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Kac-Moody Symmetries and Gauged Supergravity

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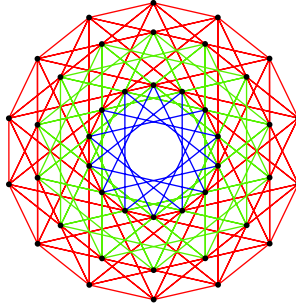
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2



Lie algebras

In this chapter the necessary mathematical concepts for describing infinite dimensional Lie algebras will be introduced. For a thorough treatment on the subject, see for example [35, 15, 50]. All the results are valid for both finite and infinite Lie algebras, unless stated otherwise.

2.1 Lie algebras

2.1.1 Basic definitions

A *Lie algebra* \mathfrak{g} is a vector space with an additional bilinear operation $[\cdot, \cdot]$ that sends two generic elements in \mathfrak{g} to another element in \mathfrak{g} :

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}. \quad (2.1)$$

This operation is called the *Lie bracket*. A defining feature of it is that the Lie bracket of an element with itself vanishes,

$$[x, x] = 0 \quad \forall x \in \mathfrak{g}. \quad (2.2)$$

Because of bilinearity the Lie bracket is automatically anti-symmetric, $[x, y] = -[y, x]$. Besides the Lie bracket, the *Jacobi identity*

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \quad \forall x, y, z \in \mathfrak{g}. \quad (2.3)$$

holds. The Lie bracket and the Jacobi identity are, as simple as they may appear, enough to endow a vector space with the rich and complex structure of a Lie algebra. In what follows, concepts and definitions indispensable to the study of Lie algebras will be introduced.

For a fixed but arbitrary element x of \mathfrak{g} the *adjoint action* is given by

$$\text{ad}_x(y) = [x, y]. \quad (2.4)$$

This is a map from \mathfrak{g} onto itself, $\text{ad}_x : \mathfrak{g} \mapsto \mathfrak{g}$. For finite Lie algebra the *Cartan-Killing form* is defined by taking traces over the adjoint action:

$$\langle x|y \rangle \propto \text{Tr}(\text{ad}_x \text{ad}_y). \quad (2.5)$$

The Cartan-Killing form is an inner product on the Lie algebra. It sends two generic elements of \mathfrak{g} to a number:

$$\langle \cdot | \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \mapsto F, \quad (2.6)$$

where F is the field over which \mathfrak{g} is a vector space.

A subspace $\mathfrak{s} \subseteq \mathfrak{g}$ of a Lie algebra \mathfrak{g} is called a Lie *subalgebra* if it is a Lie algebra in itself. In particular, it must close onto itself:

$$[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{s}, \quad (2.7)$$

which is a shorthand notation for $[x, y] \in \mathfrak{s}$ for all $x, y \in \mathfrak{s}$. If we impose the stronger condition

$$[\mathfrak{s}, \mathfrak{g}] \subseteq \mathfrak{s} \quad (2.8)$$

then \mathfrak{s} is called an *ideal* of the Lie algebra \mathfrak{g} . Proper ideals and subalgebras are those which are not equal to \mathfrak{g} or $\{0\}$.

A Lie algebra is *abelian* if its Lie bracket vanishes, $[\mathfrak{g}, \mathfrak{g}] = 0$. A *simple* Lie algebra contains no proper ideals and is not abelian. Finally, a *semi-simple* Lie algebra is a direct sum of simple ones.

The dimension $d = \dim \mathfrak{g}$ of the Lie algebra \mathfrak{g} is the dimension of \mathfrak{g} considered as a vector space. Thus we can find a basis of \mathfrak{g} consisting of d linearly independent elements t_α , which are called *generators* of \mathfrak{g} . Expanding in terms of the generators, the Lie bracket reads

$$[t_\alpha, t_\beta] = f_{\alpha\beta}{}^\gamma t_\gamma. \quad (2.9)$$

The numbers $f_{\alpha\beta}{}^\gamma$ are called the *structure constants* and characterize \mathfrak{g} completely. But because the indices α, β , and γ run over the dimension of \mathfrak{g} , the structure constants become quite untractable when dealing with infinite dimensional Lie algebras.

Instead, one usually adopts a rather different approach, in which the Lie algebra is completely defined by the so-called Cartan matrix.

Example 2.1: $\mathfrak{sl}(2)$

The Lie algebra $\mathfrak{gl}(n)$ is the vector space of $n \times n$ matrices with Lie bracket

$$[x, y] = x \cdot y - y \cdot x, \quad (2.10)$$

where \cdot stands for ordinary matrix multiplication. In this case the Jacobi identity is automatically satisfied.

If we restrict to the space of matrices with vanishing trace, the Lie algebra is denoted by $\mathfrak{sl}(n)$. The smallest non-trivial example of such a Lie algebra is $\mathfrak{sl}(2)$. One particular basis for it is

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.11a)$$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (2.11b)$$

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.11c)$$

If we calculate the Lie brackets by means of (2.10), we find

$$[h, e] = 2e, \quad (2.12a)$$

$$[h, f] = -2f, \quad (2.12b)$$

$$[e, f] = h, \quad (2.12c)$$

with all other brackets vanishing.

2.1.2 The Cartan matrix

A *Cartan matrix* $A = (A_{ij})$ is a square $n \times n$ matrix with integer values, satisfying the following conditions:

$$A_{ii} = 2, \quad (2.13a)$$

$$A_{ij} \in \mathbb{Z}_{\leq 0} \quad \text{for } i \neq j, \quad (2.13b)$$

$$A_{ij} = 0 \Leftrightarrow A_{ji} = 0, \quad (2.13c)$$

$$\det A > 0, \quad (2.13d)$$

$$M(A) > 0. \quad (2.13e)$$

Here and in the following the indices i and j run over the size of the matrix, $i, j = 1, \dots, n$. Note that the Einstein summation convention has been suspended; A_{ii} denotes the diagonal entries of the Cartan matrix, not its trace. In (2.13e) the expression $M(A)$ denotes all principal minors of A . A principal minor is the determinant of a submatrix obtained by the simultaneous removal of the same set of rows and columns. Conditions (2.13d) and (2.13e) together imply that A is positive definite. If they are dropped, the Cartan matrix is called *generalized*.

The Cartan matrix is decomposable if it can be rewritten in the form

$$A = \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix} \quad (2.14)$$

by simultaneously reordering rows and columns. Furthermore, A is assumed to be symmetrizable. This means there exists an invertible diagonal matrix $D = \text{diag}(\epsilon_1, \dots, \epsilon_n)$ such that

$$A = BD, \quad (2.15)$$

where B is a symmetric matrix.

We can associate a Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ to the Cartan matrix A by considering the $3n$ -tuple of generators $\{h_i, e_i, f_i\}$ subject to the relations

$$[h_i, h_j] = 0, \quad (2.16a)$$

$$[h_i, e_j] = A_{ji}e_j, \quad (2.16b)$$

$$[h_i, f_j] = -A_{ji}f_j, \quad (2.16c)$$

$$[e_i, f_j] = \delta_{ij}h_i, \quad (2.16d)$$

and

$$(\text{ad}_{e_i})^{1-A_{ji}}e_j = 0, \quad (2.17a)$$

$$(\text{ad}_{f_i})^{1-A_{ji}}f_j = 0. \quad (2.17b)$$

The h_i form a maximal abelian subalgebra $\mathfrak{h} \in \mathfrak{g}$ known as the *Cartan subalgebra*. The e_i and f_i are called the *Chevalley generators*. Lastly, the equations (2.17) go by the name of the *Serre relations*.

The *rank* of \mathfrak{g} is defined as the dimension of \mathfrak{h} . This in turn is equal to the size of the Cartan matrix if it is non-degenerate:

$$\text{rank}(\mathfrak{g}) = \dim \mathfrak{h} = n. \quad (2.18)$$

The full Lie algebra $\mathfrak{g}(A)$ is constructed by considering multiple commutators of the form

$$[e_i, [\dots [e_j, e_k] \dots]], \quad (2.19a)$$

$$[f_i, [\dots [f_j, f_k] \dots]]. \quad (2.19b)$$

These multiple commutators correspond to additional generators. Together with the $3n$ -tuple $\{h_i, e_i, f_i\}$ they form a basis of \mathfrak{g} . However, we must keep in mind that the Serre relations (2.17) will put certain commutators to zero, and that others are related to each other by the Jacobi identity (2.3).

A generic element of \mathfrak{g} that is not part of \mathfrak{h} is either a combination of multiple commutators of the e_i or of multiple commutators of the f_i . Thus the Lie algebra \mathfrak{g} possesses a triangular decomposition:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+. \quad (2.20)$$

The negative part \mathfrak{n}_- consists of commutators of the form (2.19b), whereas the positive part \mathfrak{n}_+ consists of commutators of the form (2.19a). The negative and positive parts Chevalley generators can be interchanged by means of the *Chevalley involution* ω , which is defined as

$$\omega(e_i) = -f_i, \quad (2.21a)$$

$$\omega(f_i) = -e_i, \quad (2.21b)$$

$$\omega(h_i) = -h_i. \quad (2.21c)$$

The Chevalley involution can consistently be extended to the whole of \mathfrak{g} , exchanging \mathfrak{n}_+ and \mathfrak{n}_- . It therefore suffices to study only one of the two.

