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## Kac-Moody Symmetries and Gauged Supergravity

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*Document Version*

Publisher's PDF, also known as Version of record

*Publication date:*

2010

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*

Nutma, T. A. (2010). *Kac-Moody Symmetries and Gauged Supergravity*. s.n.

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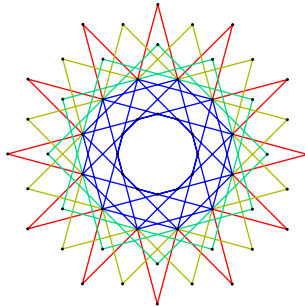
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# 5



## Supergravity

In the previous chapters we have described the generic theory and structure of Kac-Moody algebras. In this chapter we will treat a class of theories known as *supergravities* in which these infinite-dimensional symmetries appear. For an introduction to supergravity, see for example the lecture notes [89] and [31].

Supergravities are constructed by requiring that supersymmetry is a local symmetry of the theory. Very roughly, supersymmetry transforms bosons into fermions and vice-versa:

$$\delta(\text{boson}) = \text{fermion}, \quad (5.1a)$$

$$\delta(\text{fermion}) = \text{translated boson}. \quad (5.1b)$$

Because supersymmetry is required to hold locally, spacetime translations must also be a local symmetry of the theory. We therefore have diffeomorphism invariance, and expect the theory to contain gravity. Hence the name supergravity, a term which was coined even before the first theories were constructed [32].

The underlying symmetry of supergravity is captured in its superalgebra. A *superalgebra*  $\mathfrak{g}$  is a  $\mathbb{Z}_2$ -graded generalization of a Lie algebra. It splits up in two subalgebras,  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ , where  $\mathfrak{g}_0$  is the even (bosonic) subalgebra and  $\mathfrak{g}_1$  the odd

(fermionic) subalgebra. The Lie bracket of superalgebras is generalized from (2.1), and reads

$$[x, y] = -[y, x] \subseteq \mathfrak{g}_0 \quad \forall x, y \in \mathfrak{g}_0, \quad (5.2a)$$

$$\{x, y\} = +\{y, x\} \subseteq \mathfrak{g}_0 \quad \forall x, y \in \mathfrak{g}_1, \quad (5.2b)$$

$$[x, y] = -\{y, x\} \subseteq \mathfrak{g}_1 \quad \forall x \in \mathfrak{g}_0, y \in \mathfrak{g}_1. \quad (5.2c)$$

The bosonic subalgebra  $\mathfrak{g}_0$  thus constitutes an ordinary Lie algebra. Furthermore, by equation (5.2c) the fermionic subalgebra  $\mathfrak{g}_1$  carries a representation of  $\mathfrak{g}_0$ .

The superalgebra of supergravities is an extension of the usual Poincaré algebra. Its bosonic subalgebra thus always contains Lorentz generators  $M_{\mu\nu}$  and the generator of translations  $P_\mu$ . The fermionic subalgebra contains a number of supersymmetry generators  $Q_\alpha^i$ , which come in spinor representations of the Lorentz algebra. The index  $i$  runs from 1 to  $\mathcal{N}$ , the number of supersymmetry generators. The number of *supercharges*,  $Q$ , is then defined as the total number of components of all supersymmetry generators,

$$Q = \mathcal{N}q \leq 32. \quad (5.3)$$

Here  $q$  is the number of components (i.e. the dimension) of a single irreducible spinor representation. If  $Q > 32$ , acting with supersymmetry generators on the graviton (which has spin 2) will give rise to fields with spin greater than two [65]. It is thought that these fields cannot consistently couple to themselves or other fields, and hence theories with  $Q > 32$  are discarded. Supergravities that satisfy the bound in (5.3) are called *maximal* supergravities.

As is clear from Table 5.1,  $D = 11$  is the maximum number of space-time dimensions for a supergravity. We will take this theory as a starting point to see how Kac-Moody symmetries emerge in it.

