangle}] = \begin{bmatrix} 0 & -\left(\dot{\theta} + \dfrac{1}{2} r \dfrac{\partial \dot{\theta}}{\partial r}\right) & 0 \\[2ex] \dot{\theta} + \dfrac{1}{2} r \dfrac{\partial \dot{\theta}}{\partial r} & 0 & \dfrac{1}{2} r \dfrac{\partial \dot{\theta}}{\partial z} \\[2ex] 0 & -\dfrac{1}{2} r \dfrac{\partial \dot{\theta}}{\partial z} & 0 \end{bmatrix}. \tag{2.7.2}$$

The coordinate system we have chosen is fixed in space, so that the associated basis at any point in space remains one and the same (in contrast with the situation in the case of convected coordinates). However, referring to Fig. 1 it is immediately clear that for the class of deformations (2.4) where the current angle θ for a material point is a function of time, the local base vectors as measured by an observer travelling with the material continually rotate in space during the deformation process. That is, the material rates of change of the spatially fixed base vectors, $\dot{\boldsymbol{g}}_i$ and $\dot{\boldsymbol{g}}^i$ respectively, are essentially nonvanishing. This observation has important — but easily overlooked — consequences when considering material rates of change of tensors. For instance, the material time derivative of a tensor $\boldsymbol{a}$, in terms of contravariant components for instance, becomes

$$\dot{\boldsymbol{a}} = \dot{a}^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j + a^{ij}\dot{\boldsymbol{g}}_i \otimes \boldsymbol{g}_j + a^{ij}\boldsymbol{g}_i \otimes \dot{\boldsymbol{g}}_j. \tag{2.8}$$

These aspects are of course not related exclusively to the torsion problem considered here. Therefore, we shall now proceed by considering material rates of change of base vectors in a general setting, utilizing in doing so the concept of shifters.

3 Shifters and shift rates

A shifter is a two-point tensor which shifts a vector from one coordinate system to another (see, e.g., [8]). With the covariant base vectors associated with these coordinate systems denoted by $\boldsymbol{g}_i$ and $\boldsymbol{G}_\alpha$, and their reciprocals defined as usual, the shifter is defined by the components

$$g_i^{\ \alpha} = \boldsymbol{G}^\alpha \cdot \boldsymbol{g}_i \quad \text{and} \quad g_\alpha^{\ i} = \boldsymbol{g}^i \cdot \boldsymbol{G}_\alpha. \tag{3.1}$$

Applying the shifters to the base vectors themselves, we can write

$$\begin{aligned} \boldsymbol{g}_i &= g_i^{\ \alpha}\boldsymbol{G}_\alpha, \qquad & \boldsymbol{g}^i &= g_\alpha^{\ i}\boldsymbol{G}^\alpha \\ \boldsymbol{G}_\alpha &= g_\alpha^{\ i}\boldsymbol{g}_i, \qquad & \boldsymbol{G}^\alpha &= g_i^{\ \alpha}\boldsymbol{g}^i \end{aligned} \tag{3.2.1--4}$$

and the identities

$$g_i^{\ \alpha} g_\alpha^{\ j} = \delta_i^{\ j}, \qquad g_\alpha^{\ i} g_i^{\ \beta} = \delta_\alpha^{\ \beta} \tag{3.3}$$

are readily established.

Here we shall identify the $\boldsymbol{g}_i$ with the base vectors at the current position of the material point and the $\boldsymbol{G}_\alpha$ with the base vectors at its initial position measured by the coordinates X^α. Since the latter base vectors are of course constant, straightforward material time differentiation of (3.2.1, 2) gives

$$\dot{\boldsymbol{g}}_i = H_i^{\ j}\boldsymbol{g}_j, \qquad \dot{\boldsymbol{g}}^i = -H_j^{\ i}\boldsymbol{g}^j, \tag{3.4}$$

where the so-called shift rates $H_j^{\ i}$ are components of a nonsymmetric second-order tensor defined by

$$H_j^{\ i} = \dot{g}_j^{\ \alpha} g_\alpha^{\ i} = -\dot{g}_\alpha^{\ i} g_j^{\ \alpha}, \tag{3.5}$$

or

$$H_j^{\ i} = \dot{\boldsymbol{g}}_j \cdot \boldsymbol{g}^i = -\dot{\boldsymbol{g}}^i \cdot \boldsymbol{g}_j. \tag{3.6}$$

The second equality in (3.5) is obtained by differentiation of (3.3.1). We are now in a position to present general expressions for the material rates of change of the four component decompositions of an arbitrary tensor $\boldsymbol{a}$:

$$\begin{aligned} \dot{\boldsymbol{a}} &= (\dot{a}^{ij} + H_k^{\ i}a^{kj} + a^{ik}H_k^{\ j})\, \boldsymbol{g}_i \otimes \boldsymbol{g}_j \\ &= (\dot{a}_{ij} - H_i^{\ k}a_{kj} - a_{ik}H_j^{\ k})\, \boldsymbol{g}^i \otimes \boldsymbol{g}^j \\ &= (\dot{a}^i_{\cdot j} + H_k^{\ i}a^k_{\cdot j} - a^i_{\cdot k}H_j^{\ k})\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \\ &= (\dot{a}_i^{\cdot j} - H_i^{\ k}a_k^{\cdot j} + a_i^{\cdot k}H_k^{\ j})\, \boldsymbol{g}^i \otimes \boldsymbol{g}_j. \end{aligned} \tag{3.7}$$