In the Chevalley basis introduced above, the Cartan-Killing form (2.5) reads

$$\langle h_i | h_j \rangle = \epsilon_i A_{ij} = \epsilon_i \epsilon_j B_{ij}, \quad (2.22a)$$

$$\langle e_i | f_j \rangle = \delta_{ij} \epsilon_i. \quad (2.22b)$$

All other combinations vanish. Here ϵ_i is the i^{th} diagonal entry of the matrix D in equation (2.15). The Cartan-Killing form can be uniquely extended by induction to the whole of \mathfrak{g} by requiring that it is invariant, i.e. $\langle [x, y] | z \rangle = \langle x | [y, z] \rangle$.

The relations (2.16) and (2.17) summarize the structure of \mathfrak{g} in a very compact form, namely in the Cartan matrix A . The procedure of constructing the Lie algebra from the Cartan matrix is known as the *Serre construction*. A Lie algebra \mathfrak{g} obtained in this way is always finite and semi-simple. It is simple if the Cartan matrix A is indecomposable. If conditions (2.13d) and (2.13e) are dropped, the Lie algebra may become infinite. Although it is not immediately clear from the discussion here, the argument also works the other way around: *any* finite simple Lie algebra can be completely described in terms of a Cartan matrix. The task of classifying all possible finite simple Lie algebras thus boils down to finding all possible indecomposable matrices that satisfy (2.13). The result is the so-called Cartan classification. But before stating it, it is useful to introduce the concept of a Dynkin diagram.

Example 2.2: Serre construction

The simplest Cartan matrix is the one of rank 1:

$$A = (2). \quad (2.23)$$

The resulting Lie algebra has three generators, h, e , and f . This is in fact $\mathfrak{sl}(2)$, the same Lie algebra as in Example 2.1. If we inspect (2.16) more closely, we see that the building blocks of every Lie algebra of rank n are n interconnected $\mathfrak{sl}(2)$ subalgebras.

A slightly more complicated Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \quad (2.24)$$

Because its rank is equal to two, the resulting Lie algebra now has at least six generators, h_1, h_2, e_1, e_2, f_1 , and f_2 . But there are two brackets that are not put to zero by the Serre relations, which correspond to additional generators:

$$e_{1+2} \equiv [e_1, e_2], \quad (2.25a)$$

$$f_{1+2} \equiv [f_1, f_2]. \quad (2.25b)$$

If we try to take further Lie brackets, we see that they are killed by the Serre relations:

$$[e_1, e_{1+2}] = [e_1, [e_1, e_2]] = (\text{ad}_{e_1})^{1+1} e_2 = 0, \quad (2.26a)$$

$$[f_1, f_{1+2}] = [f_1, [f_1, f_2]] = (\text{ad}_{f_1})^{1+1} f_2 = 0. \quad (2.26b)$$

The Lie algebra is 8-dimensional, and isomorphic to $\mathfrak{sl}(3)$.

Dynkin diagrams

The data contained in a Cartan matrix can be neatly visualized with the help of Dynkin diagrams. A *Dynkin diagram* consists of a number of nodes that are connected by lines. Given a Cartan matrix, the rules for drawing such a diagram are simple:

- For every row of the Cartan matrix A , draw one node.
- Nodes corresponding to rows i and j ($i \neq j$) are connected if $A_{ij} \neq 0$.
- The connection consists of $\max(|A_{ij}|, |A_{ji}|)$ lines.


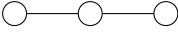
Lie algebra	Dynkin diagram	Cartan matrix
$\mathfrak{sl}(3)$		$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
$\mathfrak{sl}(4)$		$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Table 2.1: The Dynkin diagrams and Cartan matrices of $\mathfrak{sl}(3)$ and $\mathfrak{sl}(4)$.

- If $|A_{ij}| > |A_{ji}|$, the connection has an arrow pointing towards node i from node j .
- If the connection has an arrow and if $\min(|A_{ij}|, |A_{ji}|) > 1$, the connection has an additional label indicating $\min(|A_{ij}|, |A_{ji}|)$.

Furthermore, a diagram and its associated Lie algebra are called *simply laced* if the diagram contains no arrows, or equivalently, if the Cartan matrix is symmetric.

Two simple examples of Dynkin diagrams are given in Table 2.1. The complete classification of all finite-dimensional simple Lie algebras [36] is given in Figure 2.1. There are four infinite series: A_n , B_n , C_n , and D_n , where the n denotes the rank of the algebra. The A_n series are isomorphic to the $\mathfrak{sl}(n+1)$ Lie algebras, which we encountered earlier. The four series are infinite in the sense that the rank n can take on any value, while the resulting algebra remains finite. Furthermore there are five exceptional cases: E_6 , E_7 , E_8 , F_4 , and G_2 .

Besides these five isolated cases and the four infinite series, there are no other finite-dimensional simple Lie algebras. However, if we relax the conditions (2.13d) and (2.13e) and allow for generalized Cartan matrices, the range of options increases dramatically. Lie algebras associated to generalized Cartan matrices are called *Kac-Moody algebras*. When the Cartan matrix is not positive definite, they are infinite-dimensional (see also chapter 4).

2.1.3 Roots

Thus far we have seen that a Lie algebra can be characterized either by all its Lie brackets (or equivalently its structure constants), or by its associated Cartan matrix. There is a third way of describing Lie algebras that is closely linked to both previously described ways. The key object in this case is the *root system*.

A *root* α of a generator $x \in \mathfrak{g}$ is the eigenvalue of x under the adjoint operation

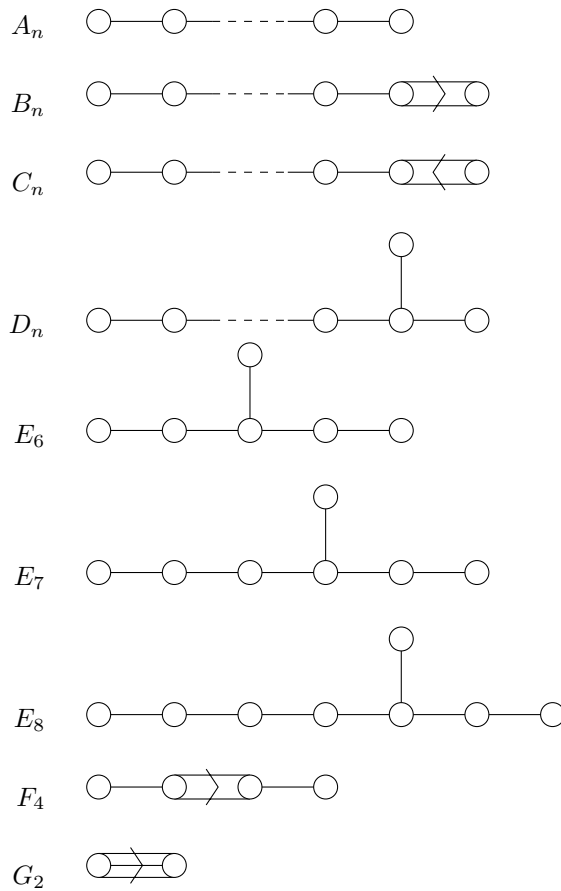


Figure 2.1: Dynkin diagrams of all finite dimensional simple Lie algebras. The subscript denotes the rank of the Lie algebra, or equivalently, the number of nodes.

of a generic element $h \in \mathfrak{h}$:

$$[h, x] = \alpha_x(h)x. \quad (2.27)$$

The root α is just a number. For a fixed element $x \in \mathfrak{g}$ it also acts as a linear function as $\alpha_x : \mathfrak{h} \mapsto F$, sending an element of the Cartan subalgebra to F , the base field over which \mathfrak{g} is a vector space. Thus roots are elements of the space dual to \mathfrak{h} ,

$$\alpha \in \mathfrak{h}^* \equiv \Phi. \quad (2.28)$$

The space Φ will be called the *root space*. The root space is a vector space with the same dimension as \mathfrak{h} , namely $\dim \Phi = n$. As a side note, the function α is only called a root if it is non-zero. Thus the generators in the Cartan subalgebra do not have a root α_h associated to them.

If we have a specific basis h_i of \mathfrak{h} , the *simple roots* α_i are defined as

$$\alpha_i(h_j) = A_{ij}, \quad (2.29)$$

where A is again the Cartan matrix. Inspection of (2.16) reveals that the simple roots are the eigenvalues of the Chevalley generators e_i under the adjoint action of h_j , that is, $\alpha_i = \alpha_{e_i}$. The simple roots also form a basis of the root space Φ . To be a bit more precise, we can consider the root of a generic Lie bracket:

$$[h, [x, y]] = (\alpha_x(h) + \alpha_y(h))[x, y]. \quad (2.30)$$

This follows straightforwardly from the Jacobi identity (2.3). The roots add up: $\alpha_{[x,y]} = \alpha_x + \alpha_y$. But because all generators are constructed of multiple brackets of the Chevalley generators (see (2.19)), this means that all roots are linear integral combinations of the simple roots:

$$Q = \sum_{i=1}^n \mathbb{Z}\alpha_i. \quad (2.31)$$

The lattice Q is called the *root lattice*. However, by courtesy of the Serre relations (2.17), not all points on the root lattice are actual roots. The set of all proper roots is called the *root system* and denoted by $\Delta (\subset Q \subset \Phi)$.