## 5.1 Maximal supergravity

Supergravity in eleven dimensions consists of the metric  $g_{\mu\nu}$ , a rank three antisymmetric gauge field  $A_{\mu\nu\rho}$  (both bosons), and the gravitino  $\psi_\mu$  (a fermion) [21]. The latter carries 128 degrees of freedom, the same as the metric and 3-form combined (44 and 84 d.o.f., respectively). In form notation, the bosonic part of the action reads

$$S = \int \star R - \frac{1}{2} \star dA_3 \wedge dA_3 - \frac{1}{6} dA_3 \wedge dA_3 \wedge A_3, \quad (5.4)$$

where  $A_3$  is the usual form-shorthand for  $\frac{1}{3!} A_{\mu\nu\rho} dx^\mu dx^\nu dx^\rho$ . This action carries no global symmetry. When it is reduced on an  $n$ -torus, one expects the resulting lower-dimensional supergravity theory to have at least a global  $GL(n)$  symmetry,

$D$	$q$	$\mathcal{N}_{\max}$
2	1	32
3	2	16
4	4	8
5	8	4
6	8	4
7	16	2
8	16	2
9	16	2
10	16	2
11	32	1
12	64	0

**Table 5.1:** The number of components of an irreducible spinor ( $q$ ) in Lorentzian space-times of dimension  $D$ , and the corresponding maximum number of supersymmetry generators ( $\mathcal{N}_{\max}$ ). Modified from [31].

which is the symmetry group of the torus. In fact, it turns out that the actual global symmetry is enhanced to the exceptional  $E_n$  series [19, 20]. The corresponding groups are given in Table 5.2, and their Coxeter projections in Figure 5.1.

The 128 degrees of freedom of the  $D = 11$  theory get upon reduction redistributed over the metric, a set of scalars, vectors, and higher-rank  $p$ -forms (see also Table 5.3). The generic form of the lower-dimensional Lagrangian is

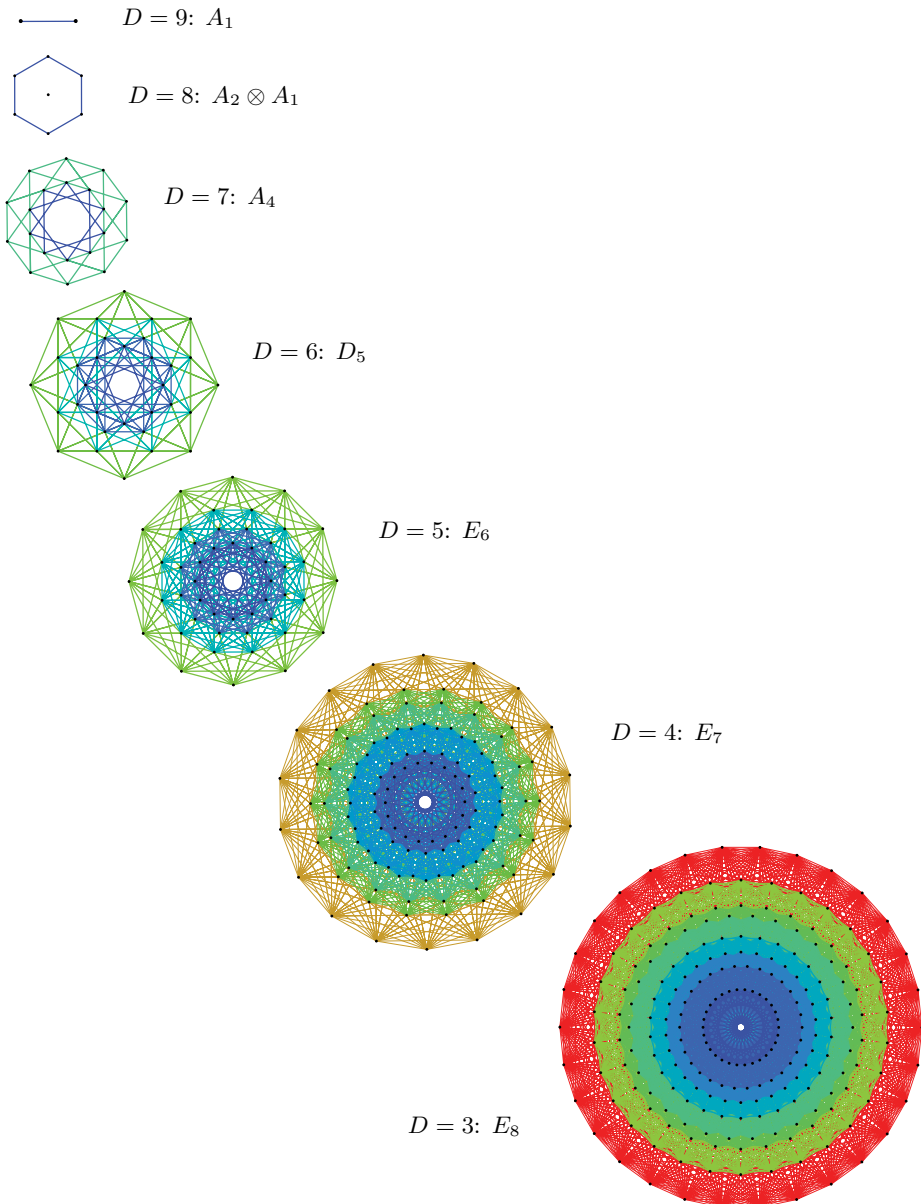
$$S = \int \star R - \frac{1}{2} G_{\alpha\beta}(\phi) \star d\phi^\alpha \wedge d\phi^\beta - \frac{1}{2} M_{MN}(\phi) \star dA_1^M \wedge dA_1^N - \dots, \quad (5.5)$$

