Kac-Moody Symmetries and Gauged Supergravity

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Introduction

1.1 Physical background

The two great successes of physics in the last century were the discovery of General Relativity by Einstein and the construction of the Standard Model. Both were guided by what may be called the principle of symmetry. For Einstein this implied that the laws of physics should be the same for all observers, whether they are upside down or not, and standing still or accelerating [86]. For the Standard Model it means that its predictions are invariant under a big set of transformations known as $SU(3) \times SU(2) \times U(1)$ [18].

Perhaps the great failure of physics of the last century has been the unsuccessfulness of combining the two into one single theory. This theory would ideally unify the four forces of nature (the strong and weak interaction, electromagnetism, and gravity) into one single description. Although not ultimately successful, one of the more promising candidates for unification is *string theory* [39, 40]. The basic idea of string theory is to replace the point-like particles of the Standard Model by one-dimensional objects known as strings. All the known elementary particles should then correspond to different vibrations of the string. The biggest attraction of string theory, besides 'smoothing out' the infinities that are inherit to the framework of
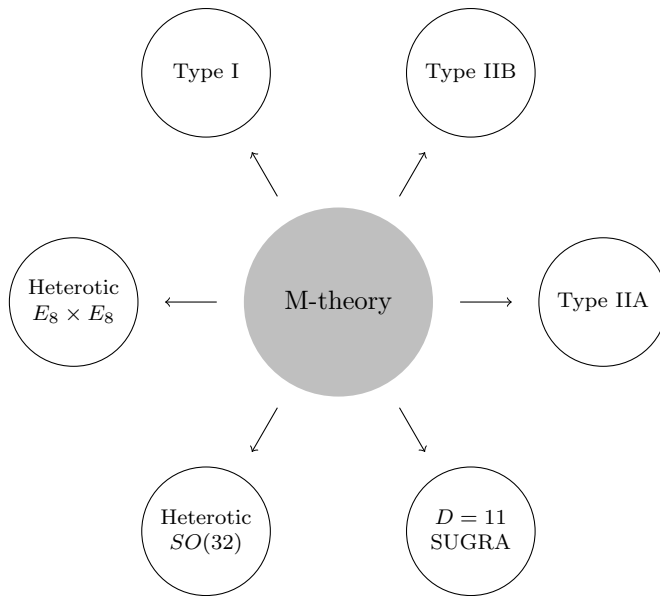
Figure 1.1: The limits of M-theory. The limits are known, M-theory is not.

the Standard Model, is that one of its vibrations gives rise to the graviton. As such, string theory naturally incorporates gravity.

For a theory that combines all the forces of nature, you would expect there to be only one. However, there is no one single unique string theory. Instead there are no less than five self-consistent string theories. They go by the (not very poetic) names of Type I, Type IIA, Type IIB, Heterotic $E_8 \times E_8$, and Heterotic $SO(32)$. They all carry some degree of supersymmetry (a symmetry between bosons (forces) and fermions (matter)), and all live in ten space-time dimensions. This did not bode well for string theory, until it was realized that these five theories are related by some form of symmetry, known as string dualities [46]. Furthermore, it was conjectured that they were all some limit of a yet unknown theory in eleven dimensions, dubbed *M-theory* [43]. Little was, and still is, known about M-theory. All we (think we) know is its low-energy limit: the unique eleven-dimensional supersymmetric gravity (supergravity) theory.

Like the theory itself, it remains a guess as to what its symmetries are. But once they are known, the principle of symmetry may guide our search for a concrete formulation of M-theory. One requirement of the symmetries of M-theory is that they should at least contain the dualities that tie the five string theories together. A further hint towards the symmetries of M-theory may come from supergravity:
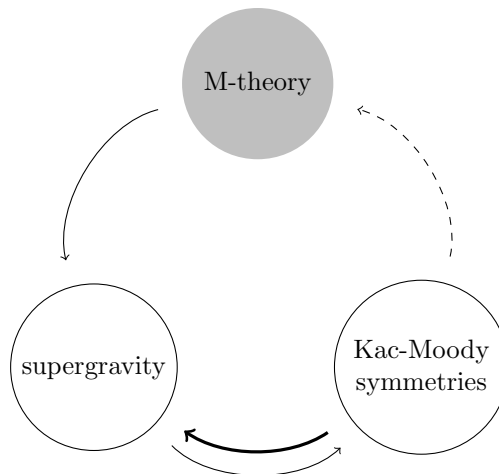
Figure 1.2: The interrelations between M-theory, supergravity, and Kac-Moody symme-
tries. This thesis focuses on the bold arrow from Kac-Moody symmetries to
supergravity.

in certain lower-dimensional limits supergravity exhibits an infinite amount of sym-
metry [42]. These symmetries go by the collective name of *Kac-Moody symmetries*.
As they contain the duality symmetries of string theory [54], Kac-Moody symme-
tries may be conjectured to describe the symmetry of M-theory [23, 87]. However,
in this thesis we will not go that far (see also Figure 1.2). Instead, we will show that
Kac-Moody symmetries play a unifying role in supergravity, and consequently may
contain little bits and pieces of information on M-theory.