A generic root α can be expanded in terms of the simple roots as

$$\alpha = \sum_{i=1}^n m^i \alpha_i. \quad (2.32)$$

The numbers m^i are collectively called the *root vector*. Because of the triangular decomposition (2.20) they are either all non-negative ($m^i \geq 0$) or all non-positive

($m^i \leq 0$). In the former case the root is said to be *positive*, and in the latter it is *negative*. Thus the root system Δ splits into two disjoint sets,

$$\Delta = \Delta_+ \cup \Delta_-, \quad (2.33)$$

where Δ_+ contains all the positive roots, and Δ_- contains all the negative roots. Because of the Chevalley involution (2.21) it holds that $\Delta_- = -\Delta_+$. It is therefore enough to study either Δ_+ or Δ_- .

The *root space* \mathfrak{g}_α of a root α is the set of generators of \mathfrak{g} that has α as an eigenvalue under \mathfrak{h} . Thus:

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}. \quad (2.34)$$

The dimension of this space is known as the *multiplicity* of the root, $\text{mult}(\alpha)$. It is the same as the number of generators that share the same root α . For finite Lie algebras the multiplicity is always equal to one, whereas for infinite Lie algebras the multiplicity can degenerate. In section 2.3 we will deal with how to calculate the root multiplicities.

By (2.30) the Lie algebra is graded by means of its roots. In particular, we have for the root spaces

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}. \quad (2.35)$$

The complete Lie algebra \mathfrak{g} can be decomposed into its root spaces as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha. \quad (2.36)$$

The reason why the Cartan subalgebra appears separately in the direct sum is that roots were defined to be non-zero.

The Cartan matrix not only specifies what the simple roots are, it also defines an inner product between them. Because the simple roots are the basis vectors of the root space Φ , the inner product acts on the whole root space:

$$(\cdot|\cdot) : \Phi \times \Phi \mapsto F. \quad (2.37)$$

It is given by the following definition,

$$A_{ij} \equiv 2 \frac{(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)}, \quad (2.38)$$

from which we can deduce the actual root space metric $B = (B_{ij})$:

$$B_{ij} \equiv (\alpha_i|\alpha_j) = \frac{A_{ij}}{\epsilon_j}. \quad (2.39)$$

Here we have set $\epsilon_i = \frac{2}{(\alpha_i|\alpha_i)}$ (compare equation (2.15)). From equation (2.22) it is clear that the inner product on the root space corresponds to the restriction of the Cartan-Killing form to the CSA. The inner product is unique, but only up to normalization. This means we have the freedom to choose a norm $\alpha_i^2 = (\alpha_i|\alpha_i)$ of one of the simple roots, after which all the others are fixed by

$$\frac{(\alpha_i|\alpha_i)}{(\alpha_j|\alpha_j)} = \frac{A_{ij}}{A_{ji}}. \quad (2.40)$$

If the Cartan matrix is decomposable, we have to fix the normalization once for every indecomposable subpart.

In the mathematical literature it is common to fix the normalization such that the simple roots have at most norm equal to 2, i.e. $\alpha_i^2 \leq 2$. However, I will adhere to the convention that the simple roots have *at least* norm equal to 2, i.e. $\alpha_i^2 \geq 2$. The reason is that in the latter case $\frac{1}{\epsilon_i}$, and thus also the metric on the root space, has only integer values. Note that if the Cartan matrix is symmetric to start with, then it coincides with the root space metric. In that case the norms of all the simple roots are equal, and the algebra is called simply laced.

The inner product between two roots α and β can easily be expanded in their respective roots vectors m^i and n^i as

$$(\alpha|\beta) = \sum_{i,j=1}^n B_{ij} m^i n^j. \quad (2.41)$$

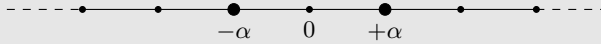
For finite Lie algebras the Cartan matrix A and the root space metric B are positive definite. This entails that root norms are always positive, $\alpha^2 > 0$. However, when conditions (2.13d) and (2.13e) are dropped, A may become indefinite, allowing for null and negative directions. This prompts us to distinguish between the following cases:

$$(\alpha|\alpha) = \begin{cases} > 0 & \text{real root,} \\ = 0 & \text{imaginary (null) root,} \\ < 0 & \text{imaginary root.} \end{cases} \quad (2.42)$$

Thus finite Lie algebras have only real roots, whereas infinite algebras also have imaginary roots.

Example 2.3: Root system of A_1 and A_2

The Lie algebra A_1 is of rank one. This means it has only one simple root α which is given by $[h, e] = \alpha e$. The root system of A_1 consists of just two roots, namely α and $-\alpha$, the latter belonging to the generator f . The root lattice is the line of integers, as depicted in the following image.

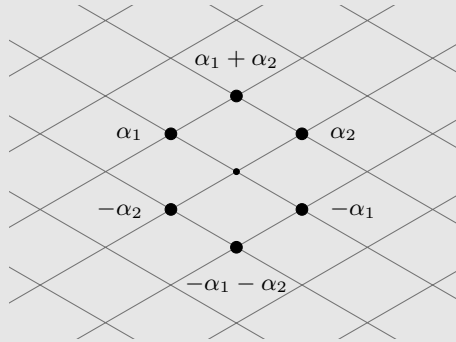


The roots are indicated with big dots, whereas other points on the root lattice that do not correspond to roots are indicated with smaller dots.

The root system of A_2 is a bit more interesting. Being of rank two, it has two simple roots α_1 and α_2 . The root space is thus a two-dimensional vector space, spanned by α_1 and α_2 . There are six roots in total: the two simple roots and their negatives, and the root of e_{1+2} and its negative (see Example 2.2). By (2.30), the root of e_{1+2} is given by

$$[h, e_{1+2}] = [h, [e_1, e_2]] = (\alpha_1 + \alpha_2)e_{1+2}. \quad (2.43)$$

The picture of the root lattice is as follows:



The roots are again indicated with the big dots, while the other vertices correspond to points on the lattice that are not roots. The small dot in the middle is the origin.

You may have noticed that the angle between the simple roots is not 90° , but 120° . The reason is that the angle is fixed by the inner product $(\cdot|\cdot)$ in the usual way: $\cos \theta = \frac{(\alpha_1|\alpha_2)}{\sqrt{\alpha_1^2\alpha_2^2}} = -\frac{1}{2}$.

2.1.4 Weights

In this section the so-called *weights* will be introduced. Their importance lies in the fact they can be used to describe representations of Lie algebras (see section 2.2). But before weights are discussed, it is convenient to associate to any root α a *coroot* α^\vee by

$$\alpha^\vee = \frac{2\alpha}{(\alpha|\alpha)}. \quad (2.44)$$

Amongst others, this simplifies the expression of the Cartan matrix in terms of the inner product (2.38) a bit:

$$A_{ij} = (\alpha_i|\alpha_j^\vee). \quad (2.45)$$

We are now in the position to define the *fundamental weights* Λ^i as the duals of the simple coroots,

$$(\Lambda^i|\alpha_j^\vee) = \delta_j^i. \quad (2.46)$$

The fundamental weights span the weight space dual to the root space Φ , and its elements are called weights. A generic weight λ can be expressed in terms of the fundamentals weights as

$$\lambda = \sum_{i=1}^n p_i \Lambda^i. \quad (2.47)$$

The coefficients p_i are called the *Dynkin labels* of the weight. The lattice P on which the Dynkin labels are strictly integers, i.e.

$$P = \sum_{i=1}^n \mathbb{Z} \Lambda^i, \quad (2.48)$$

is called the *weight lattice* and is dual to the root lattice, $P = Q^*$.

For convenience we can also introduce *cofundamental weights* $\Lambda^{\vee i}$ as the duals of the simple roots:

$$(\Lambda^{\vee i}|\alpha_j) = \delta_j^i, \quad (2.49a)$$

$$\Lambda^{\vee i} = \frac{2}{(\alpha_i|\alpha_i)} \Lambda^i. \quad (2.49b)$$

The cofundamental weights are not as important as the fundamental weights; their main use is to simplify certain notation.

When we compare equations (2.45) and (2.46) we see that the fundamental

weights and the simple roots can be expressed in terms of each other as

$$\Lambda^i = \sum_{j=1}^n (A^{-1})^{ij} \alpha_j, \quad (2.50a)$$

$$\alpha_i = \sum_{j=1}^n A_{ij} \Lambda^j. \quad (2.50b)$$

It is often of interest to know what the Dynkin labels of a particular root are. To that end, we can simply expand a root in both the simple root and the fundamental weight bases, and equate the coefficients. Upon doing so, we see that the root vector m^i and Dynkin labels p_i of a root α are related by

$$p_i = (\alpha | \alpha_i^\vee) = \sum_{j=1}^n A_{ji} m^j, \quad (2.51a)$$

$$m^i = (\alpha | \Lambda^{\vee i}) = \sum_{j=1}^n (A^{-1})^{ji} p_j. \quad (2.51b)$$

Following [53], the root vector will be written as (m^1, \dots, m^n) and the Dynkin labels as $[p_1, \dots, p_n]$ in order to distinguish between the different bases. Note that points on the root lattice always lie on the weight lattice as well, because the Cartan matrix contains only integers. However, the converse is not necessarily true, as the inverse of the Cartan matrix may contain fractional entries.

