Kac-Moody Symmetries and Gauged Supergravity

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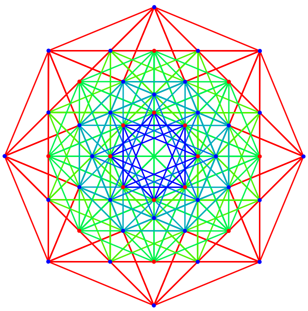


The comparison

In the previous chapter we described the spectrum of states of maximal and half-maximal supergravity. In section 6.1 we will see how they can be obtained from the Kac-Moody algebras E_{11} and D_8^{+++}, respectively. In section 6.3 we will try to compare the equations of motion of supergravity to those obtained from a non-linear sigma model of a Kac-Moody algebra. Note that for both the kinematical and dynamical analysis we will only treat bosons; fermions have been discussed in for example [28, 17, 26].

6.1 Kinematics

It was shown in [87, 81, 57] that the spectrum of physical states of the different maximal supergravity theories can be obtained from the very extended Kac-Moody algebra E_{11}. This has been extended to the set of all possible deformation and top-form potentials in [77, 1]. A similar analysis could be done for E_{10} [70, 23, 24] except for the top-form potentials. In addition, non-maximal supergravity and the associated very extended Kac-Moody algebras have been discussed in [57, 82, 2, 73, 56, 44]. We will first review how the the kinematics of maximal supergravity can be obtained, and in subsection 6.1.2 switch to half-maximal supergravity.

6.1.1 Maximal supergravity

The key idea is to decompose E_{11} with respect to subgroups that match the symmetry structure of maximal supergravity in various dimensions. The subalgebra representations resulting from this level decomposition can then be matched with the various supergravity fields. The valid decompositions for E_{11} are always of the type $G \otimes GL(D)$, where G is the duality group in D dimensions and $GL(D)$ refers to the space-time symmetries. Dynkin diagrams are a useful tool to visualize these group decompositions: the decompositions correspond to 'deleting' certain nodes of the diagram in order to obtain two disjoint parts. One of these disjoint parts is the G duality group, and the other is the $SL(D) \otimes \mathbb{R}^+ = GL(D)$ *gravity line*. The extra factor of $\mathbb{R}^+$ for the gravity line comes from the Cartan subalgebra generator of the deleted node. Furthermore, the gravity line must always include the very-extended node. All the valid $3 \leq D \leq 11$ decompositions of E_{11} are listed in Table 6.1. Note that the duality group G contains an extra $\mathbb{R}^+$ factor whenever there is a second disabled node, stemming from the Cartan subalgebra generator of the disabled node. This explains why the duality group of IIA supergravity is $\mathbb{R}^+$ and why those of IIB and $D = 11$ supergravity do not have such a factor.

After the decomposition has been fixed, the generators of E_{11} can be analyzed by means of a level decomposition (see also section 4.3). As the actual level decomposition is quite cumbersome to do by hand, we have written a computer program called SimpLie [5] to do the job. For more on SimpLie, see Appendix A.

The explicit results of the level decompositions of Table 6.1 can be found in section C.1. The spectrum is obtained by associating to each generator a supergravity field in the same representation. This leads to the following fields at each level. At the lowest levels the physical states of the supergravity we started out with (see Table 5.3) appear together with their duals. More precisely: corresponding to any p-form generator we also find a $(D-p-2)$-form. In addition there is a $(D-3, 1)$-form with mixed symmetries and possibly $(D-2)$-form generators, which are interpreted as the dual graviton [45, 87, 14, 10] and dual scalars, respectively. The duality relations themselves are not reproduced by the level decomposition: in the absence of dynamics these relations have to be imposed by hand. Beyond the level of the dual graviton we find deformation potentials and top-forms, i.e. $(D-1)$- and D-forms respectively. At the same levels as the deformation potentials and top-forms, and at higher levels still, there are the so-called 'exotic' generators. These exotic generators have a space-time symmetric structure that does not have an obvious counterpart in supergravity. Some of them may be interpreted as infinitely many exotic dual copies of the previously mentioned fields [75, 73]. A schematic representation of the level decomposition and the resulting tower of (physical) states can be found in Figure 6.1.

The $(D-1)$- and D-forms found from E_{11} are listed in Table 6.2. The interesting thing is that they exactly match the representations of the embedding tensor and its quadratic constraint as given in Table 5.5, save for an extra **248** representation in

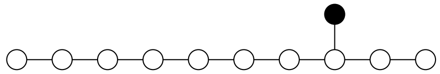
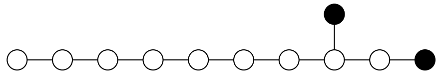
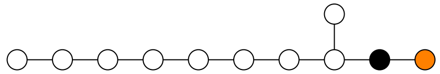
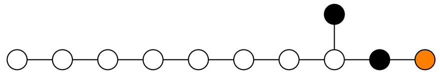
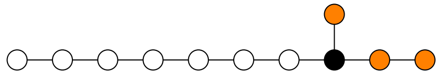
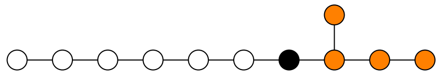
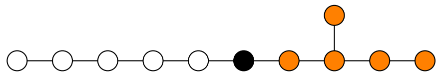
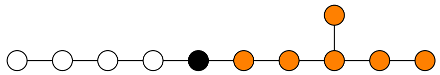
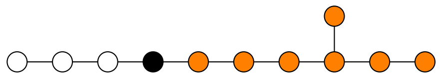
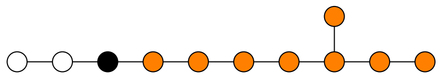
D	G	Grav. line	E_{11} decomposition
11	1	$GL(11)$	
IIA	$\mathbb{R}^+$	$GL(10)$	
IIB	$SL(2)$	$GL(10)$	
9	$GL(2)$	$GL(9)$	
8	$SL(3) \times SL(2)$	$GL(8)$	
7	$SL(5)$	$GL(7)$	
6	$SO(5,5)$	$GL(6)$	
5	E_6	$GL(5)$	
4	E_7	$GL(4)$	
3	E_8	$GL(3)$	

Table 6.1: Global symmetry groups G of all $3 \leq D \leq 11$ maximal supergravities embedded in E_{11}. The groups G can be read off from the decomposition of the Dynkin diagram of E_{11}; they correspond to the orange nodes. The white nodes together with one scaling generator from a deleted node form the gravity line $A_{D-1} \times \mathbb{R}^+ = GL(D)$.