with additional contributions of higher-rank  $p$ -forms. The scalars  $\phi^\alpha$  are described by a sigma model on the coset space  $G/K(G)$ , and thus transform non-linearly under the global symmetry group  $G$ . The  $p$ -forms carry a linear representation of  $G$ . For the vector fields this is always the fundamental representation of  $G$ , whereas the higher-rank  $p$ -forms transform in representations that are formed by taking appropriate tensor products of the fundamental representation.











The full  $G$  symmetry is only manifest when all the  $p$ -forms have been dualized by means of Hodge duality to their lowest possible rank. Standard Hodge duality relates  $p$ -forms to  $(D - p - 2)$ -forms through their field strengths. For abelian field strengths, it reads for example

$$\star dA_p = dA_{D-p-2}. \quad (5.6)$$

So after dualizing all  $p$ -forms to their lower-rank duals, the action contains  $p$ -forms whose rank is at most  $\lfloor \frac{D-2}{2} \rfloor$ , where the brackets denote the integral part. In even



**Figure 5.1:** Coxeter projections of the hidden duality groups of maximal supergravity in various dimensions.

$D$	$G$	$K(G)$	$\dim(G, K, G/K)$	diagram
11	1	1	0 0 0	
10	$\mathbb{R}^+$	1	1 0 1	
9	$GL(2)$	$SO(2)$	3 1 2	
8	$SL(3) \times SL(2)$	$SO(3) \times SO(2)$	11 4 7	
7	$SL(5)$	$SO(5)$	24 10 14	
6	$SO(5, 5)$	$SO(5) \times SO(5)$	45 20 25	
5	$E_6$	$USp(8)$	78 36 42	
4	$E_7$	$SU(8)$	133 63 70	
3	$E_8$	$SO(16)$	248 120 128	
2	$E_9$	$K(E_9)$	$\infty$ $\infty$ $\infty$	
1	$E_{10}$ ?	$K(E_{10})$	$\infty$ $\infty$ $\infty$	
	$E_{11}$ ?	$K(E_{11})$	$\infty$ $\infty$ $\infty$	

**Table 5.2:** The global symmetry group  $G$  and its maximal compact subgroup  $K(G)$  of maximal supergravity in  $D$  dimensions, as obtained from dimensional reduction. The group  $G$  is always over the real numbers and a split real form. The Dynkin diagram for a particular dimension contains the node(s) on the same horizontal axis, and all nodes above. Note that the  $D = 10$  listed here is the IIA supergravity; in  $D = 10$  there is also the IIB supergravity theory that has a global  $SL(2)$  symmetry. The IIB theory cannot be obtained from  $D = 11$ .

dimensions there can be a subtlety involving self-dual  $p$ -forms for which a Hodge dualization does not change the rank. In those cases the full  $G$  symmetry cannot be realized on the level of the action, but only on the equations of motion.