The inner product on the root space can be extended by linearity to an inner product on the weight space. Given two weights λ and μ with respective Dynkin labels p_i and q_i , it can be computed to be

$$(\lambda | \mu) = \sum_{i,j=1}^n G^{ij} p_i q_j. \quad (2.52)$$

Here G is the metric on the weight space, and is also called the *quadratic form matrix*. A short calculation reveals that it is related to the Cartan matrix by

$$G^{ij} \equiv (\Lambda^i | \Lambda^j) = \frac{1}{2} (A^{-1})^{ij} (\alpha_j | \alpha_j). \quad (2.53)$$

Similarly to the metric on the root space, this expression simplifies when the algebra is simply laced. Then the quadratic form matrix is just the inverse of the Cartan matrix.

Example 2.4: Weight lattice of A_2

The Cartan matrix of the Lie algebra A_2 is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad (2.54)$$

which has an inverse given by

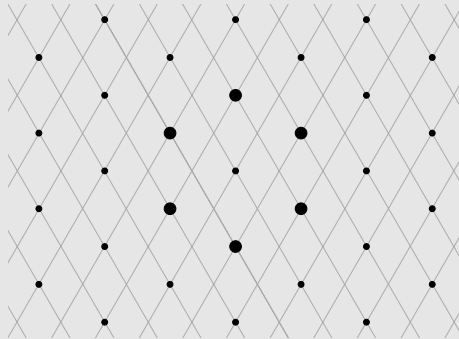
$$A^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \quad (2.55)$$

By equation (2.50a), the fundamental weights are

$$\Lambda^1 = \frac{1}{3} (2\alpha_1 + \alpha_2) \quad (2.56a)$$

$$\Lambda^2 = \frac{1}{3} (\alpha_1 + 2\alpha_2) \quad (2.56b)$$

These two fundamental weights form the basis of the weight lattice, which is drawn below. All the intersections of lines correspond to points on the weight lattice, i.e. weights with integer Dynkin labels. Superposed on the weight lattice are the roots of A_2 (indicated with big dots) and the other points on the root lattice (indicated with small dots). See also Example 2.3.



Note that not all points on the weight lattice coincide with points on the root lattice. This is due to the fact that the inverse Cartan matrix has a factor $\frac{1}{3}$ in front, stemming from its determinant.

2.1.5 The Weyl vectors and the highest root

Two distinguished elements of the weight space are the *Weyl vector* ρ and the *dual Weyl vector* ρ^\vee . They can be defined as

$$(\rho|\alpha_i^\vee) = 1, \quad (2.57a)$$

$$(\rho^\vee|\alpha_i) = 1. \quad (2.57b)$$

Expanded in the basis of fundamental weights, both read

$$\rho = \sum_{i=1}^n \Lambda^i, \quad (2.58a)$$

$$\rho^\vee = \sum_{i=1}^n \Lambda^{\vee i}. \quad (2.58b)$$

The Dynkin labels of ρ are thus all equal to one. The Weyl vector will play an important role in the analysis of representations. The dual Weyl vector on the other hand has fractional Dynkin labels when the algebra is not simply laced. Its main usage is to calculate the *height* of a root, which is simply the sum of the components of its root vector m^i :

$$\text{ht}(\alpha) = (\alpha|\rho^\vee) = \sum_{i=1}^n m^i. \quad (2.59)$$

Simple finite Lie algebras have a unique root θ that is the highest of all roots in the root system Δ . This root is called the *highest root*. Similarly, $-\theta$ is the *lowest root* of Δ . Infinite Kac-Moody algebras on the contrary do not have a highest root; their root system ‘just goes on forever’.

The components of root vector of the (dual) highest root are called the (*dual*) *Coxeter labels*, and are denoted by a^i and $a^{\vee i}$, respectively:

$$\theta = \sum_{i=1}^n a^i \alpha_i, \quad (2.60a)$$

$$\theta^\vee = \sum_{i=1}^n a^{\vee i} \alpha_i^\vee, \quad (2.60b)$$

where $\theta^\vee = \frac{2\theta}{(\theta|\theta)}$. Their respective sums plus one are known as the (*dual*) *Coxeter number* g of the Lie algebra,

$$g = 1 + (\theta|\rho^\vee) = 1 + \sum_{i=1}^n a^i, \quad (2.61a)$$

$$g^\vee = 1 + (\theta^\vee|\rho) = 1 + \sum_{i=1}^n a^{\vee i}. \quad (2.61b)$$

2.1.6 The Weyl group

As is apparent from Example 2.3, the root systems of some of the simplest of Lie algebras have already quite some symmetry. Not surprisingly, the amount of symmetry of the root system tends to increase with its size. This symmetry is captured and described in what is known as the *Weyl group* $W(\Delta)$ of the root system.

The Weyl group is a reflection group [52], generated by so-called *Weyl reflections* w_α . They are defined as

$$w_\alpha(\beta) = \beta - (\beta|\alpha^\vee)\alpha. \quad (2.62)$$

They are proper reflections in the sense that they square to the identity:

$$w_\alpha(w_\alpha(\beta)) = \beta, \quad (2.63)$$

or more succinctly, $w_\alpha^2 = \mathbb{1}$.

A Weyl reflection w_α is a reflection with respect to a hyperplane perpendicular to a fixed root α (see also Figure 2.2). For example, when a root gets reflected with respect to itself, the result is the negative of that root: $w_\alpha(\alpha) = -\alpha$. By virtue of the triangular decomposition (2.20) this is also a root. The same holds for *any* Weyl reflection: the result always lies in the root system. Thus the *orbit* $W(\alpha)$ of a root,

$$W(\alpha) = \bigcup_{w \in W} w(\alpha), \quad (2.64)$$

i.e. is its image under all the elements of the Weyl group, lies in the root system. Moreover, any points on the root lattice between β and $w_\alpha(\beta)$ are also roots:

$$\beta - q\alpha \in \Delta, \quad q \in \{0, \dots, (\beta|\alpha^\vee)\}. \quad (2.65)$$

This fact yields an iterative procedure to construct the whole root system from just the simple roots. First, consider the orbit of the simple roots. Next, consider the orbits of the ‘gaps’ of the first orbit, and so on and so forth. This procedure truncates at some point for the finite Lie algebras, but it does not for infinite algebras. In the latter case the best one can do is to calculate the root system up to some given height.

The size of the Weyl group can become rather large. For instance, the Weyl group of A_n has $(n+1)!$ elements, which makes for difficult bookkeeping of every single reflection when n increases. Luckily, any element $w \in W$ can be written as a successive combination of *fundamental reflections* w_i ,

$$w = w_{i_1} w_{i_2} \cdots w_{i_k}. \quad (2.66)$$

The *length* $l(w)$ of a Weyl reflection is the minimal number of fundamental reflections needed to write w in the above form. The fundamental reflections are Weyl reflections in the simple roots,

$$w_i \equiv w_{\alpha_i}. \quad (2.67)$$

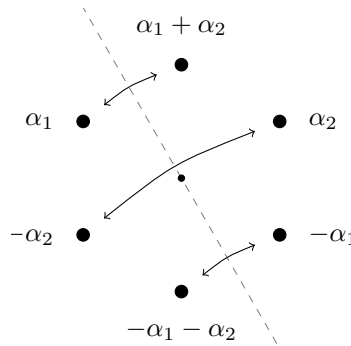


Figure 2.2: The action of $w_2 = w_{\alpha_2}$ on the root system of A_2 . The dashed line indicates the hyperplane through the origin perpendicular to α_2 . The solid lines with arrows depict the action of w_2 .

The whole Weyl group W is thus generated by just the fundamental reflections, of which there are always the same number as n , the rank of the algebra. When we let the fundamental reflections act on the simple roots, equation (2.62) becomes

$$w_i(\alpha_j) = \alpha_j - A_{ji}\alpha_i. \quad (2.68)$$

This means that for a generic root with root vector m^j , the fundamental Weyl reflections act as

$$m^j \xrightarrow{w_i} m^j - p_i \delta_i^j, \quad (2.69)$$

where p_i are the Dynkin labels of the root. Thus a Weyl reflection w_i increases the height of a root if its corresponding Dynkin label p_i is negative, decreases its height if it is positive, and leaves it invariant if it is zero.

We can therefore introduce the set P_- , which contains all roots and weights that have non-positive Dynkin labels:

$$P_- = \left\{ \lambda = \sum_{i=1}^n p_i \Lambda^i \mid p_i \leq 0 \right\}. \quad (2.70)$$

The subset $P_- \subset P$ of the weight lattice is called the *fundamental Weyl chamber*. By equation (2.69), fundamental Weyl reflections on elements of P_- never decrease their height. Furthermore, the Weyl group acts transitively on P_- . This means that by acting with the Weyl group on all the roots in the fundamental chamber, one obtains the full root system. P_- is also a fundamental domain of W ; thus on the Weyl orbit of a weight, exactly one point lies in P_- .