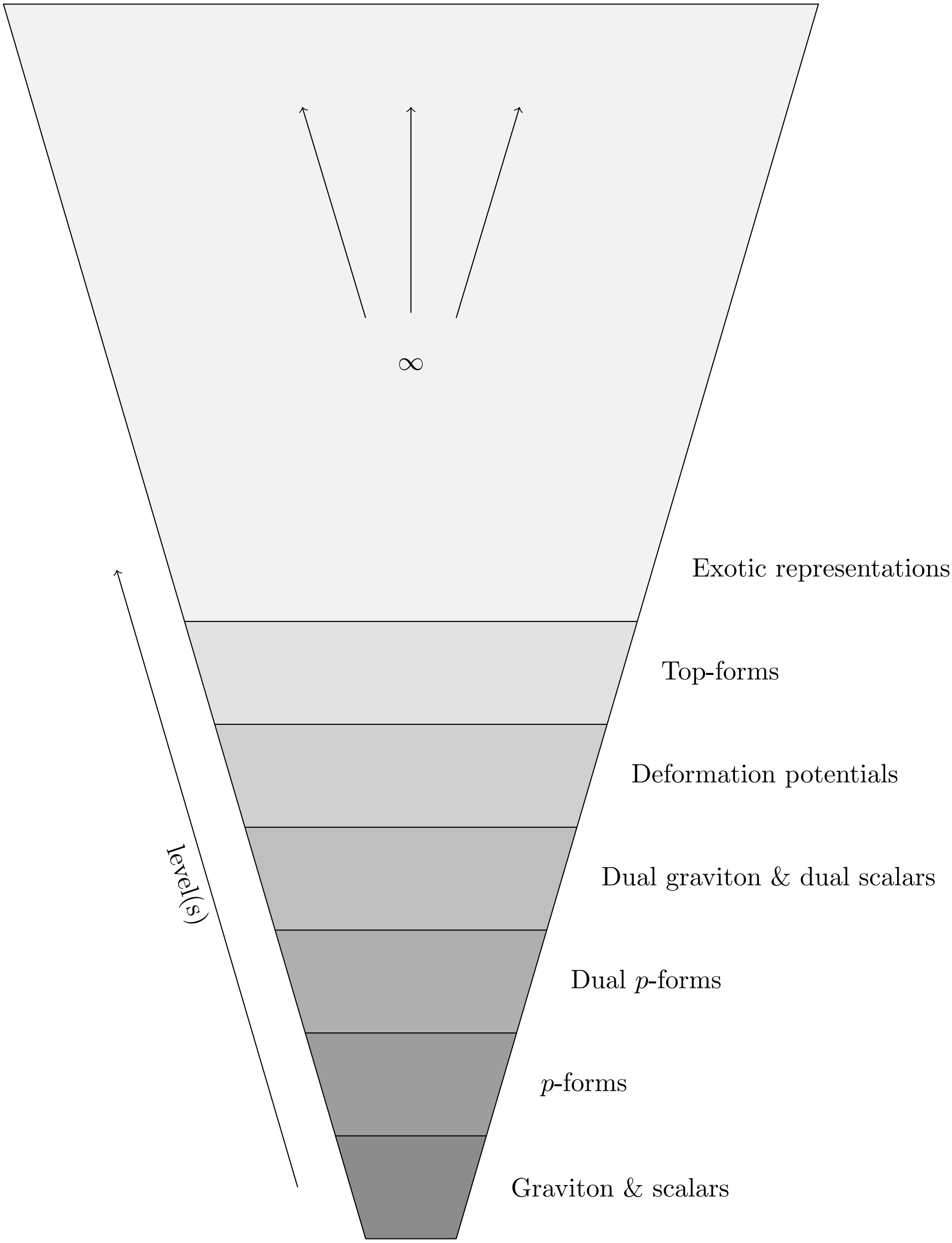


Figure 6.1: Schematic Hasse diagram of E_{11} containing the hierarchy of subalgebra representations common to all decompositions.

D	$(D-1)$-forms $p=1$	$p=2$	D-forms
IIA		$\mathbf{1}$	$\mathbf{1}\oplus\mathbf{1}$
IIB			$\mathbf{2}\oplus\mathbf{4}$
9	$\mathbf{2}\oplus\mathbf{3}$		$\mathbf{2}\oplus\mathbf{2}\oplus\mathbf{4}$
8	$(\mathbf{3},\mathbf{2})\oplus(\mathbf{6},\mathbf{2})$		$(\mathbf{3},\mathbf{1})\oplus(\mathbf{3},\mathbf{1})\oplus(\mathbf{3},\mathbf{3})\oplus(\mathbf{15},\mathbf{1})$
7	$\mathbf{15}\oplus\mathbf{40}$		$\mathbf{5}\oplus\mathbf{45}\oplus\mathbf{70}$
6	$\mathbf{144}$		$\mathbf{10}\oplus\mathbf{126}\oplus\mathbf{320}$
5	$\mathbf{351}$		$\mathbf{27}\oplus\mathbf{1728}$
4	$\mathbf{912}$		$\mathbf{133}\oplus\mathbf{8645}$
3	$\mathbf{1}\oplus\mathbf{3875}$		$\mathbf{248}\oplus\mathbf{3875}\oplus\mathbf{147250}$

Table 6.2: E_{11} predictions for deformation- and top-forms in all $3 \leq D \leq 10$ maximal supergravities. These are representations of the respective duality groups G given in Table 5.2. For the deformation-forms it is also indicated to which type-p deformation they correspond (see section 6.2).

three dimensions. Furthermore, E_{11} predicts top-forms for IIB supergravity, which were recently reconfirmed to exist in [8]. The single deformation potential for IIA supergravity corresponds to Romans' massive deformation thereof [78]. The fact that Romans' theory is not an ordinary gauged supergravity is reflected in the fact that the deformation potential is of 'type 2', which will be explained in section 6.2.

6.1.2 Half-maximal supergravity

The analogy between gauged supergravity and very-extended Kac-Moody algebras is not limited to maximal supergravity and E_{11}. Half-maximal supergravity [2], coupled to $D-10+n$ vector multiplets, reduce to the scalar coset $SO(8,8+n)/SO(8)\times SO(8+n)$ when reduced to three dimensions. In other words, the relevant groups for supergravity theories with 16 supercharges are the B and D series in the above real form. Of these, only three are of split real form, i.e. maximally non-compact, which are given by $n=-1,0,+1$. These correspond to the split forms of B_7, D_8 and B_8, respectively.

We are interested in the decomposition of the very extensions of these algebras with respect to the possible gravity lines. An exhaustive list of the possibilities for the algebras of real split form is given in table Table 6.3. As can be seen from this table, these correspond to the unique D-dimensional supergravity theory with 16 supercharges coupled to $m+n$ vector multiplets with $m=10-D$. The corresponding duality groups G in D dimensions are also given in Table 6.3. Note that there is no second disabled node and therefore no $\mathbb{R}^+$ factor in the duality group for the 6b case and in $D=3,4$.

