Kac-Moody Symmetries and Gauged Supergravity

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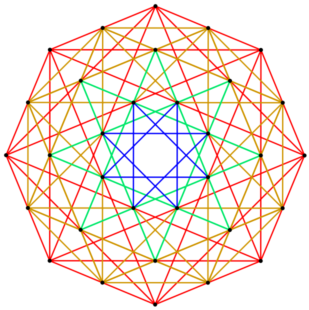


Kac-Moody algebras

A Kac-Moody algebra is a Lie algebra whose Cartan matrix A is generalized. That is, A is not necessarily positive definite (see equation (2.13)). The algebra is finite-dimensional when A is positive definite, and infinite-dimensional otherwise. This chapter will treat special kinds of the latter case. Namely, we will consider the Kac-Moody algebras that can be obtained by enlarging the Cartan matrix (or equivalently, the Dynkin diagram) of finite Lie algebras in a particular way.

4.1 Affine algebras

An *affine Kac-Moody algebra* is a Kac-Moody algebra whose Cartan matrix A has a vanishing determinant [34],

$$\det A = 0. \tag{4.1}$$

This a loosening of condition (2.13d), but condition (2.13e) remains intact: all its principal minors should remain positive. Thus if we remove any node from the associated Dynkin diagram, the remaining diagram should correspond to a finite Lie algebra.

If A is an $(n+1) \times (n+1)$ matrix, its rank is n; it has one eigenvector whose eigenvalue is zero, and is therefore positive semidefinite. The root space Φ thus has

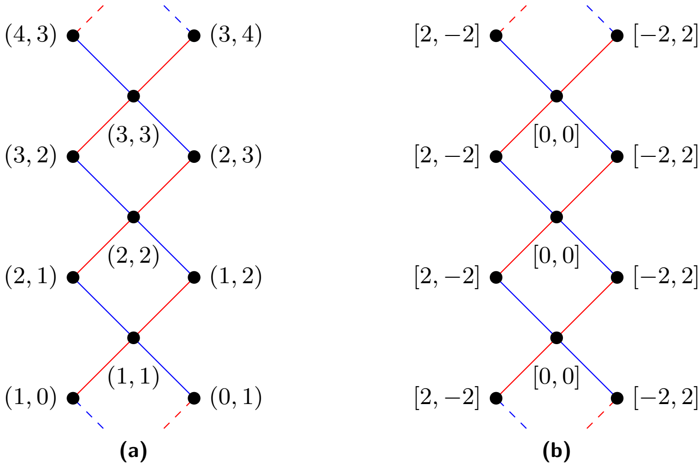


Figure 4.1: Hasse diagrams of the root system of A_1^+ up to height 7. In Figure (a) the labels are the root vectors, in Figure (b) the Dynkin labels.

one null direction, which is the reason the root system Δ becomes infinite. Another point of view is that an affine Lie algebra is infinite-dimensional because the Serre construction does not terminate when A is not positive definite.

Besides the fact root systems of affine algebras are infinite, they are also highly structured. Take for example the simplest affine algebra, known as A_1^+. Its Cartan matrix is given by

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{4.2}$$

The root system Δ of A_1^+ up to a given height is given in Figure 4.1. One feature that is immediately clear is that it is repetitive, which is in fact common to all affine Kac-Moody algebras. See for example also Figure 4.2. To be precise, there is one element $\delta \in \Delta$ for which the following holds:

$$\alpha + m\delta \in \Delta \quad \forall\, \alpha \in \Delta. \tag{4.3}$$

Here m is the smallest possible non-negative integer for which the above statement is true, and δ is the so-called *null root*, which will be defined below. Algebras for which m is equal to one are known as *untwisted* affine algebras. Otherwise they are called *twisted*. In the following we will assume that the affine algebra under consideration is always untwisted.