The images of the fundamental chamber under the Weyl group are also called Weyl chambers, albeit not fundamental. Their boundaries are hyperplanes perpen-

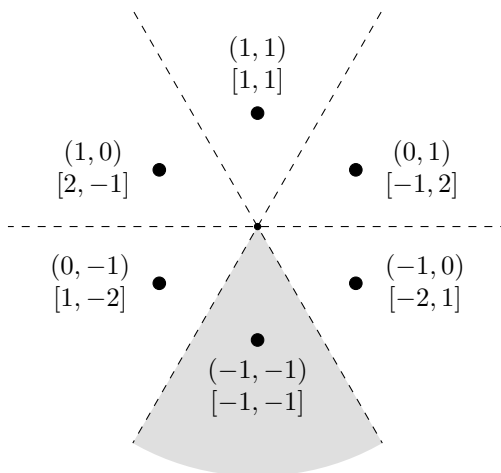


Figure 2.3: The Weyl chambers of the root system of A_2 . The fundamental chamber is shaded in gray. The numbers (m^1, m^2) form the root vector, and $[p_1, p_2]$ are the Dynkin labels. The dashed lines are the hyperplanes perpendicular to roots.

dicular to roots through the origin. Figure 2.3 shows the Weyl chambers of the Lie algebra A_2 .

As already might be apparent from the discussion above, the action of the Weyl group can be extended by linearity from roots to weights. The Weyl reflections then act on weights in a similar manner as on roots,

$$w_\alpha(\lambda) = \lambda - (\lambda|\alpha^\vee)\alpha. \quad (2.71)$$

Here λ is a generic weight. The action of the Weyl group on weights will be used in the analysis of representations.

2.1.7 A bound on root norms

In subsection 2.1.6 it was argued that the whole root system Δ can be constructed from the Weyl orbit of the simple roots, the orbits of the ‘gaps’ in the orbits of the simple roots, and so on and so forth. This, together with that fact that all Weyl reflections can be written as combinations of fundamental reflections, can be used to derive a bound on the norm of all roots in the root system.

First note that the inner product is associative with respect to Weyl reflections, i.e. $(w(\alpha)|\beta) = (\alpha|w(\beta))$. From this it follows that Weyl reflections preserve the norm of roots:

$$(w(\beta)|w(\beta)) = (\beta|\beta). \quad (2.72)$$

So we only need to worry about roots that lie on gaps of fundamental reflections. Consider a root γ that lies in a gap between the root β and its i^{th} fundamental reflection:

$$\gamma = \beta - q\alpha_i, \quad 0 < |q| < |p_i|, \quad (2.73)$$

where $p_i = (\beta|\alpha_i^\vee)$ is the i^{th} Dynkin label of β , and q has the same sign as p_i . Calculating the norm of γ , we find

$$\frac{(\gamma|\gamma) - (\beta|\beta)}{(\alpha_i|\alpha_i)} = q^2 - qp_i < 0. \quad (2.74)$$

As the norm of simple roots is always positive, $(\alpha_i|\alpha_i) > 0$, the norm of γ must be smaller than that of β . Therefore all roots $\alpha \in \Delta$ satisfy

$$(\alpha|\alpha) \leq \alpha_{\max}^2 \quad (2.75)$$

where $\alpha_{\max}^2 = \max((\alpha_i|\alpha_i))$ is the norm of the longest simple root.

2.2 Representations

So far Lie algebras have been described as objects on their own, which describe some form of symmetry. But the usefulness of describing this symmetry lies often in letting it act on objects that are symmetric. The mathematical concept of acting with a Lie algebra on another object is called a *representation* of the Lie algebra.

The object in question will be a vector space V . The fact that it carries symmetry is captured in the set $\mathfrak{gl}(V)$, which contains all linear transformations that send V to itself:

$$\mathfrak{gl}(V) : V \mapsto V. \quad (2.76)$$

Then the precise definition of ‘acting on’ is that there exists a map ψ from the Lie algebra \mathfrak{g} to $\mathfrak{gl}(V)$,

$$\psi : \mathfrak{g} \mapsto \mathfrak{gl}(V), \quad (2.77)$$

which preserves the Lie bracket structure on V ,

$$\psi_x\psi_y - \psi_y\psi_x = \psi_{[x,y]}, \quad (2.78)$$

for all $x, y \in \mathfrak{g}$. Strictly speaking, the map ψ is called the representation, and the space V the representation space or \mathfrak{g} -module. But as is common in the literature, the term ‘representation’ will refer to both in this thesis.

Concretely, a representation is a set of vectors $\{v_\lambda, v_\mu, \dots\}$ which by the action of ψ get mapped onto each other:

$$\psi_x v_\lambda = v_\mu. \quad (2.79)$$

The module V is the span of these vectors. Now, because the Cartan subalgebra \mathfrak{h} is abelian, it is possible to find a particular basis for V on which the representations of \mathfrak{h} act diagonally:

$$\psi_h v_\lambda = \lambda(h)v_\lambda. \quad (2.80)$$

Here $\lambda(h)$ is nothing more than a number, and is called the weight of the vector v_λ . For the elements in the basis of \mathfrak{h} , its specific value is given by

$$\lambda(h_i) = (\lambda|\alpha_i^\vee) = p_i. \quad (2.81)$$

Here p_i are the Dynkin labels of the weight λ (compare equation (2.47)). Similar to how the Lie algebra decomposes into a direct sum of root spaces (see equation (2.36)), the module V splits up in a direct sum of weight spaces,

$$V = \bigoplus_{\lambda \in P(V)} V_\lambda. \quad (2.82)$$

The weight spaces V_λ are the sets of vectors in V that have λ as a weight under the action of \mathfrak{h} . The object $P(V)$ is the collection of all weights of V , and is called the *weight diagram*. It must not be confused with the weight lattice P (2.48). The question of which weights are part of the weight diagram will be addressed in the next section.

Lastly, the *multiplicity* of a weight is the dimension of its weight space, that is, $\text{mult}_V(\lambda) = \dim(V_\lambda)$. It follows straightforwardly that the dimension of V is given by the sum of weight multiplicities,

$$\dim(V) = \sum_{\lambda \in P(V)} \text{mult}_V(\lambda). \quad (2.83)$$

In section 2.3 we will see how to calculate weight multiplicities.

Example 2.5: Two common representations

One particularly simple representation is the one where all elements $x \in \mathfrak{g}$ get mapped onto zero:

$$\psi_x = 0. \quad (2.84)$$

The underlying module is one-dimensional. This representation is the trivial or *singlet* representation.

Another example is the representation where the module is taken to be \mathfrak{g} itself, and the map ψ is the adjoint action:

$$\psi_x = \text{ad}_x. \quad (2.85)$$

This representation is called the *adjoint* representation.

2.2.1 Integrable lowest weight representations

All irreducible finite-dimensional modules of finite Lie algebras fall into the class of so-called *highest weight representations*. Here instead I will discuss *lowest weight representations*, which are identical in structure. The main difference from highest weight representations is that they are not classified by their highest weight, but *lowest weight vector* v_Λ that satisfies

$$\psi_h v_\Lambda = \Lambda(h)v_\Lambda, \quad (2.86a)$$

$$\psi_{f_i} v_\Lambda = 0, \quad (2.86b)$$

for all $h \in \mathfrak{h}$ and all $i = 1, \dots, n$. The lowest weight vector is thus annihilated by the action of the negative Chevalley generators. It follows that any element of \mathfrak{n}_- annihilates it, $\psi(\mathfrak{n}_-)v_\Lambda = 0$. Because the representation map preserves the Lie bracket, the weight increases with a simple root when we act on it with a positive Chevalley generator:

$$\psi_h(\psi_{e_i} v_\Lambda) \propto (\Lambda(h) + \alpha_i(h))v_\Lambda \quad (2.87)$$

Hence Λ is indeed the *lowest weight* of the module. It uniquely characterizes the module, which is therefore denoted by $V(\Lambda)$. The whole module is generated by the action of \mathfrak{n}_+ on the lowest weight vector,

$$\psi_{\mathfrak{n}_+} v_\Lambda = V(\Lambda). \quad (2.88)$$

When the underlying Lie algebra is finite, the module V itself is finite. This implies that besides a lowest weight it has also highest weight vector $v_{\Lambda'}$ which is annihilated by the positive Chevalley generators,

$$\psi_{e_i} v_{\Lambda'} = 0. \quad (2.89)$$

Thus for finite Lie algebras every lowest weight representation is simultaneously a highest weight representation. This is not the case for infinite Lie algebras, as their lowest weight representations are not bounded from above.

A representation $V(\Lambda)$ is *integrable* when the action of all positive and negative Chevalley generators is locally nilpotent. That is, the repeated action of the same step operator must yield zero at some point:

$$(\psi_{e_i})^p v_\lambda = 0, \quad (2.90a)$$

$$(\psi_{f_i})^q v_\lambda = 0, \quad (2.90b)$$

for all $v_\lambda \in V(\Lambda)$ and for some finite positive integers p and q . A necessary and sufficient condition for this is that the weight diagram of the representation lies on the weight lattice P ,

$$P(V(\Lambda)) \subset P. \quad (2.91)$$

By equation (2.87) this condition simplifies to the requirement that the lowest weight Λ lies on the weight lattice, because the roots do too. In the rest of this thesis the prefix ‘integrable’ will often be dropped, because all lowest weight representations discussed here will be integrable.

Let us now turn to the question of determining the weight diagram $P(V(\Lambda))$ of a module $V(\Lambda)$. It can be proven that $P(V(\Lambda))$ is invariant under the action of the Weyl group, which provides us with sufficient tools to determine the actual structure of $P(V(\Lambda))$.