It is noted that the expressions (3.4) are valid for any coordinate system, be it spatially fixed or embedded in the material. The difference lies only in the reason why the current base vectors $\boldsymbol{g}_i$ and the initial base vectors $\boldsymbol{G}_\alpha$ are not identical, i.e. in the physical interpretation of the shifters. In the case of convected coordinates embedded in the material, changes of the base vectors are an immediate consequence of the fact that the coordinate net becomes distorted (cf., e.g., [6]). Using a spatially fixed curvilinear coordinate system, changes of base vectors are due to the fact that the material moves relative to the coordinate net (in the case of rectangular Cartesian coordinates, the base vectors are independent of the current position and the shift rates vanish).

We proceed by deriving explicit expressions for the shift rates $H_j{}^i$ for spatially fixed coordinate systems. First of all, this may be achieved directly by writing the $\boldsymbol{g}_i$ in terms of $\boldsymbol{G}_\alpha$, evaluating the shifters according to (3.1) and finally deriving the shift rates from (3.5). A second, slightly more indirect approach is to consider the $\boldsymbol{g}_i$ at the current point x^i as vector functions of the initial coordinates X^α, i.e. $\boldsymbol{g}_i = \boldsymbol{g}_i\big(x^i(X^\alpha, t)\big)$. Then, taking the (material) time derivative we get

$$\dot{\boldsymbol{g}}_i = \frac{\partial \boldsymbol{g}_i}{\partial x^k}\, v^k = \Gamma^j_{ik} v^k \boldsymbol{g}_j,$$

and by comparing with (3.4) we conclude that

$$H_j{}^i = \Gamma^i_{jk} v^k. \tag{3.8}$$

Notice the connection with the last terms in the covariant derivative in (2.5); in fact, one can write the identity

$$L_j{}^i = \frac{\partial v^i}{\partial x^j} + H_j{}^i.$$

Another noteworthy property relates to the covariant components $H_{ij} = g_{ik} H_j{}^k$ of the shift rates, which upon substitution of e.g. (3.6) may be readily shown to satisfy

$$H_{ij} + H_{ji} = \frac{\partial g_{ij}}{\partial x^k}\, v^k. \tag{3.9}$$

This relationship implies that when the metric is independent of the position in space, the shift rate is a skewsymmetric tensor, thus representing only a rigid spin of the base vectors.

Finally, upon substitution of the particular velocity field given in (2.4), the physical components of the shift rate, $H_{\langle ij \rangle} = \sqrt{g_{ii} g^{jj}}\, H_j{}^i$, are obtained as

$$[H_{\langle ij \rangle}] = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & \dfrac{\dot{r}}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.10}$$

The $\langle 12 \rangle$ and $\langle 21 \rangle$ components are readily interpreted as representing the rigid spin of the base vectors as observed by a co-moving observer (see also Fig. 1), while the $\langle 22 \rangle$ component is related to the fact that the length of the $\boldsymbol{g}_2$ base vector changes with the current radius r. When $\dot{r} = 0$, such as in the case of fixed-end torsion of an incompressible material [1], [2], the shift rate becomes skewsymmetric, as noted already in relation to (3.9).

An alternative convenient basis for cylindrical coordinates is the set of orthonormal base vectors $\boldsymbol{e}_i$ defined by normalizing the $\boldsymbol{g}_i$ (in particular $\boldsymbol{e}_2 = \boldsymbol{g}_2/r$) such that components in that basis are physical components (cf. e.g. [2]). For the basis $\{\boldsymbol{e}_i\}$, the effect of material rate of rotation of base vectors is similar as in (3.10), but, evidently, $H_{\langle 22 \rangle} = 0$ in that case.

4 Application: stress rates in large strain plasticity

As an illustration we consider in this Section the application to the decomposition of stress rates as they appear in rate constitutive equations formulated for large strain elastoplasticity. In general, such constitutive equations can be given in the form

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{L} : \boldsymbol{D}, \tag{4.1}$$

where $\mathbf{L}$ is a fourth-order constitutive tensor and where $\overset{\circ}{\boldsymbol{\sigma}}$ is an objective (material) time-derivative of Cauchy's stress tensor $\boldsymbol{\sigma}$. This stress rate must ensure that the principle of material objectivity is satisfied. The Jaumann stress rate, defined by

$$\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{W} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \boldsymbol{W}, \tag{4.2}$$

has been most widely used; but, very recently, various other corotational (Jaumann-type) and convected stress rates have been proposed and studied in the context of constitutive equations for anisotropic hardening behavior (this is still a much debated issue in the plasticity community for which the reader is referred to the pertinent literature). Usually such other stress rates can be expressed in terms of the Jaumann rate, with deviating terms being collectively included in the right-hand side of the constitutive relationship (4.1) (see, e.g., [6]).

