Kac-Moody Symmetries and Gauged Supergravity

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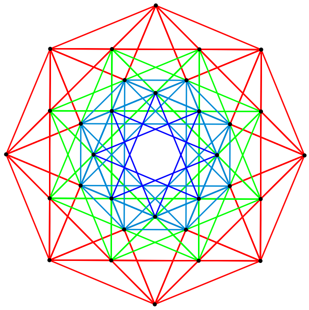


Visualizations

From time to time, it is convenient to draw a picture of a Lie algebra or one of its representations. Besides from looking nice, a picture can display their structure at a glance. For instance, both the symmetry of the root system and the height of the roots of A_2 are immediately clear from Figure 3.1, which displays the projection we already encountered in chapter 2. In fact, Figure 3.1 preserves the structure of A_2 exactly. This is possible because the root space of A_2 is two-dimensional. When the rank of the algebra is bigger than two, projections onto two dimensions of the root or weight space loose some of the information. It is then no longer possible to capture both the ordering in height and the full symmetry into one image. What one can do, however, is do one projection that preserves the ordering in height, and another that preserves (some part of) the symmetry. The former can be achieved with a *Hasse diagram*, and the latter with a *Coxeter projection*.

3.1 Hasse diagrams

A *Hasse diagram* is a graph that displays the ordering between the different elements of a set [12, 33], which in our case are the roots of a root system. An example of

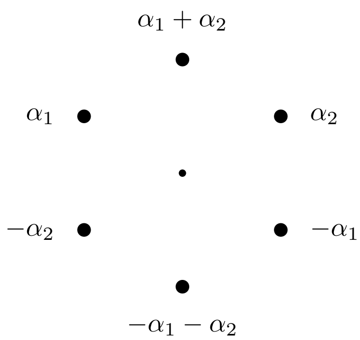


Figure 3.1: The root system of A_2.

a Hasse diagram is given in Figure 3.2. Below I will give the precise definition of a Hasse diagram, and a procedure for drawing them.

The root system Δ can be promoted to an *ordered set* $(\Delta, \geq)$ if we introduce the following partial ordering. A root α is said to be bigger than β if their difference is positive:

$$\alpha \geq \beta \quad \text{if} \quad \alpha - \beta \in Q_+. \tag{3.1}$$

Thus $\alpha - \beta$ has to be a non-negative combination of simple roots. If it is not, the two roots are incomparable. In addition to the partial ordering we need to introduce a so-called *cover relation.* A root α is said to *cover* β if there is no root γ smaller than α and bigger than β:

$$\alpha \succ \beta \quad \text{if} \quad \nexists\, \gamma \ : \ \alpha \geq \gamma \geq \beta. \tag{3.2}$$

For roots this means that one root covers the other only if their difference is one single simple root. With these two relations, a Hasse diagram of Δ can now be drawn according to the following rules:

- If $\alpha \geq \beta$ the vertical coordinate for β is less than that for α.
- If $\alpha \succ \beta$ there is a straight line connecting α and β.

Because $\alpha \geq \beta$ implies $\mathrm{ht}(\alpha) > \mathrm{ht}(\beta)$, the first criterion is satisfied if we assign the vertical coordinate according to the height of the roots. The second criterion is equivalent to drawing straight lines for every fundamental Weyl reflections, as these are used to construct the root system in the first place (see subsection 2.1.6).

What remains to be done is to determine the horizontal coordinate for each root. Although there are various algorithms with varying degree of complexity available (see for example [33]), the following simple recipe works fairly well for root systems.

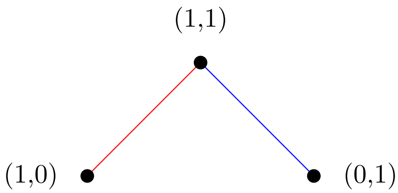


Figure 3.2: Hasse diagram of the positive roots of A_2. The numbers (m^1, m^2) denote the root vector.

