Compositional analysis and control of dynamical systems

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COMPOSITIONAL ANALYSIS AND CONTROL OF DYNAMICAL SYSTEMS

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Contents

1

Introduction

1.1. Motivation and setting

The goal of this thesis is to develop concepts and tools for compositional analysis and control of interconnected dynamical systems. We extend notions and techniques well established in computer science and apply them to dynamical systems commonly considered in systems theory and control. Our approach provides insights in various areas. Explicit connections are made with classical compositional analysis techniques in systems and control such as passivity theory and with decentralized control.

1.1.1. Compositional modeling and analysis techniques

Complexity is one of the key problems when analyzing engineering and IT processes. Many applications consist of a large number of subsystems interacting with each other, take as typical examples chemical reactors or finite element models in fluid and solid mechanics. In computer science, a similar problem is concurrency, i.e. the simultaneous execution of multiple processes e.g. in a shared memory or a network control system.

Example 1.1. A multi-product batch process is a chemical plant that processes raw materials to obtain multiple product substances. The main process units of such a batch process are the reactors where the raw materials are processed by mixing, heating, reacting with each other, etc. Both the raw materials and the products are stored in tanks. Tanks and reactors are connected through pipes. The process can be controlled by operating valves and pumps regulating the in- and outflow of tanks and reactors which are monitored by sensors, e.g. for water levels, temperature or other physical quantities. Figure 1.1 depicts a model of a two-product batch process. The storage tanks for the raw materials are modeled as subsystems $\Sigma_{P_i}, i = 1, 2, 3$, of the global plant. Each tank is controlled by pumps and valves represented by the controller systems $\Sigma_{C_i}, i = 1, 2, 3$, which are in turn interconnected with each other. Furthermore, the plant consists of two reactors, denoted by Σ_{P_4} and Σ_{P_5}, which are controlled by Σ_{C_4} and Σ_{C_5}, respectively. The product tanks are represented as the components Σ_{P_6} and Σ_{P_7}. The pipes between tanks and reactors determine the interconnection structure of the plant. The external variables $e_i, z_i, i \in \{1, 3, 6, 7\}$ can be thought of as in- and outflows of raw materials and products, respectively. Without specifying the dynamics of the

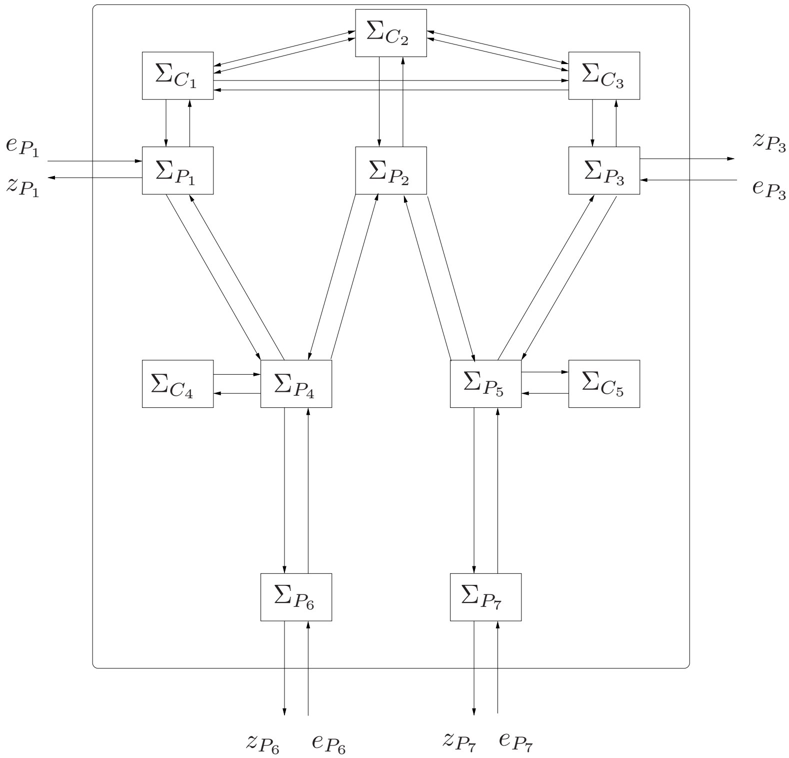

Figure 1.1.: Model of a multi-product batch process.

subsystems Σ_{P_i} and Σ_{C_i}, Example 1.1 thus illustrates how complex the model of a relatively simple industrial process can become.

Modeling these systems requires modular procedures, i.e., individual components as well as their interconnections have to be described formally and then combined to obtain a model of the whole process. Modular modeling techniques have been studied for various application, e. g. for manufacturing systems [51] and chemical processes [46, 16]. The focus of this thesis lies on compositional analysis and control techniques. Historically, the problem of how desired properties of complex systems can be verified systematically despite the "curse of dimensionality" has been addressed in the area of formal verification, a branch of computer science. In model checking [14] the design of a program represented as a transition system is verified by formulating properties in terms of a temporal logic. There exist automated procedures to evaluate logical formulas even for large systems. To reduce the complexity arising from interconnections of subsystems the modular structure can be used to decompose the global verification task into several subproblems involving components of the overall system. Based on guaranteed properties of subsystems the corresponding properties of the overall system can be inferred, thus reducing the computational effort significantly when compared to checking the interconnected system as a whole. In particular, compositional and assume-guarantee reasoning [31, 32, 58] follows this principle. To apply compositional analysis techniques, implementations (the modeled system behavior) and specifications (the desired system behavior or property) have to be represented in the same descriptive framework. If the implemented system behavior is included in (or even equivalent to) the specified one the design is guaranteed to be correct.

Formal methods like model checking and compositional reasoning are defined for labeled transition systems. This motivated researchers from computer science and systems theory to adopt some of these techniques by interpreting dynamical control systems as generalized transition systems. The results of this thesis are based on further advances achieved in recent years, in particular with respect to (bi)simulation theory for continuous-time dynamical systems using their differential equation description. Bisimulation relations were introduced as a notion of equivalence between labeled transition systems by Milner [48] and Park [57]. Intuitively, two systems are bisimilar if they cannot be distinguished by interconnecting them to a common environment. This concept of external equivalence makes bisimulations especially useful for compositional analysis. Subsystems of a network of interacting processes can be replaced by components of lower complexity which are equivalent with respect to bisimulation. In this sense, bisimulation equivalence also prescribes a reduction procedure. By contrast, the one-sided version of bisimulations – simulation relations – defines abstractions of systems. That is, an abstraction captures all the external behavior of the original system and possibly even more than that. Put reciprocally, the simulated system refines details that are absent in the abstraction. Compositional and assume-guarantee reasoning schemes often use abstraction-refinement procedures based on simulation relations [79, 25]. In particular, if a component system satisfies, i. e. is similar to, its specification it can be replaced by the latter in the overall network. In recent years, bisimulation theory has been adopted to continuous-time control systems, see [56, 55, 69]. Usually, the original concept is mimicked by defining bisimulation relations of continuous-time systems with respect to their state trajectories and external variables. However, it has been shown in [74] that the existence of a bisimulation relation can be equivalently formulated as a geometric control problem, namely the modified disturbance decoupling problem [81]. This facilitates a linear-algebraic characterization of bisimulation relations for continuous-time systems based on the differential equations describing the system model. The advantage of this structural notion of bisimulation lies in efficient and elegant algorithms for computations of maximal bisimulation relations. In this thesis, we make use of structural (bi)simulation relations to develop compositional and assume-guarantee reasoning techniques.

1.1.2. Compositional techniques in systems and control theory

Example 1.1 illustrates the need for compositional analysis techniques in the area of systems theory and control. Complexity has to be dealt with as a result of large state space dimensions. Like in the two-product batch process of Figure 1.1, interconnections between plant and controller systems or between subsystems of the overall plant are characteristics of control systems. This motivates the use of compositional techniques for control related purposes, e.g. to verify system theoretic properties of large-scale systems or to design decentralized control schemes capable of satisfying a global specification. A classical example for compositional reasoning in systems theory is the passivity theorem which states that the interconnection of two passive systems is again passive [63]. In this work, we will further develop the notion of passivity as a compositional property. To do so, passivity as a fundamental property of a control system has to be expressed in the same formalism, i.e. as a control system itself. We will show that this is possible for passivity and, to some extent, also for stability in the sense of Lyapunov. The idea is to specify properties by means of target dynamics that should be achieved by the original system. This can also be used for controller design, e.g. in the so-called immersion and invariance principle [4] where the target system is defined on an attractive and invariant manifold. Unlike in computer science, however, formulating specifications in the same language as the system model is not always part of the design process. In the context of this research, specifications are always interpreted as target systems, either explicitly defined to check passivity or stability or more abstractly to capture properties related to control performance. How to construct such specifications systematically would be an interesting topic for future research. This also holds for the problem of how to decompose global specifications into local subspecifications corresponding to subsystems of the plant. A starting point could be the decomposition strategies presented in [52, 53]. Given a transition system, these results include a procedure how to construct an isomorphic transition system given as the product of subsystems.
The next focus lies on controller design strategies for complex interconnected systems. Decentralized control [41, 17, 64] is the attempt to control the subsystems of a global plant individually by local controllers in such a way that the overall controlled system satisfies its specification. This concept has an im-

portant advantage: Restrictions due to limited communication and controller action between component systems can be incorporated naturally in the design of decentralized schemes. In the batch process depicted in Figure 1.1, the tank Σ_{P_1} is only connected to the first reactor Σ_{P_4} and similarly, Σ_{P_3} can only influence the second reactor Σ_{P_5} while tank Σ_{P_2} is connected to both reactors. The local controllers Σ_{C_1} and Σ_{C_3} only need to react to changes of the fill height of the respective reactors Σ_{P_4} and Σ_{P_5}. Likewise, distributed sensor and actuator locations like in structural monitoring [44], process control [61] and distributed robotic networks [9] also restrict communication between subsystems. Hence, the design procedures and consequently the hardware requirements for the network of all decentralized controllers should be much simpler than the corresponding global controller achieving the same performance. However, the difficulty of decentralized control is coordination, i.e. guaranteeing that the interconnection of locally controlled subsystems of the plant satisfies the desired global control target. In this respect, decentralized control can be seen as a complementary notion of compositional analysis. In this work, we therefore approach decentralized control problems using compositional analysis techniques. Different scenarios are possible, namely starting bottom-up from the level of local controllers such that a global specification is achieved as well as top-down from the level of a global specification that can be decomposed into subspecifications corresponding to subsystems of the plant.

The main aim of control theory has always been to develop procedures to construct controllers (global or decentralized) such that the closed-loop system satisfies the predefined specifications. By contrast, less is known about controller design methods for discrete transition systems, the supervisory control theory for discrete-event systems [62] being a notable exception. Recently, controller synthesis methods for classes of hybrid systems have been proposed [47, 59, 68] that follow the “correct by design”(or “correct by construction”) paradigm. That is, instead of separating the implementation from the verification step in the synthesis process, correct by design methods automatically generate implementations from previously verified abstractions. Hence, correctness is always guaranteed provided the refinement procedure is formally correct. Moreover, the problem that final implementations are hard to check due to their complexity is circumvented.

This is especially beneficial when the controller implemented in software interacts with a continuous environment. Systems that combine elements of discrete and continuous dynamics are, depending on the context, referred to as hybrid systems (which we will use throughout) or embedded systems or cyber-physical systems. Many real-life systems exhibit hybrid behavior, e.g. the valves and pumps controlling the the multi-product batch process of Example 1.1. The interplay of continuous and discrete dynamics requires to merge analysis and design methods for transition systems on the one hand (the computer science approach) and for continuous-time dynamical systems on the other hand (the systems and control approach). Within computer science, most commonly the mathematical formalism of labeled transition systems used to describe discrete-event systems is generalized to incorporate continuous variables. The resulting hybrid automata (e. g. [1, 45, 21]) then describe the continuous dynamics in terms of the trajectories. An alternative idea is to abstract hybrid by discrete dynamics, although often at the cost of very large discrete state space dimensions [2, 50, 23]. The focus lies on verification problems such as safety requirements, which have been implemented in several software packages (e.g. [22, 29]). By contrast, hybrid system models in the area of system and control often use differential-algebraic equations for the continuous part, see [77] for an overview. Starting from [43, 8] various subclasses were studied such as switching linear systems [60] or stochastic hybrid automata [13]. Analysis problems concern amongst others existence and uniqueness of solutions [27] and controller design methods. These include model-predictive and LMI-based control concepts, see e. g. [34] and [5]. In the last part of this thesis we investigate compositional analysis methods for switching linear systems combining the mathematical formalism of systems theory to benefit from structural notions of bisimulation relations and the compositional analysis techniques developed in formal verification to analyze system theoretic properties and develop decentralized control schemes.

1.2. Outline of the thesis

The main chapters of this thesis are organized as follows:

- In Chapter 2 we review simulation and bisimulation theory for both labeled transition and linear input-state-output systems. We characterize important properties of (bi)simulations and give a linear algebraic characterization. Furthermore, we give an algorithmic procedure to compute the maximal (bi)simulation relation between two linear systems. The specialization to the non-deterministic case is also dealt with. Brief examples illustrate abstractions of linear systems by simulation and reductions by bisimulation. Most of the content of this chapter is taken from [21] for labeled transition systems and [74] for linear systems.

- Chapter 3 contains a general treatment of compositional reasoning techniques for linear systems based on (bi)simulations. Two different types of interconnections are studied, namely feedback interconnections and parallel compositions. We show that compositional reasoning and both non-circular and circular assume-guarantee reasoning is sound for feedback interconnections of two linear systems. We also investigate whether the converse of these proof rules, i.e. their completeness, holds true. The results are then generalized to series of more than two feedback interconnections. Finally, we discuss compositional analysis techniques for parallel compositions of linear systems entailing algebraic constraints. In particular, a proof rule based on the decomposition of the given specification is derived. The content of this chapter is an extension of [40] and its preliminary versions [39] and [37].

- In Chapter 4 we generalize some of the previous results to analyze nonlinear input-state-output systems. The main similarity lies in the fact that also for nonlinear systems, the existence of a simulation relation can be cast as a geometric control problem. We give regularity conditions under which nonlinear simulation is a preorder. Both soundness and completeness of compositional reasoning is established. Furthermore, we investigate whether circular assume-guarantee reasoning is sound.

- Chapter 5 discusses the relationship between nonlinear simulation theory and passivity theory. We show that passivity properties of both linear and nonlinear control systems can be equivalently characterized by the existence of a nonlinear simulation relation of the system under consideration and the one-dimensional system associated with the dissipation inequality. We then apply compositional reasoning techniques to prove that the interconnection of two passive systems is again passive. The converse statement can also be shown to hold true for both linear and nonlinear systems. We obtain conditions under which the storage function of a feedback interconnection is uniquely determined as the sum of the storage functions of the components. The results of both Chapter 4 and 5 are based on [36].

- Chapter 6 shows how compositional analysis techniques can be applied to decentralized control problems. We prove that both compositional and assume-guarantee reasoning schemes are sound for feedback interconnections of locally controlled plant systems and their specifications. Thus, we obtain decentralized control schemes that guarantee fulfillment of a global specification provided conditions for the local controllers are satisfied. The problem whether there exists for a given plant and specification a controller such that the controlled plant meets its specification is characterized by achievable simulations. We combine this result, given in terms of the so-called sandwich conditions, with compositional analysis techniques and apply this to our decentralized setting. As a result, we obtain two bottom-up decentralized control schemes that contain necessary conditions for the existence of local controllers such that the overall control network satisfies a global specification. Additionally, we consider a top-down scheme based on circular assume-guarantee reasoning. Starting with a global controller satisfying the overall specification, our result gives conditions for the existence of local controllers that guarantee the same global control target. The content of this chapter is based on [35].

- Chapter 7 treats compositional analysis for a particular class of hybrid systems, namely switching linear systems. Due to the particular structure of switching linear systems, we develop a structural version of hybrid simulation which is easily checkable and is thus ideally suited for compositional analysis. We show that both compositional and non-circular assume-guarantee reasoning are sound. As to circular assume-guarantee reasoning, we give conditions under which the proof rule holds true. The material presented in this chapter can be found in [38].

- In Chapter 8 we consider a more general class of switching linear systems by adding location invariants and guard conditions. Thus, not only do the discrete dynamics influence the continuous ones through discrete events but the location invariants constraining the continuous evolution at every location (discrete state) can also trigger changes of the discrete state. Our aim is to describe equivalences of such hybrid systems by bisimulation. We therefore incorporate synchronization of guard conditions in the definition of bisimulation relations. As a first step we consider linear systems with inequality constraints representing the continuous part of the hybrid dynamics of switching linear systems. We give necessary and sufficient conditions for the existence of a bisimulation relation between two linear systems with inequality constraints and obtain, using the Farkas lemma, a linear-algebraic characterization. We then define structural hybrid bisimulation relations for switching linear systems with location invariants exploiting the dependencies between discrete and continuous dynamics.

- Chapter 9 highlights the main contributions of the thesis and outlines possible directions of future research.

2

Simulation theory: from labeled transition systems to linear systems

Many notions of equivalence, whether for discrete processes or continuous-time systems, are based on the external behavior. I.e., two processes or systems are equivalent if they cannot be distinguished with respect to their interaction with a common environment. In the theory of concurrent processes (bi)simulation theory is one of the most prominent concepts of external equivalence. In this chapter, we give a short introduction to (bi)simulation theory starting with its origin in labeled transition systems. In the second part, we describe how this concept can be adopted to linear input-state-output systems and how it is further developed using geometric control theory.

2.1. Simulation and bisimulation relations for labeled transition systems

Discrete-event systems in computer science are most commonly described as labeled transition systems.

Definition 2.1. A labeled transition system D is a triple $D = (Q, V, E)$ with

- a set of states Q,
- a set of transition labels V,
- a transition relation $E \subset Q \times V \times Q$ and

The semantics of a labeled transition system are described by executions.

Definition 2.2. An execution σ of a labeled transition system $D = (Q, V, E)$ is a finite or infinite sequence

$$\sigma = (q_0, v_0, q_1, v_1, \ldots)\ q_i, \in Q,\ v_i \in V,\ i \in \mathbb{N},$$

such that

$$(q_i, v_i, q_{i+1}) \in E$$

Executions define the external behavior of labeled transition systems. In particular, the sequence of transitions represented by their labels is used to define equivalences between labeled transition systems. Originally introduced by Park [57] and Milner [48] (bi)simulation relations formalize this idea.

Definition 2.3. For any two labeled transition systems $D_i = (Q_i, V_i, E_i), i = 1, 2$, with the same set of labels V, a relation $S \subset Q_1 \times Q_2$ is a *simulation relation* of D_1 by D_2 if and only if for all $(q_1, q_2) \in S$, $v \in V$ and $q_1' \in Q_1$ the following holds:

$$(q_1, v, q_1') \in E_1 \Longrightarrow \exists q_2' \in Q_2 : (q_2, v, q_2') \in E_2, (q_1', q_2') \in S$$

If there exists a simulation relation S of D_1 by D_2 we say that D_2 *simulates* D_1, denoted by $D_1 \preccurlyeq D_2$.
A subset $R \subset Q_1 \times Q_2$ is a *bisimulation relation* between D_1 and D_2 if and only if for all $(q_1, q_2) \in R$ it holds that

(i) for every $q_1' \in Q_1$ and every $v \in V$ such that $(q_1, v, q_1') \in E_1$ there exists a $q_2' \in Q_2$ such that $(q_2, v, q_2') \in E_2$ and $(q_1', q_2') \in R$ and conversely,

(ii) for every $q_2' \in Q_2$ and every $v \in V$ such that $(q_2, v, q_2') \in E_2$ there exists a $q_1' \in Q_1$ such that $(q_1, v, q_1') \in E_1$ and $(q_1', q_2') \in R$.

If there exists a bisimulation relation between D_1 and D_2 we say that D_1 and D_2 are *bisimilar*, denoted by $D_1 \approx D_2$.

Remark 2.4. Hence bisimilarity implies mutual similarity. However, it is a well-known fact that mutual similarity does *not* imply bisimilarity for labeled transition systems. I.e., if there exist simulation relations R_1 and R_2 of D_1 by D_2 and of D_2 by D_1, respectively, there need not exist any bisimulation relation between D_1 and D_2. This happens if one cannot construct $R_i, i = 1, 2$, in such a way that $R_1 = R_2^{-1}$, $R_2^{-1} = \{(q_1, q_2) \mid (q_2, q_1) \in R_2\}$, see e. g. [21] for a counterexample.

The notion of simulation relations is instrumental for compositional analysis since it provides a tool to relate labeled transition systems with the same transition structure. To finish this introductory section about (bi)simulation relations for labeled transition systems, we state without proof their most important properties.

- Simulation relations of labeled transition systems are preorders, i.e. they are

reflexive:	$D \preccurlyeq D$	for any D
transitive:	$\left.\begin{array}{ccc} D_1 & \preccurlyeq & D_2 \\ D_2 & \preccurlyeq & D_3 \end{array}\right\} \Longrightarrow D_1 \preccurlyeq D_3$	for any $D_i, i = 1, 2, 3$.

Bisimulation relations are equivalence relations ([14]), i.e. they are reflexive, transitive and

$$\text{symmetric:} \quad D_1 \approx D_2 \implies D_2 \approx D_1 \quad \text{for any } D_i, i = 1, 2 \ .$$

- Let D_1 and D_2 be two labeled transition systems and $(R_i)_{i \in I}$ a family of bisimulation relations with I an index set. Then the relation $R = \cup_{i \in I} R_i$ is also a bisimulation relation between D_1 and D_2, see [3]. Moreover, if $D_1 \approx D_2$ then there exists a unique *maximal* bisimulation relation $R^\star$ between D_1 and D_2 such that

$$R \quad \subset \quad R^\star$$

for any bisimulation relation R between D_1 and D_2.

2.2. Simulation and bisimulation theory for linear systems

Consider the following linear input-state-output system

$$\Sigma_i : \quad \begin{array}{rcl} \dot{x}_i & = & A_i x_i + B_i u_i + L_i d_i \\ y_i & = & C_i x_i \end{array} \tag{2.1}$$

where $x_i \in \mathcal{X}_i = \mathbb{R}^{n_i}$ represents the state variables, $u_i \in \mathcal{U}_i$ and $y_i \in \mathcal{Y}_i$ are a pair of inputs and outputs used for interconnection with other linear systems and $d_i \in \mathcal{D}_i$ is an additional input acting as a generator for non-determinism capturing e.g. unmodeled dynamics. We will therefore refer to d_i as "disturbances" influencing the modeled system behavior. Reference [55] adopted (bi)simulation theory to linear systems of the form (2.1) by interpreting them as general transition systems. Indeed, the set of states can be associated with the finite-dimensional vector space $\mathcal{X}_i$ (or in general a manifold), the transition relation is linked with the flow of the differential equation (2.1) and the set of variables corresponds to the in- and outputs u_i, y_i. Like in Definition 2.3, (bi)simulation relations for linear systems are then defined with respect to their executions, i.e. the trajectories of systems $\Sigma_i, i = 1, 2$ of the form (2.1).

Definition 2.5. Given two systems $\Sigma_i, i = 1, 2$ as in (2.1). A *simulation relation* S of Σ_1 by Σ_2 is a linear subspace of the product state space $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ with the following properties:
For any $(x_{10}, x_{20}) \in S$, any joint input function $u_1(\cdot) = u_2(\cdot)$ and any disturbance $d_1(\cdot)$ there should exist a disturbance $d_2(\cdot)$ such that the resulting state trajectories $x_i(\cdot), i = 1, 2$ with $x_i(0) = x_{i0}$ satisfy

$$\begin{array}{ll} (i): & (x_1(t), x_2(t)) \in S \quad \forall t \geq 0 \\ (ii): & C_1 x_1(t) = C_2 x_2(t) \quad \forall t \geq 0 \end{array} \tag{2.2}$$

Furthermore , Σ_1 is *simulated* by Σ_2, denoted $\Sigma_1 \preccurlyeq \Sigma_2$, if there exists a simulation relation $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ fulfilling $\Pi_{\mathcal{X}_1} S = \mathcal{X}_1$ where $\Pi_{\mathcal{X}_i} : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_i$ denotes the canonical projection on $\mathcal{X}_i$. In this case, S is called a *full simulation relation*.
A subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a *bisimulation relation* between Σ_1 and Σ_2 if it is a simulation relation of Σ_1 by Σ_2 and $R^{-1} := \{(x_2, x_1) \mid (x_1, x_2) \in R\}$ is a simulation relation of Σ_2 by Σ_1. If R satisfies $\Pi_{\mathcal{X}_i} R = \mathcal{X}_i, i = 1, 2$, then Σ_1 and Σ_2 are *bisimilar* and R is called a *full bisimulation relation*.