Let us consider the representation of the Jaumann stress rate in terms of contravariant components $\sigma^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{\sigma} \cdot \boldsymbol{g}^j$. Inserting the first of the expressions (3.7), we obtain for $\overset{\circ}{\sigma}{}^{ij} = \boldsymbol{g}^i \cdot \overset{\circ}{\boldsymbol{\sigma}} \cdot \boldsymbol{g}^j$:

$$\overset{\circ}{\sigma}{}^{ij} = \dot{\sigma}^{ij} - (W_k{}^i - H_k{}^i)\, \sigma^{kj} - \sigma^{ik}(W_k{}^j - H_k{}^j), \tag{4.3}$$

where $W_j{}^i = \boldsymbol{g}^i \cdot \boldsymbol{W} \cdot \boldsymbol{g}_j$. Now, for the class of torsion-like problems specified by (2.4) we will finally present explicit expressions in terms of physical components by substitution of (2.7.2) and (3.10). Employing an evident notation for physical components in a cylindrical coordinate system, we find

$$\overset{\circ}{\sigma}_{rr} = \dot{\sigma}_{rr} + r \frac{\partial \dot{\theta}}{\partial r} \sigma_{r\theta}, \tag{4.4.1}$$

$$\overset{\circ}{\sigma}_{r\theta} = \dot{\sigma}_{r\theta} + \frac{1}{2} r \frac{\partial \dot{\theta}}{\partial r} (\sigma_{\theta\theta} - \sigma_{rr}) - \frac{1}{2} r \frac{\partial \dot{\theta}}{\partial z} \sigma_{rz}, \tag{4.4.2}$$

$$\overset{\circ}{\sigma}_{rz} = \dot{\sigma}_{rz} + \frac{1}{2} r \frac{\partial \dot{\theta}}{\partial r} \sigma_{\theta z} + \frac{1}{2} r \frac{\partial \dot{\theta}}{\partial z} \sigma_{r\theta}, \tag{4.4.3}$$

$$\overset{\circ}{\sigma}_{\theta\theta} = \dot{\sigma}_{\theta\theta} - r \frac{\partial \dot{\theta}}{\partial r} \sigma_{r\theta} - r \frac{\partial \dot{\theta}}{\partial z} \sigma_{\theta z}, \tag{4.4.4}$$

$$\overset{\circ}{\sigma}_{\theta z} = \dot{\sigma}_{\theta z} - \frac{1}{2} r \frac{\partial \dot{\theta}}{\partial z} (\sigma_{zz} - \sigma_{\theta\theta}) - \frac{1}{2} r \frac{\partial \dot{\theta}}{\partial r} \sigma_{rz}, \tag{4.4.5}$$

$$\overset{\circ}{\sigma}_{zz} = \dot{\sigma}_{zz} + r \frac{\partial \dot{\theta}}{\partial z} \sigma_{\theta z}. \tag{4.4.6}$$

In deriving these relationships account must be given of the fact that the physical components depend on r which is a function of t (for instance, $\sigma_{r\theta} = \sigma_{\langle 12 \rangle} = r\sigma^{12}$). Time differentiation then involves terms with $\dot{r}/r$, but these terms are cancelled exactly by the $H_{\theta\theta}$ component (as expected, since we are considering a fixed coordinate system). Also notice that the rigid body spin contribution $\dot{\theta}$ to the $r\theta$ and θr continuum spin components in (2.7.2) has been cancelled out by the shift rates.

With $\dot{z} = 0$ and $\dot{\theta} = z\dot{\psi}(t)$ the governing equations for torsion with fixed ends of [2], [3] are easily recovered. Torsion between two rigid casings [5] is obtained by taking $\dot{\theta} = \dot{\theta}(r; t)$, and the expressions (4.4) reduce to

$$\mathring{\sigma}_{rr} = \dot{\sigma}_{rr} + \dot{\gamma}\sigma_{r\theta},$$

$$\mathring{\sigma}_{\theta\theta} = \dot{\sigma}_{\theta\theta} - \dot{\gamma}\sigma_{r\theta},$$

$$\mathring{\sigma}_{r\theta} = \dot{\sigma}_{r\theta} + \frac{1}{2}\dot{\gamma}(\sigma_{\theta\theta} - \sigma_{rr}),$$

$$\mathring{\sigma}_{zz} = \dot{\sigma}_{zz},$$

where $\dot{\gamma} = r\dfrac{\partial\dot{\theta}}{\partial r}$ (due to the symmetries of the problem, $\sigma_{rz} = \sigma_{\theta z} \equiv 0$). The corresponding expressions used in [5] are incorrect since the shift rates have been omitted there.

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