The first step is to distribute the simple roots evenly on a horizontal line around the origin. This is achieved by the following horizontal projection $\mathbb{P}_x$ of the simple roots α_i:

$$\mathbb{P}_x(\alpha_i) = \frac{i-1}{n-1} - \frac{1}{2} \equiv x_i, \tag{3.3}$$

where n is the rank of the Lie algebra. The horizontal position of a generic root $\alpha = m^i \alpha_i$ can now be defined as

$$\mathbb{P}_x(\alpha) = m^i\, \mathbb{P}_x(\alpha_i) = m^i x_i. \tag{3.4}$$

Note that the explicit summation of the index i has been dropped. From now on, any contracted index will be summed over. We can formalize the above a bit by introducing a *projection vector* φ that satisfies

$$(\alpha_i|\varphi) = x_i. \tag{3.5}$$

Expanded in the basis of simple co-roots, the projection vector φ explicitly reads

$$\varphi = (A^{-1})^{ij} x_j\, \alpha_i^\vee. \tag{3.6}$$

When we take its inner product with a generic root α, we see that it indeed gives us the desired projection (3.4): $(\alpha|\varphi) = m^i x_i$. The complete projection $\mathbb{P} = (\mathbb{P}_x, \mathbb{P}_y)$ can then be written as

$$\mathbb{P}_x(\alpha) = (\alpha|\varphi)\,, \tag{3.7a}$$

$$\mathbb{P}_y(\alpha) = (\alpha|\rho^\vee)\,, \tag{3.7b}$$

where the projection in the vertical coordinate y is just the height of the root.

Note that the horizontal coordinate (3.3) of a simple root α_i strongly depends on its number i. If the order of the simple roots is changed, the Hasse diagram changes shape too. The best looking diagrams are produced when the ordering of nodes in the Dynkin diagram (and thus the ordering of simple roots) matches the connections between the nodes. See also Figure 3.3.

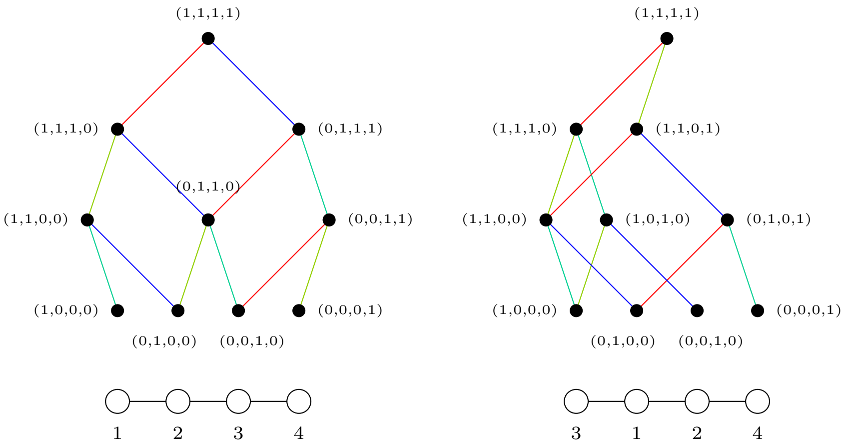


Figure 3.3: Two Dynkin diagrams (below) and Hasse diagrams (above) of the same Lie algebra, A_4. The ordering of nodes in the left Dynkin diagram, indicated with numbers below the nodes, is canonical. The ordering of nodes in the right Dynkin diagram does not match the connections between them, resulting in a Hasse diagram with crossing lines.

The lines drawn in a Hasse diagram represent the Weyl reflections in the simple roots. Say there is a root α projected to the point (x, y). Then the root $\alpha + \alpha_i$ connected to it by the line of the fundamental Weyl reflection w_i gets projected to the point $(x + x_i, y + 1)$. The line of a fundamental reflection is therefore drawn at an angle ϕ given by

$$\phi_{w_i} = \tan^{-1} \frac{1}{x_i}. \tag{3.8}$$

Because x_i is unique for all i, the n distinct fundamental reflections w_i all are drawn at different angles, and reflections in the same simple root are drawn parallel. To distinguish between them even further they will get drawn in different colors, ranging from blue (the first fundamental reflection) to red (the n^{th}).