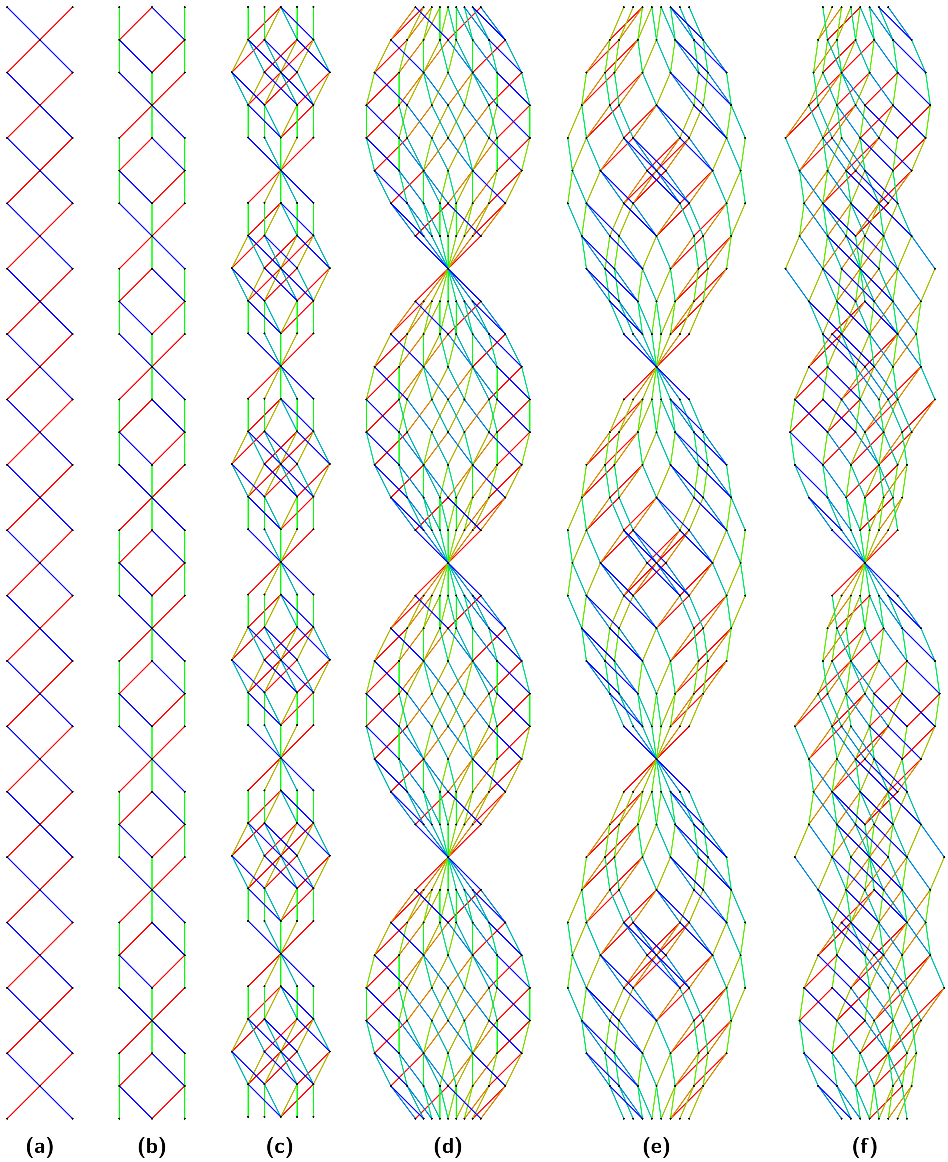


Figure 4.2: Hasse diagrams of root systems of affine Kac-Moody algebras of increasing complexity. From left to right we have: (a) A_1^+, (b) C_2^+, (c) D_4^+, (d) A_8^+, (e) D_7^+, and (f) E_7^+.

The null root δ and its dual $\delta^\vee$ are given by

$$\delta = a^i \alpha_i, \tag{4.4a}$$
$$\delta^\vee = a^{\vee i} \alpha_i^\vee. \tag{4.4b}$$

The coefficients $a^{\vee i}$ and a^i are the (dual) Coxeter labels of the affine Kac-Moody algebra. In contrast to finite algebras, they are defined as the left and right null eigenvectors of the Cartan matrix. They are normalized such that their minimum component is equal to one:

$$a^j A_{ji} = 0 \qquad \min(a^i) = 1 = a^0, \tag{4.5a}$$
$$A_{ij} a^{\vee j} = 0 \qquad \min(a^{\vee i}) = 1 = a^{\vee 0}. \tag{4.5b}$$

This particular normalization ensures that δ lies on the root lattice. For convenience, the index for which both a^i and $a^{\vee i}$ are minimal has been labeled 0. The full index i then runs over $i = 0, 1, \ldots, n$. From their definition it follows that the Coxeter labels a^i and $a^{\vee i}$ satisfy

$$a^{\vee i} = \frac{(\alpha_i|\alpha_i)}{(\alpha_0|\alpha_0)} a^i. \tag{4.6}$$

It follows straightforwardly that the null root is related to its dual by

$$\delta^\vee = \frac{2}{(\alpha_0|\alpha_0)} \delta. \tag{4.7}$$

This is not the usual definition $\delta^\vee = 2\delta / (\delta|\delta)$, as that would have been ill-defined. δ is namely truly a null root in the sense of equation (2.42); its norm $(\delta|\delta)$ is zero.

The null root of A_1^+ is $\delta = \alpha_0 + \alpha_1$. Studying Figure 4.1 in more detail, you may notice that roots differing by δ share the same Dynkin labels p_i,

$$p_i = (\alpha|\alpha_i^\vee) = A_{ji} m^j = (\alpha + \delta|\alpha_i^\vee). \tag{4.8}$$

The reason this happens is that the Dynkin labels of δ are all zero, because its root vector is the null vector of the Cartan matrix. In order to lift this degeneracy of the roots, we can add an additional Dynkin label p_{-1} that distinguishes between α and $\alpha + \delta$. For an arbitrary root α it is defined as

$$p_{-1} = (\alpha|\gamma^\vee), \tag{4.9}$$

where $\gamma^\vee = 2\frac{\gamma}{(\gamma|\gamma)}$. We will call γ the *root of derivation*. It is an element of the root space Φ that satisfies

$$(\delta|\gamma^\vee) = -1. \tag{4.10}$$

This does the trick, as $(\alpha+\delta|\gamma^\vee) = (\alpha|\gamma^\vee) - 1$. Therefore the Dynkin label p_{-1} of $\alpha+\delta$ differs from that of α by -1. Because the zeroth Coxeter label a^0 is always equal to one, the inner product of $\gamma^\vee$ with the simple roots can be defined as

$$(\alpha_i|\gamma^\vee) = -\delta_i^0, \tag{4.11}$$

With this definition γ satisfies (4.10). Note that γ cannot lie in the span of the simple roots. If that were the case, its inner product with δ would be zero. We therefore need to manually add γ to the root space,

$$\Phi = \mathrm{span}\{\alpha_0, \dots, \alpha_n\} \oplus \gamma. \tag{4.12}$$

Note that the root of derivation is not a member of the root system Δ, as all of the roots in Δ can still be expanded entirely in the basis of simple roots.

In principle the norm of the root of derivation is not fixed by the above analysis. However, things simplify a bit when we choose $(\gamma|\gamma) = (\alpha_0|\alpha_0)$. The above equation then can be written as

$$(\alpha_i^\vee|\gamma) = -\delta_i^0 = \left(-\Lambda^0|\alpha_i^\vee\right). \tag{4.13}$$

Thus γ can be identified with minus the zeroth fundamental weight, Λ^0. When the Cartan matrix is non-degenerate, the fundamental weights can be given by means of its inverse (see equation (2.50)). But when the Cartan matrix is degenerate, as is the case for affine algebras, this is not possible. The fundamental weights can be introduced for affine algebras only because the introduction of γ has lifted the degeneracy of the Cartan matrix.

The Cartan matrix A can namely be extended to a bigger matrix, A'. The extended Cartan matrix A' has one extra row and column that correspond to γ:

$$A_{(-1)i} = (\gamma|\alpha_i^\vee) = (\alpha_i^\vee|\gamma) = A_{i(-1)}. \tag{4.14}$$

The introduction of γ has effectively added a generator h_{-1} to the Cartan subalgebra, on which the action of the simple roots is (compare equation (2.29))

$$\alpha_i(h_{-1}) = A_{(-1)i}. \tag{4.15}$$

The extended Cartan matrix reads in full

$$A' = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & A_{00} & A_{10} & \cdots & A_{0n} \\ 0 & A_{10} & A_{11} & \cdots & A_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & A_{n0} & A_{n1} & \cdots & A_{nn} \end{pmatrix}. \tag{4.16}$$

This matrix is indeed non-degenerate, as its rank is $n+2$. The extended Cartan matrix can be taken as the starting point in the Serre construction of affine Kac-Moody algebras. However, one must then take some additional rules into account.

Namely, there are no Chevalley generators e_{-1} and f_{-1} associated to the extension, but only an extra generator h_{-1} for the Cartan subalgebra. The extra simple root α_{-1} is also absent. If it were present it would correspond to γ, which is, as noted above, not part of the root system. Lastly, the Weyl group is generated by the $n+1$ fundamental reflections $w_0, \ldots, w_n$. The reflection w_{-1} does not exist.