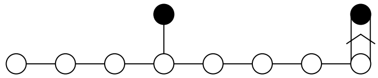
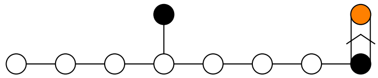
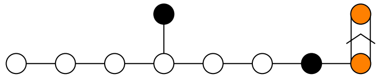
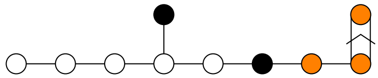
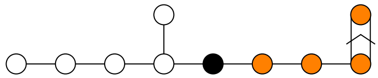
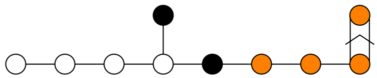
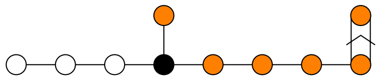
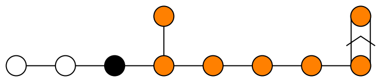
D	G	Multiplets	B_7^{+++} $(n=-1)$
10	$\mathbb{R}^+ \times SO(n)$	GV^n	–
9	$\mathbb{R}^+ \times SO(1,1+n)$	GV^{n+1}	
8	$\mathbb{R}^+ \times SO(2,2+n)$	GV^{n+2}	
7	$\mathbb{R}^+ \times SO(3,3+n)$	GV^{n+3}	
6a	$\mathbb{R}^+ \times SO(4,4+n)$	GV^{n+4}	
6b	$SO(5,5+n)$	GT^{n+4}	
5	$\mathbb{R}^+ \times SO(5,5+n)$	GV^{n+5}	
4	$SL(2) \times SO(6,6+n)$	GV^{n+6}	
3	$SO(8,8+n)$	GV^{n+7}	

Table 6.3: The decompositions of B_7^{+++}, D_8^{+++} and B_8^{+++} with respect to the possible gravity lines. The duality groups G and the multiplet structures (where G is the graviton, V the vector and T the self-dual tensor multiplet) are also given.

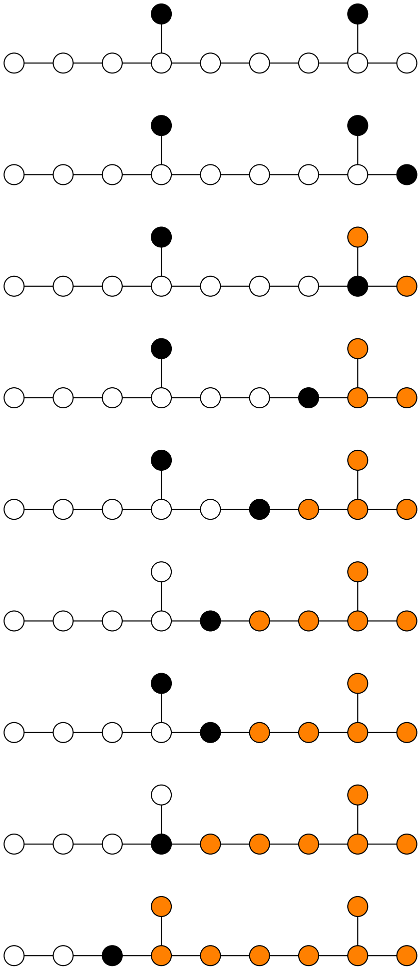
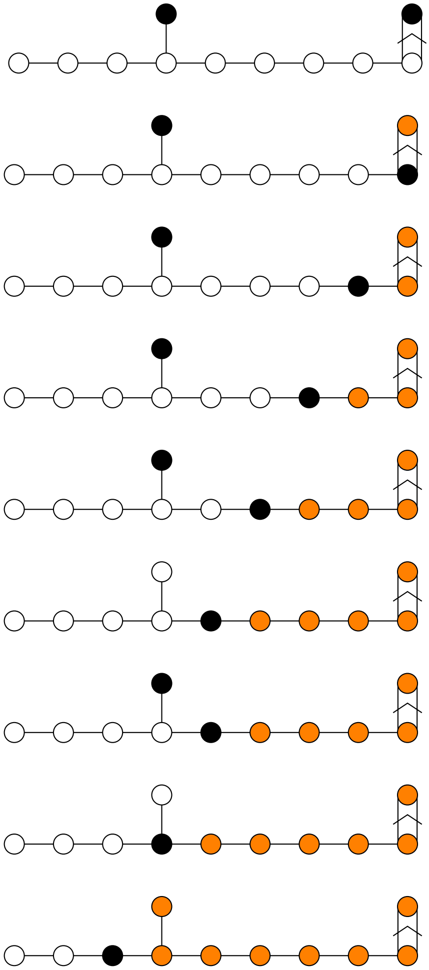
D_8^{+++} $(n = 0)$	B_8^{+++} $(n = 1)$
	

Table 6.3: Continued.

D	$(D-1)$-forms $p=1$,	2,	3.	D-forms constraints on $p=1$	other
10-8	$\yng(1)$ $\yng(1,1,1)$			1 $\yng(1,1)$ $\yng(1,1,1,1)$	
7	$\yng(1)$ $\yng(1,1,1)$		1	1 $\yng(1,1)$ $\yng(1,1,1,1)$	$\yng(1)$
6a	$\yng(1)$ $\yng(1,1,1)$	$\yng(1)$		1 $\yng(1,1)$ $\yng(1,1,1,1)$	1 1 $\yng(2)$ $\yng(1,1)$
6b					$\yng(1)$ $\yng(2,1)$
5	$\yng(1)$ $\yng(1,1)$ $\yng(1,1,1)$			1 $\yng(1)$ $\yng(1)$ $\yng(1,1)$ $\yng(2,1)$ $\yng(1,1,1)$ $\yng(1,1,1,1)$	
4	$(\mathbf{2},\yng(1))$ $(\mathbf{2},\yng(1,1,1))$			$(\mathbf{3},1)$ $(\mathbf{3},\yng(1,1))$ $(\mathbf{3},\yng(1,1,1,1))$ $\yng(1,1)$ $\yng(1,1)$ $\yng(2,1,1)$	
3	1 $\yng(2)$ $\yng(1,1,1,1)$			$\yng(2)$ $\yng(1,1)$ $\yng(1,1,1,1)$ $\yng(2,1,1)$ $\yng(2,1,1,1,1)$	

Table 6.4: The representations of deformation- and top-forms in all half-maximal supergravities. The representations refer to the duality group G given in Table 6.3. We also indicate which type p of deformations they correspond to (see also section 6.2, and to which top-forms one can associate a quadratic constraint on type 1 deformation parameters.

In section C.2 the result of the decomposition of the D_8^{+++} algebras with respect to the different $SL(D)$ subalgebras is given. It can be seen that these decompositions give rise to exactly the physical degrees of freedom [57]. In addition there are the deformation and top-form potentials in the Kac-Moody spectrum. In particular, Table 6.4 summarizes our results for the deformation and top-form potentials for half-maximal supergravity in D dimensions.

Using the embedding tensor approach, an analysis of the linear and quadratic constraints on the possible deformations has been explicitly carried out in $D = 3, 4, 5$ [83]. It turns out that the representations of the quadratic constraints exactly coincide with the representations of the possible top-forms in these dimensions.