The Hasse diagram of the full root system is symmetric around the origin, because of the Chevalley involution (2.21). It is therefore customary to draw only the positive roots in a Hasse diagram.

Following the above procedure it is straightforward, though sometimes tedious, to draw a Hasse diagrams of any root system. Figure 3.4 displays for example the Hasse diagrams of various root systems.

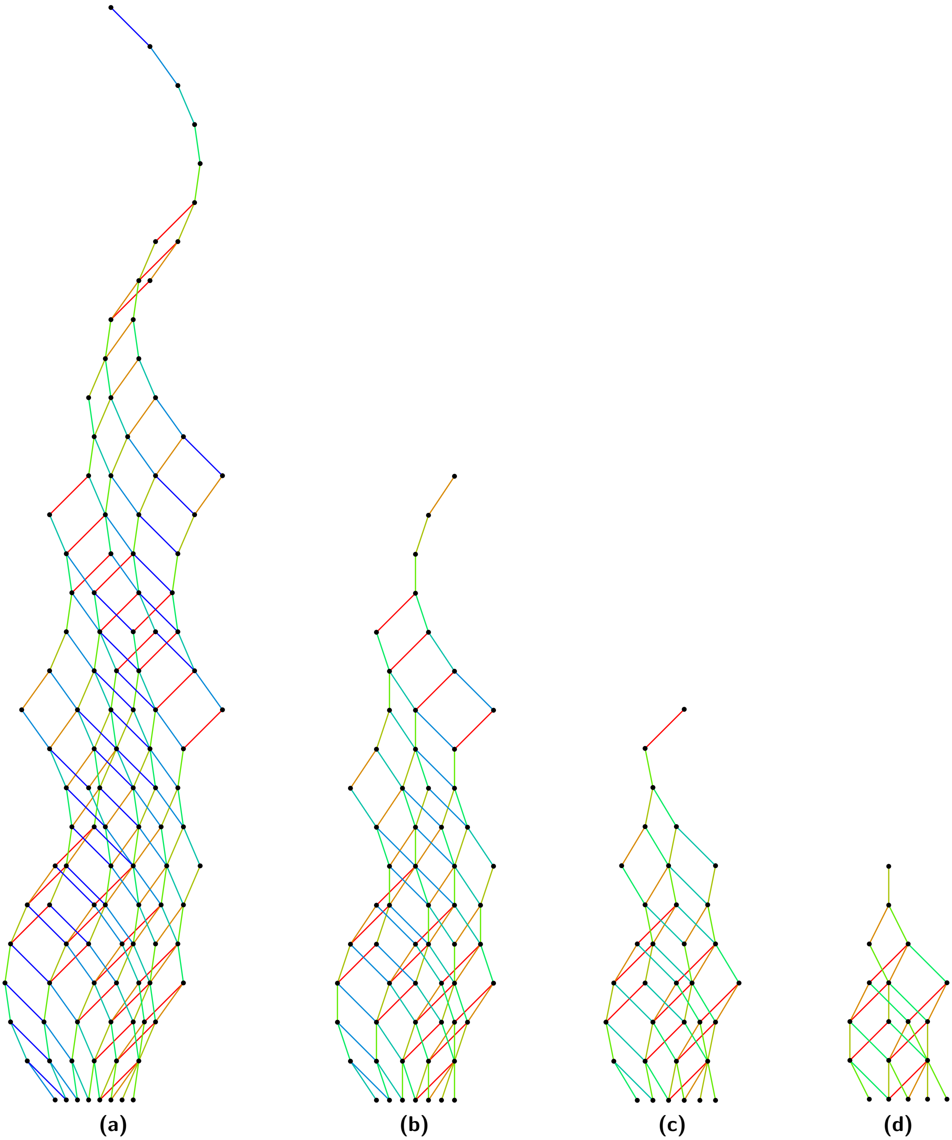


Figure 3.4: Hasse diagram of the positive roots of (a) E_8, (b) E_7, (c) E_6, and (d) D_5. The last three are subdiagrams of the E_8 diagram. The colors of the Weyl reflections are chosen such that they match their embedding within E_8.