Example 4.1: A_1^+

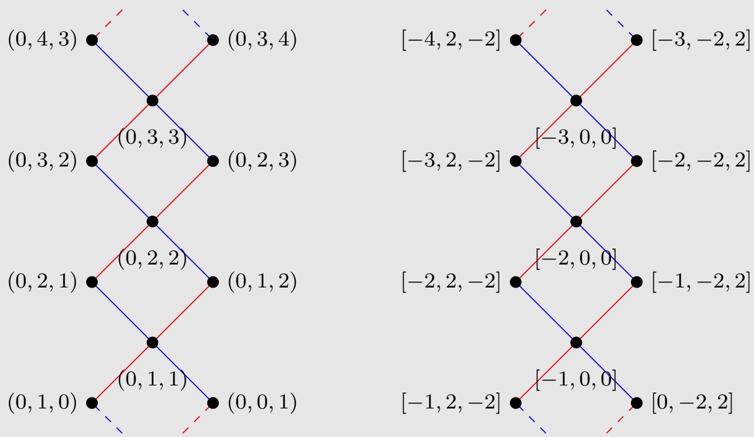
Recall that the Cartan matrix of A_1^+ is equal to

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{4.17}$$

The normalized Coxeter labels are then $a^i = (1,1)$, as $a^j A_{ji} = 0$ and $\min(a^i) = 1$. Because both Coxeter labels are equal to one, we have a freedom to choose which index will be the zeroth. If we pick the first one, the extended Cartan matrix is

$$A' = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}. \tag{4.18}$$

If we use the extended Cartan matrix to calculate the Dynkin labels of the roots via $p_i = A'_{ji} m^j$, Figure 4.1 becomes



The addition of the derivation γ to the root space, and its corresponding row and column to the Cartan matrix, has achieved what it was supposed to achieve: the Dynkin labels are no longer degenerate.

4.1.1 Affine algebras as extensions of finite algebras

Besides being defined by means of Cartan matrices with vanishing determinant, an affine Kac-Moody algebra $\mathfrak{g}$ can also be obtained from a finite semi-simple Lie algebra $\overline{\mathfrak{g}}$. The Lie algebra $\overline{\mathfrak{g}}$ then features as a particular subalgebra of $\mathfrak{g}$ whose roots $\overline{\alpha}$ all satisfy

$$(\overline{\alpha}|\gamma^\vee) = 0. \tag{4.19}$$

The subalgebra $\overline{\mathfrak{g}}$ is called the *horizontal* subalgebra of $\mathfrak{g}$. The way $\mathfrak{g}$ can be obtained from $\overline{\mathfrak{g}}$ is as follows. If the Lie bracket of $\overline{\mathfrak{g}}$ reads

$$[T^a, T^b] = f_{ab}{}^c T^c \tag{4.20}$$

a loop algebra $\overline{\mathfrak{g}}_{\text{loop}}$ can be formed by introducing a grading over $\mathbb{Z}$:

$$[T^a_m, T^b_n] = f_{ab}{}^c T^c_{m+n}. \tag{4.21}$$

Here the additional indices run over the integers: $m, n \in \mathbb{Z}$. In order to turn the loop algebra into an affine algebra we have to add two additional generators, the *central element* K and the *derivation* D. With the inclusion of K and D, the complete Lie bracket reads

$$[T^a_m, T^b_n] = f_{ab}{}^c T^c_{m+n} + m\delta_{m,-n}\eta^{ab} K, \tag{4.22a}$$

$$[D, T^a_m] = -mT^a_m, \tag{4.22b}$$

$$[K, D] = [K, T^a_m] = 0. \tag{4.22c}$$

The algebra $\mathfrak{g}$ specified by the above bracket is said to be the *affine extension* of $\overline{\mathfrak{g}}$, and is often denoted by $\overline{\mathfrak{g}}^+$. The original Lie algebra $\overline{\mathfrak{g}}$ can be identified with generators T^a_m whose bracket with D vanishes. This is the same identification as in equation (4.19).

The central element is unique up to normalization, and corresponds to the null direction of the Cartan subalgebra:

$$K = a^{\vee i} h_i. \tag{4.23}$$

Inspection then shows that K indeed commutes with all Chevalley generators e_i and f_i (see equation (2.16)), and thus also with all generators of $\mathfrak{g}^+$.

The derivation D is identical to the extra generator h_{-1} of the previous section. Like γ does for the roots, the derivation 'measures' the $\mathbb{Z}$ grading of the loop generators T^a_m. It also crucially lifts the degeneracy of the Cartan-Killing form, which is given by

$$\langle T^a_m | T^b_n \rangle = \langle T^a | T^b \rangle \delta_{m,-n}, \tag{4.24a}$$

$$\langle K | D \rangle = -1, \tag{4.24b}$$

with all other combinations vanishing. As a consequence, the derivation also lifts the degeneracy of the inner product on the root space. The root space is again defined as the dual of the Cartan subalgebra, $\Phi = \mathfrak{h}^*$, which can be given by

$$\mathfrak{h} = \overline{\mathfrak{h}} \oplus \mathrm{span}\{K, D\}. \tag{4.25}$$

The root system Δ then has the same structure as described in section 4.1. In particular, the simple roots are given by

$$\alpha_s = \overline{\alpha}_s, \tag{4.26a}$$
$$\alpha_0 = \delta - \overline{\theta}, \tag{4.26b}$$

where $\overline{\theta}$ is the highest root and $\overline{\alpha}_s$ are the simple roots of $\overline{\mathfrak{g}}$. The index s runs over the rank of $\overline{\mathfrak{g}}$, that is $i = \{0, 1, \ldots, n\} = \{0, s\}$. Note that the norm of α_0 is the same as that of $\overline{\theta}$, $(\alpha_0|\alpha_0) = \big(\overline{\theta}|\overline{\theta}\big)$.