Following the same principle, (bi)simulation relations can be defined for other classes of systems such as nonlinear ([56], [24]) or hybrid systems ([2]). From an application point of view, however, Definition 2.5 has the drawback to rely on solutions of differential equations which in general are not explicitly computable. Using geometric control theory, [74] characterized simulation relations of linear systems by the following invariance condition. [1]

Proposition 2.6. *A subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if for all $(x_1, x_2) \in S$ and all $u \in \mathcal{U}$ the following holds:*

(i): For all $d_1 \in \mathcal{D}_1$ there should exist a $d_2 \in \mathcal{D}_2$ such that

$$\begin{bmatrix} A_1 x_1 + B_1 u + L_1 d_1 \\ A_2 x_2 + B_2 u + L_2 d_2 \end{bmatrix} \in S \tag{2.3}$$

(ii):

$$C_1 x_1 = C_2 x_2 \tag{2.4}$$

A subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ defines a bisimulation relation between Σ_1 and Σ_2 if and only if for every $d_1 \in \mathcal{D}_1$ there exists a $d_2 \in \mathcal{D}_2$ such that R satisfies (2.3) *and* (2.4) *and vice versa, for every d_2 there exists a d_1 such that* (2.3) *and* (2.4) *hold.*

Conditions (2.3) and (2.4) give rise to a linear-algebraic characterization of (bi)simulation subspaces.

Theorem 2.7. *A linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if the following holds:*

1. $\operatorname{im} \begin{bmatrix} L_1 \\ 0 \end{bmatrix} \subset S + \operatorname{im} \begin{bmatrix} 0 \\ L_2 \end{bmatrix} := S_e$

2. $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} S \subset S_e$

[1] Note that in the remainder we will use the relational notation $(x_1, x_2) \in S \subset \mathcal{X}_1 \times \mathcal{X}_2$ and the subspace notation $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S$ interchangeably.

3. $\mathrm{im}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset S_e$

4. $S \subset \ker\begin{bmatrix} C_1 & -C_2 \end{bmatrix}$

A subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a bisimulation relation between Σ_1 and Σ_2 if and only if R satisfies conditions 2 – 4 and additionally

1′. $R + \mathrm{im}\begin{bmatrix} L_1 \\ 0 \end{bmatrix} = R + \mathrm{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} =: R_e$

The above subspace characterizations of (bi)simulation relations for linear systems are easy to check and to apply. Moreover, an algorithmic procedure to compute maximal (bi)simulation relations can be derived from Theorem 2.7.

Algorithm 2.8. Given two linear systems $\Sigma_i, i = 1, 2$ of the form (2.1). Define the following sequence of subspaces $S^j \subset \mathcal{X}_1 \times \mathcal{X}_2$:

$$\begin{aligned}
S^0 &= \mathcal{X}_1 \times \mathcal{X}_2 \\
S^1 &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S^0 \;\middle|\; (x_1, x_2) \in \ker\begin{bmatrix} C_1 & -C_2 \end{bmatrix} \right\} \\
S^2 &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S^1 \;\middle|\; \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathrm{im}\begin{bmatrix} L_1 \\ 0 \end{bmatrix} \subset \right. \\
&\quad \left. S^1 + \mathrm{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} \right\} \qquad (2.5) \\
&\;\;\vdots \\
S^{j+1} &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S^j \;\middle|\; \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathrm{im}\begin{bmatrix} L_1 \\ 0 \end{bmatrix} \subset \right. \\
&\quad \left. S^j + \mathrm{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} \right\}
\end{aligned}$$

Theorem 2.9. *The sequence of subspaces $S^j, j \in \mathbb{N}$, has the following properties:*

- $S^0 \supset S^1 \supset S^2 \supset \ldots \supset S^j \supset S^{j+1} \supset \ldots$
- *there exists a finite k such that $S^k = S^{k+1} := S^\star = S^i$, $i \geq k+1$.*
- *If $S^\star$ is non-empty and fulfills condition 3 in Theorem 2.7 then $S^\star$ is the maximal simulation relation of Σ_1 by Σ_2. Conversely, if $S^\star$ is empty or does not fulfill condition 3 in Theorem 2.6 then there does not exist any simulation relation of Σ_1 by Σ_2.*

Remark 2.10. Similar to (2.5), the sequence of subspaces $R^j \subset \mathcal{X}_1 \times \mathcal{X}_2, j = 0, 1, \dots,$ with

$$\begin{aligned} R^0 &= \mathcal{X}_1 \times \mathcal{X}_2, \\ R^1 &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R^0 \;\middle|\; (x_1, x_2) \in \ker \begin{bmatrix} C_1 & -C_2 \end{bmatrix} \right\}, \\ R^j &= \Bigg\{ (x_1, x_2) \in R^{j-1} \mid \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \operatorname{im} \begin{bmatrix} L_1 \\ 0 \end{bmatrix} \subset \\ & \quad R^j + \operatorname{im} \begin{bmatrix} 0 \\ L_2 \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \operatorname{im} \begin{bmatrix} 0 \\ L_2 \end{bmatrix} \subset \\ & \quad R^j + \operatorname{im} \begin{bmatrix} L_1 \\ 0 \end{bmatrix} \Bigg\}, \; j = 2, \dots \end{aligned} \tag{2.6}$$

yields a candidate for the maximal bisimulation relation $R^\star$ between Σ_1 and Σ_2. The results of Theorem 2.9 also hold for the subspaces defined by (2.6), in particular the condition to determine whether or not $R^\star$ is indeed the maximal bisimulation relation between Σ_1 and Σ_2.

Some useful relational properties of (bi)simulations of linear systems as known from their discrete counterparts are retained.

Proposition 2.11. *Simulation relations $\preccurlyeq$ for linear systems are preorders whereas bisimulation relations $\approx$ are equivalence relations.*

Proof. Consider linear systems $\Sigma_i, i \in \{1, 2, 3\}$, of the form (2.1).
Reflexivity: The relation $S = \{(x_1, x_1) \mid x_1 \in \mathcal{X}_1\}$ fulfills conditions (i) and (ii) of Definition 2.5 and therefore defines a full simulation relation of Σ_1 by Σ_1.
Transitivity: Assume S_1 defines a full simulation relation of Σ_1 by Σ_2 and S_2 of Σ_2 by Σ_3. Then $S_{12} = \{(x_1, x_3) \mid \exists x_2 : (x_1, x_2) \in S_1, (x_2, x_3) \in S_2\}$ defines a full simulation relation of Σ_1 by Σ_3.
Both properties are also valid with respect to bisimulation.
Symmetry: Suppose that Σ_1 and Σ_2 are bisimilar and R defines a full bisimulation relation between them. Then R^{-1} is a full bisimulation relation between Σ_1 and Σ_2. □

Consider now the case of linear systems Σ_i without disturbances d_i. The above results specialize as follows.

Proposition 2.12. *For any two linear systems $\Sigma_i, i = 1, 2$, of the form* (2.1) *with d_i void, there exists a simulation relation of Σ_1 by Σ_2 if and only if there exists a bisimulation relation between Σ_1 and Σ_2. Moreover,*

$$\Sigma_1 \preccurlyeq \Sigma_2 \iff \Sigma_1 \approx \Sigma_2 \tag{2.7}$$

Proposition 2.13. *Consider* $\Sigma_i, i = 1,2$, *as in* (2.1) *with* d_i *void. There exists a bisimulation relation* R *between* Σ_1 *and* Σ_2 *as in* (2.1) *with* d_i *void, if and only if the Markov parameters of* Σ_1 *and* Σ_2 *are equal, that is*

$$C_1 A_1^k B_1 \quad = \quad C_2 A_2^k B_2 \ \forall k = 0,1,2,\ldots \tag{2.8}$$

or equivalently, if their transfer matrices $G_i(s) := C_i(Is - A_i)^{-1}B_i, i = 1,2$, *are the same. Moreover, if* $\Sigma_i, i = 1,2$, *are controllable,*

$$\Sigma_1 \approx \Sigma_2 \quad \Longleftrightarrow \quad G_1(s) = G_2(s) \tag{2.9}$$

Proposition 2.13 also indicates that the maximal bisimulation relation between Σ_1 and Σ_2 is related to the unobservability space $\mathcal{O}_{12}$ of the augmented system Σ_{12}, given by

$$\Sigma_{12}: \begin{array}{rcl} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = & \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y_{12} & = & \begin{bmatrix} C_1 & -C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array}$$

Indeed, if the transfer matrices of $\Sigma_i, i = 1,2$, are the same (which by Proposition 2.13 is equivalent to the existence of a bisimulation relation), the unobservability space

$$\mathcal{O}_{12} = \ker \begin{bmatrix} C_1 & -C_2 \\ C_1 A_1 & -C_2 A_2 \\ \vdots & \vdots \\ C_1 A_1^n & -C_2 A_2^n \end{bmatrix}, \ n = \max\{n_1, n_2\} - 1$$

equals the *maximal bisimulation relation* $R^\star$, since by equality of the Markov parameters

$$\operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset \mathcal{O}_{12}$$

One of the main applications for simulation relations is to *abstract* a system of higher state space dimension by a lower dimensional one. Adopted from the discrete case, abstractions have been proposed in [55] and [74] for linear system of the form (2.1). More concretely, a system

$$\begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ y & = & \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array} \tag{2.10}$$

can be abstracted by the lower order model

$$\begin{aligned} \dot{z} &= A_{11}z + A_{12}d \\ y &= C_1 z \end{aligned} \tag{2.11}$$

in the sense that (2.10) is simulated by (2.11). Here, the internal disturbance d captures the influence of the x_2-dynamics on the the x_1-dynamics thus hiding details of the original system.
Similarly, bisimulation relations can be used to *reduce* linear systems Σ of the form (2.1). This idea was first proposed in [74] and later extended to other classes of systems, see e.g. [60] for reduction of switching linear systems. At first, the maximal bisimulation relation $R^\star_{\mathrm{id}}$ between Σ and itself is constructed. Associated with $R^\star_{\mathrm{id}}$ is the linear subspace

$$\bar{R} := \{x - y \mid (x,y) \in R^\star_{\mathrm{id}}\} \tag{2.12}$$

$\bar{R}$ can easily be seen to satisfy

$$\begin{aligned} A\bar{R} &\subset \bar{R} + \mathrm{im}G \\ \bar{R} &\subset \ker C \end{aligned} \tag{2.13}$$

The system Σ can then be reduced by factoring out $\bar{R}$ from the state space $\mathcal{X}$ using the canonical projection $\Pi_{\bar{R}} : \mathcal{X} \to \mathcal{X}/\bar{R}$. Thus, one obtains the reduced system

$$\Sigma_{\bar{R}} : \begin{aligned} \dot{x}_{\bar{R}} &= A_{\bar{R}}x_{\bar{R}} + B_{\bar{R}}u + L_{\bar{R}}d \\ y_{\bar{R}} &= C_{\bar{R}}x_{\bar{R}} \end{aligned}$$

with $A_{\bar{\mathcal{R}}}\Pi_{\bar{\mathcal{R}}} = \Pi_{\bar{\mathcal{R}}}(A + LK)$ for some matrix K computable from the first line of (2.13), $B_{\bar{\mathcal{R}}} = \Pi_{\bar{\mathcal{R}}}B$, $C = C_{\bar{\mathcal{R}}}\Pi_{\bar{\mathcal{R}}}$ and $L_{\bar{\mathcal{R}}} = \Pi_{\bar{\mathcal{R}}}L$. The following example illustrates this reduction procedure.

Example 2.14. Consider the linear system

$$\Sigma : \begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}u_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix}d_1 \\ y_1 &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Compute the maximal bisimulation relation $R^\star_{\mathrm{id}}$ between Σ and itself,

$$R^\star_{\mathrm{id}} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

The associated subspace $\bar{R}$ is given by

$$\bar{R} = \operatorname{im} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Factoring out $\bar{R}$ using the canonical projection $\Pi_{\bar{R}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} = C$ yields the reduced system $\Sigma_{\bar{R}}$ of dimension 1,

$$\Sigma_{\bar{R}} : \begin{array}{rcl} \dot{x}_{\bar{R}} & = & -x_{\bar{R}} + u + \frac{1}{2}d \\ y_{\bar{R}} & = & x_{\bar{R}} \, . \end{array}$$

3

Compositional analysis for linear systems

3.1. Introduction

Compositional and assume-guarantee reasoning [49, 31, 21] was first introduced in the area of formal verification to provide strategies to decompose a verification task for a labeled transition system into several tasks involving individual components or components restricted to a specific environment. Thus, the complexity of the original verification task can be significantly reduced. The burden of complexity is also encountered in the area of systems and control. Many engineering systems such as chemical plants or cyber-physical systems are modeled as a large number of interacting subsystems which leads to high state space dimensions. Combining formal concepts and modeling techniques from both areas offers a huge potential for analysis and controller design problems. As a first step, we develop compositional analysis techniques for linear systems in this chapter. Our approach is based on the representation of linear systems by differential equations which allows us to make use of structural notions of (bi)simulation relations. Extending previous results [37, 39, 40], we treat the following range of topics: At first, feedback interconnections are considered. We prove that compositional and assume-guarantee reasoning rules are sound and present an illustrative example from circuit theory. The complementary issue of completeness is investigated distinguishing between open and closed feedback interconnections. We also show that the proof rules for compositional and assume-guarantee reasoning are sound when replacing simulation by bisimulation relations. A generalization to interconnections of more than two systems rounds off the section on feedback interconnections. In the second part of this chapter, parallel compositions of linear systems are studied. The resulting algebraic constraints on the system variables are characteristic for models of physical processes and can be modeled in the form of DAE systems. We give a linear-algebraic characterization of simulation relations for systems with state constraints and use this notion to prove soundness of compositional and assume-guarantee reasoning. Finally, we present a special proof rule for parallel compositions based on the decomposition of the global specification. This leads to the result that verifying the global specification is equivalent to verifying each subspecification of a parallel composition.

3.2. Feedback interconnections

In this section, we want to analyze feedback interconnections of linear systems Σ_i,

$$\Sigma_i : \quad \begin{array}{rcl} \dot{x}_i & = & A_i x_i + B_i u_i + G_i e_i + L_i d_i \\ y_i & = & C_i x_i \\ z_i & = & H_i x_i \end{array} \tag{3.1}$$

where all variables belong to finite dimensional vector spaces, $x_i \in \mathcal{X}_i = \mathbb{R}^{n_i}, u_i \in \mathcal{U}_i, e_i \in \mathcal{E}_i, d_i \in \mathcal{D}_i, y_i \in \mathcal{Y}_i, z_i \in \mathcal{Z}_i$. Compared to Section 2.2, we add a pair of external variables e and z to specify performance targets[1], see Figure 3.1. These external variables remain accessible even after interconnecting two systems, see Figure 3.2. By contrast, the variables u and y are used

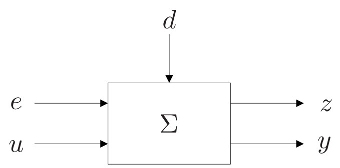

Figure 3.1.: Linear system Σ with internal and external variables.

for feedback interconnections between linear systems. As before, x represents the state variable and d an input that can be thought of as a disturbance as explained in Section 2.2. The temporal evolution of all system variables is characterized by functions of an appropriate function class, e.g. $\mathcal{C}^\infty$.

Definition 3.1. The feedback interconnection $\|$ of two linear continuous-time systems $\Sigma_i, i = 1, 2$, is defined as

$$u_2 = y_1 \ , \ u_1 = y_2 \ .$$

[1] This framework captures optimal control problems such as $H_\infty-$ optimal control where the performance target is specified as a norm bound on the transfer function between the external variables e and z.

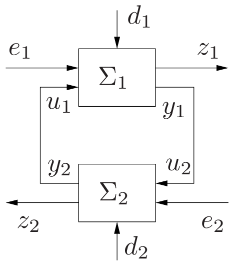

Figure 3.2.: Interconnection $\Sigma_1 \| \Sigma_2$

The dynamics of the interconnected system $\Sigma_1 \| \Sigma_2$ are then given by (see Figure 3.2)

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \\
&+ \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}$$

(Bi)simulation relations will be used to formulate proof rules relating actual systems models to models representing their desired properties. To that end, we specialize the general concept of simulation relations as introduced in Chapter 2 as follows.

Definition 3.2. A simulation relation S of Σ_1 by Σ_2 is a linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ with the following property: For any $(x_{10}, x_{20}) \in S$, any joint input function $e_1(\cdot) = e_2(\cdot) = e(\cdot)$, any joint interconnection input $u_1(\cdot) = u_2(\cdot) = u(\cdot)$ and any disturbance function $d_1(\cdot)$ there should exist a disturbance $d_2(\cdot)$ such that the resulting state trajectories $x_i(.)$ with $x_i(0) = x_{i0}, i = 1, 2$, satisfy

$$\begin{aligned}
(i) &\quad (x_1(t), x_2(t)) \in S, \ \forall t \geq 0 \\
(ii) &\quad H_1 x_1(t) = H_2 x_2(t), \ \forall t \geq 0 \\
(iii) &\quad C_1 x_1(t) = C_2 x_2(t), \ \forall t \geq 0
\end{aligned}$$

Σ_1 is simulated by Σ_2, denoted by $\Sigma_1 \preccurlyeq \Sigma_2$, if there exists a simulation relation S fulfilling $\Pi_1 S = \mathcal{X}_1$ with $\Pi_1 : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_1$ the canonical projection from $\mathcal{X}_1 \times \mathcal{X}_2$ to $\mathcal{X}_1$. In this case, S is called a *full simulation relation*.
A subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a *bisimulation relation* between Σ_1 and Σ_2 if it is a simulation relation of Σ_1 by Σ_2 and in addition $R^{-1} := \{(x_2, x_1) \mid (x_1, x_2) \in R\}$ defines a simulation relation of Σ_2 by Σ_1. Moreover, if $\Pi_i R = \mathcal{X}_i, i = 1, 2,$ then R is called a *full bisimulation relation* and Σ_1 and Σ_2 are called *bisimilar*, denoted by $\Sigma_1 \approx \Sigma_2$.

Proposition 3.3. *A subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if for all $(x_1, x_2) \in S$, all $u \in \mathcal{U}$ and all $e \in \mathcal{E}$ the following holds:*

(i): $\forall\ d_1 \in \mathcal{D}_1\ \exists\ d_2 \in \mathcal{D}_2\ :\ \begin{bmatrix} A_1x_1 + B_1u + G_1e + L_1d_1 \\ A_2x_2 + B_2u + G_2e + L_2d_2 \end{bmatrix} \in S$

(ii): $H_1x_1 = H_2x_2$

(iii): $C_1x_1 = C_2x_2$

The subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ defines a bisimulation relation between Σ_1 and Σ_2 if and only if it fulfills conditions (i) - (iii) and additionally

(iv): $\forall\ d_2 \in \mathcal{D}_2\ \exists\ d_1 \in \mathcal{D}_1\ :\ \begin{bmatrix} A_1x_1 + B_1u + G_1e + L_1d_1 \\ A_2x_2 + B_2u + G_2e + L_2d_2 \end{bmatrix} \in R$

Theorem 3.4. *A linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if the following holds:*

1. $\operatorname{im}\begin{bmatrix} L_1 \\ 0 \end{bmatrix} \subset S + \operatorname{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} := S_e$

2. $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} S \subset S_e$

3. $\operatorname{im}\begin{bmatrix} G_1 & B_1 \\ G_2 & B_2 \end{bmatrix} \subset S_e$

4. $S \subset \ker\begin{bmatrix} H_1 & -H_2 \\ C_1 & -C_2 \end{bmatrix}$

The subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a bisimulation relation between Σ_1 and Σ_2 if and only if it satisfies conditions 2 - 4 and in addition

$1'$. $R + \operatorname{im}\begin{bmatrix} L_1 \\ 0 \end{bmatrix} = R + \operatorname{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} =: R_e$

As a first compositional result, observe that feedback interconnections of linear systems as in Definition 3.1 are commutative.

Proposition 3.5. *For any two linear systems Σ_P and Σ_Q,*

$$\Sigma_P \| \Sigma_Q \approx \Sigma_Q \| \Sigma_P\ ,$$

Proof. Permuting the state vector of $\Sigma_Q \| \Sigma_P$, the relation

$$\begin{aligned} S \quad = \quad & \{((x_P, x_Q), (\bar{x}_P, \bar{x}_Q)) \mid (x_P, x_Q) \in \mathcal{X}_P \times \mathcal{X}_Q, (\bar{x}_Q, \bar{x}_P) \in \mathcal{X}_Q \times \mathcal{X}_P, \\ & x_P = \bar{x}_P, x_Q = \bar{x}_Q\} \end{aligned}$$

defines a bisimulation relation between $\Sigma_P \| \Sigma_Q$ and $\Sigma_Q \| \Sigma_P$. □

3.2.1. Soundness of compositional proof rules

Consider a complex linear plant system Σ_P which we assume to be given in the form of interconnected subsystems $\Sigma_{P_i}, i = 1, \ldots, k$, that is $\Sigma_P = \Sigma_{P_1} \| \ldots \| \Sigma_{P_k}$. We want to check whether Σ_P has the desired behavior specified by Σ_Q which again we assume to be given in the form of interconnected subspecifications Σ_{Q_i}, $\Sigma_Q = \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k}$ where the interconnection variables are elements of the same vector spaces. First, we will restrict ourselves to interconnections of two subsystems only. However, the compositional techniques described in the following can be generalized to an arbitrary number of subsystems thanks to their modular structure, which will be treated in Section 3.2.4.
Using simulation relations, the verification task can be expressed as

$$\Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{3.2}$$

If there exists a full simulation relation of the given system $\Sigma_{P_1} \| \Sigma_{P_2}$ by the specification $\Sigma_{Q_1} \| \Sigma_{Q_2}$ the behavior of the given system is included in the behavior of the specification which means that the specification will always be satisfied. In order to ensure that the given system fulfills the specification exactly, simulation in (3.2) has to be replaced by bisimulation, see Section 3.2.3. However, both Σ_P and Σ_Q are complex systems given as interconnections of subsystems. The verification task (3.2) will therefore be decomposed into two subtasks for the component systems in order to reduce its complexity. We call such a proof rule *sound* if the original verification task (3.2) can be inferred from the two subtasks it was split into.

Compositional reasoning

We start with the first pillar for compositional analysis.

Theorem 3.6. *For any four linear systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$, the* compositionality *property*

$$\left.\begin{array}{ccc} \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_2} \end{array}\right\} \implies \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{3.3}$$

holds.