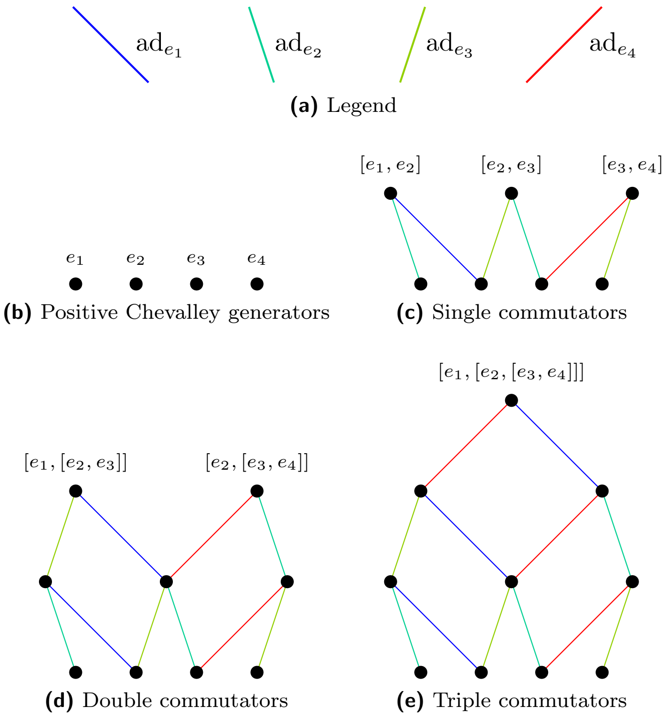


Figure 3.5: The Serre construction for A_4.

3.1.1 Visualizing the Serre construction

Hasse diagrams can serve as a neat tool to visualize the results of the Serre construction, the step-by-step construction of the full algebra from the Cartan matrix (see Example 2.2). One then has to interpret the points in the diagram not as roots, but as the generator they belong to. Furthermore, the lines can then be interpreted as the adjoint action of the respective positive Chevalley generators. Starting at the bottom, the vertical steps in the diagram then represent the steps of the Serre construction.

Figure 3.5 displays the Serre construction for the Lie algebra A_4. One starts out with just the positive Chevalley generators (Figure 3.5b). The first step is to take all (single) commutators $[e_i, e_j]$ of the positive Chevalley generators that are consistent with the Chevalley relations (2.16), the Serre relations (2.17), and the Jacobi identity (2.3), which results in Figure 3.5c. This procedure is then iterated (Figure 3.5d and 3.5e) until it no longer yields new generators.

The analogy presented above is only valid up to a certain point. The Serre construction can give you all the Lie brackets of the algebra, whereas the Hasse diagram does not contain this information. Also, if the multiplicity of a root α is greater than one, Hasse diagrams do not distinguish between the different generators of the root space $\mathfrak{g}_\alpha$.

3.2 Coxeter projections

Where Hasse diagrams try to visualize the ordering of the root system, *Coxeter projections* try to visualize its symmetry. The problem is that the full symmetry of a root system is only revealed in a space of dimension n, which is the rank of the algebra. What one can do for finite-dimensional Lie algebras, however, is project this n-dimensional space onto a carefully chosen 2-dimensional hyperplane such that the projection preserves a part of the full symmetry. The hyperplane in question is known as a *Coxeter plane* [47, 84, 16].