By equation (4.26), the Cartan matrix $\overline{A}$ of the horizontal subalgebra $\overline{\mathfrak{g}}$ can be recovered from that of $\mathfrak{g}$ by deleting the zeroth row and column. Equivalently, one can delete the zeroth node of the affine Dynkin diagram to retrieve the diagram of the horizontal subalgebra. Conversely, the Cartan matrix of the affine algebra can be obtained from $\overline{A}$ by adjoining a row and column whose entries correspond to the Dynkin labels of the lowest root of $\overline{\mathfrak{g}}$. Specifically, the row and columns to add are

$$A_{0s} = \big(\delta - \overline{\theta}|\alpha_s^\vee\big) = -a^t A_{ts}, \tag{4.27a}$$
$$A_{s0} = \frac{(\alpha_s|\alpha_s)}{\big(\overline{\theta}|\overline{\theta}\big)} A_{0s}. \tag{4.27b}$$

4.2 Over- and very-extended algebras

In section 4.1 we argued that the extended Cartan matrix (4.16) of affine Kac-Moody algebras can serve as the starting point in their construction if one takes certain rules into account. When those rules are forgotten, the resulting algebra is not affine. Using the 'ordinary' Serre construction, the result is a so-called *over-extended* Kac-Moody algebra. The nomenclature stems from the fact that an over-extended algebra is an extension of an affine algebra $\mathfrak{g}^+$, who in turn is an extension of its horizontal finite subalgebra $\mathfrak{g}$. Over-extended algebras are denoted by $\mathfrak{g}^{++}$.

By (4.16), their Dynkin diagram is obtained from that of affine algebras by adding a node to the zeroth node of the affine diagram. In Figure 4.4 the over-extended node is denoted by -1. As noted in section 4.1, the over-extended Cartan matrix is non-degenerate. If its rank is $n + 2$, it has one negative and $n + 1$ positive eigenvalues. The root space of over-extended Kac-Moody algebras has therefore a Lorentzian signature, allowing for real, null, and imaginary roots.

Example 4.2: A_1^+ as the affine extension of A_1

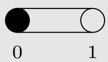
Recall from Example 2.2 and Example 2.3 that the Cartan matrix of the Lie algebra A_1 is given by

$$\overline{A} = \begin{pmatrix} 2 \end{pmatrix} . \tag{4.28}$$

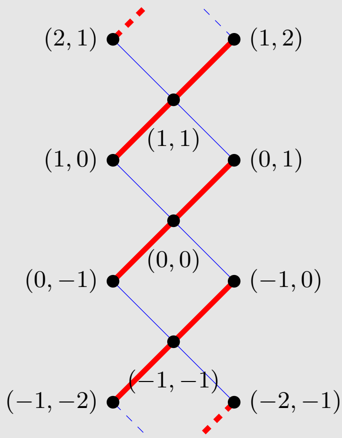
The highest root of A_1 is its only positive root, $\overline{\theta} = \overline{\alpha}_1$, whose Dynkin label is equal to 2. By equation (4.27) the Cartan matrix A of the affine extension of A_1 is thus given by

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} . \tag{4.29}$$

This is, not surprisingly, the same Cartan matrix as in equation (4.2). A_1^+ is thus truly the affine extension of A_1. Its Dynkin diagram is given by



The node labels indicate their ordering. The black zeroth node has been 'deleted': the remaining undeleted Dynkin diagram is that of the horizontal subalgebra A_1. If we again take a look at the root system of A_1^+, we can distinguish infinitely many copies of the root system of A_1 within it:



The red lines have been thickened to indicate the A_1 subalgebras. The middle roots $(0, -1)$, $(0, 0)$, and $(0, 1)$ form the horizontal root subsystem, as they all satisfy equation (4.19).

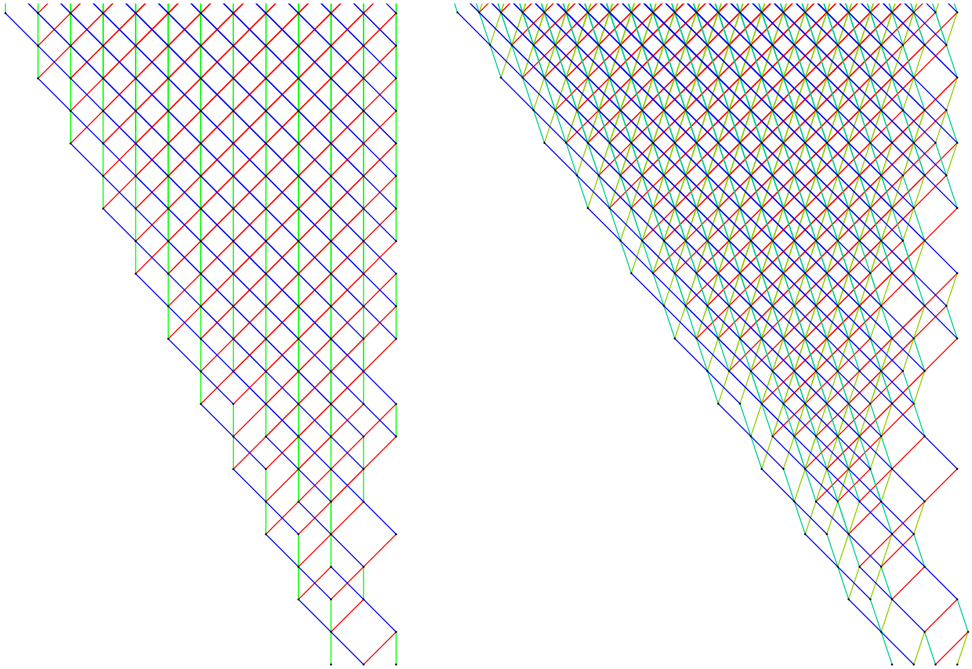


Figure 4.3: Hasse diagrams of the root systems of (a) A_1^{++} and (b) A_1^{+++} up to height 22. The number of roots of A_1^{+++} grows faster in height than that of A_1^{++}.

If we repeat the procedure and attach yet another node to the over-extended node, the resulting diagram is that of a *very-extended* Kac-Moody algebra $\mathfrak{g}^{+++}$. Figure 4.4 lists all the Kac-Moody algebras that are very-extensions of simple Lie algebras. In Figure 4.4 the very-extended node has been labeled -2. Similarly to over-extended algebras, the Cartan matrix of very-extended algebras is also Lorentzian. As opposed to affine algebras, their root systems do not grow linearly in height; both over- and very-extended root systems grow faster. Figure 4.3 shows the root systems of A_1^{++} and A_1^{+++}. Because of the extra very-extended simple root, the root system of very-extended algebras grows faster than that of the over-extended algebras.