Proof. Let S_i denote the full simulation relations of Σ_{P_i} by $\Sigma_{Q_i}, i = 1, 2$. Construct the relation

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_1}) \in S_1, (x_{P_2}, x_{Q_2}) \in S_2\} \tag{3.4}$$

Then for every $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$, every joint input $e_{P_i} = e_{Q_i} = e_i, i = 1, 2$, and every disturbances d_{P_1}, d_{P_2}, there exist disturbances d_{Q_1}, d_{Q_2} such that

$$\begin{bmatrix} A_{P_1} x_{P_1} + B_{P_1} C_{P_2} x_{P_2} + G_{P_1} e_1 + L_{P_1} d_{P_1} \\ A_{Q_1} x_{Q_1} + B_{Q_1} C_{Q_2} x_{Q_2} + G_{Q_1} e_1 + L_{Q_1} d_{Q_1} \end{bmatrix} \in S_1$$

and

$$\begin{bmatrix} A_{P_2}x_{P_2} + B_{P_2}C_{P_1}x_{P_1} + G_{P_2}e_2 + L_{P_2}d_{P_2} \\ A_{Q_2}x_{Q_2} + B_{Q_2}C_{Q_1}x_{Q_1} + G_{Q_2}e_2 + L_{Q_2}d_{Q_2} \end{bmatrix} \in S_2$$

Furthermore, since $C_{P_i}x_{P_i} = C_{Q_i}x_{Q_i}$ for all $(x_{P_i}, x_{Q_i}) \in S_i$, $H_{P_i}x_{P_i} = H_{Q_i}x_{Q_i}$. Moreover, S as defined in (3.4) is the product of the simulation relations S_1 and S_2 after reordering the vectors x_{Q_1} and x_{P_2}. Since $\Pi_{P_1}S_1 = \mathcal{X}_1$ and $\Pi_{P_2}S_2 = \mathcal{X}_2$ (because S_1 and S_2 are full) $\Pi_{P_1P_2}S = \mathcal{X}_1 \times \mathcal{X}_2$ and therefore S is full. □

Remark 3.7. In general, the converse implication in (3.3) does not hold. Take as a counterexample the following systems

$$\Sigma_{P_1}: \begin{array}{rcl} \dot{x}_{P_1} &=& 2u_{P_1} + e_{P_1} \\ y_{P_1} &=& x_{P_1} \\ z_{P_1} &=& x_{P_1} \end{array} , \qquad \Sigma_{P_2}: \begin{array}{rcl} \dot{x}_{P_2} &=& u_{P_2} + e_{P_2} \\ y_{P_2} &=& \frac{1}{2}x_{P_2} \\ z_{P_2} &=& x_{P_2} \end{array}$$

$$\Sigma_{Q_i}: \begin{array}{rcl} \dot{x}_{Q_i} &=& u_{Q_i} + e_{Q_i} \\ y_{Q_i} &=& x_{Q_i} \\ z_{Q_i} &=& x_{Q_i} \end{array} , \quad i = 1, 2 .$$

Then there exists a simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, namely

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid x_{P_1} = x_{Q_1}, x_{P_2} = x_{Q_2}\}$$

since the state space descriptions of $\Sigma_{P_1} \| \Sigma_{P_2}$ and $\Sigma_{Q_1} \| \Sigma_{Q_2}$ are identical. However, there do not exist any simulation relations of Σ_{P_1} by Σ_{Q_1} nor of Σ_{P_2} by Σ_{Q_2} since for the former

$$\text{im} \begin{bmatrix} B_{P_1} \\ B_{Q_1} \end{bmatrix} = \text{im} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \not\subseteq \ker \begin{bmatrix} C_{P_1} & -C_{Q_1} \\ H_{P_1} & -H_{Q_1} \end{bmatrix} = \text{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix} ,$$

and for the latter

$$\text{im} \begin{bmatrix} B_{P_2} \\ B_{Q_2} \end{bmatrix} = \text{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \not\subseteq \ker \begin{bmatrix} C_{P_2} & -C_{Q_2} \\ H_{P_2} & -H_{Q_2} \end{bmatrix} = \text{im} \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

We note that as a special case of compositionality, *invariance under composition* also holds:

Corollary 3.8. *Consider two linear systems $\Sigma_{P_1}, \Sigma_{Q_1}$ of the form* (3.1)*. Then for any linear system Σ_{Q_2} it holds that*

$$\Sigma_{P_1} \preccurlyeq \Sigma_{Q_1} \quad \Longrightarrow \quad \Sigma_{P_1} \| \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{3.5}$$

Assume-guarantee reasoning

In the case that one or both of the components $\Sigma_{P_i}, i = 1, 2$, do not fulfill their subspecifications Σ_{Q_i} directly, compositional reasoning cannot be applied to simplify the verification task (3.2). However, rather than considering the component systems Σ_{P_i} in isolation one can restrict them to a particular environment. More concretely, one can replace the assumption $\Sigma_{P_2} \preccurlyeq \Sigma_{Q_2}$ by $\Sigma_{Q_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}$, that is, Σ_{P_2} is compared with Σ_{Q_2} while it is already assumed that Σ_{P_1} may be replaced by Σ_{Q_1}, and similarly, $\Sigma_{P_1} \preccurlyeq \Sigma_{Q_1}$ can be replaced by $\Sigma_{P_1} \| \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}$ thus assuming that Σ_{P_2} satisfies its specification Σ_{Q_2}. This describes the general principle of *assume-guarantee reasoning*, a divide and conquer scheme based on mutual assumptions and guarantees to reduce the complexity of verification tasks. We first present two types of non-circular assume-guarantee reasoning rules, each replacing one of the assumptions $\Sigma_{P_i} \preccurlyeq \Sigma_{Q_i}$ as described. Hence, we avoid examining the more complex interconnection $\Sigma_{P_1} \| \Sigma_{P_2}$ yet are still able to guarantee that (3.2) holds.

Theorem 3.9. *For any given linear systems Σ_i,$i \in \{P_1, P_2, Q_1, Q_2\}$, non-circular assume-guarantee reasoning is sound, i.e. the following deduction scheme*

$$\left.\begin{array}{rrcl} S_1: & \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_{II}: & \Sigma_{Q_1} \| \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S : \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \quad (3.6)$$

and its symmetric counterpart

$$\left.\begin{array}{rrcl} S_2: & \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_2} \\ S_{I}: & \Sigma_{P_1} \| \Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S : \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \quad (3.7)$$

hold.

Proof. The proof only requires the relation $\preccurlyeq$ to be a preorder and the interconnection $\|$ to be invariant under composition. To prove (3.6), reflexivity of simulation and invariance under composition yield

$$\left.\begin{array}{rcl} \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ \Sigma_{P_2} & \preccurlyeq & \Sigma_{P_2} \end{array}\right\} \Longrightarrow \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{P_2}$$

which due to S_{II} and transitivity of simulation yields the desired result

$$\Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}$$

Exploiting commutativity of the interconnection, the same arguments hold for the non circular rule (3.7),

$$\Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{P_2} \| \Sigma_{P_1} \preccurlyeq \Sigma_{Q_2} \| \Sigma_{P_1} \preccurlyeq \Sigma_{P_1} \| \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}$$

□

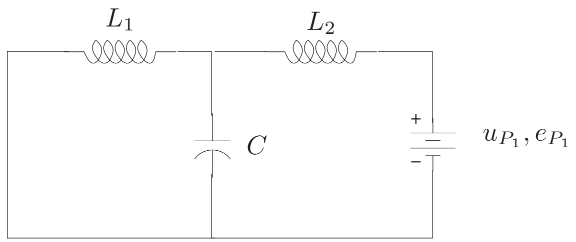

Figure 3.3.: Σ_{P_1}: $LC-$ circuit.

Example 3.10. Consider as Σ_{P_1} the $LC-$ circuit in Figure 3.3 with two inductors L_1 and L_2, one capacitor C, a voltage source as input u_{P_1} and the current over the capacitor as output y_{P_1}. The external in- and outputs are chosen to be the same as the interconnection variables, $u_{P_1} = e_{P_1}$ and $y_{P_1} = z_{P_1}$, while there are no external disturbances, i.e., d_{P_1} and d_{P_2} are absent.

Then, Σ_{P_1} is given by

$$\Sigma_{P_1}: \begin{array}{rcl} \begin{bmatrix} \dot{q}_C \\ \dot{\phi}_{L_1} \\ \dot{\phi}_{L_2} \end{bmatrix} & = & \begin{bmatrix} 0 & \frac{1}{L_1} & -\frac{1}{L_2} \\ -\frac{1}{C} & 0 & 0 \\ \frac{1}{C} & 0 & 0 \end{bmatrix} \begin{bmatrix} q_C \\ \phi_{L_1} \\ \phi_{L_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_{P_1} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e_{P_1} \\ y_{P_1} & = & \begin{bmatrix} \frac{1}{C} & 0 & 0 \end{bmatrix} x_{P_1} = z_{P_1} \end{array}$$

where $x_{P_1} = \begin{bmatrix} q_C & \phi_{L_1} & \phi_{L_2} \end{bmatrix}^T$ denotes the state vector. In the remainder, all the parameter values are set to 1 for simplicity. To stabilize the electrical circuit Σ_{P_1} we apply a simple feedback controller Σ_{P_2}, given as

$$\Sigma_{P_2}: \begin{array}{rcl} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} & = & \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_{P_2} \\ y_{P_2} & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{array}$$

Observe that $e_{P_2}, z_{P_2}, d_{P_2}$ are all void.
The verification goal is to simulate the 5-dimensional interconnection $\Sigma_{P_1} \| \Sigma_{P_2}$ by a less complex specification $\Sigma_{Q_1} \| \Sigma_{Q_2}$. The components of this specification are given by the $LC-$ circuit Σ_{Q_1} as in Figure 3.10 and an abstracted controller Σ_{Q_2}. In particular,

$$\Sigma_{Q_1}: \begin{array}{rcl} \begin{bmatrix} \dot{\phi}_{Q_1} \\ \dot{q}_{Q_1} \end{bmatrix} & = & \begin{bmatrix} 0 & -\frac{1}{C_{Q_1}} \\ \frac{1}{L_{Q_1}} & 0 \end{bmatrix} \begin{bmatrix} \phi_{Q_1} \\ q_{Q_1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{Q_1} + \\ & & + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_{Q_1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d_{Q_1} \\ y_{Q_1} & = & \begin{bmatrix} 0 & \frac{1}{C_{Q_1}} \end{bmatrix} x_{Q_1} = z_{Q_1} \end{array}$$

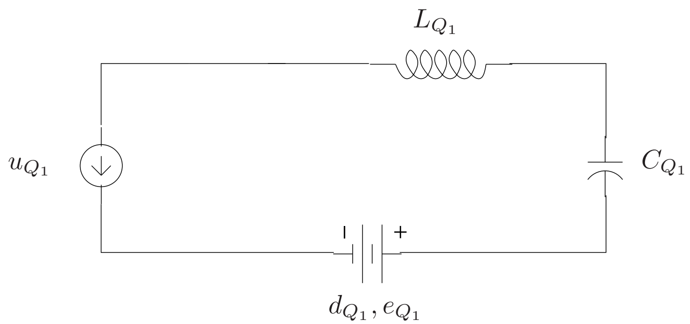

Figure 3.4.: Σ_{Q_1}

where $x_{Q_1} = \begin{bmatrix} \phi_{Q_1} & q_{Q_1} \end{bmatrix}^T$ and all parameter values are again set to 1. The controller Σ_{Q_2} is described by

$$\Sigma_{Q_2}: \quad \begin{array}{rcl} \dot{x}_{Q_2} & = & -5x_{Q_2} + u_{Q_2} + d_{Q_2} \\ y_{Q_2} & = & x_{Q_2} \end{array}$$

The first observation is that compositionality is not applicable since there does not exist any simulation relation of Σ_{P_1} by Σ_{Q_1}. The physical explanation is that the disturbance input d_{Q_1} represents a voltage source which cannot mimic the behavior of the inductor L_2. However, the controller systems Σ_{P_2} and Σ_{Q_2} can be related by means of a full simulation relation S_2 given as

$$S_2 = \{(z_1, z_2), x_{Q_2}) \mid z_1 = x_{Q_2}\} \tag{3.8}$$

Moreover, the interconnected system $\Sigma_{P_1} \| \Sigma_{Q_2}$ can be simulated by $\Sigma_{Q_1} \| \Sigma_{Q_2}$ using the simulation relation

$$\begin{array}{rcl} S_I & = & \{((q_C, \phi_{L_1}, \phi_{L_2}, x_{Q_2}), (x_1, x_2, x'_{Q_2})) \mid x_{Q_2} = x'_{Q_2}, \\ & & q_C = x_2, \phi_{L_1} = \phi_{L_2} + x_{Q_2} + x_1\} \end{array} \tag{3.9}$$

By Theorem 3.9, we can therefore conclude that there exists a full simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, given by

$$\begin{array}{rcl} S & = & \{((q_C, \phi_{L_1}, \phi_{L_2}, z_1, z_2), (x_1, x_2, x_{Q_2})) \mid z_1 = x_{Q_2}, \\ & & q_C = x_2, \phi_{L_1} = \phi_{L_2} + z_1 + x_1\} \end{array} \tag{3.10}$$

This shows that it is possible to abstract the behavior of the 5 dimensional controlled electrical circuit by a 3-dimensional electrical circuit with disturbances.

In *circular* assume-guarantee reasoning neither of the relations $S_i : \Sigma_{P_1} \preccurlyeq \Sigma_{Q_1}, i = 1, 2$, is assumed to hold. Instead, we consider interconnections of subsystems Σ_{P_i} of the plant and subspecifications Σ_{Q_i}. Thus, the behavior of the subsystems Σ_{P_i} is restricted since their interconnection variables are determined by the respective in- and outputs of Σ_{Q_j}. Replacing S_1 with the relation $S_I : \Sigma_{P_1}\|\Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2}$ and S_2 with $S_{II} : \Sigma_{Q_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2}$ results in a circular rule that is used to prove (3.2). That is, in S_I we *assume* that Σ_{P_2} fulfills Σ_{Q_2} to prove that under this assumption, Σ_{P_1} restricted by Σ_{Q_2} satisfies the specification Σ_{Q_1} (which is equally restricted by Σ_{Q_2}) while in S_{II}, we take for granted that Σ_{P_1} fulfills Σ_{Q_1} (the *guarantee*) to prove that Σ_{P_2} restricted by Σ_{Q_1} satisfies its specification Σ_{Q_2} and vice versa. Since the subspecifications are usually of much lower complexity than the corresponding subsystems of the plant the resulting reasoning scheme (3.15) is more efficient than direct verification. Due to the circular dependencies of assumptions and guarantees, circular assume-guarantee reasoning in general is only sound under additional conditions [21]. For linear systems, however, it turns out to *always* hold true. The main idea in this proof is to enlarge the simulation relations S_1 and S_2 in a suitable way and then to construct, based on these extended simulation relations, a simulation relation S of $\Sigma_{P_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$. Since the proofs of the following lemmas and the main theorem are quite technical, we defer them to the appendix.

Lemma 3.11. *Given full simulation relations $S_i, i = I, II$, of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, respectively, and define the following linear subspaces*

$$\begin{aligned}\bar{S}_I \;&:=\; \{(x_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, -x_{Q_2}) \mid x_{Q_2}, \bar{x}_{Q_2} \in \ker C_{Q_2} \cap \ker H_{Q_2}, \\ &\quad\; x_{P_1} \in \ker C_{P_1} \cap \ker H_{P_1}, x_{Q_1} \in \ker C_{Q_1} \cap \ker H_{Q_1}, \\ &\quad\; (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I\} \qquad (3.11)\\ \bar{S}_{II} \;&:=\; \{(\bar{x}_{Q_1}, x_{Q_2}, -x_{Q_1}, x_{Q_2}) \mid x_{Q_1}, \bar{x}_{Q_1} \in \ker C_{Q_1} \cap \ker H_{Q_1}, \\ &\quad\; x_{P_2} \in \ker C_{P_2} \cap \ker H_{P_2}, x_{Q_2} \in \ker C_{Q_2} \cap \ker H_{Q_2}, \\ &\quad\; (x_{Q_1}, x_{Q_1}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}\} \qquad (3.12)\end{aligned}$$

Then $S_I + \bar{S}_I$ and $S_{II} + \bar{S}_{II}$ also define full simulation relations of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, respectively.

Proof. The proof can be found in A.1. □

Lemma 3.12. *Given full simulation relations S_I and S_{II} of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, respectively, then also their symmetrical closures*

$$S_I^{sym} := S_I + \hat{S}_I, \quad S_{II}^{sym} := S_{II} + \hat{S}_{II} \qquad (3.13)$$

where

$$\begin{aligned}\hat{S}_I &:= \{(x_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I\} \qquad (3.14)\\ \hat{S}_{II} &:= \{(\bar{x}_{Q_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}\}\end{aligned}$$

define full simulation relations of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$.

Proof. The proof is very similar to the previous one and can be found in A.2. □

Adding the subspaces $\bar{S}_i, i = I, II$, and then taking the symmetrical closure ${S_i}^{\mathrm{sym}}$ ensures that the extended simulation relations $(S_i + \bar{S}_i)^{\mathrm{sym}}$ include elements of a particular form.

Lemma 3.13. *Consider full simulation relation* $(S_I + \bar{S}_I)^{sym}$ *and* $(S_{II} + \bar{S}_{II})^{sym}$ *of* $\Sigma_{P_1} \| \Sigma_{Q_2}$ *and* $\Sigma_{Q_1} \| \Sigma_{P_2}$ *by* $\Sigma_{Q_1} \| \Sigma_{Q_2}$ *as defined in the previous lemmas. Then for every* $x \in \ker C_{Q_2} \cap \ker H_{Q_2}$*,* $(0, x, 0, x) \in (S_I + \bar{S}_I)^{sym}$ *and analogously, for every* $y \in \ker C_{Q_1} \cap \ker H_{Q_1}$*,* $(y, 0, y, 0) \in (S_{II} + \bar{S}_{II})^{sym}$*.*

Proof. The proof can be found in A.3. □

Using the extended full simulation relations $(S_I + \bar{S}_I)^{\mathrm{sym}}$ and $(S_{II} + \bar{S}_{II})^{\mathrm{sym}}$ it is possible to construct a full simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, as formulated in the following main theorem.

Theorem 3.14. *For any given linear systems* Σ_i*,* $i \in \{P_1, P_2, Q_1, Q_2\}$*, circular assume-guarantee reasoning is sound, i.e. the deduction*

$$\left.\begin{array}{llll} S_I: & \Sigma_{P_1} \| \Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \\ S_{II}: & \Sigma_{Q_1} \| \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S: \ \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{3.15}$$

holds. The full simulation relation S *of* $\Sigma_{P_1} \| \Sigma_{P_2}$ *by* $\Sigma_{Q_1} \| \Sigma_{Q_2}$ *is given by*

$$\begin{array}{rcl} S & := & \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists \bar{x}_{Q_1}, \bar{x}_{Q_2} : (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in \\ & & (S_I + \bar{S}_I)^{sym}, (x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in (S_{II} + \bar{S}_{II})^{sym}\} \end{array} \tag{3.16}$$

Proof. The proof is in A.4. □

Symmetrization of simulation relations as in Lemma 3.12 has been used for labeled transition systems as well. In [21], the following lemma can be found:

Lemma 3.15. *Consider a full simulation relation* S_I *of* $\Sigma_{P_1} \| \Sigma_{Q_2}$ *by* $\Sigma_{Q_1} \| \Sigma_{Q_2}$*. Then also*

$$\tilde{S}_I := \{(x_{P_1}, x_{Q_2}, x_{Q_1}, x_{Q_2}) \mid \exists \bar{x}_{Q_2} \ : \ (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I\} \tag{3.17}$$

is a full simulation relation of $\Sigma_{P_1} \| \Sigma_{Q_2}$ *by* $\Sigma_{Q_1} \| \Sigma_{Q_2}$*. The same holds for the symmetric counterpart of a simulation relation* S_{II} *of* $\Sigma_{Q_1} \| \Sigma_{P_2}$ *by* $\Sigma_{Q_1} \| \Sigma_{Q_2}$*.*

Proof. This can be concluded simply from Definition 3.2 by substituting $\bar{x}_{Q_2}$ with x_{Q_2}. □

Lemma 3.15 gives rise to an alternative construction of a full simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2}$ in the case of circular assume-guarantee reasoning.

Proposition 3.16. *Consider any four linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ of the form (3.1). Let simulation relations S_I and S_{II} of $\Sigma_{P_1} \| \Sigma_{Q_2}$ and $\Sigma_{Q_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, respectively, be given such that the relation S, given by*

$$S := \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists \bar{x}_{Q_1}, \bar{x}_{Q_2} : (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I, (x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}\} \quad (3.18)$$

is a full simulation relation of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$. Then

$$\tilde{S} := \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_2}, x_{Q_1}, x_{Q_2}) \in \tilde{S}_I, (x_{Q_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in \tilde{S}_{II}\} \quad (3.19)$$

with $\tilde{S}_i, i = I, II$ as in (3.17)*, defines the same simulation relation of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, i.e. $S = \tilde{S}$.*

Proof. Observe first that due to Theorem 3.14, there always exist relations S_I and S_{II} large enough such that S as in (3.16) is a full simulation relation of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$. Clearly, $\tilde{S} \subset S$ since the components $\bar{x}_{Q_2}$, $\bar{x}_{Q_1}$ do not play any role in constructing S. For the converse, recall that $\tilde{S}_i, i = I, II$, as defined in (3.17) are full simulation relations. Take any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. Then there exists $\bar{x}_{Q_1}, \bar{x}_{Q_2}$ such that $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I$ and $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}$. But then $(x_{P_1}, x_{Q_2}, x_{Q_1}, x_{Q_2}) \in \tilde{S}_I$ and similarly, $(x_{Q_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in \tilde{S}_{II}$. This in turn implies that $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in \tilde{S}$. □

3.2.2. Completeness considerations

So far we used compositional analysis techniques to split a global verification task into several less complex tasks for the components involved. In other words, verifying properties of component systems allowed us to infer properties of their interconnection. This gives rise to the question whether the converse strategy also works, i.e., if we know that the overall system satisfies a certain property can we then conclude that all the components have this property? In the terminology of formal verification, *completeness* means that the converse implication of a compositional proof rule holds true as well. In this section, we will investigate completeness of both compositional and assume-guarantee reasoning.

Completeness of compositional reasoning with respect to open and closed interconnections

Recall the earlier Example 3.10 of two controlled $LC-$circuits. We showed that the interconnection $\Sigma_{P_1} \| \Sigma_{P_2}$ did satisfy the lower-dimensional specification $\Sigma_{Q_1} \| \Sigma_{Q_2}$ even though $\Sigma_{P_1} \not\preceq \Sigma_{Q_1}$. Thus, compositionality cannot be complete in general. As it turns out, it depends on the presence of additional inputs in the feedback interconnection whether or not compositional reasoning is complete. To illustrate this, we modify Definition 3.1 by adding inputs v_i to the feedback interconnection.

Definition 3.17. Given two linear systems $\Sigma_i, i = 1, 2$, of the form

$$\Sigma_i : \begin{array}{rcl} \dot{x}_i & = & A_i x_i + B_i u_i + L_i d_i \\ y_i & = & C_i x_i \end{array} \tag{3.20}$$

The *open* feedback interconnection, denoted by $\Sigma_1 \parallel_o \Sigma_2$, is given by

$$u_1 = y_2 + v_1 \quad , \quad u_2 = y_1 + v_2$$

yielding the closed loop dynamics

$$\Sigma_1 \|_o \Sigma_2 : \begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} A_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ & & + \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} & = & \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array} \tag{3.21}$$

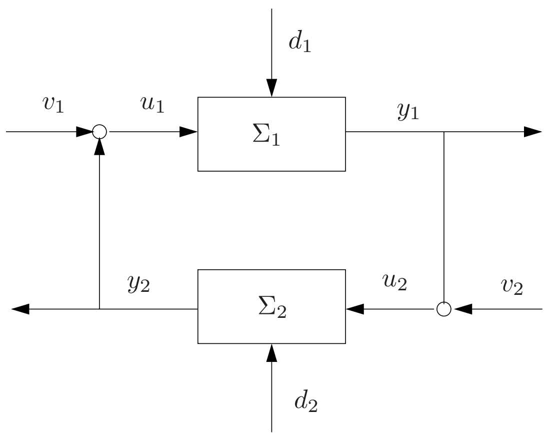

Figure 3.5.: Open feedback interconnection $\Sigma_1 \|_o \Sigma_2$.