3.2.1 The Coxeter plane

The Coxeter plane can only be defined for finite-dimensional Lie algebras. In order to introduce it, we must first define a distinguished element of the Weyl group, known as the *Coxeter element*, w_c. It is given by the product of all fundamental Weyl reflections:

$$w_c = \prod_{i=1}^{n} w_i. \tag{3.9}$$

The Coxeter element is not unique, but depends on the choice of basis of the root system and the ordering of the above product. However, all Coxeter elements are conjugate to each other in the Weyl group, which implies they share the same properties. In particular, the order of the Coxeter element is always equal to the Coxeter number g of the Lie algebra (2.61a). Thus g is the smallest possible integer such that

$$(w_c)^g = \mathbb{1}. \tag{3.10}$$

Furthermore, it can be shown that w_c has exactly one eigenvalue equal to $e^{\frac{2\pi i}{g}}$ [16]. The corresponding (complex) eigenvector will be denoted by z:

$$w_c(z) = e^{\frac{2\pi i}{g}} z \tag{3.11}$$

Recall that the inner product is associative with respect to Weyl reflections, that is, $(w(\alpha)|\beta) = (\alpha|w(\beta))$. So upon considering the inner product between z and the action of w_c on a generic root, it follows that

$$(w_c(\alpha)|z) = (\alpha|w_c(z)) = e^{\frac{2\pi i}{g}} (\alpha|z) \,. \tag{3.12}$$

Thus when projected onto z, the Coxeter element acts as $(\frac{1}{g})^{\text{th}}$ of a rotation on all roots. This leads us to the concept of a Coxeter plane C, which is spanned by the real and imaginary parts of z:

$$C = \mathbb{R}x_c + \mathbb{R}y_c, \tag{3.13}$$

where

$$x_c = \operatorname{Re} z, \tag{3.14a}$$

$$y_c = \operatorname{Im} z. \tag{3.14b}$$

A Coxeter projection is the projection of a root system onto its Coxeter plane. Its horizontal and vertical components are respectively given by

$$\mathbb{P}_x(\alpha) = (\alpha|x_c)\,, \tag{3.15a}$$

$$\mathbb{P}_y(\alpha) = (\alpha|y_c)\,. \tag{3.15b}$$

Following [84] we will draw lines between roots that are nearest neighbors. That is, we will draw a line between roots α and β if their distance $(\alpha - \beta|\alpha - \beta)$ is minimal. The coloring of the lines depends only on their maximal distance from the origin in the projected graph.

Coxeter projections preserve the g-fold rotational symmetry of the root system, which is generically only a small part of its complete symmetry. Nonetheless the resulting graph can display a rich structure, as for example the E_8 Coxeter plane does (Figure 3.6). Note that the Coxeter projection is always mirror symmetric in the origin, because the negative roots project as $\mathbb{P}(-\alpha) = -\mathbb{P}(\alpha)$. This effectively doubles the rotational symmetry from g-fold to $2g$-fold for Lie algebras whose Coxeter number is odd. For more Coxeter projections, see Appendix B.