In contrast to affine algebras, the root systems of over- and very-extended Kac-Moody algebras cannot (yet) be described fully in closed form. However, it is worth noting that there is a class of Lorentzian Kac-Moody algebras for which one can make general statements about the root system. These Kac-Moody algebras are called *hyperbolic*, and they are characterized by the fact their Cartan matrix contains

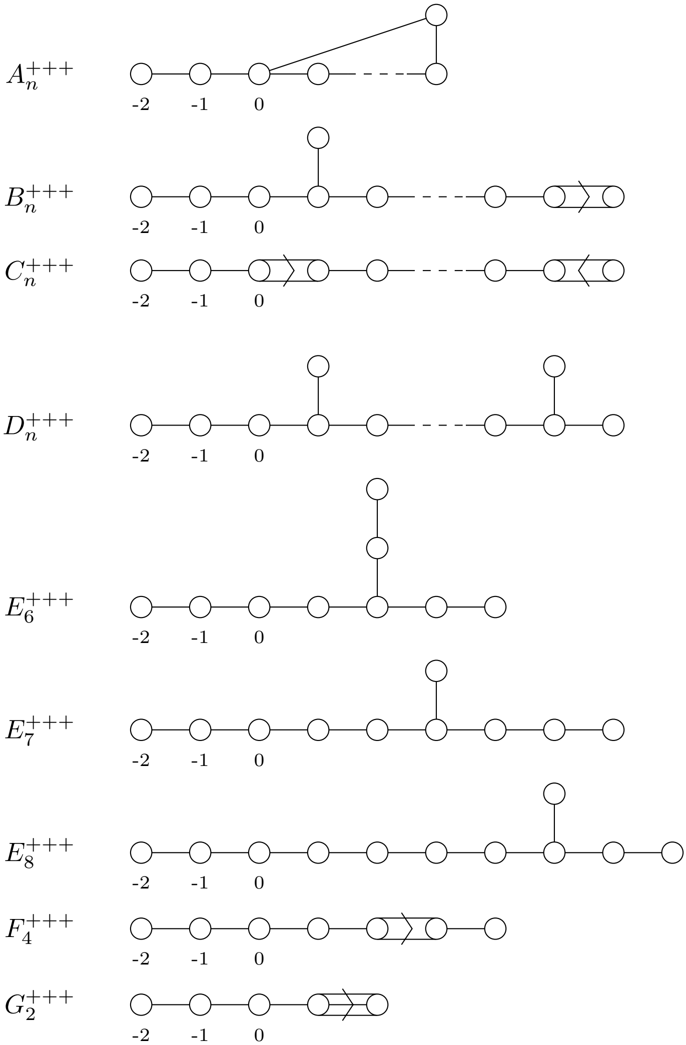


Figure 4.4: Dynkin diagrams of all Kac-Moody algebras that are very-extensions of finite simple Lie algebras. Deleting node -2 results in all of the over-extended Kac-Moody algebras g^{++}, and also deleting node -1 in all of the extended (affine) algebras g^{+}. In all cases the subscript denotes the rank of the unextended algebra.

only positive definite or positive semidefinite submatrices. Thus upon removing any node from their Dynkin diagram, we are left with a diagram of either a finite or an affine algebra. Very-extended Kac-Moody algebras are not hyperbolic; removing the very-extended node yields a Lorentzian over-extended algebra. In contrast, over-extended algebras can be hyperbolic. The root system Δ of hyperbolic algebras consists of all the points on the root lattice whose norm is smaller than norm of the longest simple root, α^2_{max} [50]:

$$\Delta = \big\{\alpha \in \Delta \,\big|\, (\alpha|\alpha) \leq \alpha^2_{\mathrm{max}}\big\}. \tag{4.30}$$

Unfortunately, this tells us nothing about the root multiplicities. Besides, we would also like to analyze the root system of very-extended Kac-Moody algebras. So for the general case one has to resort to methods that analyze the root system part by part, such as the level decomposition.

4.3 Level decomposition

A *level decomposition* is a way to chop up a Lie algebra $\mathfrak{g}$ in terms of one of its subalgebras $\mathfrak{s}$ [23, 70, 53, 57]. More precisely, under a level decomposition the adjoint representation of $\mathfrak{g}$ branches into a number of representations of $\mathfrak{s}$. This is particularly useful for over- and very-extended Kac-Moody algebras, as their infinite adjoint representation can be described in terms of finite representations of a finite subalgebra. One can of course also perform a level decomposition of finite algebras; see Figure 4.5 for a few examples.

The subalgebra $\mathfrak{s}$ will always be chosen to be *regular*. That is, all simple roots of $\mathfrak{s}$ are also simple roots of $\mathfrak{g}$. Put differently, the Cartan matrix of $\mathfrak{s}$ is obtained from that of $\mathfrak{g}$ by simultaneously deleting certain rows and columns. Equivalently, the Dynkin diagram of $\mathfrak{s}$ is a subdiagram of that of $\mathfrak{g}$ and can be recovered by deleting one or more nodes. The number of nodes that are deleted will be denoted by m. In order to indicate which of the simple roots belong to $\mathfrak{s}$ we will split the indices as $i = (s, a)$. They respectively run over

$$\begin{aligned} i, j, \ldots &: \text{Rank of full algebra, } n,\\ s, t, \ldots &: \text{Rank of regular subalgebra, } n - m,\\ a, b, \ldots &: \text{Number of deleted nodes, } m. \end{aligned} \tag{4.31}$$

Any root $\alpha \in \mathfrak{g}$ can then be written as

$$\alpha = m^i \alpha_i = m^s \alpha_s + l^a \alpha_a, \tag{4.32}$$

where $l^a \equiv m^a$ are called the *levels* of a root. The algebra $\mathfrak{g}$ then splits up into *level*