It is worth noting that there exists a formulation of supergravity in which all the  $p$ -forms appear with their duals [9]. This is known as the *democratic formulation* of supergravity. The highest rank forms in that case are the duals of scalars, which are  $(D - 2)$ -forms.

Three is the lowest number of space-time dimensions for which the global symmetry group is still finite. In that case it is  $E_8$  [48, 61], the largest finite exceptional Lie group, and the dimension of the coset space  $E_8/SO(16)$  is exactly 128. This matches the fact that all the degrees of freedom are contained in the scalars, as the metric carries no propagating degrees of freedom in three dimensions.

In two dimensions the global symmetry group is enlarged to the affine extension of the three-dimensional group. This was first noted for plain general relativity [38, 37], but it also holds for supergravity. For maximal supergravity the two-dimensional

$D$	$g_{\mu\nu}$	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$
11	$44 \times \mathbf{1}$				$84 \times \mathbf{1}$	
IIA	$35 \times \mathbf{1}$	$1 \times \mathbf{1}$	$8 \times \mathbf{1}$	$28 \times \mathbf{1}$	$56 \times \mathbf{1}$	
IIB	$35 \times \mathbf{1}$	$1 \times \mathbf{2}$		$28 \times \mathbf{2}$		$70 \times \frac{1}{2} \mathbf{1}$
9	$27 \times \mathbf{1}$	$1 \times \mathbf{3}$	$7 \times (\mathbf{1} + \mathbf{2})$	$21 \times \mathbf{2}$	$35 \times \mathbf{1}$	
8	$20 \times (\mathbf{1}, \mathbf{1})$	$1 \times \mathbf{7}$	$6 \times (\mathbf{3}, \mathbf{2})$	$15 \times (\mathbf{3}, \mathbf{1})$	$20 \times \frac{1}{2} (\mathbf{1}, \mathbf{2})$	
7	$14 \times \mathbf{1}$	$1 \times \mathbf{14}$	$5 \times \mathbf{10}$	$10 \times \mathbf{5}$		
6	$9 \times \mathbf{1}$	$1 \times \mathbf{25}$	$4 \times \mathbf{16}$	$6 \times \frac{1}{2} \mathbf{10}$		
5	$5 \times \mathbf{1}$	$1 \times \mathbf{42}$	$3 \times \mathbf{27}$			
4	$2 \times \mathbf{1}$	$1 \times \mathbf{70}$	$2 \times \frac{1}{2} \mathbf{56}$			
3	$- \times \mathbf{1}$	$1 \times \mathbf{128}$				

**Table 5.3:** The physical states of all  $3 \leq D \leq 11$  maximal supergravities. The  $p$ -columns indicate which  $p$ -form potentials are present. All entries except  $p = 0$  are of the form ‘physical d.o.f.  $\times G$  representation,’ where  $G$  is the duality group. For  $p = 0$  the entries read ‘physical d.o.f.  $\times$  number of scalars.’ The factor  $\frac{1}{2}$  in  $D = 4, 6, 8$  ensures the correct counting of self-dual  $p$ -forms.

symmetry becomes  $E_9 = E_8^+$  [49, 66, 69], the affine extension of  $E_8$ .

Seeing as the rank of the group  $G$  increases with every step in the reduction, it is tempting to conjecture that this also happens for the reduction to one space-time dimension. The global symmetry group would then be  $G^{++}$ , the over-extension of the  $D = 3$  group. In the case of maximal supergravity, this would be  $E_{10} = E_8^{++}$  [49, 63]. Taking the conjecture one step further, one can formally reduce to zero dimensions and hope to obtain the very-extension  $E_{11} = E_8^{+++}$  [88, 87], which is conjectured to be a symmetry of M-theory.