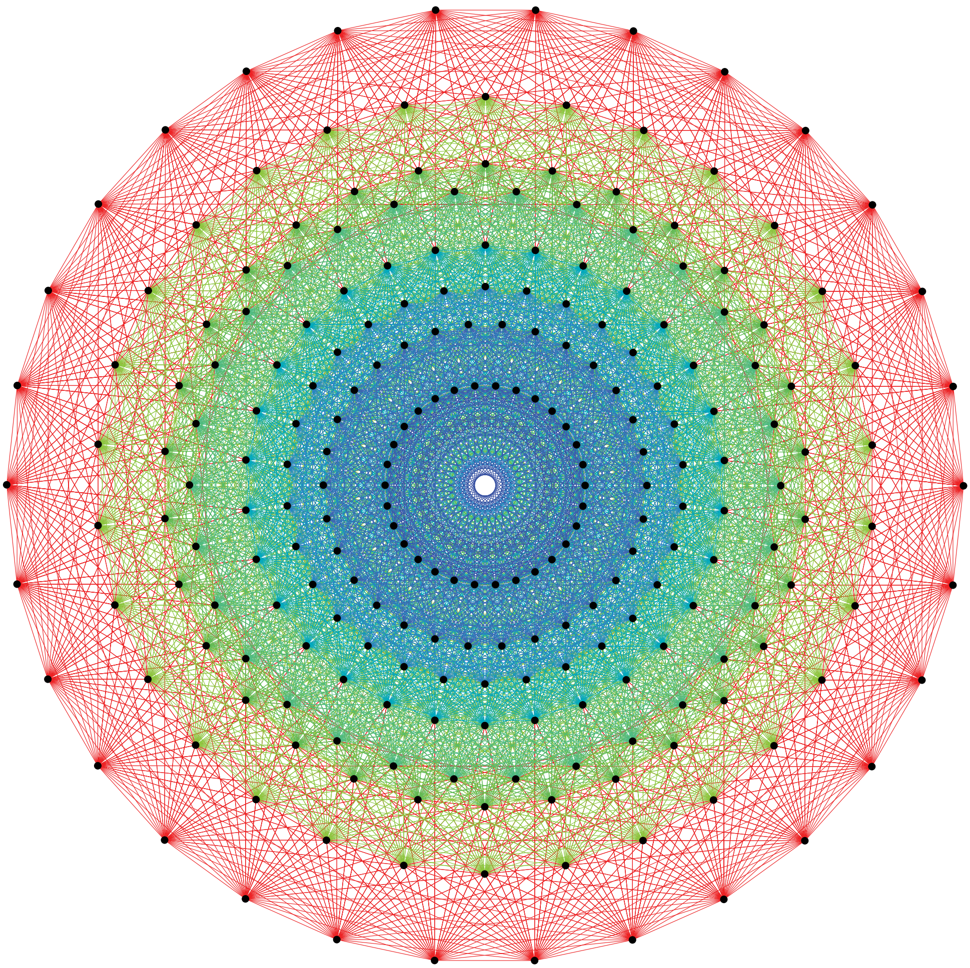


Figure 3.6: Coxeter projection of the roots of E_8.

Example 3.1: Coxeter projection of A_2

When acting on a root vector m^i, the Weyl reflections can be written as $n \times n$ matrices. By equation (2.69), the two fundamental Weyl reflections of A_2 are

$$w_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{3.16a}$$

$$w_2 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}. \tag{3.16b}$$

The Coxeter element w_c is the product of the two,

$$w_c = w_1 w_2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}. \tag{3.17}$$

We're looking for an eigenvector of w_c that has an eigenvalue of $e^{\frac{2\pi i}{3}}$, since the Coxeter number of A_2 is $g = 1 + (\rho^\vee|\alpha_1 + \alpha_2) = 3$. The eigenvector z is

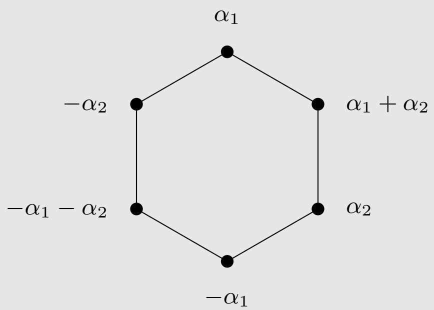
$$z = \begin{pmatrix} 1 + i\sqrt{3} \\ 2 \end{pmatrix}. \tag{3.18}$$

Expanded in terms of components, the horizontal and vertical projections of a root α with root vector m^i then respectively read

$$\mathbb{P}_x(\alpha) = A_{ij} m^i x_c^j = 3m^2, \tag{3.19a}$$

$$\mathbb{P}_y(\alpha) = A_{ij} m^i y_c^j = \sqrt{3}(2m^1 - m^2). \tag{3.19b}$$

Doing this projection for all roots of A_2 results in the following picture:



Not surprisingly, this is the same old picture of the root system we have seen before, but now rotated over an angle of 60 degrees. The lines between the roots indicate the nearest neighbour pairs.