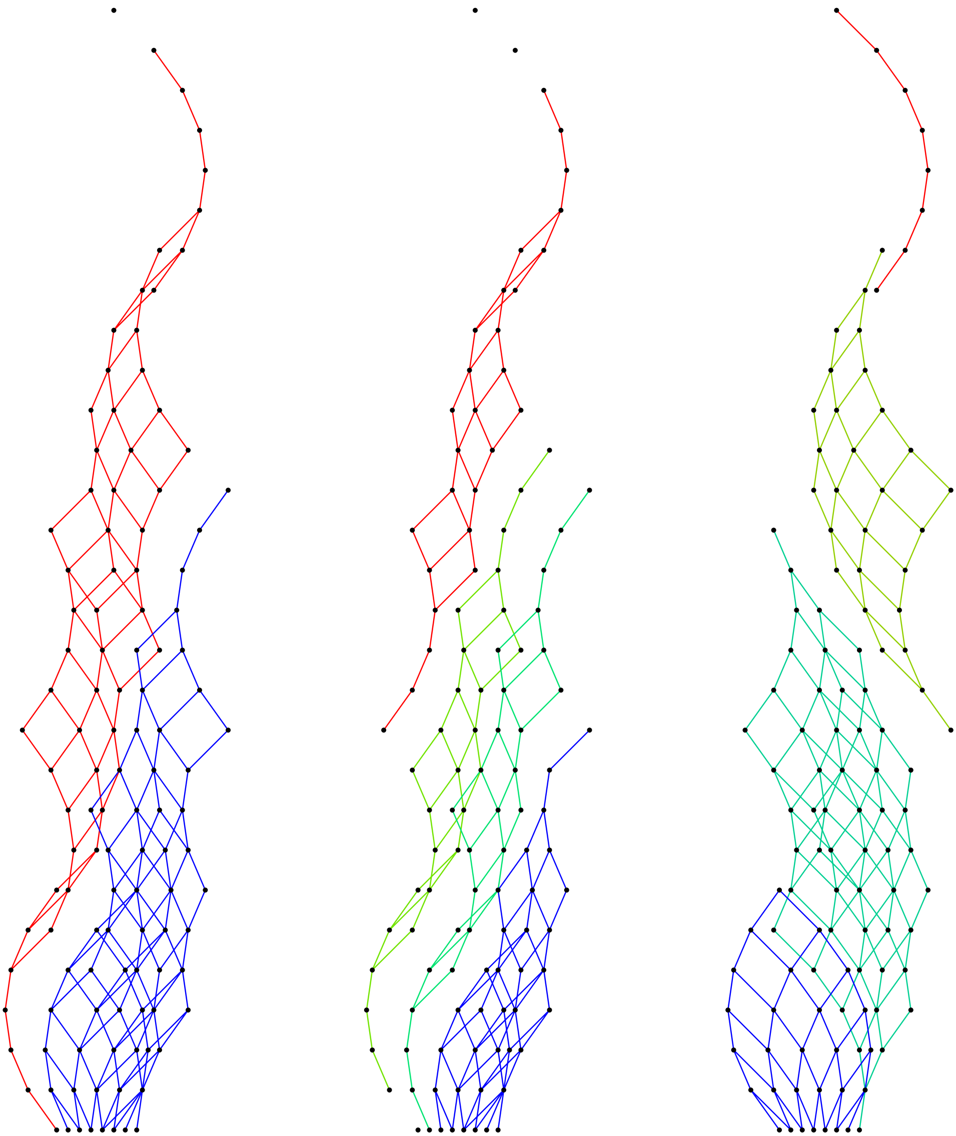


Figure 4.5: Hasse diagrams of the root system of E_8 branched with respect to different regular subalgebras. The subalgebras are: (a) E_7, (b) E_6, and (c) A_7. Compare also Figure 3.4.

spaces $\mathfrak{g}_{l^a}$. They are direct sums of root spaces $\mathfrak{g}^\alpha$ whose root is of level l^a:

$$\mathfrak{g}_{l^a} = \bigoplus_{(\alpha|\Lambda^{\vee a})=l^a} \mathfrak{g}_\alpha. \tag{4.33}$$

The whole algebra $\mathfrak{g}$ can then be written as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{l^a \in \mathbb{Z}^m} \mathfrak{g}_{l^a}. \tag{4.34}$$

Because $\mathfrak{g}$ is graded by means of its roots (see equation (2.35)), the levels also induce a grading on $\mathfrak{g}$:

$$[\mathfrak{g}_{l_1^a}, \mathfrak{g}_{l_2^a}] \subseteq \mathfrak{g}_{l_1^a + l_2^a}. \tag{4.35}$$

The roots of the regular subalgebra $\mathfrak{s}$ all have level zero by construction. This implies that the adjoint action of $\mathfrak{s}$ on a level space cannot change the level,

$$[\mathfrak{s}, \mathfrak{g}_{l^a}] \subseteq \mathfrak{g}_{l^a}. \tag{4.36}$$

Owing to the triangular decomposition (2.20), a level space $\mathfrak{g}_{l^a} \neq \mathfrak{s}$ sits either in the positive part ($\mathfrak{g}_{l^a} \in \mathfrak{n}_+$) or in the negative part ($\mathfrak{g}_{l^a} \in \mathfrak{n}_-$) of $\mathfrak{g}$. In the former case the adjoint action of $\mathfrak{s}$ on $\mathfrak{g}_{l^a}$ can never reach $\mathfrak{n}_-$. The level space $\mathfrak{g}_{l^a}$ is therefore bounded from below. This in turn implies there exist generators $x \in \mathfrak{g}_{l^a}$ that are annihilated by the adjoint action of the negative Chevalley generators of $\mathfrak{s}$:

$$\mathrm{ad}_{f_s} x = 0. \tag{4.37}$$

Comparing with equation (2.86), we see that the generators x are lowest weight vectors in lowest weight representations of $\mathfrak{s}$. The level space is thus completely reducible in terms of lowest weight representations. On the other hand, if $\mathfrak{g}_{l^a} \in \mathfrak{n}_-$ it is reducible in terms of highest weight representations. In the final case that $\mathfrak{g}_{l^a} = \mathfrak{s}$ the level space is given by the adjoint representation of $\mathfrak{s}$. We will focus only on the decomposition of $\mathfrak{n}_+$, as the decomposition of $\mathfrak{n}_-$ follows from it using the Chevalley involution (2.21).

As we saw in subsection 2.2.1, if x is a lowest weight vector of $\mathfrak{s}$ its Dynkin labels with respect to $\mathfrak{s}$ are non-positive. By equation (2.51a) it can be easily checked if this is the case. The Dynkin labels of any positive root $\alpha = m^i \alpha_i$ are

$$p_s = A_{ts} m^t + A_{as} l^a. \tag{4.38}$$

Using the above equation, we can check every root to see if it is a lowest weight of $\mathfrak{s}$. It can also be inverted to give

$$m^s = \left(A^{-1}_{\mathrm{sub}}\right)^{ts} \left(p_t - l^a A_{at}\right), \tag{4.39}$$

where A_{sub} is the Cartan matrix of $\mathfrak{s}$. This, together with the requirement that the root norm is bounded from above (2.75), gives us an algorithm to scan for potential lowest weights. The bound (2.75) can be written as

$$\alpha^2 = G^{st}_{\text{sub}} \left(p_s p_t - A_{as} A_{bt} l^a l^b \right) + B_{ab} l^a l^b \leq \alpha^2_{\text{max}}. \tag{4.40}$$

Note that for this formula to be valid, we have to make sure that a long (or short) root in the full algebra is also a long (short) root in the subalgebra, which in general is not automatically the case. Luckily we are always free to choose a normalization such that the root lengths match.