Proposition 3.18. *For linear systems* (3.1) *interconnected by open feedback* (3.19), *the following equivalence holds:*

$$\left. \begin{array}{rcl} \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_2} \end{array} \right\} \Longleftrightarrow \Sigma_{P_1} \|_o \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \|_o \Sigma_{Q_2} \tag{3.22}$$

Proof. "$\Longrightarrow$": The proof is the same as for Theorem 3.6.
"$\Longleftarrow$": Let a simulation relation S of $\Sigma_{P_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2}$ be given, i.e. for any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$, any joint v_1, v_2 and any d_{P_1}, d_{P_2} there exists d_{Q_1}, d_{Q_2} such that

$$(i): \begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}(v_1 + C_{P_2}x_{P_2}) + G_{P_1}d_{Q_1} \\ A_{P_2}x_{P_2} + B_{P_2}(v_2 + C_{P_1}x_{P_1}) + G_{P_2}d_{P_2} \\ A_{Q_1}x_{Q_1} + B_{Q_1}(v_1 + C_{Q_2}x_{Q_2}) + G_{Q_1}d_{Q_1} \\ A_{Q_2}x_{Q_2} + B_{Q_2}(v_2 + C_{Q_1}x_{Q_1}) + G_{Q_2}d_{Q_2} \end{bmatrix} \in S \tag{3.23}$$
$$(ii): C_{P_1}x_{P_1} = C_{Q_1}x_{Q_1}\ ,\ C_{P_2}x_{P_2} = C_{Q_2}x_{Q_2}$$

Define the relation $S_1 := \{(x_{P_1}, x_{Q_1}) \mid \exists\, x_{P_2}, x_{Q_2} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S\}$. It has to be shown that for any $(x_{P_1}, x_{Q_1}) \in S_1$, any u and any d_1 there exists a d_2 such that

$$(i): \begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u + G_{P_1}d_1 \\ A_{Q_1}x_{Q_1} + B_{Q_1}u + G_{Q_1}d_2 \end{bmatrix} \in S_1$$
$$(ii): C_{P_1}x_{P_1} = C_{Q_1}x_{Q_1}$$

Take any $(x_{P_1}, x_{Q_1}) \in S_1$ and fix u and d_1. Since (3.23) holds for any v_1, v_2 and d_{P_1}, d_{P_2}, in particular it holds for $v_1 = u - C_{P_2}x_{P_2}$ and $d_{P_1} = d_1$. Thus, S_1 defines a simulation relation of Σ_{P_1} by Σ_{Q_1}. The same arguments also hold for a relation $S_2 := \{(x_{P_2}, x_{Q_2}) \mid \exists\, x_{P_1}, x_{Q_1} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S\}$.
□

The consequence of Proposition 3.18 is immediate. For open feedback interconnections, the problem of checking $\Sigma_{P_1}\|_o\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_o\Sigma_{Q_2}$ can be reduced to (and, in fact, is equivalent to)

$$\Sigma_{P_1} \preccurlyeq \Sigma_{Q_1}\ ,\ \Sigma_{Q_1} \preccurlyeq \Sigma_{Q_2}$$

Hence, it is enough to verify properties of the individual components to ensure that these properties hold for the interconnection. Stated differently, assume-guarantee reasoning is not needed for *open* feedback interconnections since one can always resort to compositional reasoning which is less complex. This result also simplifies the computation of the maximal simulation relation for linear systems without disturbance inputs.

Proposition 3.19. *Consider four linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$, of the form* (3.1) *with d_i void and without external variables, $e_i = z_i \equiv 0$. Assume that the transfer matrices of the open feedback interconnections$\Sigma_{P_1}\|_o\Sigma_{P_2}$and $\Sigma_{Q_1}\|_o\Sigma_{Q_2}$ are* *the same. Then the maximal simulation relation* $R^{\star}_{P_1P_2Q_1Q_2}$ *between* $\Sigma_{P_1}\|_o\Sigma_{P_2}$ *and* $\Sigma_{Q_1}\|_o\Sigma_{Q_2}$ *is given by*

$$R^{\star}_{P_1P_2Q_1Q_2} = \ker \begin{bmatrix} C_{P_1} & 0 & -C_{Q_1} & 0 \\ 0 & C_{P_2} & 0 & C_{Q_2} \\ C_{P_1}A_{P_1} & 0 & C_{Q_1}A_{Q_1} & 0 \\ 0 & C_{P_2}A_{P_2} & 0 & C_{Q_2}A_{Q_2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{P_1}A^n_{P_1} & 0 & C_{Q_1}A^n_{Q_1} & 0 \\ 0 & C_{P_2}A^n_{P_2} & 0 & C_{Q_2}A^n_{Q_2} \end{bmatrix} \tag{3.24}$$

where $n = \max\{n_{P_1}, n_{P_2}, n_{Q_1}, n_{Q_2}\} - 1$.

Proof. Observe first that the maximal simulation relation $R^{\star}_{P_1P_2Q_1Q_2}$ can be written as

$$R^{\star}_{P_1P_2Q_1Q_2} = \bigcap_{i=0}^{n} \ker \Big[C_{P_1P_2}\left(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2}\right)^i \quad - C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^i\Big]$$

where

$$A_{j_1j_2} = \begin{bmatrix} A_{j_1} & 0 \\ 0 & A_{j_2} \end{bmatrix}, B_{j_1j_2} = \begin{bmatrix} 0 & -B_{j_1} \\ B_{j_2} & 0 \end{bmatrix}, C_{j_1j_2} = \begin{bmatrix} C_{j_1} & 0 \\ 0 & C_{j_2} \end{bmatrix},$$

$j \in \{P, Q\}$. Secondly, for all $(x_{P_1P_2}, x_{Q_1Q_2})$, $x_{j_1j_2} = (x_{j_1}, x_{j_2})$, such that

$$C_{P_1P_2}x_{P_1P_2} = C_{Q_1Q_2}x_{Q_1Q_2}$$

the following equivalence holds for all $k = 1, 2, \ldots$:

$$\begin{aligned} C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^k x_{P_1P_2} &= C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^k x_{Q_1Q_2} \\ &\Longleftrightarrow \\ C_{P_1P_2}A^k_{P_1P_2}x_{P_1P_2} &= C_{Q_1Q_2}A^k_{Q_1Q_2}x_{Q_1Q_2} \end{aligned} \tag{3.25}$$

Indeed, for $k = 1$, we obtain due to equality of the Markov parameters $C_{P_1P_2}B_{P_1P_2} = C_{Q_1Q_2}B_{Q_1Q_2}$ that

$$\begin{aligned} C_{P_1P_2}A_{P_1P_2}x_{P_1P_2} - C_{P_1P_2}B_{P_1P_2}C_{P_1P_2}x_{P_1P_2} &= C_{Q_1Q_2}A_{Q_1Q_2}x_{Q_1Q_2} - C_{Q_1Q_2}B_{Q_1Q_2}C_{Q_1Q_2}x_{Q_1Q_2} \\ &\Longleftrightarrow \\ C_{P_1P_2}A_{P_1P_2}x_{P_1P_2} &= C_{Q_1Q_2}A_{Q_1Q_2}x_{Q_1Q_2} \end{aligned}$$

Assuming that (3.25) holds for a certain k, we conclude

$$C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^{k+1}x_{P_1P_2} = C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^{k+1} x_{Q_1Q_2}$$
$$\Longleftrightarrow$$
$$C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^{k}A_{P_1P_2}x_{P_1P_2} - C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^{k} B_{P_1P_2}C_{P_1P_2}x_{P_1P_2} = C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^{k}A_{Q_1Q_2}x_{Q_1Q_2} - C_{Q_1Q_2} (A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^{k}B_{Q_1Q_2}C_{Q_1Q_2}x_{Q_1Q_2}$$
$$\Longleftrightarrow$$
$$C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^{k}A_{P_1P_2}x_{P_1P_2} = C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^{k} A_{Q_1Q_2}x_{Q_1Q_2}$$
$$\Longleftrightarrow$$
$$C_{P_1P_2}A_{P_1P_2}^{k+1}x_{P_1P_2} = C_{Q_1Q_2}A_{Q_1Q_2}^{k+1}x_{Q_1Q_2}$$

where we made use of

$$C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^{k}B_{P_1P_2}C_{P_1P_2}x_{P_1P_2} = C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^{k}B_{Q_1Q_2}C_{Q_1Q_2}x_{Q_1Q_2}.$$

As before,

$$C_{P_1P_2}\tilde{x}_{P_1P_2} := C_{P_1P_2}B_{P_1P_2}C_{P_1P_2}x_{P_1P_2} = C_{Q_1Q_2}B_{Q_1Q_2}C_{Q_1Q_2}x_{Q_1Q_2} =: C_{Q_1Q_2}\tilde{x}_{Q_1Q_2}$$

so that we can apply the hypothesis (3.25) on

$$C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^{k}\tilde{x}_{P_1P_2} = C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^{k}\tilde{x}_{Q_1Q_2}\ .$$

Equation (3.26) then reduces to the equality of the $k+1-$th Markov parameters. The maximal simulation relation $R^{\star}_{P_1P_2Q_1Q_2}$ can thus be constructed as the product of the maximal simulation relations $R^{\star}_{P_1Q_1}$ and $R^{\star}_{P_2Q_2}$ between Σ_{P_1} and Σ_{Q_1} on the one hand and Σ_{P_2} and Σ_{Q_2} on the other hand since the influence of the feedback terms vanishes. By Proposition 2.12 $R^{\star}_{P_1P_2Q_1Q_2}$ is also the maximal bisimulation relation between $\Sigma_{P_1}\|_o\Sigma_{P_2}$ and $\Sigma_{Q_1}\|_o\Sigma_{Q_2}$. □

Another interesting fact about open feedback interconnections becomes apparent if we consider a version of simulation relations that does not require the inputs $v_i, i = 1, 2$, to be the same. In this case Definition 3.2 is modified such that *for any* input $u_1(\cdot)$ *there should exist* an input $u_2(\cdot)$ such that (i) and (iii) hold. Then, for any two linear systems $\Sigma_i, i = 1, 2$, of the form (3.20) the open feedback interconnection $\Sigma_1\|_o\Sigma_2$ is in fact bisimilar to the two systems running in parallel [2], denoted by $\Sigma_1 \vDash \Sigma_2$ and given as

$$\Sigma_1 \vDash \Sigma_2 :$$
$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} + \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$
$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}.$$

In fact, the inputs v_i of the open feedback interconnection $\Sigma_1\|_o\Sigma_2$ and $\tilde{v}_i$ of the interconnection $\Sigma_1 \vDash \Sigma_2$ are related,

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} C_2 x_2 \\ C_1 x_1 \end{bmatrix} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} \tag{3.26}$$

Construct the bisimulation relation S between $\Sigma_1\|_o\Sigma_2$ and $\Sigma_1 \vDash \Sigma_2$ as

$$S = \{((x_1, x_2), (\tilde{x}_1, \tilde{x}_2)) \mid x_i = \tilde{x}_i,\ i = 1, 2\}\ .$$

It follows immediately that for any input $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ there always exists an input $\begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}$ given by (3.26) such that conditions (i) and (iii) in Definition 3.2 are fulfilled and vice versa. Hence,

$$\Sigma_1\|_o\Sigma_2 \approx \Sigma_1 \vDash \Sigma_2$$

Thus, the open feedback interconnection does not restrict the behavior of the individual components since the influence of the feedback terms can always be compensated by the additional inputs v_i.

Completeness of circular assume-guarantee reasoning

Circular assume-guarantee reasoning is known not to be complete for labeled transition systems, see e.g. the counterexample in [21]. We will show, however, that circular assume-guarantee reasoning is indeed complete for linear systems.

Theorem 3.20. *Circular assume-guarantee reasoning is complete, i.e. for any given linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ such that there exists a full simulation relation S of $\Sigma_{P_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, there also exist full simulation relations S_I and S_{II} of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, respectively.*
The relations S_I and S_{II} are given by

$$\begin{aligned} S_I &= \{(x_{P_1}, x_{Q_2}, x_{Q_1}, x_{Q_2}) \mid \exists x_{P_2} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S\} + \\ &\quad +\{(0, x, 0, x) \mid x \in \mathcal{X}_{Q_2}\} \\ S_{II} &= \{(x_{Q_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists x_{P_1} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S\} + \\ &\quad +\{(x, 0, x, 0) \mid x \in \mathcal{X}_{Q_1}\} \end{aligned} \tag{3.27}$$

[2]This result was contributed by Paulo Tabuada, whose authorship is gratefully acknowledged

Proof. The proof is deferred to Appendix B. □

3.2.3. Proof rules involving bisimulation relations

In the previous sections, we presented compositional analysis techniques for linear systems using simulation relations to reduce the complexity of the verification task. The target system in these deduction schemes represents an abstraction of the actual system model as illustrated by Example 3.10. By contrast, bisimulation relations constitute *equivalence relations*. Hence, they can be used to exactly reduce system models by quotienting out parts of the state space, see Example 2.14. In this section, we want to investigate whether the deduction schemes of Section 3.2 also work when replacing simulation with bisimulation.

Proposition 3.21. *Consider four linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ of the form* (3.1). *Then the following deduction schemes are sound:*

1. *(compositional reasoning)*

$$\left.\begin{array}{lrcl} S_1: & \Sigma_{P_1} & \approx & \Sigma_{Q_1} \\ S_2: & \Sigma_{P_2} & \approx & \Sigma_{Q_2} \end{array}\right\} \implies S : \Sigma_{P_1} \| \Sigma_{P_2} \approx \Sigma_{Q_1} \| \Sigma_{Q_2}$$

2. *(non-circular assume-guarantee reasoning)*

$$\left.\begin{array}{lrcl} S_1: & \Sigma_{P_1} & \approx & \Sigma_{Q_1} \\ S_{II}: & \Sigma_{Q_1} \| \Sigma_{P_2} & \approx & \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array}\right\} \implies S : \Sigma_{P_1} \| \Sigma_{P_2} \approx \Sigma_{Q_1} \| \Sigma_{Q_2}$$

3. *(circular assume-guarantee reasoning)*

$$\left.\begin{array}{lrcl} S_I: & \Sigma_{P_1} \| \Sigma_{Q_2} & \approx & \Sigma_{Q_1} \| \Sigma_{Q_2} \\ S_{II}: & \Sigma_{Q_1} \| \Sigma_{P_2} & \approx & \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array}\right\} \iff S : \Sigma_{P_1} \| \Sigma_{P_2} \approx \Sigma_{Q_1} \| \Sigma_{Q_2}$$

Proof. Notice that in all three cases we only need to check condition (iv) in Proposition 3.3 (respectively (4) in Theorem 3.4) and fullness of the candidate relation. For rule 1, observe that the construction of S, namely taking the product $S = S_1 \times S_2$ after reordering the state variables, ensures that $\Pi_{Q_1 Q_2} S = \mathcal{X}_{Q_1} \times \mathcal{X}_{Q_2}$ since both S_1 and S_2 are full bisimulation relations. Moreover, condition (iv) in Proposition 3.3 is fulfilled since for every d_{P_2} there exists a d_{P_1} such that (3.5) holds and similarly, for every d_{Q_2} there exists a d_{Q_1} such that (3.5) holds.
Since bisimulation relations are equivalence relations, they are also transitive which together with rule 1 guarantees soundness of rule 2.
Rule 3 will be proved more explicitly. Use the same construction of S as in (3.16). To show condition (4) of Theorem 3.4, observe that since $(S_I + \bar{S}_I)^{\mathrm{sym}}$ is

a full bisimulation relation, for every $d_{Q_2} \in \mathrm{im} L_{Q_1}$ there exists $x_{P_1} \in \mathrm{im} L_{P_1}$, $x_{Q_2} \in \mathrm{im} L_{Q_2} \cap \ker C_{Q_2} \cap \ker H_{Q_2}$ such that

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ d_{Q_2} \end{bmatrix} = \begin{bmatrix} x_{P_1} \\ x_{Q_2} \\ 0 \\ d_{Q_2} \end{bmatrix} + \mathrm{im} \begin{bmatrix} L_{P_1} & 0 \\ 0 & L_{Q_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Due to symmetrization, also

$$\begin{bmatrix} x_{P_1} \\ d_{Q_2} \\ 0 \\ x_{Q_2} \end{bmatrix} \in (S_I + \bar{S}_I)^{\mathrm{sym}} \; \forall d_{Q_2} \in \mathrm{im} L_{Q_2}$$

Since $(S_{II} + \bar{S}_{II})^{\mathrm{sym}}$ is also a full bisimulation relation, for every $d_{Q_2} \in \mathrm{L}_{Q_2}$ there exists $x_{Q_1} \in \mathrm{im} L_{Q_1} \cap \ker C_{Q_1} \cap \ker H_{Q_1}$ such that

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ d_{Q_2} \end{bmatrix} = \begin{bmatrix} x_{Q_1} \\ x_{P_2} \\ 0 \\ d_{Q_2} \end{bmatrix} + \mathrm{im} \begin{bmatrix} L_{Q_1} & 0 \\ 0 & L_{P_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_{Q_1} \\ x_{P_2} \\ 0 \\ d_{Q_2} \end{bmatrix} \in (S_{II} + \bar{S}_{II})^{\mathrm{sym}}$$

Due to Lemma 3.13, also $(x_{Q_1}, 0, x_{Q_1}, 0) \in (S_{II} + \bar{S}_{II})^{\mathrm{sym}}$ and therefore

$$\begin{bmatrix} x_{P_1} \\ d_{Q_2} \\ 0 \\ x_{Q_2} \end{bmatrix} \in (S_I + \bar{S}_I)^{\mathrm{sym}}, \quad \begin{bmatrix} x_{Q_1} \\ x_{P_2} \\ 0 \\ d_{Q_2} \end{bmatrix} - \begin{bmatrix} x_{Q_1} \\ 0 \\ x_{Q_1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ x_{P_2} \\ -x_{Q_1} \\ d_{Q_2} \end{bmatrix} \in (S_{II} + \bar{S}_{II})^{\mathrm{sym}}$$

Hence, for any $d_{Q_2} \in \mathrm{im} L_{Q_2}$,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ d_{Q_2} \end{bmatrix} = \underbrace{\begin{bmatrix} x_{P_1} \\ x_{P_2} \\ 0 \\ d_{Q_2} \end{bmatrix}}_{\in S} + \mathrm{im} \begin{bmatrix} L_{P_1} & 0 \\ 0 & L_{P_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Analogously, one can proof that $\mathrm{im} \begin{bmatrix} 0 \\ 0 \\ L_{Q_1} \\ 0 \end{bmatrix} \in S + \mathrm{im} \begin{bmatrix} L_{P_1} & 0 \\ 0 & L_{P_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

To show that $\Pi_{Q_1 Q_2} S = \mathcal{X}_{Q_1} \times \mathcal{X}_{Q_2}$, note that for every every x_{Q_1}, x_{Q_2} there exists $\bar{x}_{P_1}, \bar{x}_{Q_2}$ such that $(\bar{x}_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, x_{Q_2}) \in (S_I + \bar{S}_I)^{\mathrm{sym}}$. Due to Lemma 3.12, then also $(\bar{x}_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in (S_I + \bar{S}_I)^{\mathrm{sym}}$. Similarly, since $(S_{II} + \bar{S}_{II})^{\mathrm{sym}}$ is full, for every x_{Q_1}, x_{Q_2} there exists $\hat{x}_{Q_1}, \hat{x}_{Q_1}$ such that $(\hat{x}_{Q_1}, \hat{x}_{P_2}, x_{Q_1}, x_{Q_2}) \in (S_{II} + \bar{S}_{II})^{\mathrm{sym}}$, and by symmetry also $(x_{Q_1}, \hat{x}_{P_2}, \hat{x}_{Q_1}, x_{Q_2}) \in (S_{II} + \bar{S}_{II})^{\mathrm{sym}}$. Hence, $(\bar{x}_{P_1}, \hat{x}_{P_2}, x_{Q_1}, x_{Q_2}) \in S$ for every x_{Q_1}, x_{Q_2}. □

Remark 3.22. Since bisimulation implies mutual simulation, Proposition 3.21 also proves soundness of the following type of reasoning:

$$\left.\begin{array}{ccccc} \underline{\Sigma_{Q_1}} & \preccurlyeq & \Sigma_{P_1} & \preccurlyeq & \overline{\Sigma_{Q_1}} \\ \underline{\Sigma_{Q_2}} & \preccurlyeq & \Sigma_{P_2} & \preccurlyeq & \overline{\Sigma_{Q_2}} \end{array}\right\} \Longrightarrow \left(\underline{\Sigma_{Q_1}} \| \underline{\Sigma_{Q_2}}\right) \preccurlyeq (\Sigma_{P_1} \| \Sigma_{P_2}) \preccurlyeq \left(\overline{\Sigma_{Q_1}} \| \overline{\Sigma_{Q_2}}\right)$$

Here, $\underline{\Sigma_{Q_i}}, i = 1, 2$, could represent an under-approximation of a specification or a *refinement* of Σ_{P_i} and $\overline{\Sigma_{Q_i}}$ an over-approximation or *abstraction*. A complementary notion of abstraction, refinements express distinct features of a model in more detail thus adding additional information about the system, which in the case of control systems usually involves more state variables and differential equations. Some verification schemes e.g. in [42] use both abstraction and refinement.

3.2.4. Generalization to k systems

Having completed our analysis of compositional techniques for feedback interconnections of two systems, a natural generalization is to consider interconnections of *more than two* systems. As before, the interconnection variables u and y will be used to interconnect linear systems with each other. Furthermore, we want to consider series interconnections. Hence, we partition the in- and output matrices related to the variables u_i, y_i as follows:

$$B_i = \left[\begin{array}{cc} B_i^- & B_i^+ \end{array}\right] \quad , \quad C_i = \left[\begin{array}{c} C_i^- \\ C_i^+ \end{array}\right], i = 1, \ldots, k$$

Every system $\Sigma_i, i = 2, \ldots, k-1$, is thus connected to a predecessor Σ_{i-1} on the left and a successor system Σ_{i+1} on the right:

$$\begin{array}{ccccccc} y_i^- & = & u_{i-1}^+ & , & y_i^+ & = & u_{i+1}^- \\ u_i^- & = & y_{i-1}^+ & , & u_i^+ & = & y_{i+1}^- \end{array}$$

The interconnection variables $u_1 = u_1^-, y_1 = y_1^-$ of the first and the variables $u_k^+ = u_k, y_k^+ = y_k$ of the last system Σ_k remain accessible, see Figure 3.6.

Remark 3.23. In the remainder of this chapter we restrict ourselves to series of feedback interconnections of linear systems. However, all the results presented here also hold for other network topologies, e.g. for the full interconnection case where each system is interconnected to every other systems by feedback.

Compositional reasoning for k systems can then be derived inductively based on the result of Section 3.2.1.

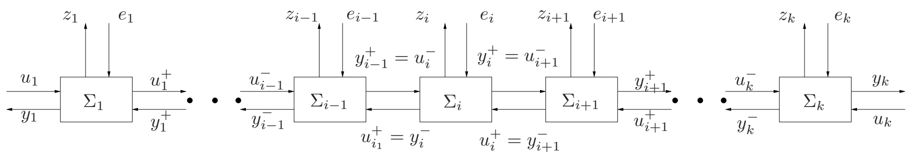

Figure 3.6.: Series interconnection $\Sigma_1\|\ldots\|\Sigma_k$.

Theorem 3.24. *Consider k linear systems Σ_{P_i} and $\Sigma_{Q_i}, i = 1\ldots, k$, of the form* (3.1). *Then compositional reasoning is sound for series interconnections of k systems, i.e.*

$$\begin{aligned} \Sigma_{P_i} \quad &\preccurlyeq \quad \Sigma_{Q_i} \qquad\qquad \forall\, i = 1,\ldots,k \\ &\Longrightarrow \\ \Sigma_{P_1}\|\ldots\|\Sigma_{P_k} \quad &\preccurlyeq \quad \Sigma_{Q_1}\|\ldots\|\Sigma_{Q_k} \end{aligned} \tag{3.28}$$

Proof. The proof uses induction over k. Theorem 3.6 contains the proof for the case $k = 2$. Assume now that the series interconnection Σ_P of k plant-controller systems,

$$\Sigma_P := \Sigma_{P_1}\|\ldots\|\Sigma_{P_k}$$

fulfills a composed specification Σ_Q of the form

$$\Sigma_S := \Sigma_{S_1}\|\ldots\|\Sigma_{Q_k}\ ,$$

i.e.,

$$\Sigma_P \quad \preccurlyeq \quad \Sigma_Q \tag{3.29}$$

Moreover, let there exist a full simulation relation of $\Sigma_{P_{k+1}}$ by $\Sigma_{Q_{k+1}}$,

$$\Sigma_{P_{k+1}} \quad \preccurlyeq \quad \Sigma_{Q_{k+1}} \tag{3.30}$$

Taking the product of the full simulation relations for (3.29) and (3.30) yields, after reordering the state components, a full simulation relation of

$$\Sigma_P\|\Sigma_{P_{k+1}} \preccurlyeq \Sigma_Q\|\Sigma_{Q_{k+1}} \tag{3.31}$$

This proves the induction step. □

In contrast to this straightforward generalization, proving soundness of circular assume-guarantee reasoning for k linear systems is more involved. We need an auxiliary result that extends the proof rule of Theorem 3.14 by interconnecting arbitrary systems from the left and right.