3.2.2 Projections to subalgebras

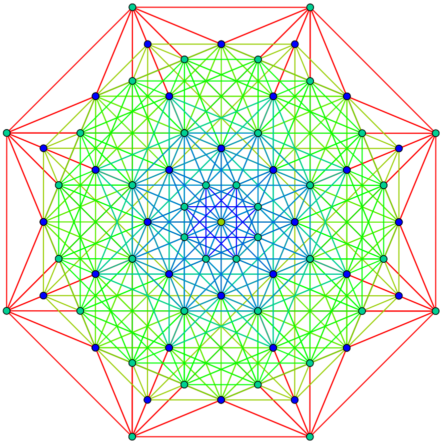
The discussion in subsection 3.2.1 is only valid for finite-dimensional Lie algebras. Although the Coxeter element can be defined for infinite-dimensional Lie algebras in a similar way, it no longer has the nice properties its finite-dimensional counterpart has. For instance, it does not have an eigenvalue of $e^{\frac{2\pi i}{g}}$. One reason for this is that the Coxeter number is ill-defined, because infinite Lie algebras do not have a highest root. A notable exception are the affine Lie algebras: despite the fact that they are infinite, one is still able to define a Coxeter number and do a Coxeter projection. More on this in section 4.1.

However, it is possible to project the root system of an infinite-dimensional Lie algebra onto the Coxeter plane of a *finite* subalgebra $\mathfrak{s}$. The finite subalgebra can be specified by picking a subset α_s $(s = 1, \ldots, n-m)$ of the simple roots α_i such that α_s generate a finite root system. The Coxeter element of $\mathfrak{s}$, denoted by w_c^{sub}, is then

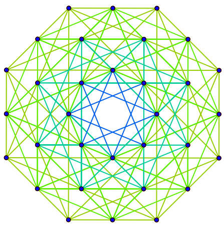
$$w_c^{\text{sub}} = \prod_{s=1}^{n-m} w_s. \tag{3.20}$$

Its order is equal to g_{sub}, the Coxeter number of $\mathfrak{s}$. The rest of analysis follows the same lines as that of subsection 3.2.1.

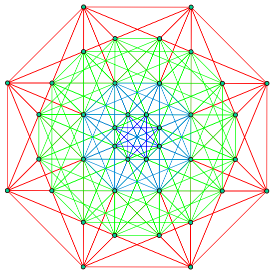
A projection onto a Coxeter plane of a subalgebra can be viewed as a level decomposition of the whole Coxeter projection. For more on level decompositions, see section 4.3. The resulting graph consists of Coxeter projections of representations of the subalgebra, stacked on top of each other. This procedure is of course not limited to infinite Lie algebras, but can also be done for finite cases. For example, in Figure 3.7 the root system of E_8 is projected onto the Coxeter plane of an A_7 subalgebra. Subalgebra projections display the g_{sub}-fold rotational symmetry of the subalgebra. As $g > g_{\text{sub}}$, the resulting picture is less symmetric than the full projection.



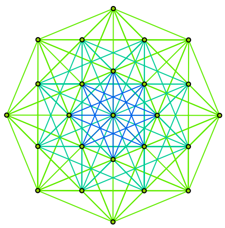
(a) The full subalgebra projection.



(b) Level 0



(c) Level 1



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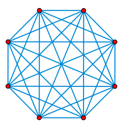

(e) Level 3

Figure 3.7: Projection of E_8 onto the Coxeter plane of an A_7 subalgebra, split into the contributions of A_7 representations at different levels.