First, it is convenient to define the *height* of a weight λ by the amount of simple roots it differs from the lowest weight Λ :

$$\text{ht}_\Lambda(\lambda) = (\lambda - \Lambda | \rho^\vee). \quad (2.92)$$

By this definition, the lowest weight has height zero. But in order for it to be truly a lowest weight, it has to lie in the fundamental Weyl chamber: then all Weyl reflections increase its height. This is most easily seen when we look at the action of a fundamental Weyl reflection on a generic weight:

$$w_i(\lambda) = \lambda - (\lambda | \alpha_i^\vee) \alpha_i = \lambda - p_i \alpha_i, \quad (2.93)$$

where p_i are the Dynkin labels of λ . So indeed, when the Dynkin labels are all non-positive, the height of the weight can only increase under Weyl reflections. Weights that lie in the fundamental Weyl chamber are called *dominant* weights.

Having established that Λ is dominant, the full weight diagram can now be constructed from Λ by considering its orbit under the Weyl group. Weights that lie on this orbit belong to $P(V(\Lambda))$,

$$W(\Lambda) \subseteq P(V(\Lambda)). \quad (2.94)$$

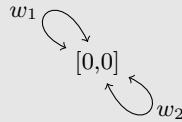
Moreover, weights that lie ‘in between’ reflections also belong to the weight diagram, although not to the same orbit. Thus for a weight $\lambda \in P(V(\Lambda))$, the points on the line between λ and $w_i(\lambda)$ that differ by a single simple root α_i are part of the weight diagram:

$$\lambda - q\alpha_i \in P(V(\Lambda)), \quad q \in \{0, \dots, (\lambda | \alpha_i^\vee)\}. \quad (2.95)$$

This is enough information to construct $P(V(\Lambda))$. First, consider the orbit of the lowest weight. Next, consider the orbits of the ‘gaps’ in the first orbit, and so on and so forth until the weight diagram closes. For infinite Lie algebras, one cannot calculate the full weight diagram. There one has to be content to calculate it up to a given height.

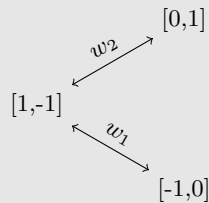
Example 2.6: Weight diagrams of A_2

Let's have a look at some weight diagram of representations of A_2 . The smallest diagram is that of the trivial or *singlet* representation. This is the representation with zero lowest weight (i.e. all Dynkin labels are zero). This is also its only weight. The weight diagram is thus as follows:



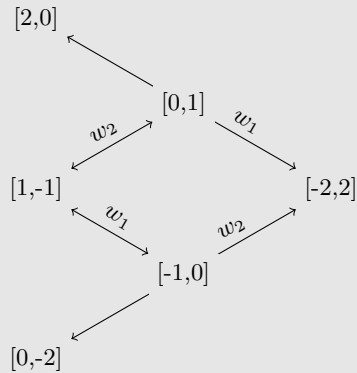
The lines with arrows depict the fundamental Weyl reflections, and the numbers $[p_1, p_2]$ are the Dynkin labels of a weight. w_1 and w_2 map the $[0, 0]$ weight onto itself, because all its Dynkin labels are zero. In the following, Weyl reflections that leave weights invariant will not be drawn.

The next simplest weight diagram is that of the $\Lambda = -1\Lambda^1 + 0\Lambda^2$ lowest weight representation. The Weyl orbit of the lowest weight consists of three weights in total:



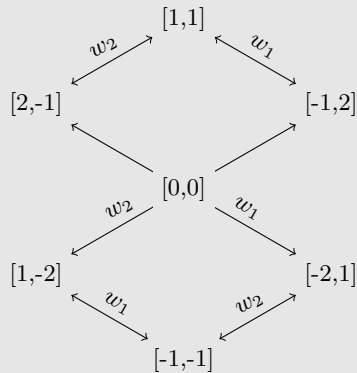
There are no gaps in the reflections, which means the weight diagram consists of one single orbit. Note that reflection w_i flips the sign of the i th Dynkin label.

A more involved weight diagram is that of the $\Lambda = 0\Lambda^1 - 2\Lambda^2$ lowest weight representation:



Here the w_2 reflection brings the lowest weight $[0, -2]$ to $[-2, 2]$. This is indicated by the missing arrows on the $[-1, 0]$ weight; the w_2 reflection ‘passes through’ it. By equation (2.95), $[-1, 0]$ is then also a weight of the diagram, although it doesn’t lie on the same orbit. Hence this diagram consists of two Weyl orbits, one of which is the $[-1, 0]$ weight diagram.

Yet another example is weight diagram of the lowest weight representation associated to the lowest root of A_2 , which is $\Lambda = -\theta = -\Lambda^1 - \Lambda^2$.



This weight diagram is identical to the root system of A_2 . This not a coincidence of A_2 , but is actually valid for all finite Lie algebras if the lowest weight is taken to be the lowest root. The resulting representation is then the adjoint.

2.3 Multiplicities

So far we have studied the question which points on the root and weight lattice are elements of the root system and weight diagram, respectively. What remains to be done is to calculate their multiplicity, that is, the degeneracy of the root and weight spaces.

A convenient way to store the final answer for lowest weight representations is its *character* \mathcal{X} ,

$$\mathcal{X}_V = \sum_{\lambda \in P(V)} \text{mult}_V(\lambda) e^\lambda. \quad (2.96)$$

This is a sum over the formal exponents of all the weights in the weight diagram. It can be shown [51] that the character satisfies the *Weyl-Kac character formula*

$$\mathcal{X}_{V(\Lambda)} = \frac{\sum_{w \in W} \epsilon(w) e^{w(\Lambda + \rho) - \rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult } \alpha}}. \quad (2.97)$$

Here the function ϵ is given by $\epsilon(w) = (-1)^{l(w)}$, with l being the length of the Weyl reflection. Not only do weight multiplicities follow from the Weyl-Kac character formula, it can also be used to calculate root multiplicities. One then has to evaluate it for the singlet representation, $\Lambda = 0$, whose character is equal to one. The result is

$$\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult } \alpha} = \sum_{w \in W} \epsilon(w) e^{w(\rho) - \rho}. \quad (2.98)$$

This is known as the *denominator identity*.

Although the denominator identity can in principle be used to calculate root multiplicities, a shorter and faster way is to use the *Peterson recursion formula* [71], which is derived from it. It reads

$$(\alpha|\alpha - 2\rho) c_\alpha = \sum_{\substack{\beta, \gamma \in Q_+ \\ \beta + \gamma = \alpha}} (\beta|\gamma) c_\beta c_\gamma. \quad (2.99)$$

The coefficient c_α is given by

$$c_\alpha = \sum_{k \geq 1} \frac{1}{k} \text{mult} \left(\frac{\alpha}{k} \right), \quad (2.100)$$

and will be called the *co-multiplicity* of the root α . The factors in the co-multiplicity for which α/k is not a root do not contribute to the sum, as their multiplicity is zero. As indicated in (2.99), the factors β and γ of α do not have to be roots. But

in order for them to contribute to the sum, they do need to be integer multiples of roots. Otherwise their co-multiplicity would vanish.

Using the Peterson recursion formula, it is possible to calculate the multiplicities of all roots. Starting from the simple roots, which have multiplicity one, you can inductively work your way up in the root system. For higher and higher roots the calculation will get more involved, as the number of contributions to the sum will increase.

For the weight multiplicities a similar recursion formula can be deduced, this time from the Weyl-Kac character formula [71]. It reads

$$\text{mult}_\Lambda(\lambda) = \frac{2 \sum_{\alpha \in \Delta_+} \text{mult}(\alpha) \sum_{k \geq 1} (-\alpha | \lambda - k\alpha) \text{mult}_\Lambda(\lambda - k\alpha)}{(\Lambda - \rho | \Lambda - \rho) - (\lambda - \rho | \lambda - \rho)}. \quad (2.101)$$

This is a generalization of the Freudenthal recursion formula for finite Lie algebras, in which case the root multiplicities are all equal to one. Starting from the lowest weight Λ , which has multiplicity one, all other weight multiplicities can be calculated by induction on height.

Luckily it is not necessary to calculate the multiplicities for all roots and weights. It can namely be shown that Weyl reflections preserve multiplicities:

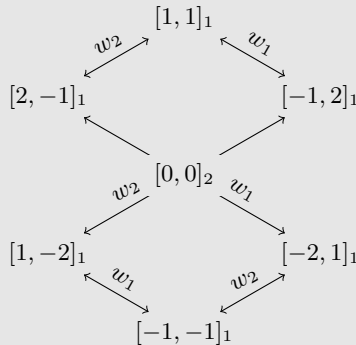
$$\text{mult}(w(\alpha)) = \text{mult}(\alpha), \quad (2.102a)$$

$$\text{mult}_\Lambda(w(\lambda)) = \text{mult}_\Lambda(\lambda). \quad (2.102b)$$

It therefore suffices to calculate the multiplicity once for every Weyl orbit.

Example 2.7: Weight multiplicities of the adjoint A_2 representation

Recall that the weight diagram of the adjoint representation of A_2 was as follows (see Example 2.6):



The subscripts on the Dynkin labels are the multiplicities of the weight. By definition the multiplicity of the lowest weight $[-1, -1]$ is one. As Weyl reflections preserve the multiplicity, all other weights on its orbit also have multiplicity one.