Assuming $\mathfrak{s}$ is finite, the quadratic form matrix G^{st}_{sub} of $\mathfrak{s}$ only has positive entries. Therefore α^2 is a monotonically increasing function of $|p_s|$ at fixed levels l^a. The number of distinct Dynkin labels p_s that respect the bound is thus finite. Furthermore, the root vector associated to p_s has to lie on the root lattice, implying that m^s should be non-negative integers. Hence equation (4.39) restricts the possible values of p_s even more. All in all, there are a finite number of subalgebra representations at a given level if $\mathfrak{s}$ is finite.

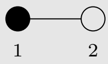
After scanning for possible valid Dynkin labels at fixed levels using equations (4.39) and (4.40), we still have to determine whether the representation $V(\Lambda)$ with lowest weight $\Lambda = p_s \Lambda^s$ actually occurs at level l^a. Let n_{l^a} be the number of distinct Dynkin labels found at level l^a using the above scanning technique. In principle there could be n_{l^a} distinct representations V_i at level l^a. Say a root α occurs as a weight within the weight diagrams of some of the representations V_i. Then the sum of its multiplicity as a weight in the different representations has to add up to its multiplicity as a root:

$$\text{mult}(\alpha) = \sum_{i=1}^{n_{l^a}} \mu_{l^a}(V_i)\, \text{mult}_{V_i}(\alpha). \tag{4.41}$$

The number $\mu_{l^a}(V_i)$ counts how often the representation V_i occurs at level l^a, and is called the *outer multiplicity* of the representation. If one calculates $\text{mult}(\alpha)$ by means of the Peterson recursion formula (2.99) and $\text{mult}_{V_i}(\alpha)$ by means of the Freudenthal recursion formula (2.101), it is relatively straightforward to compute the outer multiplicities.

Example 4.3: Level decomposition of A_2

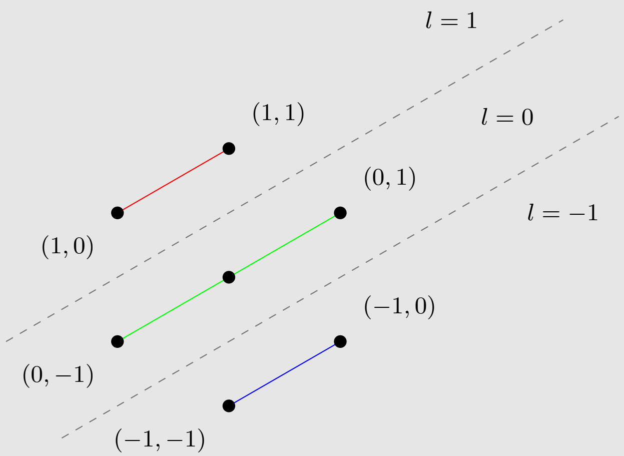
Let us see how the root system of A_2 splits up under the level decomposition with respect to an A_1 regular subalgebra. If we delete the first node, the Dynkin diagram looks like



The simple root of the deleted (black) node will count the level. The root system splits up according to

$$\alpha = l\alpha_1 + m\alpha_2. \tag{4.42}$$

Graphically, it looks like the following:

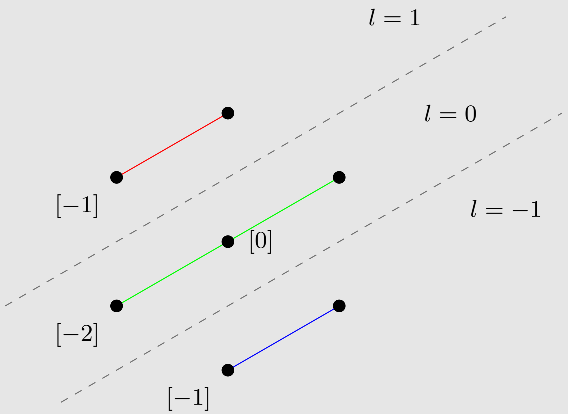


The colored lines are Weyl reflections of the A_1 subalgebra. The different colors indicate the three levels -1, 0, and 1. From the above picture it is fairly obvious which roots are lowest or highest weights of A_1, but we can also employ the scanning method to find them. Equations (4.39) and (4.40) simplify to

$$m = \tfrac{1}{2}(p + l) \tag{4.43a}$$

$$\alpha^2 = \tfrac{1}{2}\left(p^2 + 3l^2\right) \leq 2. \tag{4.43b}$$

The last equation has no solution for $l \geq 2$, so, as expected, we only have to look at level -1,0, and 1. For $l = \pm 1$, the only allowed values are $p = \pm 1$. For $l = 0$ the options are $p = \{-2, 0, 2\}$. Since we're looking for lowest weights, the positive values of p can be discarded. All in all, the different Dynkin labels we have found are:



Each representation occurs with outer multiplicity one, including the singlet [0] representation. The singlet corresponds to the Cartan subalgebra generator of the deleted node, h_1. Generically, for every deleted node there is one singlet representation. The results of the decomposition can be collected in a table:

l	$-p$	m^i	α^2	dim	mult(α)	μ
-1	1	-1 -1	2	2	1	1
0	2	0 -1	2	3	1	1
0	0	0 0	0	1	2	1
1	1	1 0	2	2	1	1

The rows in the columns are the various representations of the subalgebra A_1 at the different levels. Here l denotes the level, p the Dynkin label, m^i the root vector of the lowest weight, dim the dimension of the representation, mult(α) the multiplicity of the root of the lowest weight, and μ the outer multiplicity of the representation.

To summarize, the level decomposition splits the adjoint A_2 representations into four A_1 representations over three different levels. In terms of the dimensions of the representations, we have made the following branching: $\mathbf{8} \rightarrow \mathbf{3} + \mathbf{1} + \mathbf{2} + \mathbf{2}$.

4.4 Non-linear realizations

For infinite-dimensional Kac-Moody algebras one can in principle construct a non-linear realization. But as there are an infinite amount of generators, the parameterization of the group element $V(t)$ (2.127) will contain an infinite amount of terms. This makes it impossible to compute the Maurer-Cartan form (2.124) in full generality. What is possible, however, is to make a consistent truncation of the formally defined action (2.126) for which the solutions of the equations of motion are also solutions to the full model [24, 42]. This can be done by truncating not the group element, but the Maurer-Cartan form either at a given root height or at a given level of a level decomposition. These two methods are similar in spirit, and we will discuss only the latter.