We will see later in chapter 6 how pieces of the conjectured  $E_{10}$  and  $E_{11}$  Kac-Moody symmetries nicely tie in with the structure of supergravity. Surprisingly, it turns out the Kac-Moody algebras encode information on possible gauge deformations of the supergravity theory. This is remarkable, as the ungauged theory was the starting point for the conjecture. We will discuss gaugings of supergravity in section 5.3, but first we will briefly touch upon half-maximal supergravity.

## 5.2 Half-maximal supergravity

Supergravities that have 16 superchargers are called *half-maximal* supergravities. From Table 5.1 one easily deduces that the maximum number of spacetime dimensions is 10 for a half-maximal supergravity. In contrast to maximal supergravity, (ungauged) half-maximal supergravity is not unique in its highest dimension, nor in lower dimensions. Besides the usual graviton multiplet, one can add an arbitrary

$D$	Mult.	$p = 0$	$p = 1$	$p = 2$
10	$GV^n$	$\mathbf{1} \times \mathbf{1}$	$\mathbf{8} \times \mathbf{n}$	$\mathbf{28} \times \mathbf{1}$
9	$GV^{n+1}$	$\mathbf{1} \times \left(1 + (1n + 1)\right)$	$\mathbf{7} \times (\mathbf{n} + \mathbf{2})$	$\mathbf{21} \times \mathbf{1}$
8	$GV^{n+2}$	$\mathbf{1} \times \left(1 + (2n + 4)\right)$	$\mathbf{6} \times (\mathbf{n} + \mathbf{4})$	$\mathbf{15} \times \mathbf{1}$
7	$GV^{n+3}$	$\mathbf{1} \times \left(1 + (3n + 9)\right)$	$\mathbf{5} \times (\mathbf{n} + \mathbf{6})$	$\mathbf{10} \times \mathbf{1}$
6a	$GV^{n+4}$	$\mathbf{1} \times \left(1 + (4n + 16)\right)$	$\mathbf{4} \times (\mathbf{n} + \mathbf{8})$	$\mathbf{6} \times \mathbf{1}$
6b	$GT^{n+4}$	$\mathbf{1} \times (5n + 25)$		$\mathbf{6} \times \frac{1}{2}(\mathbf{10} + \mathbf{n})$
5	$GV^{n+5}$	$\mathbf{1} \times \left(1 + (5n + 25)\right)$	$\mathbf{3} \times (\mathbf{n} + \mathbf{11})$	
4	$GV^{n+6}$	$\mathbf{1} \times \left((1, 2) + (6n + 36, 1)\right)$	$\mathbf{2} \times (\mathbf{n} + \mathbf{12})$	
3	$GV^{n+7}$	$\mathbf{1} \times (8n + 64)$		

**Table 5.4:** The physical states apart from the graviton of all  $D = 10 - m$  half-maximal supergravities coupled to  $m + n$  vector multiplets. The multiplet structures are also given:  $G$  is the graviton multiplet,  $V$  the vector multiplet and  $T$  the self-dual tensor multiplet. The physical states of the graviton can be found in Table 5.3.

number of vector multiplets to the theory. We will describe both these multiplets in turn.

The bosonic part of the graviton multiplet of  $D$ -dimensional half-maximal supergravity consists of a metric,  $m$  vector gauge fields (with  $m = 10 - D$ ), a two-form gauge field and a single scalar which is the dilaton. It has a global  $SO(m)$  symmetry, under which the vectors transform in the fundamental representation.

The other possible multiplet in generic dimensions is the vector multiplet, which contains a vector and  $m$  scalars. The effect of adding  $m + n$  vector multiplets is to enlarge the symmetry group from  $SO(m)$  to  $SO(m, m + n)$ . The scalars parameterize the corresponding scalar coset while the vectors transform in the fundamental representation. In four dimensions the symmetry becomes  $SL(2, \mathbb{R}) \times SO(6, 6 + n)$  while in three dimensions it is given by  $SO(8, 8 + n)$ . In the latter case there is symmetry enhancement due to the equivalence between scalars and vectors. The entire theory can be described in terms of the corresponding scalar coset (coupled to gravity).