There is just one weight that does not lie on the orbit of $[-1, -1]$, namely $[0, 0]$. Since this weight corresponds to the Cartan subalgebra of A_2 , its multiplicity is bound to be equal to two. But let us invoke the Freudenthal recursion formula (2.101) to prove it. First we need to determine for which roots and k the weight $\lambda - k\alpha$ for $\lambda = 0$ still lies in the weight diagram. They are

$$0 - \alpha_2 = \Lambda^1 - 2\Lambda^2, \quad (2.103a)$$

$$0 - \alpha_1 = -2\Lambda^1 + \Lambda^2, \quad (2.103b)$$

$$0 - \alpha_1 - \alpha_2 = -\Lambda^1 - \Lambda^2. \quad (2.103c)$$

The integer k is one in all cases, as otherwise the resulting weight would lie outside the diagram. The double sum in (2.101) simplifies to one that runs over the roots α_1, α_2 , and $\alpha_1 + \alpha_2$. The multiplicity of $\lambda = 0$ can then be evaluated to give

$$\begin{aligned} \text{mult}_{-\Lambda^1 - \Lambda^2}(0) &= 2 \frac{(\alpha_1|\alpha_1) + (\alpha_2|\alpha_2) + (\alpha_1 + \alpha_2|\alpha_1 + \alpha_2)}{(\Lambda^1 + \Lambda^2|\Lambda^1 + \Lambda^2) + 2(\Lambda^1 + \Lambda^2|\rho)} \\ &= \frac{2(2 + 2 + 2)}{2 + 4} = 2. \end{aligned} \quad (2.104)$$

2.4 Real forms

Up to this point the field F over which \mathfrak{g} is a vector space has not been specified. In the classification of Lie algebras one assumes that F is algebraically closed, and one usually takes the complex numbers, $F = \mathbb{C}$. In that case, the Cartan matrix A uniquely characterizes the Lie algebra $\mathfrak{g}(A)$. However, the Lie algebras that pop up in the context of supergravities (see chapter 5) are vector spaces over the real numbers, \mathbb{R} . As \mathbb{R} is not algebraically closed, there can be multiple non-isomorphic Lie algebras associated to one single Cartan matrix. These different real Lie algebras are called the various *real forms* of the complex algebra.

A real form can be characterized by the signature of its Cartan-Killing form. Note that this wouldn't make sense for an algebra over the complex numbers, as rescaling generators with a factor of i would effectively change the sign of their norm. The generators of real forms can be classified by the sign of their norm: *compact* generators have negative norm, and *non-compact* generators have positive norm. Real forms of Lie algebras are often denoted by $X_{n(m)}$, where X is the type of algebra (A, B, \dots), n is its rank, and m the difference between the number of non-compact and compact generators.

It can be shown that the Cartan-Killing form pairs the root spaces of a root and its negative in a non-degenerate manner:

$$\langle \mathfrak{g}_\alpha | \mathfrak{g}_\beta \rangle = 0 \text{ if } \alpha + \beta \neq 0. \quad (2.105)$$

The Cartan-Killing form on the full algebra in the triangular decomposition (2.20) can therefore be written as

$$\langle \cdot | \cdot \rangle = \begin{pmatrix} DBD & 0 & 0 \\ 0 & 0 & C \\ 0 & C & 0 \end{pmatrix} \begin{matrix} \mathfrak{h} \\ \mathfrak{n}_+ \\ \mathfrak{n}_- \end{matrix}. \quad (2.106)$$

Here DBD is the Cartan-Killing form on \mathfrak{h} (see equation (2.22)), and C is a positive diagonal matrix:

$$C = \text{diag}(\langle y | y^T \rangle, \dots, \langle z | z^T \rangle). \quad (2.107)$$

The elements $\{y, \dots, z\}$ are a basis of \mathfrak{n}_+ , and their transpose $\{y^T, \dots, z^T\}$ a basis of \mathfrak{n}_- . The (generalized) transpose is defined by means of the Chevalley involution (2.21),

$$x^T = -\omega(x). \quad (2.108)$$

As the Cartan-Killing form on \mathfrak{h} has the same signature as the Cartan matrix A , we only need to calculate the signature on the subspace $\mathfrak{n}_+ \oplus \mathfrak{n}_-$. We can diagonalize that part of the Cartan-Killing form by taking the combinations $x + x^T$ and $x - x^T$, which results in

$$\langle \cdot | \cdot \rangle = \begin{pmatrix} DBD & 0 & 0 \\ 0 & 2C & 0 \\ 0 & 0 & -2C \end{pmatrix} \begin{matrix} \mathfrak{h} \\ \mathfrak{p} \ominus \mathfrak{h} \\ \mathfrak{l} \end{matrix}. \quad (2.109)$$

The subspaces \mathfrak{p} and \mathfrak{l} are respectively the odd and even eigenspaces of \mathfrak{g} under the Chevalley involution:

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l}, \quad (2.110)$$

where

$$\mathfrak{p} = \{x \in \mathfrak{g} \mid \omega(x) = -x\}, \quad (2.111a)$$

$$\mathfrak{l} = \{x \in \mathfrak{g} \mid \omega(x) = x\}. \quad (2.111b)$$

The space \mathfrak{p} is spanned by elements of the form $x + x^T$, and \mathfrak{l} by elements of the form $x - x^T$.

From (2.109) we can easily read of the signature of the Cartan-Killing form. For finite Lie algebras the Cartan matrix is always positive definite. Thus if we simply restrict the field F to be \mathbb{R} , then the signature is

$$(\#\text{non-compact}, \#\text{compact}) = \left(\frac{\dim \mathfrak{g} + n}{2}, \frac{\dim \mathfrak{g} - n}{2} \right), \quad (2.112)$$

where n is the rank of the finite algebra. This particular real form is called the *split real form* or the *maximal non-compact* real form. The difference between the number of non-compact and compact generators is always equal to the rank of the algebra. The split real form of an algebra X_n can thus be denoted by $X_{n(+n)}$.

The split real form is not the only real form of a Lie algebra. This can be seen by inspecting the decomposition (2.110). The Lie brackets of the subspaces read

$$[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}, \quad (2.113a)$$

$$[\mathfrak{l}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad (2.113b)$$

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{l}. \quad (2.113c)$$

Thus \mathfrak{l} is not just a subspace of \mathfrak{g} , but also a subalgebra. From (2.109) it is clear that the Cartan-Killing form on \mathfrak{l} is negative definite. Because all the generators of \mathfrak{l} are compact, it is called the *maximal compact subalgebra* of \mathfrak{g} . Real forms in which all the generators are compact, are not surprisingly called *compact* real forms.

The split and compact real forms are the two ‘extreme’ real forms of a Lie algebra. The former has the maximal number of non-compact generators, while the latter has the maximal number of compact generators (namely all). In between these two options there usually lies an array of other possibilities. However, in this thesis we will only encounter the split real form and the compact real form.

Example 2.8: Split real form of A_2

Recall from Example 2.2 that the 8 generators of A_2 are

$$h_1, h_2 \in \mathfrak{h}, \quad (2.114a)$$

$$e_1, e_2, e_{1+2} \in \mathfrak{n}_+, \quad (2.114b)$$

$$f_1, f_2, f_{2+1} \in \mathfrak{n}_-, \quad (2.114c)$$

where $e_{1+2} = [e_1, e_2]$ and $f_{2+1} = [f_2, f_1]$. To work out the Cartan-Killing form, we can either explicitly calculate the adjoint action of all generators and take traces (equation (2.5)), or extend it by invariance from \mathfrak{h} to the whole of \mathfrak{g} (equation (2.22)). In either case, we get

$$\langle \cdot | \cdot \rangle = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} h_1 \\ h_2 \\ e_1 \\ e_2 \\ e_{1+2} \\ f_1 \\ f_2 \\ f_{2+1} \end{matrix} \quad (2.115)$$

The generalized transpose of the generators are

$$\begin{aligned} h_1^T &= h_1, & e_1^T &= f_1, & e_{1+2}^T &= -[\omega(e_1), \omega(e_2)] \\ h_2^T &= h_2, & e_2^T &= f_2, & &= +f_{2+1}. \end{aligned} \quad (2.116a)$$

Going to the basis (2.111) of the $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l}$ decomposition, the Cartan-Killing form becomes

$$\langle \cdot | \cdot \rangle = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix} \begin{matrix} h_1 \\ h_2 \\ e_1 + f_1 \\ e_2 + f_2 \\ e_{1+2} + f_{2+1} \\ e_1 - f_1 \\ e_2 - f_2 \\ e_{1+2} - f_{2+1} \end{matrix} \quad (2.117)$$