Lemma 3.25. *Consider six linear control systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2, L, R\}$ of the form* (3.1)*. Then the following reasoning is sound:*

$$\begin{array}{rrcl} S_I: & \Sigma_L\|\Sigma_{P_1}\|\Sigma_{Q_2}\|\Sigma_R & \preccurlyeq & \Sigma_L\|\Sigma_{Q_1}\|\Sigma_{Q_2}\|\Sigma_R \\ S_{II}: & \Sigma_L\|\Sigma_{Q_1}\|\Sigma_{P_2}\|\Sigma_R & \preccurlyeq & \Sigma_L\|\Sigma_{Q_1}\|\Sigma_{Q_2}\|\Sigma_R \\ & & \Downarrow & \\ S: & \Sigma_L\|\Sigma_{P_1}\|\Sigma_{P_2}\|\Sigma_R & \preccurlyeq & \Sigma_L\|\Sigma_{Q_1}\|\Sigma_{Q_2}\|\Sigma_R \end{array} \tag{3.32}$$

Proof. In order to prove this lemma, we make use of Theorem 3.14 and Proposition 3.16. That is, we extend $S_i, i = I, II$, in two steps. First, consider

$$\begin{aligned} S'_I &:= \{(x_L, x_{P_1}, x_{Q_2}, x_R, x_L, x_{Q_1}, \bar{x}_{Q_2}, x_R) \mid \exists x'_L, x'_R : \\ &\quad (x_L, x_{P_1}, x_{Q_2}, x_R, x'_L, x_{Q_1}, \bar{x}_{Q_2}, x'_R) \in S_I\} \\ S'_{II} &:= \{(x_L, x_{Q_1}, x_{P_2}, x_R, x_L, \bar{x}_{Q_1}, x_{Q_2}, x_R) \mid \exists x'_L, x'_R : \\ &\quad (x_L, x_{Q_1}, x_{P_2}, x_R, x'_L, \bar{x}_{Q_1}, x_{Q_2}, x'_R) \in S_{II}\} \end{aligned}$$

The fact that $S'_i, i = I, II$, are also full simulation relations is a consequence of Proposition 3.16 and will not be proved explicitly. Second, add suitable subspaces to obtain the relations $\left(S'_i + \bar{S}'_i\right)^{\text{sym}}$ with

$$\begin{aligned} \bar{S}'_I &= \{(x_L, x_{P_1}, \bar{x}_{Q_2}, x_R, x_L, x_{Q_1}, -x_{Q_2}, x_L) \mid \bar{x}_{Q_2} \in \ker H_{Q_2} \cap \ker C_{Q_2}, \\ &\quad x_i \in \ker H_i \cap \ker C_i, i \in \{L, P_1, Q_2, Q_1, R\}, \\ &\quad (x_L, x_{P_1}, x_{Q_2}, x_R, x_L, x_{Q_1}, \bar{x}_{Q_2}, x_L) \in S'_I\} \\ \bar{S}'_{II} &= \{(x_L, \bar{x}_{Q_1}, x_{P_2}, x_R, x_L, -x_{Q_1}, x_{Q_2}, x_L) \mid \bar{x}_{Q_1} \in \ker H_{Q_1} \cap \ker C_{Q_1}, \\ &\quad x_i \in \ker H_i \cap \ker C_i, i \in \{L, P_2, Q_1, Q_2, R\}, \\ &\quad (x_L, x_{Q_1}, x_{P_2}, x_R, x_L, \bar{x}_{Q_1}, x_{Q_2}, x_L) \in S'_{II}\} \end{aligned}$$

and

$$\begin{aligned} \left(S'_O + \bar{S}'_I\right)^{\text{sym}} &= \{(x_L, x_{P_1}, \bar{x}_{Q_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_L) \mid \\ &\quad (x_L, x_{P_1}, x_{Q_2}, x_R, x_L, x_{Q_1}, \bar{x}_{Q_2}, x_L) \in \left(S'_I + \bar{S}'_I\right)\} \\ \left(S'_{II} + \bar{S}'_{II}\right)^{\text{sym}} &= \{(x_L, \bar{x}_{Q_1}, x_{Q_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_L) \mid \\ &\quad (x_L, x_{Q_1}, x_{P_2}, x_R, x_L, \bar{x}_{Q_1}, x_{Q_2}, x_L) \in \left(S'_{II} + \bar{S}'_{II}\right)\} \end{aligned}$$

Similarly as before, construct S as

$$\begin{aligned} S &= \{(x_L, x_{P_1}, x_{P_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_R) \mid \exists\, \bar{x}_{Q_1}, \bar{x}_{Q_2} : \\ &\quad (x_L, x_{P_1}, \bar{x}_{Q_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_R) \in \left(S'_I + \bar{S}'_I\right)^{\text{sym}}, \\ &\quad (x_L, \bar{x}_{Q_1}, x_{P_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_R) \in \left(S'_{II} + \bar{S}'_{II}\right)^{\text{sym}}\} \end{aligned} \tag{3.33}$$

- The proofs that $S'_i + \bar{S}'_i, i = I, II$, and $(S'_i)^{\text{sym}}$ are full simulation relations of $\Sigma_L\|\Sigma_{P_1}\|\Sigma_{Q_2}\|\Sigma_R$ and of $\Sigma_L\|\Sigma_{Q_1}\|\Sigma_{P_2}\|\Sigma_R$ by $\Sigma_L\|\Sigma_{Q_1}\|\Sigma_{Q_2}\|\Sigma_R$, respectively, are analogous to the proofs of Lemma 3.11 and Lemma 3.12,

- The proof that for every $x \in \ker H_{Q_2} \cap C_{Q_2}, y \in \ker H_{Q_1} \cap \ker C_{Q_1}$, there exist elements $(0,0,x,0,0,0,x,0) \in \left(S'_I + \bar{S}'_I\right)^{\text{sym}}$ and $(0,y,0,0,0,y,0,0) \in \left(S'_{II} + \bar{S}'_{II}\right)^{\text{sym}}$ is similar to the proof of Lemma 3.13.

Finally, we have to prove that S as constructed by (3.33) is indeed a full simulation relation of $\Sigma_L \| \Sigma_{P_1} \| \Sigma_{P_2} \| \Sigma_R$ by $\Sigma_L \| \Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_R$. To do so, the proof of Theorem 3.14 can be repeated in all its steps. □

We are now in the position to prove soundness of circular assume-guarantee reasoning for feedback interconnections of arbitrarily yet finitely many linear systems.

Theorem 3.26. *Consider k linear systems Σ_{P_i} and $\Sigma_{Q_i} i = 1,\ldots,k$ of the form* (3.1). *Let k circularly dependent conditions*

$$\begin{array}{llcl} S_I : & \Sigma_{P_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{Q_k} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{Q_k} \\ S_{II} : & \Sigma_{Q_1} \| \Sigma_{P_2} \| \ldots \| \Sigma_{Q_k} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{Q_k} \\ \vdots & \vdots & \vdots & \vdots \\ S_k : & \Sigma_{Q_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{P_k} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{Q_k} \end{array} \tag{3.34}$$

be fulfilled. Then the global interconnected plant

$$\Sigma_P := \Sigma_{P_1} \| \ldots \| \Sigma_{P_k}$$

fulfills its global specification

$$\Sigma_Q := \Sigma_{Q_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{Q_k} \ ,$$

i.e.,

$$S : \ \Sigma_P \ \preccurlyeq \ \Sigma_Q$$

Proof. The idea of this proof is to successively combine the k conditions (3.34) and apply Lemma 3.25 at every step. Let k simulation relations $S_i, i = I,\ldots,k$ as in (3.34) be given, i.e.

$$S_i : \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_{i-1}} \| \Sigma_{P_i} \| \Sigma_{Q_{i+1}} \| \ldots \| \Sigma_{Q_k} \preccurlyeq \Sigma_Q \tag{3.35}$$

Starting with $i = 1$, consider $\lfloor \frac{k}{2} \rfloor$ pairs of two relations S_i and $S_{i+1}, i = 1,\ldots,k-1$ where $\lfloor x \rfloor$ is the greatest natural number less or equal to x. Apply Lemma 3.25 to each of the $\lfloor \frac{k}{2} \rfloor$ pairs to obtain simulation relations $S_{i,i+1}$ of the form

$$S_{i,i+1} : \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_{i-1}} \| \Sigma_{P_i} \| \Sigma_{P_{i+1}} \| \ldots \| \Sigma_{Q_k} \preccurlyeq \Sigma_Q \tag{3.36}$$

After this first step, $\lceil \frac{k}{2} \rceil$ simulation relations are left where $\lceil x \rceil$ is the smallest natural number greater or equal to x.

$$\begin{cases} \{S_{i,i+1}, i = 1, \ldots, k-1\}, & \text{if } \lfloor \frac{k}{2} \rfloor = \frac{k}{2} \\ \{\{S_{i,i+1}, i = 1, \ldots, k-2\}, S_k\}, & \text{otherwise} \end{cases} \tag{3.37}$$

Continue by forming $\lfloor \lceil \frac{k}{2} \rceil \rfloor$ pairs of two simulation relations $S_j, S_j \in$ (3.37) to apply Lemma 3.25 on them. Repeating this procedure $\lceil \frac{k}{2} \rceil$- times in total, the desired result follows in the last step. We formalize this approach in the following

Algorithm 3.27. Compute S from k simulation relations $S_i, i = 1, \ldots, k$ of the form (3.34)

$k = \lfloor \frac{N}{2} \rfloor$
$R = \{S_i, i = I, \ldots, k\}$
for $i = 1$ to $\lceil \frac{k}{2} \rceil$ **do**
  $k = |R|$
  for $j = 1$ to $\lfloor \frac{k}{2} \rfloor$ **do**
    apply Lemma 3.25 to $S_{2j-1}, S_{2j}, S_j \in R$ to obtain the relations $S_{2j-1,2j}$ as given by (3.36)
  end for
  if $\frac{k}{2} == \lfloor \frac{k}{2} \rfloor$ **then**
    $R = \{S_{2j-1,2j}, j = 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$
  else
    $R = \left\{\{S_{2j-1,2j}, j = 1, \ldots, \lfloor \frac{k}{2} \rfloor\}, S_k\right\}$
  end if
end for
$S = R$

□

We illustrate Theorem 3.26 and Algorithm 3.27 with the following example.

Example 3.28. Let the linear systems $\Sigma_{P_i}, \Sigma_{Q_i}, i = 1, \ldots, 5$, of the form (3.1) be given such that there exist full simulation relations S_i as follows:

$$\begin{array}{rlcl}
S_I: & \Sigma_{P_1} \| \Sigma_{Q_2} \| (\Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5}) & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \| (\Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5}) \\
S_{II}: & \Sigma_{Q_1} \| \Sigma_{P_2} \| (\Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5}) & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \| (\Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5}) \\
S_{III}: & (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| \Sigma_{P_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5} & \preccurlyeq & (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| \Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5} \\
S_{IV}: & (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| \Sigma_{Q_3} \| \Sigma_{P_4} \| \Sigma_{Q_5} & \preccurlyeq & (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| \Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5} \\
S_V: & (\Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \| \Sigma_{Q_4}) \| \Sigma_{P_5} & \preccurlyeq & (\Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \| \Sigma_{Q_4}) \| \Sigma_{Q_5}
\end{array}$$

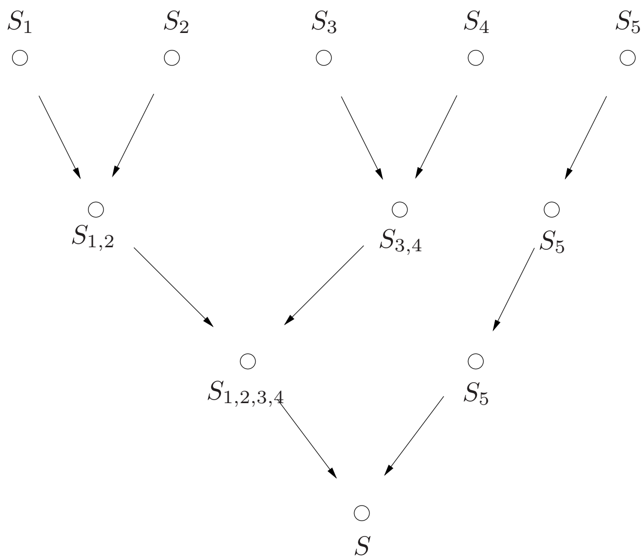

Figure 3.7.: Algorithm 3.27 applied to Example 3.28.

In the first step, we pair the relations S_I, S_{II} and S_{III}, S_{IV}. Applying Lemma 3.25 to each pair, we conclude that there exist full simulation relations $S_{I,II}$ and $S_{III,IV}$ such that

$$S_{I,II} : (\Sigma_{P_1} \| \Sigma_{P_2}) \| (\Sigma_{Q_3} \| \Sigma_{Q_4}) \| \Sigma_{Q_5} \preccurlyeq (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| (\Sigma_{Q_3} \| \Sigma_{Q_4}) \| \Sigma_{Q_5}$$
$$S_{III,IV} : (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| (\Sigma_{P_3} \| \Sigma_{P_4}) \| \Sigma_{Q_5} \preccurlyeq (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| (\Sigma_{Q_3} \| \Sigma_{Q_4}) \| \Sigma_{Q_5}$$

In the second step, Lemma 3.25 can now be applied to $S_{I,II}$ and $S_{III,IV}$ to obtain

$$S_{I,II,III,IV} : (\Sigma_{P_1} \| \Sigma_{P_2} \| \Sigma_{P_3} \| \Sigma_{P_4}) \| \Sigma_{Q_5} \quad \preccurlyeq \quad (\Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \| \Sigma_{Q_4}) \| \Sigma_{Q_5}$$

In the third step, consider the relations $S_{I,II,III,IV}$ and S_V. As a special case of Lemma 3.25, they fulfill the circular assume-guarantee rule of Theorem 3.14. Hence, there indeed exists a full simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2} \| \Sigma_{P_3} \| \Sigma_{P_4} \| \Sigma_{P_5}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5}$,

$$S : \Sigma_{P_1} \| \Sigma_{P_2} \| \Sigma_{P_3} \| \Sigma_{P_4} \| \Sigma_{P_5} \quad \preccurlyeq \quad \Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5}$$

Remark 3.29. Without formalizing it in form of a theorem it is worth pointing out that apart from compositional and circular assume-guarantee reasoning rules also triangular proof rules based on non-circular assume-guarantee reasoning can be developed for more than two systems. Like in Theorem 3.9, soundness of non-circular assume-guarantee reasoning is ensured by compositionality of series interconnections (Theorem 3.24) and transitivity of simulation. Example 6.8 in Chapter 6 describes a decentralized control scheme based on a triangular rule for 3 systems.

Finally, we investigate whether circular assume-guarantee reasoning for k linear systems is also complete.

Proposition 3.30. *Let S be a full simulation relation of $\Sigma_{P_1}\|\dots\|\Sigma_{P_k}$ by $\Sigma_{Q_1}\|\dots\|\Sigma_{Q_k}$. Then it holds for every $i=1,\dots,k$ that*

$$\Sigma_{Q_1}\|\dots\Sigma_{Q_{i-1}}\|\Sigma_{P_i}\|\Sigma_{Q_{i+1}}\|\dots\|\Sigma_{P_k} \preccurlyeq \Sigma_{Q_1}\|\dots\|\Sigma_{Q_k}\ , \tag{3.38}$$

where the full simulation relation S_i of $\Sigma_{Q_1}\|\dots\Sigma_{Q_{i-1}}\|\Sigma_{P_i}\|\Sigma_{Q_{i+1}}\|\dots\|\Sigma_{P_k}$ by $\Sigma_{Q_1}\|\dots\|\Sigma_{Q_k}$ can be constructed as

$$\begin{aligned} S_i \quad = \quad & \{(x_{Q_1},\dots,x_{Q_{i-1}},x_{P_i},x_{Q_{i+1}},\dots,x_{Q_k},x_{Q_1},\dots,x_{Q_k}) \mid \exists x_{P_j}, \\ & j=1,\dots,k, j\neq i : (x_{P_1},\dots,x_{P_k},x_{Q_1},\dots,x_{Q_k})\in S\} \\ & +\{(a_1,\dots,a_{i-1},0,a_{i+1},\dots,a_k,a_1,\dots,a_{i-1},0,a_{i+1},\dots,a_k), a_j\in\mathcal{X}_{Q_j}\} \end{aligned} \tag{3.39}$$

Proof. This statement can be proved in the same way as Theorem 3.20. □

It follows immediately that circular assume-guarantee reasoning for k systems is indeed complete.

Theorem 3.31. *Given k linear systems Σ_{P_i} and Σ_{Q_i}, $i=1,\dots,k$, of the form* (3.1). *Suppose there exists a full simulation relation S of $\Sigma_{P_1}\|\dots\|\Sigma_{P_k}$ by $\Sigma_{Q_1}\|\dots\|\Sigma_{Q_k}$. Then the following holds:*

$$S: \qquad \Sigma_{P_1}\|\dots\|\Sigma_{P_k} \preccurlyeq \Sigma_{Q_1}\|\dots\|\Sigma_{Q_k}$$

$$\Downarrow$$

$$\begin{array}{rrcl} S_I: & \Sigma_{P_1}\|\Sigma_{Q_2}\|\dots\|\Sigma_{Q_k} & \preccurlyeq & \Sigma_{Q_1}\|\Sigma_{Q_2}\|\dots\|\Sigma_{Q_k} \\ S_{II}: & \Sigma_{Q_1}\|\Sigma_{P_2}\|\dots\|\Sigma_{Q_k} & \preccurlyeq & \Sigma_{Q_1}\|\Sigma_{Q_2}\|\dots\|\Sigma_{Q_k} \\ \vdots & \vdots & \vdots & \vdots \\ S_k: & \Sigma_{Q_1}\|\Sigma_{Q_2}\|\dots\|\Sigma_{P_k} & \preccurlyeq & \Sigma_{Q_1}\|\Sigma_{Q_2}\|\dots\|\Sigma_{Q_k} \end{array} \tag{3.40}$$

3.3. Interconnections with algebraic constraints

In many applications, the interconnection of subsystems introduces *algebraic constraints* on the state variables. In this section, we consider *parallel compositions* of linear systems as a specific type of interconnection inducing algebraic constraints on the states. One way to incorporate these constraints is to use the representation of a DAE system. We first show that the notions of simulation relations for parallel compositions derived from the original definition for linear systems (Definition 2.5) on the one hand and from the definition for DAE systems [75] on the other hand are equivalent. Once this relation is established, we investigate compositional and assume-guarantee reasoning rules for parallel compositions. Specifically for parallel compositions a reasoning scheme based on the decomposition of the global specification into subspecifications is shown to hold true to round off this section.

Definition 3.32. Given two linear dynamical systems Σ_i, $i = 1, 2$, of the form

$$\Sigma_i : \begin{array}{rcl} \dot{x}_i & = & A_i x_i + B_i u_i + L_i d_i \\ y_i & = & C_i x_i \end{array} \tag{3.41}$$

where $x_i \in \mathcal{X}_i = \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^p, d_i \in \mathcal{D}_i$ and $y_i \in \mathbb{R}^q$.
Then the parallel composition $\Sigma_1\|_{\mathrm{pc}}\Sigma_2$ is given by

$$\Sigma_1\|_{\mathrm{pc}}\Sigma_2 : \begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \\ & & + \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ y & = & C_1 x_1 = C_2 x_2 \end{array} \tag{3.42}$$

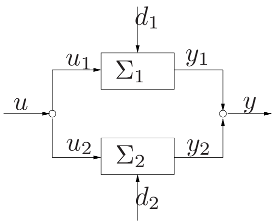

Figure 3.8.: $\Sigma_1 \|_{\mathrm{pc}} \Sigma_2$

Since parallel composition entails the algebraic constraint $C_1x_1 = C_2x_2$, depicted in Figure 3.8, the equations (3.42) can be rewritten in differential-algebraic form as

$$\Sigma_{12} : \begin{array}{rcl} E_{12}\dot{z}_{12} & = & A_{12}z_{12}, z_{12} \in \mathcal{Z}_{12} \\ w_{12} & = & C_{12}z_{12} \end{array} \tag{3.43}$$

where the matrices E_{12}, A_{12}, C_{12} and the state and output vectors z_{12} and w_{12} are given by

$$\begin{aligned} z_{12} &= \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}, A_{12} = \begin{bmatrix} L_1^{\perp}A_1 & 0 & L_1^{\perp}B_1 \\ 0 & L_2^{\perp}A_2 & L_2^{\perp}B_2 \\ C_1 & -C_2 & 0 \end{bmatrix}, \\ w_{12} &= \begin{bmatrix} y \\ u \end{bmatrix}, E_{12} = \begin{bmatrix} L_1^{\perp} & 0 & 0 \\ 0 & L_2^{\perp} & 0 \\ 0 & 0 & 0 \end{bmatrix}, C_{12} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \end{aligned} \tag{3.44}$$

respectively, where $L_1^{\perp}$ and $L_2^{\perp}$ are left annihilating matrices of L_1 and L_2 of maximal rank. The set of states and inputs consistent with these constraints is defined by the consistent subspace.

Definition 3.33. Consider a system Σ_{12} of the form (3.43). Then the consistent subspace $\mathcal{V}_{12}^{\star}$ for Σ_{12} is the largest subspace $\mathcal{V}_{12} \subset \mathcal{Z}_{12}$ such that

$$A_{12}\mathcal{V}_{12} \subset E_{12}\mathcal{V}_{12} \tag{3.45}$$

Furthermore, denote by $\mathcal{W}_{12}^{\star}$ and $\mathcal{U}_{12}^{\star}$ the projections

$$\mathcal{W}_{12}^{\star} = \Pi_{\mathcal{X}_1\mathcal{X}_2}\mathcal{V}_{12}^{\star} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \,\middle|\, \exists u : \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} \in \mathcal{V}_{12}^{\star} \right\} \tag{3.46}$$

$$\mathcal{U}_{12}^{\star} = \Pi_{\mathcal{U}}\mathcal{V}_{12}^{\star} = \left\{ u \,\middle|\, \exists x_1, x_2 : \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} \in \mathcal{V}_{12}^{\star} \right\} \tag{3.47}$$

This allows us to specialize the general definition of simulation relations (Definition 3.2) to linear systems of the form (3.42) respectively (3.43), compare also with [75].

Definition 3.34. Given two linear systems Σ_i, $i = \{P_1P_2, Q_1Q_2\}$ of the form (3.43) with consistent subspaces $\mathcal{V}_i^{\star}$. Then a subspace $\tilde{S} \subset \mathcal{Z}_{P_1P_2} \times \mathcal{Z}_{Q_1Q_2}$ with $\Pi_{P_1P_2}\tilde{S} \subset \mathcal{V}_{P_1P_2}^{\star}$ is a simulation relation of $\tilde{\Sigma}_{P_1P_2}$ by $\tilde{\Sigma}_{Q_1Q_2}$ if and only if for all $(z_{P_1P_2}, z_{Q_1Q_2}) \in S$,

1. for all $v_{P_1P_2} \in \mathcal{V}_{P_1P_2}^{\star}$ such that $E_{P_1P_2}v_{P_1P_2} = A_{P_1P_2}z_{P_1P_2}$ there should exist a $v_{Q_1Q_2} \in \mathcal{V}_{Q_1Q_2}^{\star}$ such that $E_{Q_1Q_2}v_{Q_1Q_2} = A_{Q_1Q_2}z_{Q_1Q_2}$ and $(v_{P_1P_2}, v_{Q_1Q_2}) \in \tilde{S}$
2. $C_{P_1P_2}z_{P_1P_2} = C_{Q_1Q_2}z_{Q_1Q_2}$

The simulation relation $\tilde{S}$ is *full*, denoted by $\Sigma_{P_1P_2} \preccurlyeq \Sigma_{Q_1Q_2}$, if the projection on $\mathcal{Z}_{P_1P_2}$ equals the consistent subspace, that is, $\Pi_{P_1P_2}\tilde{S} = \mathcal{V}_{12}^{\star}$.

The linear algebraic characterization is derived similarly to Proposition 3.3 and Theorem 3.4.