1.2   From symmetry to groups

Symmetry. Not only makes it our world round, but it's also what makes it *go* round.
From the perfect circular wheels on our bikes and cars that deliver an enjoyable ride,
to the error-correction protocols that keep e-mails from turning into junk [64]; it's
literally all around us. It's also symmetry that dictates the laws of nature. On
the small scale the symmetry group $SU(3) \times SU(2) \times U(1)$ of the Standard Model
controls the interactions in molecules, atoms, and nuclei. On the large scale gravity
is governed by Einstein's symmetry principle of our space-time.

But what is symmetry exactly? Let us first consider the principle of symmetry in
physics. It can be formulated as the fact that a physical process remains a physical
process after it has been transformed by a symmetry. In Figure 1.3 person 1 is
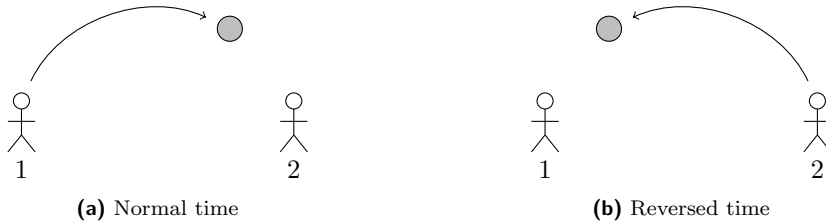
(a) Normal time   **(b)** Reversed time

Figure 1.3: Person 1 throwing a ball to person 2 (a), and the other way around if we flip the time (b). Both are valid physical processes.

throwing a ball to person 2. If we reverse the time direction, the process is altered: it is now person 2 who's throwing a ball to person 1. A different process, but a valid one nonetheless. We can therefore say that Newton's laws of physics have a time reversal symmetry.

In the more mathematical sense, symmetry is an action on an object that, once you're done performing it, does not change that object. This is a very abstract definition, but we can try to illustrate it with a simple equilateral triangle. The triangle (see Figure 1.4a) has 6 symmetries. There are two different rotations (over $120°$ and $240°$), three reflections, and finally the action of doing nothing at all, called the *identity*. After performing any of these actions you end up with the same triangle in the same position.

What's more, if we perform any two of these actions in a row, we will always end up with a third action. This is known as *closure* of the symmetry actions. For instance, if we rotate first over $120°$ and then over $240°$, the net result is the identity operation. A concise way to write this is

$$a \cdot b = e, \tag{1.1}$$

where e is the identity, a and b are rotations over $120°$ and $240°$, respectively. The result of all possible combinations of rotational symmetries are summarized in Figure 1.4b.

In fact, the combined symmetry actions form a mathematical object known as a *group*. A group G has four defining characteristics:

1. Identity element There exists an element $e \in G$, such that for all elements $a \in G$, the equation $e \cdot a = a \cdot e = a$ holds.

2. Closure For all $a, b \in G$, their product $a \cdot b$ is also in G.

3. Inverse For any $a \in G$ there exists an element a^{-1} such that $a \cdot a^{-1} = e$.

4. Associativity The equation $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds for any $a, b, c \in G$.

·	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

(a)   (b)

Figure 1.4: The equilateral triangle (a) and the group multiplication table (b) of its rotational symmetries. e is the identity, a and b are rotations over $120°$ and $240°$, respectively.

We have already seen that closure holds for the triangle; it is not hard to see that the other three properties also hold. But this is not only restricted to the symmetries of the triangle. The importance of group theory lies in the fact that any symmetry you can think of can be described as a group, and that conversely all groups describe a symmetry.

1.3 From groups to algebras

If we examine the symmetries of the circle (see Figure 1.5), you will notice that a rotation over any arbitrary angle leaves it invariant. This means that the circle has an infinite amount of rotational symmetry. The mathematical object that describes these symmetries is still a group, but no longer a discrete (i.e. finite) one. The symmetry group of the circle is *continuous*: every angle between e.g. $120°$ and $240°$ corresponds to a symmetry. This is not so for the triangle: in that case there are 'gaps' between the rotations. The symmetry of the triangle is therefore called *discrete*.

Continuous groups are known as *Lie groups*. They contain an infinite amount of elements. But because they're continuous we can parameterize the elements in one or more parameters. For the circle we can write any rotation $R(\theta)$ over an angle θ as

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{1.2}$$

which is the rotation matrix in two dimensions. With a bit of work it follows that

$$R(\theta_1) \cdot R(\theta_2) = R(\theta_1 + \theta_2). \tag{1.3}$$

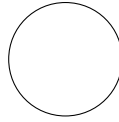This is what you would expect if you were to perform two rotations in a row: namely, the angles simply add up.

Figure 1.5: The circle.