6.2 Fundamental p-forms and type-p deformations

Recall from section 4.3 that a level decomposition always induces a grading on the decomposed algebra $\mathfrak{g}$:

$$[\mathfrak{g}_{l_1}, \mathfrak{g}_{l_2}] \subseteq \mathfrak{g}_{l_1+l_2}, \tag{6.1}$$

where, for simplicity, we have assumed there is only one level. This grading implies that all the higher levels can be recovered from the repeated adjoint action of the level one generators on themselves:

$$\underbrace{[\mathfrak{g}_1, \dots [\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]] \dots]}_{l \text{ times}} \subseteq \mathfrak{g}_l. \tag{6.2}$$

For the supergravity decompositions of Kac-Moody algebras considered in this chapter, the subalgebra representations at level 1 always correspond to one or more p-forms. We will call these level 1 p-forms *fundamental p-forms.* The fundamental p-forms correspond to the positive simple roots of the disabled nodes in the Kac-Moody algebra. From the decomposed Dynkin diagram one can thus deduce the number and type of these fundamental p-forms: any disabled node connected to the n^{th} node of the gravity line (counting from the very extended node) gives rise to a fundamental $(D-n)$-form. Furthermore, if the disabled node in question is also connected to a node of the duality group the $(D-n)$-form carries a non-trivial representation of the duality group.

The level 1 p-forms are fundamental in the sense that by virtue of (6.2) their commutators generate the other p-forms in the decomposition. Say A_p is a fundamental p-form, and A_q is a p-form that occurs higher in the decomposition. The latter can then be written as

$$\underbrace{[A_p, \dots [A_p, [A_p, A_p]] \dots]}_{l \text{ times, } lp=q} \subseteq A_q. \tag{6.3}$$

This allows us to distinguish between deformations potentials that correspond to gauged and massive deformations as follows.

The most familiar class of deformed supergravities are the gauged supergravities. They are special in the sense that the deformations can be seen as the result of gauging a subgroup G_0 of the duality group G (see section 5.3). Not all deformed supergravities can be viewed as gauged supergravities. In the case of maximal supergravity there is one exception: massive IIA supergravity cannot be obtained by gauging the $\mathbb{R}^+$ duality group [78]. The gauged supergravities can be seen as the first in a series of *type-p deformations*. There is a simple criterion that defines to which type of deformation parameter each deformation potential gives rise to. The central observation is that to each $(D-1)$-form A_{D-1} one can associate a unique commutator

$$[A_p, A_{D-p-1}] = A_{D-1}\,, \tag{6.4}$$

where A_p corresponds to a fundamental p-form. The deformation potential corresponding to such a deformation generator gives rise to a type-p deformation parameter.

We observe that each type p deformation is characterized by the fact that a fundamental p-form gauge field becomes massive. For $p = 1$ this leads to gauged supergravities, in which a vector can become massive by absorbing a scalar degree of freedom. Note that other non-fundamental gauge fields may become massive as well. The case $p = 2$ entails a fundamental two-form that becomes massive by 'eating' a vector. The prime example of this is massive IIA supergravity in ten dimensions [78]. Another example is the non-chiral half-maximal supergravity in six dimensions [79]. An example of a $p = 3$ deformation is the half-maximal supergravity theory of [85] where a fundamental three-form potential acquires a topological mass term.

As is evident from Table 6.2 and Table 6.4 the Kac-Moody algebras E_{11} and D_8^{+++} correctly reproduce the type-p deformation potentials for known non-gauged supergravity deformations in ten [78], seven [85], and six [79] dimensions.

6.3 Dynamics

Up to this point we have been concerned with matching the kinematics of (gauged) supergravity and certain Kac-Moody algebras. We will now attempt to take the correspondence one step further, and try to compare the dynamics, i.e. the equations of motion.

6.3.1 E_{10} or E_{11}?

There are two proposals for implementing Kac-Moody symmetries in supergravity theories. In the case of maximal supergravity, one employs the very-extended Kac-Moody algebra E_{11} [88, 87], while the other uses the over-extended E_{10} [23]. Both use a non-linear realization to implement the infinite symmetry. The former approach realizes the E_{11} symmetry directly in the dimension of the corresponding supergravity theory by including an infinite amount of coordinates [76, 74, 72].

In this thesis we will focus on the latter approach, which realizes the E_{10} symmetry by reducing the coordinate dependence to only time. This is inspired by the behavior of gravity near a spacetime singularity, where the spatial points dynamically decouple [7]. The equations of motion in that regime only depend on time, and exhibit chaotic behavior [62] that may be interpreted as taking place in the fundamental Weyl chamber of a hyperbolic Kac-Moody algebra [25, 22]. For maximal supergravity, the hyperbolic Kac-Moody algebra is E_{10}.

In this approach, the idea is to compare a suitably reduced supergravity theory to a dynamic model based on an over-extended Kac-Moody algebra. To compare

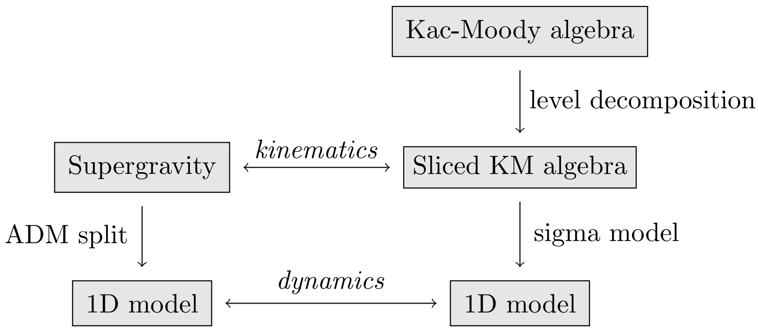


Figure 6.2: Comparing supergravity with a Kac-Moody algebra. The former has to be reduced to one dimension with an ADM-like split, and the latter has to be sliced with a level decomposition and then realized as a non-linear sigma model.

both sides of this correspondence, on the one hand one has to truncate the supergravity fields and break spacetime covariance by choosing an ADM gauge [6] in order to be amenable to a one-dimensional language. On the Kac-Moody side, on the other hand, one has to perform a level decomposition and put the results on a one-dimensional non-linear sigma model (see section 4.4). The correspondence has been schematically depicted in Figure 6.2. Because the non-linear sigma model already provides the time dimension, the Kac-Moody target space has to provide the spatial dimensions. Thus the gravity line to which to decompose the Kac-Moody algebra must be $GL(D-1)$ for the D-dimensional decomposition. In order to get the correct duality group, it is clear that the relevant Kac-Moody algebras are not of the very-extended type, but must be over-extended. In particular, the correct Kac-Moody algebra for a dynamical comparison in this particular way to maximal supergravity is E_{10}, and not E_{11} (see also Figure 6.3).

As one attractive scenario it has been suggested [23, 70, 24] that the higher levels of the over-extended Kac-Moody algebra encode the spatial gradients of the supergravity fields, and so by including all of these states one should finally recover the full unrestricted supergravity in D dimensions (though in an 'unconventional' formulation). This is in contrast to the E_{11} approach, where, as already mentioned, some of the higher level states can be interpreted as dual representations of lower level states [75].