The above multiplets belong to non-chiral half-maximal supergravity and are the correct and complete story in generic dimensions. In six dimensions, however, the half-maximal theory can be chiral or non-chiral, similar to the maximal theory in ten dimensions. The non-chiral theory is denoted by  $D = 6a$  and follows the above pattern. The chiral theory,  $D = 6b$ , instead has different multiplets. In particular, the graviton multiplet contains the graviton, five scalars and five self-dual plus one



anti-self-dual two-form gauge fields. The global symmetry is given by  $SO(5, 1)$ . The other possible multiplet is that of the tensor, which contains an anti-self-dual two-form and five scalars. Adding  $4 + n$  of such tensor multiplets to the graviton multiplet enhances the symmetry to  $SO(5, 5 + n)$ .

Upon dimensional reduction over a circle, the graviton multiplet splits up into a graviton multiplet plus a vector multiplet. A vector (or tensor) multiplet reduces to a vector multiplet in the lower dimensions. This was the reason for adding  $m + n$  instead of  $n$  vector or tensor multiplets in any dimension; it can easily be seen that  $n$  remains invariant under dimensional reduction. That is, a theory with a certain value of  $n$  reduces to a theory with the same value of  $n$  in lower dimensions.

The bosonic physical states of the various half-maximal supergravities have been listed in Table 5.4.

## 5.3 Gaugings

In this section we will quickly review gauge deformations of supergravity theories. For a more in-depth review, see [80]. Besides the global symmetry  $G$  described in the previous section, ungauged supergravities have a local symmetry  $U(1)^n$ . Here  $n$  is the number of vector fields of the supergravity. This local symmetry corresponds to the abelian gauge symmetry of the vector fields:

$$\delta A_1^M = d\Lambda_0^M, \quad (5.7)$$

where  $\Lambda_0^M = \Lambda_0^M(x)$  is a coordinate-dependent 0-form parameter.

A *gauging* or *gauge deformation* of the supergravity turns this abelian local symmetry into a non-abelian local symmetry. Another point of view is that it promotes a subgroup  $G_0 \in G$  to a local symmetry. This is achieved by introducing covariant derivatives

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu - gA_\mu^M X_M, \quad (5.8)$$

where  $g$  is the gauge coupling constant, and  $X_M$  are the generators of the subgroup  $G_0$ . All the possible choices of a consistent gauge group  $G_0$  are best described in an object that is known as the embedding tensor [68, 67, 94, 95, 91].

### 5.3.1 The embedding tensor

Because the gauge group  $G_0$  is embedded within  $G$ , its generators are a linear combination of the generators  $t_\alpha$  of  $G$ :

$$X_M \equiv \Theta_M^\alpha t_\alpha. \quad (5.9)$$

The constant object  $\Theta_M^\alpha$  is called the *embedding tensor*, and encodes the gauging completely.

$D$	Linear constraint	Quadratic constraint
7	<b>15</b> $\oplus$ <b>40</b>	<b>5</b> $\oplus$ <b>45</b> $\oplus$ <b>70</b>
6	<b>144</b>	<b>10</b> $\oplus$ <b>126</b> $\oplus$ <b>320</b>
5	<b>351</b>	<b>27</b> $\oplus$ <b>1728</b>
4	<b>912</b>	<b>133</b> $\oplus$ <b>8645</b>
3	<b>1</b> $\oplus$ <b>3875</b>	<b>3875</b> $\oplus$ <b>147250</b>

**Table 5.5:** The linear and quadratic constraint representations in various dimensions ( $D$ ). Adapted from [90].

The embedding tensor is a tensor with two indices that both take values in representations of  $G$ . The first index,  $M$ , takes values in the fundamental representation  $\mathcal{R}_{\text{fund}}$ , and the second,  $\alpha$ , in the adjoint representation  $\mathcal{R}_{\text{adj}}$ . The embedding tensor therefore ‘lives’ in the tensor product of these two representations,

$$\Theta \in \mathcal{R}_{\text{fund}} \otimes \mathcal{R}_{\text{adj}}. \quad (5.10)$$