Because of this property, we can go from the identity to a rotation over an arbitrary angle θ by repeatedly applying rotations over an infinitesimally small angle $d\theta$. In fact, the amount of change described by such a small rotation at the identity encodes almost all the information we need to describe the full group. This particular amount of change is encoded in a new object T:

$$T \equiv \left.\frac{dR(\theta)}{d\theta}\right|_{\theta=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.4}$$

Indeed, we recover all rotations by exponentiating T,

$$R(\theta) = e^{\theta T}. \tag{1.5}$$

We say that T generates the symmetry group of the circle. It is therefore also called a *generator*. This single object captures almost all of the important properties of the infinite symmetry group. It is not part of the group, but lives in the tangent space at the identity, known as the *Lie algebra* of the group.

Now the group of rotational symmetries of the circle is a particularly simple one. But every continuous group, even if it is horrendously complicated, has an associated Lie algebra. Thus studying the Lie algebra of a particular symmetry is sufficient to uncover most of its properties.

1.4 Infinite Lie algebras

The Lie algebras studied in this thesis, known as *Kac-Moody algebras*, are slightly more complicated than the one described above. While the Lie algebra of the rotational symmetries of the circle has only one generator, T, Kac-Moody algebras have an infinite number of them. Bear in mind that every single generator 'generates' an infinite amount of symmetry by means of the exponential mapping (1.5). This means that Kac-Moody algebras describe a symmetry that is infinitely many times infinite.

It is then not so surprising that most of the interesting Kac-Moody algebras are hard to describe in full. What one usually does is focus on a small portion (a *subalgebra*), and see how the whole algebra behaves with respect to that.
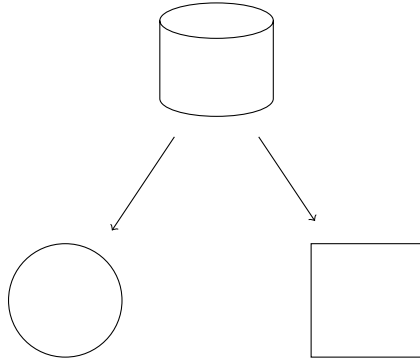
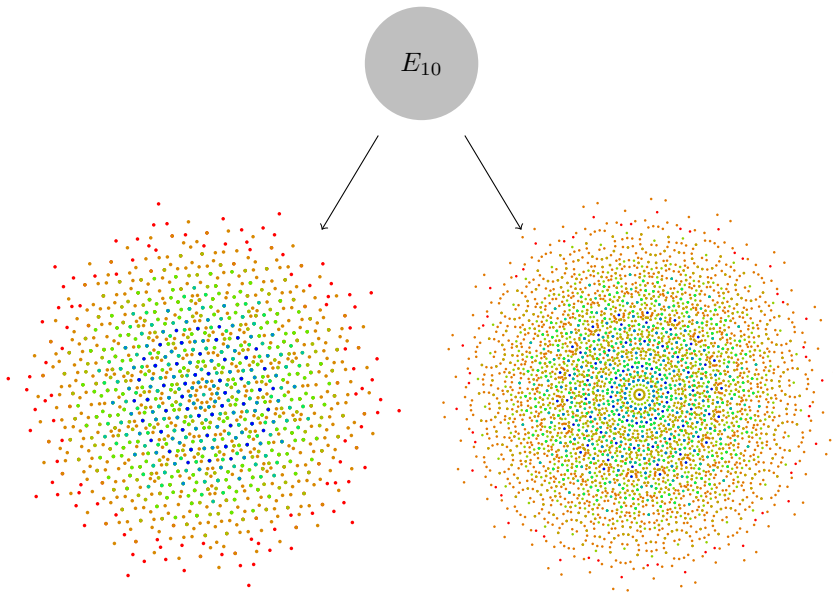**Figure 1.6:** Two different projections of the same cylinder.

Figure 1.7: Two different slices of the Kac-Moody algebra E_{10} produce two different projections.

To illustrate the concept, take a look at for example a cylinder (Figure 1.6). The cylinder can be thought of as a stack of circles on top of each other, and thus it carries the symmetry of the circle. But rotating the circles is not the only symmetric operation we can perform on the cylinder. We can also interchange the circles in the stack. These two distinct symmetries can be thought of as two different projections of the cylinder: one produces a circle, and the other a square. Both are aspects of the full symmetry of the cylinder.

This is a simplification of what happens for Kac-Moody algebras. Because we cannot describe their symmetry in full, we must resort to finite subalgebras and 'slice' with respect to those (see Figure 1.7). The resulting projections then tell us something of what the full Kac-Moody algebra looks like.

Things get interesting when the subalgebras with respect to which we slice are chosen such that they match symmetries of physical theories, namely supergravities. Not only do the slicings then tell us something about the full Kac-Moody algebras, they can then also be used to construct maps from the Kac-Moody side to the physical side. The maps in question relate the various physical fields, for instance the graviton, to sets of generators of the Kac-Moody algebra.

How to do these slicings, and the process of matching Kac-Moody algebras to physical theories, is the main topic of this thesis.