Instead we will interpret part of the higher levels (i.e. the deformation potentials and top-forms) as deformations of pure maximal supergravity. In [55, 41] it has been shown that Romans' massive supergravity in ten dimensions [78], which deforms type IIA supergravity with a mass parameter, is contained in the E_{10} model, upon taking a certain 9-form representation into account. We will try to do the same for $D = 3$ maximal supergravity [3].

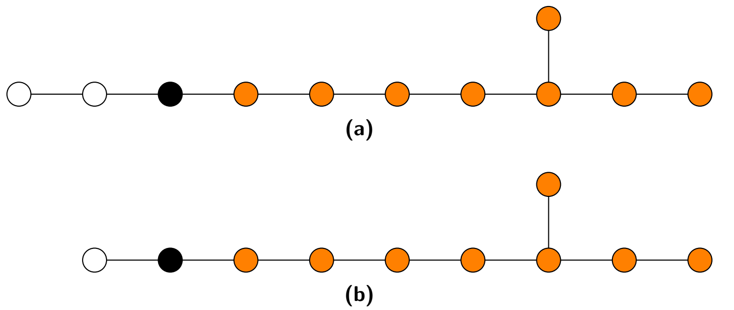


Figure 6.3: Both E_{11} (a) and E_{10} (b) in a $D = 3$ E_8 decomposition. For the very-extended E_{11}, the gravity line has to encode the space-time symmetries, and for the over-extended E_{10} it has to encode the spatial symmetries.

6.3.2 The $E_{10}/K(E_{10})$ coset model

In this section we will focus on gauged supergravity in three dimensions. The advantage of this case is that E_8 is the largest finite-dimensional duality group. As a consequence, the E_{10} equations of motion truncated to level $l = 0$ already match ungauged supergravity reduced to a one-dimensional system. Thus, this model allows a clear distinction between the 'manifest' aspects of the E_{10} conjecture at level $l = 0$ and the more speculative features related to higher levels, such as spatial gradients or gauge couplings.

In order to make contact with three-dimensional supergravity we perform a level decomposition of E_{10} with respect to the subgroup of spatial diffeomorphisms and the duality group:

$$E_{10} \supset SL(2) \otimes E_8. \tag{6.5}$$

This corresponds to deleting the black node in the Dynkin diagram in Figure 6.3b. The representations occurring in this level decomposition can be calculated using the computer program SimpLie [5]. Up to level $l < 3$ we find the $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{e}_8$ representations in Table 6.5, where we indicated the corresponding generators with their symmetries.

We denote by $a, b = 1, 2$ the fundamental indices of $GL(2, \mathbb{R})$ and by $\mathcal{A}, \mathcal{B} = 1, 2 \ldots, 248$ the adjoint indices of E_8. The fields associated to the $l = 0$ generators are the spatial zweibein and the coset scalars. The $l = 1$ fields can be interpreted as gauge vectors. We will take the $l = 2$ fields to be the embedding tensor components θ and $\tilde{\Theta}$. At the negative levels we have the conjugate representations, i.e. the transposed generators of those at the positive levels.

We will now construct the non-linear realization of $E_{10}/K(E_{10})$, in a similar spirit in which the much simpler $A_2/K(A_2)$ has been constructed in Example 4.4 in section 4.4. Much of the details can be found in [3].

Level ℓ	$SL(2) \otimes E_8$ representation	Generator	Interpretation
0	$(\mathbf{1} \oplus \mathbf{3}, \mathbf{1})$	$K^a{}_b$	metric
0	$(\mathbf{1}, \mathbf{248})$	$t^{\mathcal{A}}$	scalars
1	$(\mathbf{2}, \mathbf{248})$	$E^a{}_{\mathcal{A}}$	gauge vectors
2	$(\mathbf{1}, \mathbf{1})$	E	θ
2	$(\mathbf{1}, \mathbf{3875})$	$E_{\mathcal{AB}} = E_{(\mathcal{AB})}$	$\tilde{\Theta}_{\mathcal{MN}}$
2	$(\mathbf{3}, \mathbf{248})$	$E^{ab}{}_{\mathcal{A}} = E^{(ab)}{}_{\mathcal{A}}$	

Table 6.5: $SL(2) \otimes E_8$ representations within E_{10} up to level 2.

The local $K(E_{10})$ invariance allows us to choose a suitable gauge for the E_{10}-valued group element V. In the Borel gauge, we can write V as a product

$$V = V_l V_0 = e^X e^h e^{\mathcal{H}}, \tag{6.6}$$

where V_l and V_0 are group elements corresponding to $l > 0$ and $l = 0$, respectively. Thus we can expand the corresponding algebra elements in the basis of $\mathfrak{e}_{10}$ as

$$X = A_m{}^{\mathcal{M}} E^m{}_{\mathcal{M}} + B_{mn}{}^{\mathcal{M}} E^{mn}{}_{\mathcal{M}} + BE + B^{\mathcal{MN}} E_{\mathcal{MN}} + \cdots, \tag{6.7a}$$

$$h = h_a{}^b K^a{}_b, \tag{6.7b}$$

$$\mathcal{H} = \mathcal{H}_{\mathcal{A}} t^{\mathcal{A}}, \tag{6.7c}$$

where the dots stand for higher-level contributions. Here and in the following, $m, n, \ldots = 1, 2$ and $\mathcal{M}, \mathcal{N} \ldots = 1, 2, \ldots, 248$ denote curved $GL(2)$ and E_8 indices, respectively. This means that they are 'world' indices indicating rigid transformations from the left, while $\mathcal{A}$ and a are flat indices.

The Maurer-Cartan form J can then be computed to give

$$J = V^{-1} \partial V = J_0 + J_1 + J_2 + \cdots. \tag{6.8}$$

The dots stand for the higher-level contributions, which will be truncated. The derivative is with respect to time, i.e. $\partial \equiv \partial_t$. The low-level contributions that will be kept are

$$J_0 = P_{\mathcal{A}} t^{\mathcal{A}} + \tfrac{1}{2} P_a{}^b K^a{}_b, \tag{6.9a}$$

$$J_1 = P_a{}^{\mathcal{A}} E^a{}_{\mathcal{A}}, \tag{6.9b}$$

$$J_2 = PE + P^{\mathcal{AB}} E_{\mathcal{AB}} + P_{ab}{}^{\mathcal{A}} E^{ab}{}_{\mathcal{A}}, \tag{6.9c}$$

where the individual components read

$$P_a{}^b = \tfrac{1}{2}(e_a{}^m \partial e_m{}^b + e_b{}^m \partial e_m{}^a), \tag{6.10a}$$
$$P_{\mathcal{A}} = \tfrac{1}{2}(\mathcal{E}^{-1}\partial\mathcal{E})_{\mathcal{A}}, \tag{6.10b}$$
$$P_a{}^{\mathcal{A}} = \tfrac{1}{2} e_a{}^m \mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}} DA_m{}^{\mathcal{M}}, \tag{6.10c}$$
$$P = \tfrac{1}{2}(\det e)^{-1} DB, \tag{6.10d}$$
$$P^{\mathcal{AB}} = \tfrac{1}{2}(\det e)^{-1}\mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}}\mathcal{E}^{\mathcal{B}}{}_{\mathcal{N}} DB^{\mathcal{MN}}, \tag{6.10e}$$
$$P_{ab}{}^{\mathcal{A}} = \tfrac{1}{2} e_a{}^m e_b{}^n \mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}} DB_{mn}{}^{\mathcal{M}}. \tag{6.10f}$$