Proposition 3.35. *There exists a simulation relation* $S \subset \mathcal{X}_{P_1} \times \mathcal{X}_{P_2} \times \mathcal{X}_{Q_1} \times \mathcal{X}_{Q_2}$ *of* $\Sigma_{P_1} \|_{pc} \Sigma_{P_2}$ *by* $\Sigma_{Q_1} \|_{pc} \Sigma_{Q_2}$ *if and only if for all* $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$ *and all* $u \in \mathcal{U}^{\star}_{P_1 P_2}$ *the following holds:*

1. $\forall \begin{bmatrix} d_{P_1} \\ d_{P_2} \end{bmatrix} \in \mathcal{D}_{P_1} \times \mathcal{D}_{P_2} \quad \exists \quad \begin{bmatrix} d_{Q_1} \\ d_{Q_2} \end{bmatrix} \in \mathcal{D}_{Q_1} \times \mathcal{D}_{Q_2} \ :$
$\begin{bmatrix} A_{P_1} x_{P_1} + B_{P_1} u + L_{P_1} d_{P_1} \\ A_{P_2} x_{P_2} + B_{P_2} u + L_{P_2} d_{P_2} \\ A_{Q_1} x_{Q_1} + B_{Q_1} u + L_{Q_1} d_{Q_1} \\ A_{Q_2} x_{Q_2} + B_{Q_2} u + L_{Q_2} d_{Q_2} \end{bmatrix} \in S$

2. $C_{P_1} x_{P_1} = C_{P_2} x_{P_2} = C_{Q_1} x_{Q_1} = C_{Q_2} x_{Q_2}$

Proof. With the system matrices (3.44), condition 2. in Definition 3.34 yields

$$u_{P_1} = u_{Q_1} \tag{3.48}$$

and

$$C_{P_1} x_{P_1} = C_{Q_1} x_{Q_1} \tag{3.49}$$

Writing out condition 1. from Definition 3.34 results in

$$\begin{bmatrix} A_{P_1} x_{P_1} + B_{P_1} u_{P_1} + L_{P_1} d_{P_1} \\ A_{P_2} x_{P_2} + B_{P_2} u_{P_1} + L_{P_2} d_{P_2} \\ A_{Q_1} x_{Q_1} + B_{Q_1} u_{Q_1} + L_{Q_1} d_{Q_1} \\ A_{Q_2} x_{Q_2} + B_{Q_2} u_{Q_1} + L_{Q_2} d_{Q_2} \end{bmatrix} \in S \tag{3.50}$$

and

$$C_{P_1} x_{P_1} = C_{P_2} x_{P_2}, C_{Q_1} x_{Q_1} = C_{Q_2} x_{Q_2} \tag{3.51}$$

for all $(x_{P_1}, x_{P_2}, u_{P_1}, x_{Q_1}, x_{Q_2}, u_{Q_1}) \in \tilde{S}$ and

$$u_{P_1} \in \mathcal{U}^{\star}_{P_1 P_2} \tag{3.52}$$

Thus, equations (3.48) – (3.52) are equivalent to the conditions 1. and 2. in Definition 3.34. □

Proposition 3.36. *There exists a simulation relation* $S \subset \mathcal{X}_{P_1} \times \mathcal{X}_{P_2} \times \mathcal{X}_{Q_1} \times \mathcal{X}_{Q_2}$ *of* $\Sigma_{P_1} \|_{pc} \Sigma_{P_2}$ *by* $\Sigma_{Q_1} \|_{pc} \Sigma_{Q_2}$ *if and only if the following conditions hold:*

1. $\mathit{diag}\{A_{P_1}, A_{P_2}, A_{Q_1}, A_{Q_2}\} S \subset S + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}$

2. $\operatorname{im}\begin{bmatrix} L_{P_1} & 0 \\ 0 & L_{P_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \subset S + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}$

3. $\operatorname{im}\begin{bmatrix} B_{P_1} \\ B_{P_2} \\ B_{Q_1} \\ B_{Q_2} \end{bmatrix} \cap \left(\mathcal{W}^\star_{P_1P_2} \times \mathcal{W}^\star_{Q_1Q_2}\right) \subset S + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}$

4. $S \subset \ker\begin{bmatrix} C_{P_1} & -C_{P_2} & 0 & 0 \\ 0 & 0 & C_{Q_1} & -C_{Q_2} \\ C_{P_1} & 0 & -C_{Q_1} & 0 \end{bmatrix}$

Proof. Condition 2 in Proposition 3.35 is equivalent to condition 4 in Proposition 3.36. Condition 1 in Proposition 3.35 results in

$$\operatorname{diag}\{A_{P_1}, A_{P_2}, A_{Q_1}, A_{Q_2}\}S + \operatorname{im}\begin{bmatrix} B_{P_1} \\ B_{P_2} \\ B_{Q_1} \\ B_{Q_2} \end{bmatrix} \subset S + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}.$$

Since u is restricted to $u \in \mathcal{U}^\star_{P_1P_2}$, the image of the input map has to be restricted to the subspace of all admissible inputs, which is given by

$$\{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists u : (x_{P_1}, x_{P_2}, u) \in \mathcal{V}^\star_{P_1P_2}, (x_{Q_1}, x_{Q_2}, u) \in \mathcal{V}^\star_{Q_1Q_2}\} = \mathcal{W}^\star_{P_1P_2} \times \mathcal{W}^\star_{Q_1Q_2}$$

Therefore, conditions 1 – 3 in Proposition 3.36 are equivalent to condition 1 in Proposition 3.35. ☐

Proposition 3.37. *Simulation of systems interconnected by parallel composition defines a preorder.*

Proof. (*reflexivity*): Clearly, the identity relation

$$S_{\text{id}} = \{((x_P, x_Q), (x_p, x_Q)) \mid (x_P, x_Q) \in \mathcal{W}_{PQ}\}$$

defines a full simulation relation of $\Sigma_P \|_{\text{pc}} \Sigma_Q$ by itself.
(*transitivity*): Let S_1 and S_2 be full simulation relations of $\Sigma_{P_1}\|_{\text{pc}}\Sigma_{P_2}$ by $\Sigma_{Q_1}\|_{\text{pc}}\Sigma_{Q_2}$ and of $\Sigma_{Q_1}\|_{\text{pc}}\Sigma_{Q_2}$ by $\Sigma_{R_1}\|_{\text{pc}}\Sigma_{R_2}$, respectively. Then

$$\begin{aligned} S \;:=\; & \{(x_{P_1}, x_{P_2}, x_{R_1}, x_{R_2}) \mid \exists x_{Q_1}, x_{Q_2} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S_1, \\ & (x_{Q_1}, x_{Q_2}, x_{R_1}, x_{R_2}) \in S_2\} \end{aligned}$$

defines a full simulation relation of $\Sigma_{P_1}\|_{\text{pc}}\Sigma_{P_2}$ by $\Sigma_{R_1}\|_{\text{pc}}\Sigma_{R_2}$. Take any $(x_{P_1}, x_{P_2}, u) \in \mathcal{V}_{P_1P_2}$. Then there exists a (x_{Q_1}, x_{Q_2}) such that $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in$

S_1 and $(x_{Q_1}, x_{Q_2}, x_{R_1}, x_{R_2}) \in S_2$. Thus, for every d_{P_1}, d_{P_2} there exist d_{Q_1}, d_{Q_2} such that

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u + L_{P_1}d_{P_1} \\ A_{P_2}x_{P_2} + B_{P_2}u + L_{P_2}d_{P_2} \\ A_{Q_1}x_{Q_1} + B_{Q_1}u + L_{Q_1}d_{Q_1} \\ A_{Q_2}x_{Q_2} + B_{Q_2}u + L_{Q_2}d_{Q_2} \end{bmatrix} \in S_1$$
$$C_{P_1}x_{P_1} = C_{P_2}x_{P_2} = C_{Q_1}x_{Q_1} = C_{Q_2}x_{Q_2}, (x_{Q_1}, x_{Q_2}, u) \in \mathcal{U}^{\star}_{Q_1Q_2}$$

since S_1 is full. Moreover, since S_2 is also full, for the same u and d_{Q_1}, d_{Q_2} there exist d_{R_1}, d_{R_2} such that

$$\begin{bmatrix} A_{Q_1}x_{Q_1} + B_{Q_1}u + L_{Q_1}d_{Q_1} \\ A_{Q_2}x_{Q_2} + B_{Q_2}u + L_{Q_2}d_{Q_2} \\ A_{R_1}x_{R_1} + B_{R_1}u + L_{R_1}d_{R_1} \\ A_{R_2}x_{R_2} + B_{R_2}u + L_{R_2}d_{R_2} \end{bmatrix} \in S_2$$
$$C_{R_1}x_{R_1} = C_{R_2}x_{R_2} = C_{Q_1}x_{Q_1} = C_{Q_2}x_{Q_2}, (x_{R_1}, x_{R_2}, u) \in \mathcal{U}^{\star}_{Q_1Q_2}$$

and hence $(x_{P_1}, x_{P_2}, x_{R_1}, x_{R_2}) \in S$ as well as

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u + L_{P_1}d_{P_1} \\ A_{P_2}x_{P_2} + B_{P_2}u + L_{P_2}d_{P_2} \\ A_{R_1}x_{R_1} + B_{R_1}u + L_{R_1}d_{R_1} \\ A_{R_2}x_{R_2} + B_{R_2}u + L_{R_2}d_{R_2} \end{bmatrix} \in S$$
$$C_{R_1}x_{R_1} = C_{R_2}x_{R_2} = C_{P_1}x_{P_1} = C_{P_2}x_{P_2}$$

Moreover, since this holds for any $(x_{P_1}, x_{P_2}, u) \in \mathcal{V}^{\star}_{P_1P_2}$, S is also full. □

3.3.1. Compositional reasoning

We begin our analysis of parallel compositions by examining the compositionality property.

Theorem 3.38. *Given any four systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$ of the form (3.41) respectively* (3.43). *Then parallel composition is compositional, i.e.*

$$\left.\begin{array}{ll} S_1: & \Sigma_{P_1} \preccurlyeq \Sigma_{Q_1} \\ S_2: & \Sigma_{P_2} \preccurlyeq \Sigma_{Q_2} \end{array}\right\} \implies \Sigma_{P_1} \|_{pc} \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \|_{pc} \Sigma_{Q_2} \tag{3.53}$$

Proof. Construct the relation S from given full simulation relations S_1 and S_2 of Σ_{P_1} and Σ_{P_2} by Σ_{Q_1}, respectively Σ_{Q_2}, as the product relation

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_1}) \in S_1, (x_{P_2}, x_{Q_2}) \in S_2\} \tag{3.54}$$

Then for any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$, any joint input $u \in \mathcal{U}^{\star}_{P_1P_2}$ and any d_{P_1}, d_{P_2} there exist d_{Q_1}, d_{Q_2} such that

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u + G_{P_1}d_{P_1} \\ A_{P_2}x_{P_2} + B_{P_2}u + G_{P_2}d_{P_2} \\ A_{Q_1}x_{Q_1} + B_{Q_1}u + G_{Q_1}d_{Q_1} \\ A_{Q_2}x_{Q_2} + B_{Q_2}u + G_{Q_2}d_{Q_2} \end{bmatrix} \in S \ ,$$

since for any d_{P_1} there exists a d_{Q_1} such that

$$\begin{bmatrix} A_{P_i}x_{P_i} + B_{P_i}u + G_{P_1}d_{P_1} \\ A_{Q_i}x_{Q_i} + B_{P_i}u + G_{Q_1}d_{Q_1} \end{bmatrix} \in S_i, i = 1, 2$$

for all $u \in \mathcal{U}$. Moreover, since $y_{P_1} = y_{Q_1}$ due to S_1 and $y_{P_2} = y_{Q_2}$ due to S_2 and $y_{P_1} = y_{P_2}$ as well as $y_{Q_1} = y_{Q_2}$ enforced by parallel composition, condition (ii) in Proposition 3.35 is also fulfilled which proves that S is indeed a simulation relation of $\Sigma_{P_1}\|_{\mathrm{pc}}\Sigma_{P_2}$ by $\Sigma_{Q_1}\|_{\mathrm{pc}}\Sigma_{Q_2}$.
To show that S as defined in (3.54) is full, observe that (3.55) holds for all u. Since both S_1 and S_2 are full, we can find for every $u \in \mathcal{U}^{\star}_{P_1P_2}$ and every $(x_{P_1}, x_{P_2}) \in \mathcal{W}^{\star}_{P_1P_2}$ elements x_{Q_1}, x_{Q_2} such that $(x_{P_i}, x_{Q_i}) \in S_i, i = 1, 2$ and thus $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. ☐

The converse is in general not true since the consistent subspace $\mathcal{V}^{\star}_{P_1P_2}$ restricts the choice of inputs u depending on the states x_{P_1}, x_{P_2}. Thus, contrary to the open feedback interconnection, compositional reasoning is not complete for parallel compositions.

3.3.2. Assume-guarantee reasoning

Since compositional reasoning is not complete for parallel compositions, assume-guarantee reasoning schemes are to be investigated.

Theorem 3.39. *Consider four linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ of the form* (3.41)*. For parallel compositions, the non-circular assume-guarantee reasoning schemes*

$$\left.\begin{array}{lrcl} S_1: & \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_2: & \Sigma_{Q_1} \,\|_{pc}\, \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \,\|_{pc}\, \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S : \Sigma_{P_1}\|_{pc}\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_{pc}\Sigma_{Q_2}$$

and

$$\left.\begin{array}{lrcl} S_1': & \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_2} \\ S_2': & \Sigma_{P_1} \,\|_{pc}\, \Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1} \,\|_{pc}\, \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S : \Sigma_{P_1}\|_{pc}\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_{pc}\Sigma_{Q_2}$$

are sound.

Proof. Compositionality of parallel composition, see Theorem 3.38, and transitivity of simulation, c. f. Proposition 3.37, prove this result. ☐

Finally, circular assume-guarantee reasoning also works for parallel compositions.

Theorem 3.40. *Consider four linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ of the form* (3.41). *Circular assume-guarantee reasoning is sound for parallel compositions,*

$$\left.\begin{array}{lllll} S_1 : & \Sigma_{P_1} \|_{pc} \Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1} \|_{pc} \Sigma_{Q_2} \\ S_2 : & \Sigma_{Q_1} \|_{pc} \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \|_{pc} \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S : \Sigma_{P_1} \|_{pc} \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \|_{pc} \Sigma_{Q_2} \quad (3.55)$$

Proof. Performing the same steps as in the proof for soundness of circular assume-guarantee reasoning for linear systems (Theorem 3.14 and Lemmas 3.11 – 3.13) yields the desired result. □

3.3.3. Verification of the global specification by decomposition

Throughout this chapter we have assumed that the given overall specification Σ_Q can be decomposed into subspecifications $\Sigma_{Q_i}, i = 1, \ldots k,$, in the same way as the modeled system Σ_P consists of interconnected components $\Sigma_{P_i}, i = 1, \ldots, k$. For parallel compositions, this decomposition of the specification makes it possible to verify the global specification by verifying all the subspecifications. In other words, it is enough to prove that Σ_P is simulated by each of the subspecifications Σ_{Q_i} to guarantee that it also fulfills the overall specification $\Sigma_Q = \Sigma_{Q_1} \|_{\mathrm{pc}} \Sigma_{Q_2} \|_{\mathrm{pc}} \ldots \|_{\mathrm{pc}} \Sigma_{Q_k}$. Since each subspecification is expected to be of lower complexity than the global specification, this reasoning scheme reduces the complexity of the overall verification task (3.2). Before proving the main result we collect some basic facts about parallel compositions.

Proposition 3.41. *For any system Σ_P it holds that*

$$\Sigma_P \preccurlyeq \Sigma_P \|_{pc} \Sigma_P \quad (3.56)$$

Proof. Construct a simulation relation S by setting all state variables equal to each other,

$$S = \{(x_1, (x_2, x_3)) \mid x_1 = x_2 = x_3 \in \Sigma_P\}$$

Then, S defines a full simulation relation of Σ_P by $\Sigma_P \| \Sigma_P$ since the evolution remains within the constrained subspace $Cx_1 = Cx_2 = Cx_3$ for all times. □

Proposition 3.42. *For any two systems Σ_P, Σ_Q, it holds that*

$$\Sigma_P \|_{pc} \Sigma_Q \preccurlyeq \Sigma_P \quad (3.57)$$

Proof. The relation

$$S = \{((x_P, x_Q), \bar{x}_P) \mid x_P = \bar{x}_P, (x_P, x_Q) \in \mathcal{W}^{\star}_{PQ}\}$$

defines a full simulation relation of $\Sigma_P \|_{\mathrm{pc}} \Sigma_Q$ by Σ_P. □

The main result to decompose a given global specification Σ_Q into an interconnection of local specifications Σ_{Q_1} and Σ_{Q_2} can now be stated as follows:

Theorem 3.43. *Given a system Σ_P and specifications $\Sigma_{Q_i}, i = 1, 2$, of the form (3.41). Then*

$$\Sigma_P \preccurlyeq \Sigma_{Q_1} \|_{pc} \Sigma_{Q_2} \tag{3.58}$$

if and only if

$$\Sigma_P \preccurlyeq \Sigma_{Q_1} \text{ and } \Sigma_P \preccurlyeq \Sigma_{Q_2} \tag{3.59}$$

Proof. $\Longrightarrow$: Given a full simulation relation of Σ_P by $\Sigma_{Q_1} \|_{\mathrm{pc}} \Sigma_{Q_2}$, Proposition 3.42 allows us to conclude that

$$\Sigma_P \preccurlyeq \Sigma_{Q_1} \|_{\mathrm{pc}} \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1} \Longrightarrow \Sigma_P \preccurlyeq \Sigma_{Q_1}$$

and by symmetry,

$$\Sigma_P \preccurlyeq \Sigma_{Q_1} \|_{\mathrm{pc}} \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_2} \|_{\mathrm{pc}} \Sigma_{Q_1} \preccurlyeq \Sigma_{Q_2} \Longrightarrow \Sigma_P \preccurlyeq \Sigma_{Q_2}$$

$\Longleftarrow$: Compositionality and Proposition 3.41 yield

$$\Sigma_P \|_{\mathrm{pc}} \Sigma_P \preccurlyeq \Sigma_{Q_1} \|_{\mathrm{pc}} \Sigma_{Q_2}, \Sigma_P \preccurlyeq \Sigma_P \|_{\mathrm{pc}} \Sigma_P \quad \Longrightarrow \quad \Sigma_P \preccurlyeq \Sigma_{Q_1} \|_{\mathrm{pc}} \Sigma_{Q_2}$$

□

3.4. Conclusions

The aim of this chapter is to demonstrate how compositional techniques that were developed in computer science for verification of labeled transition systems can be extended to linear systems. We analyzed two types of interconnections, namely feedback interconnections and parallel compositions of linear systems. For each one, we studied compositional and assume-guarantee reasoning rules and proved their soundness. In the feedback case, we also showed that these proof rules are complete. Furthermore, they could be extended firstly by considering bisimulation instead of simulation relations and secondly by allowing for interconnections of more than two systems. For parallel compositions we derived a proof rule based on the decomposition of the specification.
The next step will be to generalize this methodology to other classes of systems; in Chapter 7 results for switching linear systems will be presented. Further generalizations could also be obtained using the more abstract framework presented in [67]. Compositional analysis and decentralized control are strongly related. We will therefore apply the techniques developed in this

chapter to a decentralized control setting in Chapter 6. Another important direction of research is to investigate how to formulate system properties such as stability by means of simulations. In Chapter 5 we will establish a link between simulation and passivity theory that promotes the idea of model checking for linear and nonlinear systems.

4

Simulation relations and compositional analysis for nonlinear systems

Compositional reasoning for nonlinear systems generalizes the main results obtained for linear systems in the previous chapter. To start with, the idea to formulate the existence of a simulation relation between two systems as a geometric control problem can be adopted to the nonlinear case, as outlined in [74]. However, the differential geometric characteristics of submanifolds impose additional technical conditions compared to the linear case. For example, nonlinear simulation is in general (without extra technical conditions) not transitive. Constant rank assumptions on the relations between the defining submanifolds have to be made in order to ensure that nonlinear simulation is a preorder, and that compositional and assume-guarantee reasoning is sound.

4.1. Nonlinear simulation relations

Consider the class of input affine nonlinear systems Σ

$$\Sigma_i : \quad \begin{array}{rcl} \dot{x}_i & = & f_i(x_i) + g_i(x_i)u_i \\ y_i & = & h_i(x_i), \end{array} \tag{4.1}$$
$$x_i \in \mathcal{X}_i, \quad u_i \in \mathcal{U}_i, \quad y_i \in \mathcal{Y}_i,$$

with $\mathcal{X}_i$ an n_i- dimensional manifold, and $\mathcal{U}_i$ and $\mathcal{Y}_i$ the linear input and output spaces of dimension m_i and p_i, respectively. We assume smoothness of all the vector fields f_i, g_i and mappings h_i.
Nonlinear simulation relations are treated in several publications, see [74, 69, 24] for an overview. Here we follow [74] where simulation relations are defined as regular submanifolds.

Definition 4.1. A subset S of a manifold N of dimension n is a regular submanifold of dimension k if for every $p \in S$ there is a coordinate neighborhood $(U, \varphi) = (U, x^1, \dots, x^n)$ of N around p such that

$$U \cap S := \{q \in U \mid x^i(p) = x^i(q),\ i = k+1, \dots, n\}$$

This implies that S is a manifold in its own right, with coordinate chart $\phi_{|S}$ on $U \cap S$ given by

$$\phi_{|S} = (x^1, \dots, x^k, 0, \dots, 0)$$

From now on, "submanifold" will always mean "regular submanifold". We define nonlinear simulation relations as the following invariance condition [74]:

Definition 4.2. A submanifold $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2, $\Sigma_i, i = 1, 2$, as defined in (4.1), if and only if for all $(x_1, x_2) \in S$ and all $u \in \mathcal{U}$ the following properties are satisfied:

$$\begin{aligned} &(1) \quad (f_1(x_1) + g_1(x_1)u, f_2(x_2) + g_2(x_2)u) \in T_{(x_1,x_2)}S \\ &(2) \qquad\qquad h_1(x_1) = h_2(x_2) \end{aligned} \tag{4.2}$$

where $T_{(x_1,x_2)}S$ denotes the tangent space to S at the point $(x_1, x_2) \in S$. S is called a *full* simulation relation, denoted by $\Sigma_1 \preccurlyeq \Sigma_2$, if $\Pi_1 S = \mathcal{X}_1$. In this case, Σ_1 *is simulated by* Σ_2.

Definition 4.2 can be reformulated as

Proposition 4.3. *A submanifold $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if for all $(x_1, x_2) \in S$ the following properties hold:*

$$\begin{aligned} (i) &\qquad \begin{bmatrix} f_1(x_1) \\ f_2(x_2) \end{bmatrix} &\in&\ T_{(x_1,x_2)}S \\ (ii) &\quad \operatorname{im} \begin{bmatrix} g_1(x_1) \\ g_2(x_2) \end{bmatrix} &\subset&\ T_{(x_1,x_2)}S \\ (iii) &\qquad h_1(x_1) &=&\ h_2(x_2) \end{aligned} \tag{4.3}$$

Remark 4.4. Note that we defined nonlinear bisimulation relations directly as the invariance condition (4.2), and not as a trajectory-based notion as in the linear case. It is easily seen that a trajectory-based notion of bisimulation *implies* the invariance condition. For the converse, however, we need an extra constant rank assumption, see [24] for details.

4.2. Compositional analysis for nonlinear systems

Like in Section 3.2.2, open and closed feedback interconnections are defined in the following way: For any two nonlinear systems $\Sigma_i, i = 1, 2$, let

- $\Sigma_1 \|_o \Sigma_2$ denote the *open* feedback interconnection where

$$u_1 = y_2 + e_1 \quad , \quad u_2 = y_1 + e_2$$

  and

- $\Sigma_1 \|_{cl} \Sigma_2$ denote the *closed* feedback interconnection where

$$u_1 = y_2 \quad , \quad u_2 = y_1$$

To develop a theory for nonlinear compositional reasoning, we collect some important facts from differential geometry that can be found in any textbook such as [65].