Recall that under a level decomposition the Lie algebra $\mathfrak{g}$ splits into level $\mathfrak{g}_{l^a}$ according to (4.34). The group element (2.127) can thus be written as

$$V(t) = \prod_{l^a \geq 0} V_{l^a}(t) = \prod_{l^a \geq 0} \exp\big(\phi_{l^a}(t) \cdot E_{l^a}\big). \tag{4.44}$$

Here E_{l^a} denotes all the representations of the subalgebra that occur at level l^a. The negative levels do not enter, as we can use the Borel gauge (2.128) to parameterize $V(t)$. Consequently, the Maurer-Cartan form splits as

$$J = V^{-1}\partial V = \sum_{l^a \geq 0} J_{l^a}, \tag{4.45}$$

where the summand schematically reads

$$J_{l^a} = R_{l^a} \cdot E_{l^a}. \tag{4.46}$$

The coefficients R_{l^a} contracting E_{l^a} can depend on scalar fields $\phi_{l'^a}$ whose level is at most l^a, i.e. $l'^a \leq l^a$. The exact form of R_{l^a} depends strongly on the graded structure of $\mathfrak{g}$. Before truncating, the sum in the expansion of J is infinite for Kac-Moody algebras. The truncated version can simply be cut off at any given level t^a,

$$\tilde{J} = \sum_{0 \leq l^a \leq t^a} J_{l^a}. \tag{4.47}$$

This truncated Maurer-Cartan form $\tilde{J}$ can then be used to calculate a (truncated) coset element $\tilde{P}$ and its corresponding action.

Example 4.4: Non-linear realization of $A_1 \subset A_2$

In Example 4.3 we decomposed A_2 with respect to A_1. In this example we will show how to do a non-linear realization of the fields recovered in the level decomposition. Recall that the results of the level decomposition were

l	$-p$	α^2	dim	mult(α)	μ	representation
-1	1	2	2	1	1	F_a
0	2	2	3	1	1	$K^a{}_b$
0	0	0	1	2	1	K
1	1	2	2	1	1	E^a

The representations introduced in the last column correspond to the various generators as follows:

$$l = 0: \qquad h_1 = K^1{}_1 + 2K^2{}_2, \qquad e_2 = K^1{}_2, \tag{4.48a}$$

$$h_2 = K^1{}_1 - K^2{}_2, \qquad f_2 = K^2{}_1, \tag{4.48b}$$

$$l = 1: \qquad e_1 = E^2, \qquad [e_1, e_2] = E^1, \tag{4.48c}$$

$$l = -1: \qquad f_1 = F_2, \qquad [f_2, f_1] = F_1. \tag{4.48d}$$

Like in Example 2.10, $K^a{}_b$ are 2×2 matrices, and E^a and F_b are vectors,

$$(K^a{}_b)^i{}_j = \delta^{ai}\delta_{bj}, \tag{4.49a}$$

$$(E^a)^i = \delta^{ai}, \tag{4.49b}$$

$$(F_a)_i = \delta_{ai}. \tag{4.49c}$$

All indices run from one to two. As we already know the Cartan-Killing norm on the generators from Example 2.8, we can straightforwardly cast it in the form

$$\langle K^a{}_b | K^c{}_d \rangle = \delta^a_d \delta^c_b - \tfrac{1}{3}\delta^a_b \delta^c_d, \tag{4.50a}$$

$$\langle E^a | F_b \rangle = \delta^a_b. \tag{4.50b}$$

All other combinations vanish. A convenient parameterization of the group element V is

$$V(t) = V_1(t)V_0(t), \tag{4.51a}$$

$$V_0(t) = e^{h_a{}^b(t) K^b{}_a}, \tag{4.51b}$$

$$V_1(t) = e^{A_a(t) E^a}. \tag{4.51c}$$

The Maurer-Cartan form splits into a level 0 part J_0 and a level 1 part J_1,

$$J = J_0 + J_1, \tag{4.52a}$$
$$J_0 = e_a{}^m \partial e_m{}^b K^a{}_b, \tag{4.52b}$$
$$J_1 = e_a{}^m \partial A_m E^a, \tag{4.52c}$$

where $e_m{}^a$ is again the matrix exponential of $h_m{}^a$ and plays the role of the vielbein, and $e_a{}^m$ is its inverse. The coset element splits in a similar fashion:

$$P = P_0 + P_1, \tag{4.53a}$$
$$P_0 = e_a{}^m \partial e_m{}^b S^a{}_b, \tag{4.53b}$$
$$P_1 = e_a{}^m \partial A_m S^a, \tag{4.53c}$$

where the coset basis elements S are given by

$$S^a{}_b = \tfrac{1}{2}\left(K^a{}_b + K^b{}_a\right), \tag{4.54a}$$
$$S^a = \tfrac{1}{2}\left(E^a + F_a\right). \tag{4.54b}$$

Note that E^a and F_a are each others transposeds. If we evaluate the Cartan-Killing norm on the coset basis, we find

$$\langle S^a | S^b \rangle = \tfrac{1}{2}\delta^{ab}, \tag{4.55a}$$
$$\langle S^a{}_b | S^c{}_d \rangle = \delta^a_d \delta^c_b - \tfrac{1}{3}\delta^a_b \delta^c_d. \tag{4.55b}$$

Upon introducing the metric $g_{mn} = \delta_{ab} e_m{}^a e_n{}^b$ the action finally becomes

$$S = -\tfrac{1}{4}\int dt\, n(t)^{-1}\Big(\left(g^{mp}g^{nq} - \tfrac{1}{3}g^{mn}g^{pq}\right)\partial g_{mn}\partial g_{pq} + 2g^{mn}\partial A_m \partial A_n\Big). \tag{4.56}$$

This action should of course also follow from a direct reduction of the non-linear sigma model action of a non-decomposed A_2. If we recall from Example 2.10 that action was

$$S = -\tfrac{1}{4}\int dt\, n(t)^{-1} G^{MP} G^{NQ} \partial G_{MN} \partial G_{PQ}, \tag{4.57}$$

and the metric G_{MN} was constrained to have unit determinant. The capital indices run from one to three. It turns out that if we use the Ansatz

$$G_{MN} = \frac{1}{\det(g_{mn})^{1/3}} \begin{pmatrix} g_{mn} + A_m A_n & A_m \\ A_n & 1 \end{pmatrix} \tag{4.58}$$

the action (4.57) exactly reduces to (4.56), just as expected.