The split real form $A_{2(+2)}$ of A_2 thus has 5 non-compact and 3 compact generators. The subspace \mathfrak{l} is indeed a subalgebra, because its commutators close:

$$[e_1 - f_1, e_2 - f_2] = e_{1+2} - f_{2+1}, \quad (2.118a)$$

$$[e_{1+2} - f_{2+1}, e_1 - f_1] = e_2 - f_2, \quad (2.118b)$$

$$[e_{1+2} - f_{2+1}, e_2 - f_2] = e_1 - f_1. \quad (2.118c)$$

2.5 Cosets and non-linear sigma models

Consider a real Lie group G and its maximal compact subgroup $K(G) \in G$. The (left) *coset* is then defined as the set of equivalence classes of elements $g \in G$ [30]

$$g \sim g' \quad \text{iff} \quad g' = gk, \quad (2.119)$$

with $k \in K(G)$. That is, elements g and g' of G are identified if they are related by an element of the subgroup $K(G)$. Thus the coset is a quotient space, and is denoted by $G/K(G)$. Its dimension is given by

$$\dim G/K(G) = \dim G - \dim K(G). \quad (2.120)$$

We can build a dynamical theory on the coset space as follows. Let ϕ^α be local coordinates on the coset, and $G_{\alpha\beta}(\phi)$ the metric. In addition we have a ‘space-time’ manifold M with coordinates x^μ and metric $\gamma_{\mu\nu}$. If we let $\phi^\alpha = \phi^\alpha(x)$ be functions of x^μ , we can view them as maps from the space-time M to the target space $G/K(G)$. See also Figure 2.4. The dynamics of these mappings are governed by an action of the form

$$S = - \int_M dx \sqrt{\gamma} \gamma^{\mu\nu} G_{\alpha\beta}(\phi) \partial_\mu \phi^\alpha \partial_\nu \phi^\beta. \quad (2.121)$$

For historical reasons, this particular dynamical realization is called a *non-linear sigma model*. In the analysis below we will consider the case when M is one-dimensional and parameterized by the coordinate t . The action then simplifies to

$$S = - \int dt n(t)^{-1} G_{\alpha\beta}(\phi) \partial \phi^\alpha \partial \phi^\beta, \quad (2.122)$$

where $\partial \equiv \partial_t$. The function $n(t) = \sqrt{\gamma}$ ensures reparameterization invariance in the coordinate t .

The action (2.122) can be constructed explicitly by introducing a t dependent group element $V(t) \in G$ that transforms as

$$V(t) \longrightarrow V'(t) = gV(t)k(t), \quad (2.123)$$

where $g \in G$ and $k(t) \in K(G)$. Dropping the explicit coordinate dependence, the Maurer-Cartan form of the group element V reads

$$J = V^{-1} \partial V \in \mathfrak{g}. \quad (2.124)$$

The Maurer-Cartan form is Lie algebra-valued. If \mathfrak{g} is the algebra of G , then the algebra of $K(G)$ is \mathfrak{l} , the maximal compact subalgebra of \mathfrak{g} . Hence J can be decomposed in a part that belongs to \mathfrak{l} , and a part that belongs to its complement, \mathfrak{p} (see

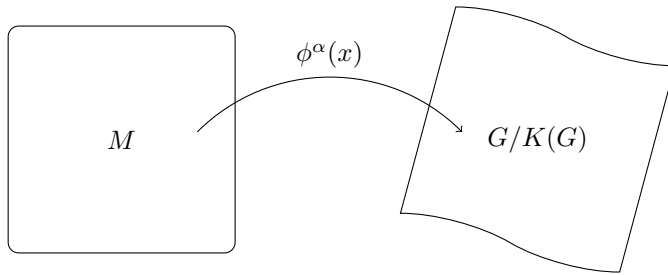


Figure 2.4: The coordinates ϕ^α of the coset as maps from the space-time M to the coset.

equation (2.110)):

$$P = \frac{1}{2} (J + J^T) \in \mathfrak{p}, \quad (2.125a)$$

$$Q = \frac{1}{2} (J - J^T) \in \mathfrak{l}. \quad (2.125b)$$

Here the transpose $()^T$ is the generalized transpose (2.108). The subspace \mathfrak{p} can be interpreted as the ‘coset-part’ of the algebra \mathfrak{g} . The action (2.122) can then be written as

$$S = - \int dt n(t)^{-1} \langle P|P \rangle, \quad (2.126)$$

where $\langle \cdot | \cdot \rangle$ is the usual Cartan-Killing form on \mathfrak{g} . This action has a global (rigid) G invariance and a local $K(G)$ gauge invariance. The gauge invariance allows us to parameterize $V(t)$ as

$$V(t) = e^{\phi^\alpha(t)t_\alpha}, \quad (2.127)$$

where the generators t_α only take values in the so-called *Borel gauge* (compare (2.20)),

$$t_\alpha \in \mathfrak{h} \oplus \mathfrak{n}_+. \quad (2.128)$$

The Borel gauge is particularly convenient if we want to calculate the Maurer-Cartan form (2.124) explicitly. This can be done with the help of some Baker-Campbell-Hausdorff formulas,

$$e^{-A} \partial e^A = \partial A + \frac{1}{2} [\partial A, A] + \frac{1}{3!} [[\partial A, A], A] + \dots, \quad (2.129a)$$

$$e^{-A} B e^A = B + [B, A] + \frac{1}{2} [B, A] + \dots. \quad (2.129b)$$

To illustrate the above analysis, we will conclude with two simple examples.

Example 2.9: Non-linear realization of A_1

The three generators of A_1 were h, e , and f . Thanks to the Borel gauge, we only need two of them to write down our group element:

$$V(t) = e^{\psi(t)e} e^{\phi(t)h}. \quad (2.130)$$

This group element is related via a coordinate transformation to a group element of the form $e^{\psi'e + \phi'h}$. We will use the former because in that case the Maurer-Cartan form is easier to calculate. It reads

$$J = V^{-1} \partial V = h \partial \phi + e \exp(-2\phi) \partial \psi. \quad (2.131)$$

The coset element P becomes

$$P = \frac{1}{2}(J + J^T) = h \partial \phi + \frac{1}{2}(e + f) \exp(-2\phi) \partial \psi. \quad (2.132)$$

Finally, the action is

$$S = \int dt n(t)^{-1} \left(-2\partial\phi\partial\phi - \frac{1}{2}e^{-4\phi}\partial\psi\partial\psi \right). \quad (2.133)$$

In the next example we will treat the non-linear realization of A_2 . In that case we could in principle use the same approach as in the previous example. However, for future use it will be more convenient to tackle this problem in a slightly different way.

Example 2.10: Non-linear realization of A_2

In the adjoint representation the generators of A_2 can be written as traceless 3×3 matrices. Specifically,

$$h_1 = K^1_1 - K^2_2, \quad e_1 = K^1_2, \quad f_1 = K^2_1, \quad [e_1, e_2] = K^1_3, \quad (2.134a)$$

$$h_2 = K^2_2 - K^3_3, \quad e_2 = K^2_3, \quad f_2 = K^3_2, \quad [f_2, f_1] = K^3_1, \quad (2.134b)$$

where the 3×3 matrices K^a_b are given by

$$(K^a_b)^i_j = \delta^{ai} \delta_{bj} - \frac{1}{3} \delta_b^a \delta_j^i. \quad (2.135)$$

The indices a, b label the different matrices, and the i, j indices indicate the rows and columns of the matrices. Both sets run from one to three. The Lie bracket then reads

$$[K^a{}_b, K^c{}_d] = \delta_b^c K^a{}_d - \delta_d^a K^c{}_b, \quad (2.136)$$

and the Cartan-Killing form becomes

$$\langle K^a{}_b | K^c{}_d \rangle = \text{Tr}(K^a{}_b \cdot K^c{}_d) = \delta_d^a \delta_b^c - \frac{1}{3} \delta_b^a \delta_d^c. \quad (2.137)$$

The group element V can then be written as

$$V(t) = \exp\left(h_a{}^b(t) K^a{}_b\right) = e_a{}^b(t) K^a{}_b, \quad (2.138)$$

where $h_a{}^b$ is a generic traceless matrix, and $e_a{}^b = \exp(h_a{}^b)$ is its standard matrix exponential. Hence, the latter has unit determinant. Because V transforms under global G transformations from the left, and local $K(G)$ transformations from the right, the upper and lower indices of $e_a{}^b$ transform differently. To indicate this, we write

$$V = e_m{}^a K^m{}_a. \quad (2.139)$$

The index m transforms under global G transformations, and the index a under local $K(G)$ transformations. Hence $e_m{}^a$ behaves as a vielbein on the coset space. With this distinction in place, we proceed to calculate the Maurer-Cartan form and the coset element:

$$J = e_a{}^m \partial e_m{}^b K^a{}_b, \quad (2.140a)$$

$$P = e_a{}^m \partial e_m{}^b S^a{}_b, \quad (2.140b)$$

where $e_a{}^m$ is the inverse vielbein, and $S^a{}_b$ the basis elements of the coset:

$$S^a{}_b = \frac{1}{2} \left(K^a{}_b + K^b{}_a \right). \quad (2.141)$$

Note that here the generalized transposed is equal to the ordinary matrix transpose. The inner product of the coset basis is easily evaluated to give

$$\langle S^a{}_b | S^c{}_d \rangle = \delta_d^a \delta_b^c - \frac{1}{3} \delta_b^a \delta_d^c. \quad (2.142)$$

If we introduce the metric $g_{mn} = \delta_{ab} e_m{}^a e_n{}^b$, the action reads

$$\begin{aligned} S &= -\frac{1}{4} \int dt n(t)^{-1} \left(g^{mp} g^{nq} - \frac{1}{3} g^{mn} g^{pq} \right) \partial g_{mn} \partial g_{pq} \\ &= -\frac{1}{4} \int dt n(t)^{-1} g^{mp} g^{nq} \partial g_{mn} \partial g_{pq}. \end{aligned} \quad (2.143)$$

Here g^{mn} is the inverse metric. The last step follows from the fact that, like the vielbein, both the metric and its inverse have unit determinant.