Proposition 4.5. *Let $S_i, i = 1, 2$, be regular submanifolds of the manifolds M_i with dimensions r_i. Then $S_1 \times S_2$ is a regular submanifold of $M_1 \times M_2$ of dimension $r_1 + r_2$.*

Given how simulation relations of nonlinear systems are characterized by Proposition 4.3, the following theorem assures that under a regularity assumption, zero level sets of smooth functions define smooth submanifolds.

Theorem 4.6. *Let $f_1, \dots, f_m, m \leq n$, be smooth functions on $\mathbb{R}^n$. Define*

$$N := \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_m(x) = 0\} \tag{4.4}$$

and assume $N \neq \varnothing$. Suppose that

$$\text{rank} \left[\frac{\partial f_i}{\partial x_j}(x) \right] = m, \quad \forall\, x \in N \tag{4.5}$$

Then N is a smooth submanifold of $\mathbb{R}^n$ of dimension $n - m$. Moreover, all submanifolds of $\mathbb{R}^n$ can be represented in this way.

In the remainder, we assume that all simulations relations will be defined by submanifolds of the form (4.4). The following lemma, taken from [71], shows how by means of a coordinate transform, variables can be partially eliminated from a set of nonlinear equations without changing the solution set.

Lemma 4.7 (compare with Lemma 2.1 in [71])**.** *Let $f_1(x, z), \dots, f_m(x, z)$ be smooth functions of $x \in \mathbb{R}^n, z \in \mathbb{R}^k$ and let $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$ be such that $f_1(a, b) = \ldots = f_m(a, b) = 0$. Suppose that*

$$\text{rank} \left[\frac{\partial f_i}{\partial x_j}(x, z) \right]_{\substack{i=1,\dots,m, \\ j=1,\dots,n}} = m - p, \ \forall\, (x, z) \in U(a, b),$$

with $U(a,b)$ are neighborhood of (a,b).
Then there exist functions $\psi_1(y,z),\ \ldots,\ \psi_p(y,z)$, $y \in \mathbb{R}^m,\ z \in \mathbb{R}^k$, defined around a neighborhood $V(0,b)$ and independent as functions of y, such that

$$\begin{aligned} f_1(x,z) &= \cdots = f_m(x,z) = 0 \\ &\Longleftrightarrow \\ &\left\{ \begin{array}{rcl} \psi_i(f_1(x,z),\ldots,f_m(x,z),z) &=& 0, \qquad i=1,\ldots,p \\ f_j(x,z) &=& 0, \quad j=p+1,\ldots,m \end{array} \right. \end{aligned} \tag{4.6}$$

for all (x,z) around (a,b) where

$$\frac{\partial \psi_i}{\partial x_j}(f_1(x,z),\ldots,f_m(x,z),z) = 0\ ,\ i=1,\ldots,p,\ j=1,\ldots,n$$

and, after permuting the functions $f_1,\ldots,f_m$,

$$\operatorname{rank}\left[\ \frac{\partial f_{p+i}}{\partial x_j}\ \right]_{\substack{i=1,\ldots,m-p \\ j=1,\ldots,m}} = m-p\ .$$

Remark 4.8. We want to illustrate Lemma 4.7 and its use for nonlinear simulation theory with the following example. Consider three linear systems $\Sigma_i, i=1,2,3$, and suppose that S_{12} and S_{23} are linear simulation relations of Σ_1 by Σ_2 and of Σ_2 by Σ_3, respectively, each given as

$$\begin{aligned} S_{12} &= \{(x_1,x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid F_1x_1 + F_2x_2 = 0,\ F_i \in \mathbb{R}^{f\times n_i},\ i=1,2\} \\ S_{23} &= \{(x_2,x_3) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \mid G_2x_2 + G_3x_3 = 0,\ G_j \in \mathbb{R}^{g\times n_j},\ j=2,3\}\ . \end{aligned}$$

Combining S_{12} and S_{23} as a system of $f+g$ linearly independent equations we obtain

$$S(x_1,x_2,x_3) = \left\{ (x_1,x_2,x_2) \mid \begin{bmatrix} F_1 & F_2 & 0 \\ 0 & G_2 & G_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}\ . \tag{4.7}$$

Assume now that $\operatorname{rank}\begin{bmatrix} F_2 \\ G_2 \end{bmatrix} = f+g-p$. Then there exists an annihilating matrix $\psi = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \in \mathbb{R}^{p\times(f+g)}$ of rank p such that

$$\psi(S(x_1,x_2,x_3)) = \{(x_1,x_3) \mid M_1F_1x_1 + M_2G_3x_3 = 0\} \tag{4.8}$$

since

$$\begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} F_2 \\ G_2 \end{bmatrix} = 0\ .$$

Hence, (4.8) is independent of x_2. In fact, (4.8) defines the subspace $S_{13} := \{(x_1,x_3) \mid \exists x_2 :\ (x_1,x_2) \in S_{12}, (x_2,x_3) \in S_{23}\}$ and is therefore a simulation relation of Σ_1 by Σ_3. Moreover, the solution subspace of (4.7) is equivalent to the solution of (4.8) combined with $f + g - p$ equations of (4.7). If the systems $\Sigma_i, i = 1, 2, 3$, and the corresponding simulations are nonlinear, Lemma 4.7 can be applied accordingly to eliminate the variables x_2. However, the resulting system $\psi_i(f(x_1, x_2), g(x_2, x_3), x_1, x_3) = 0,\ i = 1, \ldots, p$, need not have submanifold properties as Example 4.14 will show.

The first main result for nonlinear compositional analysis is that nonlinear simulation relations – as their linear counterparts – are compositional.

Theorem 4.9. *Consider four nonlinear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ of the form* (4.1). *Assume there exists full nonlinear simulation relations S_i of Σ_{P_i} by $\Sigma_{Q_i}, i = 1, 2$. Then nonlinear simulation is compositional under both open and closed feedback, i.e.*

$$\left.\begin{array}{lcc} S_1 : & \Sigma_{P_1} \preccurlyeq & \Sigma_{Q_1} \\ S_2 : & \Sigma_{P_2} \preccurlyeq & \Sigma_{Q_2} \end{array}\right\} \implies S :\ \Sigma_{P_1} \|_{o,cl} \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \|_{o,cl} \Sigma_{Q_2} \tag{4.9}$$

Proof. Construct the nonlinear simulation relation S by taking the product $S_1 \times S_2$ with the state variables reordered afterwards,

$$S = \{x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_1}) \in S_1, (x_{P_2}, x_{Q_2}) \in S_2\}$$

Since S_1 and S_2 are both submanifolds, their product also defines a submanifold according to Proposition 4.5. Condition 2 in Definition 4.2 is fulfilled by construction while

$$\begin{bmatrix} f_{P_1}(x_{P_1}) + g_{P_1}(x_{P_1})u_1 \\ f_{Q_1}(x_{Q_1}) + g_{Q_1}(x_{Q_1})u_2 \\ f_{P_2}(x_{P_2}) + g_{P_2}(x_{P_2})u_1 \\ f_{Q_2}(x_{Q_2}) + g_{Q_2}(x_{Q_2})u_2 \end{bmatrix} \in T_{x_{P_1}x_{Q_1}} S_1 \times T_{x_{P_2}x_{Q_2}} S_2$$

where $u_1 = e_1 + h_{P_2}(x_{P_2}) = e_1 + h_{Q_2}(x_{Q_2})$, $u_2 = e_2 + h_{P_1}(x_{P_1}) = e_2 + h_{Q_1}(x_{Q_1})$ for open and $u_1 = h_{Q_2}(x_{Q_2}) = h_{Q_2}(x_{Q_2})$, $u_2 = h_{Q_1}(x_{Q_1}) = h_{P_1}(x_{P_1})$ for closed feedback interconnections, respectively. Fullness of S follows from taking the construction as the product of S_1 and S_2, both full relations themselves. □

The converse of Theorem 4.9 does not hold in general as we know already from the linear case, see Remark 3.7. For open feedback interconnections of nonlinear systems, however, compositional reasoning is again, as in the linear case (cf. Proposition 3.18), complete. To obtain this result, the maximal simulation relation of two open feedback interconnections has to be expressed explicitly. Recall that as a consequence of Proposition 2.13, the maximal simulation relation of two linear systems without disturbance inputs d_i equals the unobservability subspace of their augmented system. The nonlinear version of this result has been proved in [74].

Proposition 4.10. *Let $\Sigma_i, i = 1, 2$, be two nonlinear systems of the form* (4.1). *A submanifold S is a simulation relation of Σ_1 by Σ_2 if and only if all $(x_1, x_2) \in S$ satisfy*

$$L_{X_1^1} L_{X_1^2} \dots L_{X_1^k} h_1(x_1) = L_{X_2^1} L_{X_2^2} \dots L_{X_2^k} h_2(x_2) \ , k = 1, 2 \dots , \tag{4.10}$$

where $X_i^j \in \{f_i, g_i\}, i = 1, 2, \ j = 1, 2, \dots$.
Moreover, the maximal bisimulation relation $S^\star$ is given by

$$S^\star \quad = \quad \{(x_1, x_2) \mid (x_1, x_2) \text{ satisfy } (4.10)\} \tag{4.11}$$

Condition (4.10) can be interpreted as the nonlinear analogon of the equality of the Markov parameters (2.8). Proposition 4.10 now allows us to explicitly compute the maximal simulation relation for open feedback interconnections of nonlinear systems.

Lemma 4.11. *Let $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$, be four nonlinear input affine systems of the form* (4.1) *such that*

$$\Sigma_{P_1} \|_o \Sigma_{P_2} \quad \preccurlyeq \quad \Sigma_{Q_1} \|_o \Sigma_{Q_2} \tag{4.12}$$

Then the maximal simulation relation $S^\star$ of $\Sigma_{P_1} \|_o \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_o \Sigma_{Q_2}$ is given by

$$\begin{aligned} S^\star \quad = \quad & \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \\ & L_{X^1_{P_1}} \dots L_{X^k_{P_1}} h_{P_1}(x_{P_1}) = L_{X^1_{Q_1}} \dots L_{X^k_{Q_1}} h_{Q_1}(x_{Q_1}), \\ & L_{X^1_{P_2}} \dots L_{X^k_{P_2}} h_{P_2}(x_{P_2}) = L_{X^1_{Q_2}} \dots L_{X^k_{Q_2}} h_{Q_2}(x_{Q_2}), \ k = 1, 2, \dots\} \end{aligned} \tag{4.13}$$

Proof. By Proposition 4.10, the maximal simulation relation $S^\star$ of $\Sigma_{P_1} \|_o \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_o \Sigma_{Q_2}$ is the set of points

$$\begin{aligned} S^\star = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) | & L_{X^1_{P_1 P_2}} \dots L_{X^k_{P_1 P_2}} h_{P_1 P_2}(x_{P_1}, x_{P_2}) \\ & = L_{X^1_{Q_1 Q_2}} \dots L_{X^k_{Q_1 Q_2}} h_{Q_1 Q_2}(x_{Q_1}, x_{Q_2})\} \end{aligned}$$

where $X_{j_1 j_2} \in \left\{ \begin{bmatrix} f_{j_1} + g_{j_1} h_{j_2} \\ f_{j_2} + g_{j_2} h_{j_1} \end{bmatrix}, \begin{bmatrix} g_{j_1} \\ g_{j_2} \end{bmatrix} \right\}, h_{j_1 j_2} = \begin{bmatrix} h_{j_1} \\ h_{j_2} \end{bmatrix}, j \in \{P, Q\}$. We claim that the repeated Lie derivatives in (4.14) can be split into smooth functions depending on either x_{P_1}, x_{Q_1} or x_{P_2}, x_{Q_2}, respectively. Start with condition (iii) in Theorem 4.3, i.e.

$$h_{P_1}(x_{P_1}) = h_{Q_1} x_{Q_1} \quad , \quad h_{P_2}(x_{P_2}) = h_{Q_2} x_{Q_2} \tag{4.14}$$

The first time derivative becomes

$$L_{f_{P_1} + g_{P_1} h_{P_2}} h_{P_1} + L_{g_{P_1}} h_{P_1} \cdot e_1 = L_{f_{Q_1} + g_{Q_1} h_{Q_2}} h_{Q_1} + L_{g_{Q_1}} h_{Q_1} \cdot e_1$$

which is equivalent to

$$L_{f_{P_1}} h_{P_1} - L_{f_{Q_1}} h_{Q_1} = (L_{g_{Q_1}} h_{Q_1} - L_{g_{P_1}} h_{P_1})(h_{P_2} + e_1) \tag{4.15}$$

using the fact that $h_{P_2} = h_{Q_2}$. Since (4.15) has to hold for all e_1, we obtain

$$L_{f_{P_1}} h_{P_1} = L_{f_{Q_1}} h_{Q_1} \tag{4.16}$$

from setting $e_1 = h_{P_2}$ and consequently also

$$L_{g_{P_1}} h_{P_1} = L_{g_{Q_1}} h_{Q_1} \tag{4.17}$$

Similarly, the second equation in (4.14) yields

$$L_{f_{P_2}} h_{P_2} = L_{f_{Q_2}} h_{Q_2} \quad , \quad L_{g_{P_2}} h_{P_2} = L_{g_{Q_2}} h_{Q_2} \tag{4.18}$$

Further time derivatives of (4.16), (4.17) and (4.17) result in (4.13). □

Remark 4.12. For linear systems without disturbance inputs d_i, Example 3.19 contains a similar result as (4.13).

Lemma 4.11 means that the submanifold $S^\star$ defining the maximal simulation relation of $\Sigma_{P_1} \|_o \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_o \Sigma_{Q_2}$ can be written as the zero-level set of smooth functions

$$\begin{aligned} v &: \mathcal{X}_{P_1} \times \mathcal{X}_{Q_1} \to \mathbb{R}^v \\ w &: \mathcal{X}_{P_2} \times \mathcal{X}_{Q_2} \to \mathbb{R}^w \end{aligned} \tag{4.19}$$

as

$$\begin{aligned} S^\star = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) | v_i(x_{P_1}, x_{Q_1}) = 0, \ i = 1, \ldots, v, \\ w_j(x_{P_2}, x_{Q_2}) = 0, \ j = 1, \ldots, w\} . \end{aligned} \tag{4.20}$$

Since the maximal simulation relation of the interconnections $\Sigma_{P_1} \|_o \Sigma_{P_2}$ and $\Sigma_{Q_1} \|_o \Sigma_{Q_2}$ can be constructed simply as the product of the maximal simulation relations of Σ_{P_1} by Σ_{Q_1} and of Σ_{P_2} by Σ_{Q_2}, the converse also holds true, i.e. that the simulation relations of the components can be reconstructed from $S^\star$.

Proposition 4.13. *Consider four nonlinear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ such that*

$$\Sigma_{P_1} \|_o \Sigma_{P_2} \quad \preccurlyeq \quad \Sigma_{Q_1} \|_o \Sigma_{Q_2} \tag{4.21}$$

with the maximal simulation relation $S^\star$ of $\Sigma_{P_1} \|_o \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_o \Sigma_{Q_2}$ given by (4.20). *Then there exist full simulation relations $S_i, i = 1, 2$, of Σ_{P_1} by Σ_{Q_1} and of Σ_{P_2} by Σ_{Q_2}, respectively.*

Proof. Construct the relations $S_i, i = 1, 2$, as follows:

$$\begin{aligned} S_1 &= \{(x_{P_1}, x_{Q_1}) \mid v_i(x_{P_1}, x_{Q_1}) = 0\ ,\ i = 1, \ldots, v\} \\ S_2 &= \{(x_{P_2}, x_{Q_2}) \mid w_j(x_{P_2}, x_{Q_2}) = 0\ ,\ j = 1, \ldots, w\} \end{aligned}$$

where v_i and w_j are the smooth functions defining $S^\star$. By (4.20), $S^\star$ is defined as the zero-level set of smooth functions v_i, w_j, which according to Theorem 4.6 means that $S^\star$ is a submanifold. Hence, the rank of the Jacobian Jac$S^\star$ is constant:

$$\text{rank} \begin{bmatrix} \frac{\partial f_i}{\partial x_{P_1}} & 0 & \frac{\partial f_i}{\partial x_{Q_1}} & 0 \\ 0 & \frac{\partial g_j}{\partial x_{P_2}} & 0 & \frac{\partial g_j}{\partial x_{Q_2}} \end{bmatrix}_{\substack{i=1,\ldots,l,\\ j=1,\ldots,m}} (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) = \text{const} \tag{4.22}$$

for all $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S^\star$. Furthermore, (4.22) is equivalent to

$$\begin{aligned} \text{rank} \begin{bmatrix} \frac{\partial f_i}{\partial x_{P_1}} & \frac{\partial f_i}{\partial x_{Q_1}} \end{bmatrix}_{i=1,\ldots,l} (x_{P_1}, x_{Q_1}) = \text{const.} \\ \text{rank} \begin{bmatrix} \frac{\partial g_j}{\partial x_{P_2}} & \frac{\partial g_j}{\partial x_{Q_2}} \end{bmatrix}_{j=1,\ldots,m} (x_{P_2}, x_{Q_2}) = \text{const.} \end{aligned}$$

for all $\{(x_{P_i}, x_{Q_i}) \mid \exists\, x_{P_j}, x_{Q_j} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S^\star\}$, $(i, j) \in \{(1, 2), (2, 1)\}$. By Theorem 4.6, both S_1 and S_2 are therefore submanifolds. Since by Lemma 4.11, the functions f_i, g_j are given by all the independent repeated Lie derivatives,

$$\begin{aligned} f_i(x_{P_1}, x_{Q_1}) &= L_{X^1_{P_1}} \ldots L_{X^k_{P_1}} h_{P_1}(x_{P_1}) - L_{X^1_{Q_1}} \ldots L_{X^k_{Q_1}} h_{Q_1}(x_{Q_1})\ , \\ g_j(x_{P_2}, x_{Q_2}) &= L_{X^1_{P_2}} \ldots L_{X^k_{P_2}} h_{P_2}(x_{P_2}) - L_{X^1_{Q_2}} \ldots L_{X^k_{Q_2}} h_{Q_2}(x_{Q_2}) \end{aligned}$$

the relations S_1 and S_2 in fact define the maximal simulation relations $S_1^\star$ and $S_2^\star$ of Σ_{P_1} by Σ_{Q_1} and of Σ_{P_2} by Σ_{Q_2}, respectively, see Proposition 4.10. Finally, fullness of $S^\star$ ensures fullness of $S_i, i = 1, 2$. □

Thus, the results for soundness and completeness of compositional reasoning of linear systems also hold for nonlinear input-affine systems. However, more effort is needed as far as other properties are concerned. E.g., simulation for nonlinear systems cannot be expected to be a preorder in general as the following example illustrates.

Example 4.14. Consider the nonlinear systems

$$\Sigma_P : \begin{array}{rcl} \dot{x}_P &=& 0 \\ y_P &=& x_P^3 \end{array}, \qquad \Sigma_Q : \begin{array}{rcl} \dot{x}_Q &=& 0 \\ y_Q &=& x_Q \end{array}, \qquad \Sigma_R : \begin{array}{rcl} \dot{x}_R &=& 0 \\ y_R &=& x_R^2 \end{array}$$

The simulation relations S_{PQ} and S_{QR} are given by

$$\begin{aligned} S_{PQ} &= \{(x_P, x_Q) \mid x_P^3 - x_Q = 0\} \\ S_{QR} &= \{(x_Q, x_R) \mid x_Q - x_R^2 = 0\} \end{aligned}$$

where both S_{PQ} and S_{QR} are smooth submanifolds of $\mathbb{R}^2$. We construct a relation S_{PR} by setting

$$S_{PR} = \{(x_P, x_R) \mid x_P^3 - x_R^2 = 0\} \tag{4.23}$$

S_{PR} as in (4.23) fulfills the conditions (i)–(iv) of Theorem 4.3 but is not smooth in $(0,0)$. Hence, S_{PR} is not a submanifold and therefore does not define a simulation relation of Σ_P by Σ_R according to Definition 4.2.

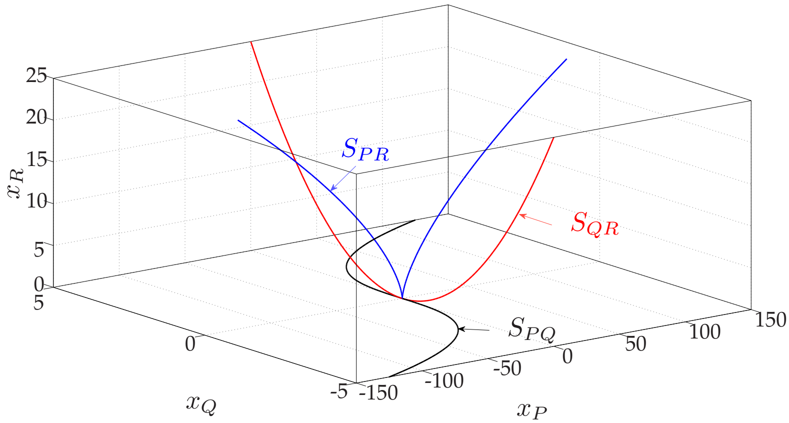

Figure 4.1.: Simulation relations S_{PQ}, S_{QR} and S_{PR}.

In order to obtain results that are valid at least locally we need assumptions on the regularity of the composition of the submanifolds that define simulation relations.

Proposition 4.15. *Consider the nonlinear systems $\Sigma_i, i \in \{P, Q, R\}$. Let there exist full simulation relations S_{PQ} and Σ_{QR} of Σ_P by Σ_Q and of Σ_Q by Σ_R each given as the zero-level sets of smooth functions $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^k, g : \mathbb{R}^q \times \mathbb{R}^r \to \mathbb{R}^l$ in the following way:*

$$\begin{aligned} S_{PQ} &= \{(x_P, x_Q) \in \mathcal{X}_P \times \mathcal{X}_Q \mid f_1(x_P, x_Q) = \cdots = f_k(x_P, x_Q) = 0\} \\ S_{QR} &= \{(x_Q, x_R) \in \mathcal{X}_P \times \mathcal{X}_Q \mid g_1(x_Q, x_R) = \cdots = g_l(x_Q, x_R) = 0\} \end{aligned} \tag{4.24}$$

Suppose that

$$\text{rank} \begin{bmatrix} \frac{\partial f}{\partial x_Q} \\ \frac{\partial g}{\partial x_Q} \end{bmatrix} (x_P, x_q, x_r) = k + l - c, \ \forall (x_P, x_Q) \in S_{PQ}, (x_Q, x_R) \in S_{QR} \tag{4.25}$$

Then there exist c functions $\psi_i(y, x_P, x_R), i = 1, \ldots, c, y \in \mathbb{R}^{k+l}$ with

$$\frac{\partial \psi_i}{\partial x_Q}((f(x_P, x_Q), g(x_Q, x_R)), x_Q, x_R) = 0, i = 1, \ldots, c, \tag{4.26}$$

Moreover, if the rank assumption

$$\operatorname{rank}\begin{bmatrix} \frac{\partial \psi}{\partial x_P} & \frac{\partial \psi}{\partial x_R} \end{bmatrix}(x_P, x_R) = \text{const.} \tag{4.27}$$

holds for all x_P, x_R *such that there exists a* x_Q *such that* $(x_p, x_Q) \in S_{PQ}, (x_Q, x_R) \in S_{PR}$*, then*

$$S_{PR} := \{(x_P, x_R) \mid \psi_i(f(x_P, x_Q), g(x_Q, x_R), x_P, x_R) = 0, i = 1, \dots, c\} \tag{4.28}$$

defines a full simulation relation of Σ_P *by* Σ_R*.*

Proof. Let S_{PQ} and S_{QR} be simulation relations of Σ_P by Σ_Q and of Σ_Q by Σ_R, respectively, as in (4.24). Consider the relation

$$S_{PQR} = \{(x_P, x_Q, x_R) \mid (x_P, x_Q) \in S_{PQ}, (x_Q, x_R) \in S_{QR}\}$$

If assumption (4.25) holds, then Lemma 4.7 ensures that there exist c functions $\psi_i : \mathbb{R}^{k+l} \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^c$ such that (4.26). Since $\psi_i((f(x_P, x_Q), g(x_Q, x_R)), x_p, x_R), i = 1, \dots, c$, do not depend on x_Q, take them as the candidate simulation S_{PR} (4.28) of Σ_P by Σ_R. First of all, due to assumption (4.27), S_{PR} indeed defines a submanifold of $\mathcal{X}_P \times \mathcal{X}_R$. Moreover, due to (4.6), Lemma 4.7 implies that

$$S_{PR} = \{(x_P, x_R) \mid \exists x_Q : (x_P, x_Q) \in S_{PQ}, (x_Q, x_R) \in S_{QR}\}$$

Hence conditions (1) and (2) of Definition 4.2 are fulfilled by construction. Furthermore, $\Pi_{\mathcal{X}_P} S_{PR} = \mathcal{X}_P$ due to fullness of S_{PQ} and S_{QR}. Thus, S_{PR} indeed defines a full nonlinear simulation relation of Σ_P by Σ_R. □

It is now clear that the non-circular assume guarantee reasoning rule holds true for nonlinear simulation relations under regularity conditions.