Here we have introduced two 'vielbeine' $\exp h$ and $\exp \mathcal{H}$, which are group elements of $\mathfrak{gl}(2)$ and $\mathfrak{e}_8$, respectively. We denote the components of these group elements by $e_m{}^a$ and $\mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}$, and their inverses by $e_a{}^m$ and $\mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}}$. Furthermore, we have introduced the 'covariant derivatives'

$$DA_m{}^{\mathcal{M}} = \partial A_m{}^{\mathcal{M}}, \tag{6.11a}$$
$$DB_{mn}{}^{\mathcal{P}} = \partial B_{mn}{}^{\mathcal{P}} + \tfrac{1}{2} f_{\mathcal{MN}}{}^{\mathcal{P}} A_{(m}{}^{\mathcal{M}}\partial A_{n)}{}^{\mathcal{N}}, \tag{6.11b}$$
$$DB = \partial B - \tfrac{1}{4}\varepsilon^{ab}\eta_{\mathcal{MN}} A_m{}^{\mathcal{M}}\partial A_n{}^{\mathcal{N}}, \tag{6.11c}$$
$$DB^{\mathcal{MN}} = \partial B^{\mathcal{MN}} - \tfrac{1}{2}\varepsilon^{mn}\mathbb{P}_{\mathcal{PQ}}{}^{\mathcal{MN}} A_m{}^{\mathcal{P}}\partial A_n{}^{\mathcal{Q}}. \tag{6.11d}$$

Here the $\mathbb{P}_{\mathcal{AB}}{}^{\mathcal{CD}}$ projector projects onto the **3875** representation of E_8. It reads [58]

$$\mathbb{P}_{\mathcal{AB}}{}^{\mathcal{CD}} = \tfrac{1}{7}\delta_{(\mathcal{A}}{}^{\mathcal{C}}\delta_{\mathcal{B})}{}^{\mathcal{D}} - \tfrac{1}{56}\eta_{\mathcal{AB}}\eta^{\mathcal{CD}} - \tfrac{1}{14} f^{\mathcal{E}}{}_{\mathcal{A}}{}^{(\mathcal{C}} f_{\mathcal{EB}}{}^{\mathcal{D})}. \tag{6.12}$$

After computing the coset element $\mathcal{P}(t) = \frac{1}{2}(J + J^T)$, it can be plugged into the action to give

$$\begin{aligned} S &= \frac{1}{4}\int dt\, n(t)^{-1}\langle\mathcal{P}(t)|\mathcal{P}(t)\rangle \\ &= \int dt\,(\mathcal{L}_0 + \mathcal{L}_{12})\,. \end{aligned} \tag{6.13}$$

The 'level zero' and 'higher level' Lagrangians $\mathcal{L}_0$ and $\mathcal{L}_{12}$ can be written as

$$\mathcal{L}_0 = n^{-1}P^A P^A + \tfrac{1}{4}n^{-1}\left(P_a{}^b P_a{}^b - P_a{}^a P_b{}^b\right), \tag{6.14a}$$
$$\mathcal{L}_{12} = \tfrac{1}{2}n^{-1}\big(P_a{}^{\mathcal{A}} P_a{}^{\mathcal{A}} + P_{ab}{}^{\mathcal{A}} P_{ab}{}^{\mathcal{A}} + PP + 14 P^{\mathcal{AB}} P^{\mathcal{AB}}\big), \tag{6.14b}$$

Note that the index A runs here only over the $E_8/SO(16)$ coset, and not over the whole of E_8. The above Lagrangians can also be written out completely as

$$\mathcal{L}_0 = \tfrac{1}{16} n^{-1}\Big(\partial g_{mn}\partial g_{pq}(g^{mp}g^{nq} - g^{mn}g^{pq}) + \tfrac{1}{60}\partial\mathcal{G}^{\mathcal{MN}}\partial\mathcal{G}^{\mathcal{PQ}}\mathcal{G}_{\mathcal{MP}}\mathcal{G}_{\mathcal{NQ}}\Big),$$

$$\mathcal{L}_{12} = \tfrac{1}{8} n^{-1}\Big(g^{mp}g^{nq}\mathcal{G}_{\mathcal{MN}}DB_{mn}{}^{\mathcal{M}}DB_{pq}{}^{\mathcal{N}} + g^{mn}\mathcal{G}_{\mathcal{MN}}DA_m{}^{\mathcal{M}}DA_n{}^{\mathcal{N}}$$
$$+ \big(14\mathcal{G}_{\mathcal{MP}}\mathcal{G}_{\mathcal{NQ}}DB^{\mathcal{MN}}DB^{\mathcal{PQ}} + DBDB\big)(\det g)^{-1}\Big). \qquad (6.15)$$

Here we have introduced the respective $GL(2)$ and E_8 metrics

$$g^{mn} = \delta^{ab}\, e_a{}^m e_b{}^n, \qquad (6.16a)$$
$$\mathcal{G}_{\mathcal{MN}} = \delta_{\mathcal{AB}}\,\mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}}\mathcal{E}^{\mathcal{B}}{}_{\mathcal{N}}. \qquad (6.16b)$$

6.3.3 Gauged supergravity in three dimensions

In this section we review gauged three-dimensional supergravity in a formulation suitable for comparison with the E_{10} analysis of the preceding section.

The bosonic sector of ungauged maximal supergravity in three dimensions contains 128 propagating scalars transforming in the coset $E_8/(Spin(16)/\mathbb{Z}_2)$ and a vielbein $\mathtt{e}_\mu{}^\alpha$ that carries no dynamical degrees of freedom [61]. The scalars can also be described by an (internal) vielbein which we denote by $\mathtt{E}^{\mathcal{M}}{}_{\mathcal{A}}$ (which was denoted $\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}$ in [67]). We will use the 'typewriter' font for supergravity variables in order to distinguish them from the corresponding E_{10} quantities. The inverses will be written as $\mathtt{e}_\alpha{}^\mu$ and $\mathtt{E}^{\mathcal{A}}{}_{\mathcal{M}}$. The curved indices are written as Greek indices $\mu, \nu, \ldots = (t, m)$ and the flat indices are $\alpha, \beta, \ldots = 0, 1, 2$.