If we take  $D = 4$  maximal supergravity as a concrete example, the embedding tensor a priori can take values in

$$\Theta \in \mathbf{56} \otimes \mathbf{133} = \mathbf{56} \oplus \mathbf{912} \oplus \mathbf{6480}. \quad (5.11)$$

However, there are two sets of constraints that restrict the possible values of the embedding tensor. The first comes from demanding that the gauging is consistent with supersymmetry, although in almost all cases it can also be derived from purely bosonic arguments [93]. Either way, it poses a linear restriction on  $\Theta$ , and is therefore called the *linear constraint*. Say  $\mathbb{P}_{\text{lin}}$  is an operator that projects onto the forbidden components of  $\Theta$  (i.e. those components that are inconsistent with supersymmetry). Then the linear constraint can be written as

$$\mathbb{P}_{\text{lin}} \Theta_M^\alpha = 0. \quad (5.12)$$

The linear constraint is usually indicated by the allowed representations of the embedding tensor. By the above equation, they live in the kernel of  $\mathbb{P}_{\text{lin}}$ . For  $D = 4$  maximal supergravity, the linear constraint leaves only the **912** representation:

$$\Theta \in \mathbf{56} \otimes \mathbf{133} = \mathbf{56} \oplus \mathbf{912} \oplus \mathbf{6480}. \quad (5.13)$$

The linear constraint has to be worked out on a case-by-case basis, and some results are collected in Table 5.5.

The second constraint restricting the possible values of the embedding tensor even further comes from demanding that the gauge algebra is self-consistent. In

particular  $\Theta$  must be gauge invariant, or equivalently, the generators  $X_M$  must close into an algebra,

$$[X_M, X_N] = -\Theta_M^\alpha (t_\alpha)_N^P X_P. \quad (5.14)$$

This poses restrictions on the symmetric tensor product of  $\Theta$ , and is therefore called the *quadratic constraint*. Again, say  $\mathbb{P}_{\text{quad}}$  is an operator that projects onto the components of  $\Theta_M^\alpha \Theta_N^\beta$  that are inconsistent with the quadratic constraint. The constraint itself can then be written as

$$\mathbb{P}_{\text{quad}} \left( \Theta_M^\alpha \Theta_N^\beta \right) = 0. \quad (5.15)$$

Therefore the allowed components of the symmetric tensor product of  $\Theta$  live in the kernel of  $\mathbb{P}_{\text{quad}}$ . In contrast to the linear constraint, the quadratic constraint is usually labeled by the representations in the complement of the kernel, i.e. those components that are *not* allowed. For  $D = 4$  maximal supergravity, the quadratic constraint evaluates with the help of LiE [60] to

$$\Theta \otimes_s \Theta \in \mathbf{912} \otimes_s \mathbf{912} = \mathbf{133} \oplus \mathbf{1463} \oplus \mathbf{8645} \oplus \mathbf{152152} \oplus \mathbf{253935}, \quad (5.16)$$

where the **133** and **8645** representations constitute the components of the symmetric tensor product that are not allowed. The analysis of the components of the quadratic constraints is a purely group-theoretical problem, and some results are listed in Table 5.5.

Note that the embedding tensor formalism relies on the presence of vectors fields. In  $D = 3$  they are absent, as the only physical fields are scalars. In order to proceed, one introduces vectors fields that are not independent but are related to the scalars by a duality relation [68].

### 5.3.2 The tensor hierarchy

The implementation of the linear and quadratic constraints are necessary for a consistently gauged supergravity. However, they are not sufficient. The introduction of the embedding tensor also forces one to introduce a whole tower of anti-symmetric  $p$ -forms that transform in specific representations of the global symmetry group  $G$ . This tower is called the  *$p$ -form hierarchy* [90, 92], and we will briefly describe it below.