The bosonic Lagrangian of three-dimensional maximal gauged supergravity is [68, 67]

$$L = L_0 + L_g, \qquad (6.17)$$

where the 'ungauged' part of the action L_0 and the 'gauged' part L_g read

$$L_0 = +\,\mathtt{e}\left(\tfrac{1}{4}R - \mathtt{P}_\mu{}^A\mathtt{P}^{\mu A}\right), \qquad (6.18a)$$
$$L_g = -\,\mathtt{e}V - \tfrac{1}{4}g\varepsilon^{\mu\nu\rho}\Theta_{\mathcal{MN}}\mathtt{A}_\mu{}^{\mathcal{M}}\partial_\nu\mathtt{A}_\rho^{\mathcal{N}}$$
$$- \tfrac{1}{12}g^2\varepsilon^{\mu\nu\rho}\Theta_{\mathcal{MN}}\Theta_{\mathcal{PQ}}f^{\mathcal{MP}}{}_{\mathcal{R}}\mathtt{A}_\mu{}^{\mathcal{N}}\mathtt{A}_\nu{}^{\mathcal{Q}}\mathtt{A}_\rho{}^{\mathcal{R}}, \qquad (6.18b)$$

with $\mathtt{e} = \det(\mathtt{e}_\mu{}^\alpha)$. Since there is no kinetic term for them, the gauge fields $\mathtt{A}_\mu{}^{\mathcal{M}}$ do not contain propagating degrees of freedom. The gauging also introduces an indefinite scalar potential, which can be written in the form

$$V = \tfrac{1}{32}g^2 G^{\mathcal{MN},\mathcal{KL}}\Theta_{\mathcal{MN}}\Theta_{\mathcal{KL}}, \qquad (6.19)$$

where [4]

$$G^{\mathcal{MN},\mathcal{KL}} = \tfrac{1}{14}\mathtt{G}^{\mathcal{MK}}\mathtt{G}^{\mathcal{NL}} + \mathtt{G}^{\mathcal{MK}}\eta^{\mathcal{NL}} - \tfrac{3}{14}\eta^{\mathcal{MK}}\eta^{\mathcal{NL}} - \tfrac{4}{6727}\eta^{\mathcal{MN}}\eta^{\mathcal{KL}}, \qquad (6.20)$$

with the metric $\mathtt{G}^{\mathcal{MN}}$ defined in (6.16), but here with respect to the supergravity E_8 vielbein $\mathtt{E}^{\mathcal{M}}{}_{\mathcal{A}}$.

We now effectively reduce the three-dimensional gauged supergravity theory to a one-dimensional time-like system. For this we perform the ADM-like split of the vielbein

$$\mathtt{e}_\mu{}^\alpha = \begin{pmatrix} N & 0 \\ 0 & \mathtt{e}_m{}^a \end{pmatrix}, \tag{6.21}$$

in which everything depends only on one coordinate $x^0 = t$ and we have split curved indices as $\mu = (t, m)$ and flat ones as $\alpha = (0, a)$ (with signature $(-++)$). Here we have chosen a gauge with vanishing shift N^m, which turns out to be necessary in order to match the E_{10} coset. As stressed before, gauge fixing is crucial for comparing the E_{10} sigma model to supergravity. The field $\mathtt{e}_m{}^a$ denotes the internal 'spatial' vielbein, i.e. an element of $GL(2,\mathbb{R})/SO(2)$.

For the reduction of the gauge fields we choose a temporal gauge

$$\mathtt{A}_t{}^{\mathcal{M}} = 0. \tag{6.22}$$

The reduced Lagrangian (6.17) then reads

$$L^{D=1} = L_0^{D=1} + L_g^{D=1}, \tag{6.23}$$

with

$$L_0^{D=1} = +\,\mathtt{n}^{-1}\mathtt{P}_t{}^A\mathtt{P}_t{}^A + \tfrac{1}{4}\mathtt{n}^{-1}\left(\mathtt{P}_{ab}\mathtt{P}_{ab} - \mathtt{P}_{aa}\mathtt{P}_{bb}\right), \tag{6.24a}$$

$$L_g^{D=1} = -\,\tfrac{1}{8}g^2\mathtt{e}\mathtt{g}^{mn}(\mathtt{G}^{\mathcal{MN}} + \eta^{\mathcal{MN}})\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}\mathtt{A}_m{}^{\mathcal{K}}\mathtt{A}_n{}^{\mathcal{L}} - \mathtt{e}V + \tfrac{1}{4}g\Theta_{\mathcal{MN}}\varepsilon^{mn}\mathtt{A}_m{}^{\mathcal{M}}\partial_t\mathtt{A}_n{}^{\mathcal{N}}. \tag{6.24b}$$

Here we have defined the quantities

$$\mathtt{n} = N(\det(\mathtt{e}_m{}^a))^{-1}, \tag{6.25a}$$

$$\mathtt{P}_{ab} = -N^{-1}\mathtt{e}_{(a}{}^m\partial_t\mathtt{e}_{m|b)}. \tag{6.25b}$$

Furthermore, we have written the Lagrangian entirely in terms of the E_8 'metric' $\mathtt{G}^{\mathcal{MN}}$. For this we have used the identity

$$\mathtt{E}^{\mathcal{M}}{}_A\mathtt{E}^{\mathcal{N}A} = \tfrac{1}{2}\left(\mathtt{G}^{\mathcal{MN}} + \eta^{\mathcal{MN}}\right), \tag{6.26}$$

which follows from the fact that the Cartan-Killing metric $\eta^{\mathcal{MN}}$ differs from $\mathtt{G}^{\mathcal{MN}}$ by a relative sign in the non-compact part.

6.3.4 The correspondence

It is clear that the E_{10} 'level zero' Lagrangian (6.14a) has exactly the same form as its supergravity counterpart (6.24a). We therefore focus on equating the E_{10} 'higher level' Lagrangian $\mathcal{L}_{12}$ (6.15) and the supergravity 'gauged' Lagrangian L_g (6.18b), or rather their equations of motion.

The first step is to eliminate the mixed-symmetry field $B_{mn}{}^{\mathcal{M}}$ on the Kac-Moody side, as it has no obvious counterpart in supergravity. Similar mixed-symmetrical objects have been found to be in a one-to-one correspondence with trombone gaugings [59], but their space-time mixed symmetry lacks a solid understanding. We therefore consistently truncate it by setting its covariant derivate equal to zero,

$$DB_{mn}{}^{\mathcal{M}} = 0. \tag{6.27}$$

If we vary the action (6.13) with respect to the remaining level two fields, we obtain

$$0 = \partial(n^{-1}(\det g)^{-1}DB), \tag{6.28a}$$

$$0 = \partial(n^{-1}(\det g)^{-1}\mathcal{G}_{\mathcal{M}\mathcal{P}}\mathcal{G}_{\mathcal{N}\mathcal{Q}}DB^{\mathcal{P}\mathcal{Q}}). \tag{6.28b}$$

These two equations can be be integrated and identified with the **1** and **3875** components of the embedding tensor:

$$c_1 g\theta = n^{-1}(\det g)^{-1}DB, \tag{6.29a}$$

$$c_2 g\tilde{\Theta}_{\mathcal{M}\mathcal{N}} = n^{-1}(\det g)^{-1}\mathcal{G}_{\mathcal{M}\mathcal{P}}\mathcal{G}_{\mathcal{N}\mathcal{Q}}DB^{\mathcal{P}\mathcal{Q}}, \tag{6.29b}$$

where c_1 and c_1 are two arbitrary constants. The above equations can then be subsequently used to write the equation of motion of $A_m{}^{\mathcal{M}}$ from (6.13) as

$$\partial\Big(n^{-1}g^{mn}\mathcal{G}_{\mathcal{M}\mathcal{N}}\partial A_n{}^{\mathcal{N}} + g\varepsilon^{mn}\big(\eta_{\mathcal{M}\mathcal{N}}\theta + \tilde{\Theta}_{\mathcal{M}\mathcal{N}}\big)A_n{}^{\mathcal{N}}\Big) = 0. \tag{6.30}$$

Here we have already chosen the integration constants to be $c_1 = 2$ and $c_2 = \frac{1}{14}$. This allows us to combine the **1** and **3875** components of the embedding tensor into Θ, and integrate the above equation to

$$n^{-1}g^{mn}\mathcal{G}_{\mathcal{M}\mathcal{N}}\partial A_n{}^{\mathcal{N}} = g\varepsilon^{mn}\Theta_{\mathcal{M}\mathcal{N}}A_n{}^{\mathcal{N}} + \Xi^m{}_{\mathcal{M}}. \tag{6.31}$$

$\Xi^m{}_{\mathcal{M}}$ denotes an integration constant. This integration constant cannot be set to zero without breaking the symmetries. The situation is analogous to the integration leading to the embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ in (6.29), which generically breaks the global E_8 symmetry once $\Theta_{\mathcal{M}\mathcal{N}}$ is constant. Correspondingly, the E_{10} shift symmetry leaves this first-order equation only invariant if the integration constant also transforms as a shift,

$$\delta_\Lambda \Xi^m{}_{\mathcal{M}} = -g\varepsilon^{mn}\Theta_{\mathcal{M}\mathcal{N}}\Lambda_n{}^{\mathcal{N}}, \tag{6.32}$$

which is consistent with the time-independence of Ξ. Thus, fixing it to any specific value (as zero) breaks the symmetry, and in this sense supergravity may at best be viewed as a broken phase of E_{10}. After setting $\Xi = 0$ and contracting with $\Theta_{\mathcal{MN}}$, (6.31) implies

$$g\Theta_{\mathcal{MN}}\varepsilon^{mn}\partial A_n{}^{\mathcal{N}} + g^2 eN\mathcal{G}^{\mathcal{KL}}g^{mn}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}A_n{}^{\mathcal{N}} = 0. \tag{6.33}$$

This matches the duality relation obtained from the supergravity Lagrangian (6.23) by varying with respect to $\mathtt{A}_m{}^{\mathcal{M}}$, which reads

$$g\Theta_{\mathcal{MN}}\varepsilon^{mn}\partial\mathtt{A}_n{}^{\mathcal{N}} + \tfrac{1}{2}g^2\mathtt{e}(\mathtt{G}^{\mathcal{KL}} + \eta^{\mathcal{KL}})\mathtt{g}^{mn}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}\mathtt{A}_n{}^{\mathcal{N}} = 0. \tag{6.34}$$

Finally, we compare the 'Einstein equations' and the equations of motions for the scalars on both side. For supergravity, they read

$$\begin{aligned}0 = \frac{\delta L_0}{\delta \mathtt{g}^{mn}} &+ \tfrac{1}{2}\mathtt{e}\mathtt{g}_{mn}V \\ &+ \tfrac{1}{16}g^2\mathtt{e}(\mathtt{G}^{\mathcal{MN}} + \eta^{\mathcal{MN}})\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}\left(\mathtt{g}_{mn}\mathtt{g}^{kl}\mathtt{A}_k{}^{\mathcal{K}}\mathtt{A}_l{}^{\mathcal{L}} - 2\mathtt{A}_m{}^{\mathcal{K}}\mathtt{A}_n{}^{\mathcal{L}}\right),\end{aligned} \tag{6.35a}$$

$$\begin{aligned}0 = \frac{\delta L_0}{\delta \mathtt{G}^{\mathcal{MN}}} &- \tfrac{1}{8}g^2\mathtt{e}\mathtt{g}^{mn}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}\mathtt{A}_m{}^{\mathcal{K}}\mathtt{A}_n{}^{\mathcal{L}} \\ &- \tfrac{1}{7\cdot 32}\mathtt{e}g^2\mathtt{G}^{\mathcal{KL}}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}} - \tfrac{1}{16}\mathtt{e}g^2\eta^{\mathcal{KL}}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}},\end{aligned} \tag{6.35b}$$

whereas the equations of motion that follow from the E_{10} coset model are given by

$$\begin{aligned}0 = \frac{\delta \mathcal{L}_0}{\delta g^{mn}} &+ \tfrac{1}{8}g^2\mathtt{e}G^{\mathcal{MN}}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}\left(g_{mn}g^{kl}A_k{}^{\mathcal{K}}A_l{}^{\mathcal{L}} - A_m{}^{\mathcal{K}}A_n{}^{\mathcal{L}}\right) \\ &+ \tfrac{1}{2}g^2\mathtt{e}g_{mn}\left(\tfrac{1}{56}\mathcal{G}^{\mathcal{MK}}\mathcal{G}^{\mathcal{NL}}\tilde{\Theta}_{\mathcal{MN}}\tilde{\Theta}_{\mathcal{KL}} + \theta^2\right),\end{aligned} \tag{6.36a}$$

$$0 = \frac{\delta \mathcal{L}_0}{\delta \mathcal{G}^{\mathcal{MN}}} - \tfrac{1}{8}g^2\mathtt{e}g^{mn}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}A_m{}^{\mathcal{K}}A_n{}^{\mathcal{L}} - \tfrac{1}{56}g^2\mathtt{e}\mathcal{G}^{\mathcal{KL}}\tilde{\Theta}_{\mathcal{MK}}\tilde{\Theta}_{\mathcal{NL}}. \tag{6.36b}$$

By comparing (6.35) with (6.36) we observe that the equations are structure-wise the same, but differ in the details. For one thing, on the E_{10} side we generically have just $\mathcal{G}^{\mathcal{MN}}$ instead of $\frac{1}{2}(\mathcal{G}^{\mathcal{MN}} + \eta^{\mathcal{MN}})$. Apart from that, the indefinite contributions to the supergravity potential are not reproduced, but only the leading term quadratic in $\mathcal{G}^{\mathcal{MN}}$.

Summarizing, we find that the gauging appears exclusively as a consequence of 'switching on' certain higher level degrees of freedom in the level expansion of the Cartan form and the coset equations of motion. The embedding tensor appears naturally in the coset model by integrating the one-dimensional equations of motion. The same holds for duality relation between the scalars and vectors, and the scalar potential. However, the latter is not fully reproduced by E_{10}, but only the positive definite contributions. This is due to the fact that in supergravity the scalar potential is indefinite [90], while the corresponding 2-forms appearing in the E_{10} coset model necessarily enter with a positive definite kinetic term.