Besides the gauge vectors  $A_1^M$  transforming the fundamental representation of  $G$ , the embedding tensor also requires the existence of higher-rank  $p$ -forms. The whole set of  $p$ -forms can be denoted by

$$A_1^M, A_2^{[MN]}, A_3^{[M[NP]]}, A_4^{[M[N[PQ]]]}, A_5^{[M[N[P[QR]]]]}, \dots \quad (5.17)$$

The tower truncates at  $D$ -forms, which are the anti-symmetric forms whose rank is the highest possible in  $D$  dimensions. The  $G$  representations in which the  $p$ -forms

live are tensor products of the fundamental representation  $\mathcal{R}_{\text{fund}}$ . For instance, the 2-form carries indices  $MN$  that reside in the first symmetrized tensor product of the fundamental representation:

$$MN : \mathcal{R}_{\text{fund}} \otimes \mathcal{R}_{\text{fund}}. \quad (5.18)$$

The extra brackets  $[MN]$  indicate that the two-form does not take values in the whole tensor product. Rather, it lives in a restricted subspace defined by some projector  $\mathbb{P}_2$  whose precise form depends on the embedding tensor,

$$[MN] : \mathbb{P}_2 (\mathcal{R}_{\text{fund}} \otimes \mathcal{R}_{\text{fund}}) \equiv \mathcal{R}_{\text{fund}}^{\otimes 2}. \quad (5.19)$$

Here we have defined the representation  $\mathcal{R}_{\text{fund}}^{\otimes p}$  to be the representation of the  $p$ -form in question. The same holds for the higher-rank  $p$ -forms: their  $G$  representation can be evaluated by taking the tensor product of the fundamental representation with the  $(p-1)$ -representation, and projecting with the correct operator:

$$\mathcal{R}_{\text{fund}}^{\otimes p} = \mathbb{P}_p \left( \mathcal{R}_{\text{fund}}^{\otimes (p-1)} \otimes \mathcal{R}_{\text{fund}} \right). \quad (5.20)$$

It turns out that if the various  $p$ -forms in the hierarchy are related by Hodge-duality, then their  $G$  representations are each other's conjugate. Thus the  $(D-2)$ -forms transform in the conjugate adjoint representation, the  $(D-3)$ -forms in the conjugate fundamental representation, and so on and so forth. Furthermore, for these forms one finds exactly the same representation as one would find in a supersymmetry analysis (i.e. Table 5.3 plus their Hodge duals).

The  $(D-1)$ - and  $D$ -forms are special in the sense that they do not have a Hodge dual. Furthermore, they do not carry any propagating degrees of freedom. For those two  $p$ -forms at the end of the hierarchy one finds that they transform in the representation conjugate to the embedding tensor and conjugate to its quadratic constraint, respectively. This suggest they may be employed as Lagrange multipliers, enforcing the constancy of  $\Theta$  and the quadratic constraint [90, 96]:

$$\mathcal{L}_{\text{constraints}} = g A_{(D-1)\alpha}^M \wedge Q_M^\alpha + g^2 A_D^{MN}{}_{\alpha\beta} \wedge Q_{MN}{}^{\alpha\beta}, \quad (5.21)$$

where

$$Q_M^\alpha = D_\mu \Theta_M^\alpha dx^\mu, \quad (5.22a)$$

$$Q_{MN}{}^{\alpha\beta} = \mathbb{P}_{\text{quad}} \left( \Theta_M^\alpha \Theta_N^\beta \right). \quad (5.22b)$$

Upon adding  $\mathcal{L}_{\text{constraints}}$  to the action and varying with respect to  $A_{(D-1)\alpha}^M$ , we find that  $\Theta$  should be a constant tensor. The linear constraint (5.12) is enforced automatically by only letting those components of  $\Theta$  enter the action that satisfy

it. Varying with respect to  $A_D^{MN}{}_{\alpha\beta}$  tells us that  $\Theta$  should also satisfy the quadratic constraint (5.15).

Because the  $(D - 1)$ -forms couple to the embedding tensor, we will call them *deformation potentials*. And, as the  $D$ -forms have the highest rank possible, we will call them *top-form potentials* [1].

The remarkable property of the Kac-Moody algebra  $E_{11}$ , and to a lesser extent  $E_{10}$ , is that it encodes all the representations of all the  $p$ -forms in a unified manner. How this comes about exactly, we will see in the next chapter.