Theorem 4.16. *Consider four nonlinear systems* $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$*, of the form* (4.1)*. Let* S_1 *be a full nonlinear simulation relation of* Σ_{P_1} *by* Σ_{Q_1}*, given as the zero level set of k smooth functions* $f_i, i = 1, \dots, k$*,*

$$S_1 = \{(x_{P_1}, x_{Q_1}) \mid f_1(x_{P_1}, x_{Q_1}) = \cdots = f_k(x_{P_1}, x_{Q_1}) = 0\}\ ,$$

and S_{II} *a full simulation relation of* $\Sigma_{Q_1} \|_{cl} \Sigma_{P_2}$ *by* $\Sigma_{Q_1} \|_{cl} \Sigma_{Q_2}$*, given as the zero level set of l smooth functions* $g_j, j = 1, \dots, l$*,*

$$S_{II} = \{(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \mid g_j(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) = 0, j = 1, \dots, l\}.$$

If the rank assumption

$$\operatorname{rank}\begin{bmatrix} \frac{\partial f}{\partial x_{Q_1}} & 0 \\ \frac{\partial g}{\partial x_{Q_1}} & \frac{\partial g}{\partial x_{P_2}} \end{bmatrix} = k + l - c = \text{const.} \tag{4.29}$$

holds for all (x_{Q_1}, x_{P_2}) *such that there exist* $x_{P_1}, \bar{x}_{Q_1}, x_{Q_2}$ *such that* $(x_{P_1}, x_{Q_1}) \in S_1$ *and* $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}$*, then there exist* c *smooth functions* $\psi(x_{P_1}, x_{P_2}, y, \bar{x}_{Q_1}, x_{Q_2})$ *such that for all* $i = 1, \ldots, c$

$$\begin{aligned} \frac{\partial \psi_i}{\partial x_{Q_1}}(x_{P_1}, x_{P_2}, (f(x_{P_1}, x_{Q_1}), g(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2})), \bar{x}_{Q_1}, x_{Q_2}) &= 0 \\ \frac{\partial \psi_i}{\partial x_{P_2}}(x_{P_1}, x_{P_2}, (f(x_{P_1}, x_{Q_1}), g(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2})), \bar{x}_{Q_1}, x_{Q_2}) &= 0 \end{aligned} \tag{4.30}$$

If moreover

$$\operatorname{rank} \begin{bmatrix} \frac{\partial \psi}{\partial x_{P_1}} & \frac{\partial \psi}{\partial x_{P_2}} & \frac{\partial \psi}{\partial \bar{x}_{Q_1}} & \frac{\partial \psi}{\partial x_{Q_2}} \end{bmatrix} (x_{P_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) = \text{const.} \tag{4.31}$$

for all $x_{P_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}$ *such that there exists a* x_{Q_1} *such that* $(x_{P_1}, x_{Q_1}) \in S_1$ *and* $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}$*, then the non-circular assume-guarantee reasoning rule holds for nonlinear systems.*

Proof. First, we use the fact that nonlinear simulation is compositional with respect to feedback interconnection, see Theorem 4.9. Hence,

$$\Sigma_{P_1} \preccurlyeq \Sigma_{Q_1} \implies \Sigma_{P_1} \|_{\text{cl}} \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \|_{\text{cl}} \Sigma_{P_2}$$

with the full simulation relation S_I of $\Sigma_{P_1} \|_{\text{cl}} \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_{\text{cl}} \Sigma_{P_2}$ given by

$$S_I = \{(x_{P_1}, x_{P_2}, x_{Q_1}, \bar{x}_{P_2}) \mid (x_{P_1}, x_{Q_1}) \in S_1, x_{P_2} = \bar{x}_{P_2}\}$$

Next, consider the relation

$$\begin{aligned} S' = \{&(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, \bar{x}_{P_2}, \bar{x}_{Q_1}) \mid (x_{P_1}, x_{P_2}, x_{Q_1}, \bar{x}_{P_2}) \in S_I, \\ &(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}\} \end{aligned}$$

with

$$\operatorname{rank} S_I = \operatorname{rank} \begin{bmatrix} \frac{\partial f}{\partial x_{P_1}} & 0 & \frac{\partial f}{\partial x_{Q_1}} & 0 & 0 & 0 \\ 0 & I & 0 & -I & 0 & 0 \\ 0 & 0 & \frac{\partial g}{\partial x_{Q_1}} & \frac{\partial g}{\partial x_{P_2}} & \frac{\partial g}{\partial \bar{x}_{Q_1}} & \frac{\partial g}{\partial x_{Q_2}} \end{bmatrix}$$

We now want to apply Proposition 4.15. If

$$\operatorname{rank} \begin{bmatrix} \frac{\partial f}{\partial x_{Q_1}} & 0 \\ \frac{\partial g}{\partial x_{Q_1}} & \frac{\partial g}{\partial x_{P_2}} \end{bmatrix} (x_{Q_1}, x_{P_2}) = k + l - c$$

for all x_{Q_1}, x_{P_2} such that there exist $x_{P_1}, \bar{x}_{Q_1}, x_{Q_2}$ such that $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, \bar{x}_{P_2}, \bar{x}_{Q_1}) \in S'$, then there exist smooth functions $\psi_i, i = 1, \ldots, c$, fulfilling (4.30) due to Lemma 4.7. Taking

$$\begin{aligned} S := \{(x_{P_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \mid \exists \bar{x}_{P_2}, x_{Q_1} : \psi_i(&x_{P_1}, x_{P_2}, (f(x_{P_1}, x_{Q_1}), \\ &g(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2})), \bar{x}_{Q_1}, x_{Q_2}) = 0, i = 1 \ldots, c, \} \end{aligned}$$

as the candidate relation, the rank assumption (4.30) ensures that S indeed defines a submanifold. Hence, Proposition 4.15 can be applied to conclude

$$\Sigma_{P_1}\|_{\mathrm{cl}}\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_{\mathrm{cl}}\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_{\mathrm{cl}}\Sigma_{Q_2}$$

due to transitivity of nonlinear simulation. □

For circular assume-guarantee reasoning , we only prove the existence of a nonlinear simulation relation but not its fullness.

Proposition 4.17. *Consider four nonlinear control systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ such that there exist simulation relation S_I, S_{II}, each given as the zero level sets of smooth functions $f_i, i = 1, \ldots, k$ and $g_j, j = 1, \ldots, l$, as follows:*

$$\begin{array}{llcl} S_I: & \Sigma_{P_1}\|_{cl}\Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1}\|_{cl}\Sigma_{Q_2} \\ S_{II}: & \Sigma_{Q_1}\|_{cl}\Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1}\|_{cl}\Sigma_{Q_2} \end{array} \tag{4.32}$$

with

$$\begin{aligned} S_I &:= \{x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \mid f_i(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) = 0, i = 1, \ldots, k\} \\ S_{II} &:= \{x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \mid g_j(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) = 0, j = 1, \ldots, l\} \end{aligned}$$

If

$$\mathit{rank}\begin{bmatrix} 0 & \frac{\partial f}{\partial \bar{x}_{Q_2}} \\ \frac{\partial g}{\partial \bar{x}_{Q_1}} & 0 \end{bmatrix}(\bar{x}_{Q_1}, \bar{x}_{Q_2}) = k + l - c \tag{4.33}$$

then there exist smooth functions $\psi_i(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, y), i = 1, \ldots, c$, such that

$$\frac{\partial \psi_i}{\partial \bar{x}_{Q_j}}(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, (f(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}), g(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}))) = 0$$
$$i = 1, \ldots, c,\ j = 1, 2$$

Moreover, if

$$\operatorname{rank}\begin{bmatrix} \frac{\partial \psi}{\partial x_{P_1}} & \frac{\partial \psi}{\partial x_{P_2}} & \frac{\partial \psi}{\partial x_{Q_1}} & \frac{\partial \psi}{\partial x_{Q_2}} \end{bmatrix}(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) = \text{const.} \tag{4.34}$$

for all $x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}$, then

$$\begin{aligned} S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \psi_i((x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, (f(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}), \\ g(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}))) = 0, i = 1, \ldots, c\} \end{aligned} \tag{4.35}$$

defines a simulation relation of $\Sigma_{P_1}\|_{cl}\Sigma_{P_2}$ by $\Sigma_{Q_1}\|_{cl}\Sigma_{Q_2}$.

Proof. The proof uses the same arguments as in the previous theorem. Consider first the relation

$$S' = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, \bar{x}_{Q_1}, \bar{x}_{Q_2}) \mid f_i(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) = 0, \\ i = 1, \ldots, k, g_j(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) = 0, j = 1, \ldots, l\}$$

By Lemma 4.7, assumptions (4.33) ensures that there exist smooth functions $\psi_i(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, y), i = 1, \ldots, c$ such that 4.34 holds. Defining the candidate relation S as in (4.35), the full rank assumption (4.34) guarantees that S is a smooth submanifold. By the equality of the solution sets of S' on the one hand and S plus the remaining $k + l - c$ equations of S' on the other hand, S can be rewritten as

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists\, \bar{x}_{Q_1}, \bar{x}_{Q_2} : (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I, \\ (x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}\}$$

Hence,

$$\begin{bmatrix} f_{P_1}(x_{P_1}) - g_{P_1}(x_{P_1})h_{Q_2}(x_{Q_2}) \\ f_{P_2}(x_{P_2}) + g_{P_2}(x_{P_2})h_{Q_1}(x_{Q_1}) \\ f_{Q_1}(x_{Q_1}) - g_{Q_1}(x_{Q_1})h_{P_2}(x_{P_2}) \\ f_{Q_2}(x_{Q_2}) + g_{Q_2}(x_{Q_2})h_{P_1}(x_{P_1}) \end{bmatrix} \in T_{x_{P_1}x_{P_2}x_{Q_1}x_{Q_2}}S$$

while $h_{P_1}x_{P_1} = h_{Q_1}(x_{Q_1}) = h_{Q_1}(\bar{x}_{Q_1})$ and $h_{P_2}x_{P_2} = h_{Q_2}(x_{Q_2}) = h_{Q_2}(\bar{x}_{Q_2})$ which proves the claim. □

Recall that in the linear case, we could prove fullness of S by ensuring that the relations S_I and S_{II} were large enough. To that end, suitable subspaces were added, see Theorem 3.14. Since we define nonlinear simulation relations as zero-level sets of smooth functions, this idea cannot be immediately generalized.

Example 4.18. Consider the systems

$$\Sigma_{P_1}: \begin{array}{rcl} \dot{x}_{P_1} &=& x_{P_1} \\ y_{P_1} &=& x_{P_1} \end{array}, \ \Sigma_{P_2}: \begin{array}{rcl} \dot{x}_{P_2} &=& x_{P_2} \\ y_{P_2} &=& x_{P_2}^2 \end{array}, \ \Sigma_i \ \begin{array}{rcl} \dot{x}_i &=& x_i \\ y_i &=& 0 \end{array}, i \in \{Q_1, Q_2\}\,.$$

Let the simulation relations S_1 and S_2 be given as the solution sets

$$S_I: \ x_{P_1} - x_{Q_1} + x_{Q_2} = 0 \quad , \quad S_{II}: \ x_{Q_1} + x_{P_2}^2 - x_{Q_2} = 0$$

Clearly, both $S_i, i \in \{I, II\}$, define submanifolds since $\mathrm{rank} S_I = 1 = \mathrm{rank} S_{II}$ for all $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I$ and all $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}$, respectively. Moreover, both S_I and S_{II} are full as for every (x_{P_1}, x_{Q_2}) there exists a $\bar{x}_{Q_2}$ and a $x_{Q_1} = x_{P_2} + x_{Q_2}$ such that $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I$ and similarly, for every $(x_{Q_1}, x_{P_2}$ there exists a $\bar{x}_{Q_1}$ and $x_{Q_2} = x_{Q_1} + x_{P_2}^2$ such that $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}$. The rank condition (4.33) is fulfilled,

$$\mathrm{rank}\begin{bmatrix} 0 & \frac{\partial S_I}{\partial \bar{x}_{Q_2}} \\ \frac{\partial S_{II}}{\partial \bar{x}_{Q_1}} & 0 \end{bmatrix} = 0 \,,$$

and

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid x_{P_1} - x_{Q_1} + x_{Q_2} = 0 = x_{Q_1} + x_{P_2}^2 - x_{Q_2}\} \quad (4.36)$$

has full rank, $\mathrm{rank} S = 2$ for all $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. Hence, S defines a simulation relation of $\Sigma_{P_1} \|_{\mathrm{cl}} \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_{\mathrm{cl}} \Sigma_{Q_2}$. However, S is not full since $x_{P_1} = -x_{P_2}^2$ is imposed by (4.36), and thus $x_{P_1} \leq 0$.

In the next chapter we will apply nonlinear simulation theory to passivity.

5

Passivity theory and compositional analysis

The classical passivity theorem states that the negative feedback interconnection of two passive systems is again passive with respect to the same supply rate, cf. [63]. Clearly, this can be regarded as compositional reasoning for passive systems. In this chapter, a new approach towards passivity is developed using nonlinear simulation relations. Associating the storage function with the state variable and the inputs and outputs of the underlying dynamical system with the external variables, the differential dissipation inequality can be interpreted as a one-dimensional state space control system. Passivity is then equivalent to the existence of a nonlinear simulation relation between the original control system and the one-dimensional system given by the dissipation inequality. Formulating passivity properties by means of simulation relations allows us to apply some of the compositional reasoning techniques developed in the previous chapters. The passivity theorem mentioned above can thus be reinterpreted as compositionality of passivity. This is verified for both open and closed feedback interconnections of linear systems. We then show that passivity is also complete under open feedback, i.e. if the open feedback interconnection of two control systems is passive then the individual components must be passive as well. As a result, the structure of the storage function of the interconnected system can be specified in terms of the storage functions of the components. These results are then generalized to nonlinear systems. Thus, a strong relation between passivity and simulation theory has been established demonstrating that properties of interconnected control systems – in this case passivity – can be verified efficiently using compositional reasoning techniques. Due to its link with Lyapunov stability theory some of the results can, under mild assumptions on the storage functions, be reformulated as compositional stability analysis, in particular for closed feedback interconnections. This motivates further research on the topic of passivity based controller design methods, see e.g. [72].

5.1. A brief introduction to passivity theory

Passivity as a system theoretic concept can be defined in great generality. We consider continuous-time state-space systems with state variables x taking values in an $n-$ dimensional manifold $\mathcal{X} \subset \mathbb{R}^n$ and external variables u and y related by a set of differential and algebraic equations or inequalities

$$\Sigma : \quad F(x, \dot{x}, u, y) \ \leq \ 0 \ , \tag{5.1}$$

The input and output spaces $\mathcal{U}$ and $\mathcal{Y}$ are assumed to be dual spaces, i.e. $\mathcal{Y} = \mathcal{U}^{\star}$, in order to define the product $u^T y$.

Definition 5.1. A state space system Σ is *passive* if there exists a function $V : \mathcal{X} \to \mathbb{R}^+$, called the storage function, such that for all $x_0 \in \mathcal{X}$, all $t_1 \geq t_0$, and all input functions u

$$V(x(t_1)) \leq V(x(t_0)) + \int_{t_0}^{t_1} u^T(t) y(t) dt \tag{5.2}$$

where $x_0 = x(t_0)$.
If (5.2) holds with equality, then Σ is *lossless*.

If V is differentiable, the differential version of the dissipation inequality (5.2) is given by

$$V_x(x)\dot{x} \leq u^T y \tag{5.3}$$

for all $(x, \dot{x}, u, y)$ satisfying (5.1). Here $V_x(x)$ denotes the row vector of partial derivatives

$$V_x(x) = \left(\begin{array}{ccc} \frac{\partial V}{\partial x_1}(x) & \ldots & \frac{\partial V}{\partial x_n}(x) \end{array} \right)$$

The next proposition expresses passivity properties of a state space system in terms of the existence of a nonlinear simulation relation.

Proposition 5.2. *For a continuous-time state space system Σ of the form* (4.1)*, the following two statements are equivalent:*

- *Σ is passive (lossless)*
- *Σ is simulated by the one-dimensional system*

$$\Xi : \quad \dot{\xi} \quad \leq (=) \quad u^T y, \quad \xi \in \mathcal{X}_{\xi} = \mathbb{R}^+ \tag{5.4}$$

The non-linear simulation relation S of Σ by Ξ is given by

$$S = \{(x, \xi) \,|\, \xi = V(x)\} \tag{5.5}$$

Proof. $\mathcal{S}$ as given by (5.5) is defined on the product manifold $\mathcal{X} \times \mathcal{X}_\xi$ and has the submanifold property. Indeed, it is given as the zero level set of a smooth function $f(x, \xi) := \xi - V(x)$ such that the Jacobian $\text{Jac}(f) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial \xi} \end{bmatrix} = \begin{bmatrix} V_x(x) & 1 \end{bmatrix}$ has full rank for all $(x, \xi) \in S$. Hence, by Theorem 4.6, S is a submanifold. Moreover, the external behavior of Σ and Ξ is identical. Thus, for any pair of states $(x_0, \xi_0) \in S$, the state trajectories $x(t), \xi(t) = V(x(t))$ remain within the submanifold $S \subset \mathcal{X} \times \mathcal{X}_\xi$ for all times $t \geq 0$ while the external variables $u(t)$ and $y(t)$ are the same. Moreover, S is full since for every x there exists a $\xi = V(x)$. □

5.2. Compositional reasoning for passive systems

The motivation for using simulation theory to characterize passivity properties of control systems is to analyze interconnections of such systems. We first want to study *linear* state space systems before generalizing the results to nonlinear input-affine systems.
To simplify the exposition, we consider linear systems of the form

$$\Sigma_i : \begin{array}{rcl} \dot{x}_i & = & A_i x_i + B_i u_i, \quad x_i \in \mathcal{X}_i \\ y_i & = & C_i x_i \end{array}, \; i = 1, 2, \tag{5.6}$$

without feedthrough terms.
Based on the differential dissipation inequality (5.3), the Kalman-Yakubovic conditions [80] characterize passivity in terms of the state space representation of linear systems.

Proposition 5.3. *A linear system Σ is passive (lossless) if there exists a quadratic storage function $V(x) = \frac{1}{2}x^T Qx, Q = Q^T \geq 0$ such that*

$$\begin{array}{rcl} A^T Q + QA & \leq (=) & 0 \\ B^T Q & = & C \end{array} \tag{5.7}$$

Moreover, if the system Σ is passive with a continuous and differentiable storage function V, then there exists a quadratic storage function $V = \frac{1}{2}x^T Qx$ with $Q = Q^T \geq 0$ such that Q fulfills (5.7).
If the system Σ is lossless and controllable, the storage function is unique (up to a constant) and given by a quadratic function $V(x) = \frac{1}{2}x^T Qx$.

The specialization of Proposition 5.2 to the linear case leads to the following

Corollary 5.4. *A linear system Σ of the form (5.6) is passive (lossless) if and only if there exists a full simulation relation S of Σ by Ξ, Ξ as in (5.4), where*

$$S = \{(x, \xi) \mid \xi = \frac{1}{2}x^T Qx, Q = Q^T \geq 0, \; Q \text{ fulfills } (5.7)\} \tag{5.8}$$

It is a well-known fact in electrical network and systems theory that the negative feedback interconnection of two passive systems is again passive [63]. Capturing passivity as nonlinear simulation this result can be reformulated as compositional reasoning.

Theorem 5.5. *Given two linear systems $\Sigma_i, i = 1,2$ of the form (5.6) and two systems Ξ_i of the form (5.4). Then passivity (losslessness) is compositional under open negative feedback interconnection $\|_o$, i.e.*

$$\left.\begin{array}{lccc} S_1: & \Sigma_1 & \preccurlyeq & \Xi_1 \\ S_2: & \Sigma_2 & \preccurlyeq & \Xi_2 \end{array}\right\} \implies S: \ \Sigma_1\|_o\Sigma_2 \ \preccurlyeq \ \Xi_1\|_o\Xi_2 \tag{5.9}$$

Moreover, $\Xi_1\|_o\Xi_2 \approx \Xi$ where

$$\Xi: \quad \dot{\xi} \ \leq \ e_1^T y_1 + e_2^T y_2 \tag{5.10}$$

and thus

$$\tilde{S}: \quad \Sigma_1\|_o\Sigma_2 \preccurlyeq \Xi \tag{5.11}$$

Proof. We will prove the result for passive systems, the same arguments also hold for lossless systems. Construct the simulation relation S by setting

$$S := \left\{ ((x_1,x_2),(\xi_1,\xi_2)) \mid \xi_1 + \xi_2 = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\} \tag{5.12}$$

Since $S_i, i = 1,2$ is a simulation relation of Σ_i by Ξ_i, we know that there exists a quadratic storage function $\xi_i = \frac{1}{2} x_i^T Q_i x_i, Q_i = Q_i^T \geq 0$, such that

$$\dot{\xi}_i = \frac{1}{2} x_i^T (A_i^T Q_i + Q_i A_i) x_i + u_i^T B_i^T Q_i x_i \leq u_i^T y_i = u_i^T C_i x_i$$

The open negative feedback interconnection $u_1 = e_1 - y_2, u_2 = e_2 + y_1$ yields the closed loop systems

$$\Sigma_1\|_o\Sigma_2: \quad \begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} & = & \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array} \tag{5.13}$$

and

$$\Xi_1\|_o\Xi_2: \ \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} u_1^T y_1 \\ u_2^T y_2 \end{bmatrix} = \begin{bmatrix} (e_1 - y_2)^T y_1 \\ (e_2 + y_1)^T y_2 \end{bmatrix} \tag{5.14}$$

From (5.12), we compute

$$\begin{aligned} \dot{\xi}_1 + \dot{\xi}_2 &= \frac{1}{2}\frac{d}{dt}\left(x_1^T Q_1 x_1 + x_2^T Q_2 x_2\right) \leq \\ &\leq (e_1 - y_2)^T y_1 + (e_2 + y_1)^T y_2 = e_1^T y_1 + e_2^T y_2 \end{aligned} \tag{5.15}$$

and thus, $\Sigma_1 \|_o \Sigma_2$ is also passive with respect to the supply rate $s(e_1, e_2, y_1, y_2) = e_1^T y_1 + e_2^T y_2$.
Next, consider the system

$$\Xi : \quad \dot{\xi} = e^T z$$

with $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ and $z = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and a relation $\tilde{\mathcal{S}}$ given by

$$\tilde{\mathcal{S}} := \{((\xi_1, \xi_2), \xi) \mid \xi = \xi_1 + \xi_2\}$$

Again, $\tilde{S} \subset \mathcal{X}_\xi \times \mathcal{X}_{\xi_1} \times \mathcal{X}_{\xi_2}$ is defined as the zero level set of the smooth function $g(\xi, \xi_1, \xi_2) := \xi - \xi_1 - \xi_2$ such that the Jacobian $\mathrm{Jac}(g(\xi, \xi_1, \xi_2)) = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$ has full rank for all $(\xi, \xi_1, \xi_2) \in \tilde{S}$. Hence, $\tilde{S}$ is indeed a submanifold. Take now a set of initial states $((\xi_1(0), \xi_2(0)), \xi(0)) \in \tilde{S}$. Starting from $\xi_1(0) + \xi_2(0) = \xi(0)$, the trajectories by definition satisfy

$$\xi_1(t) + \xi_2(t) = \xi(t), \quad \forall t \geq 0$$

while the corresponding external variables are the same,

$$e^T(t) z(t) = e_1^T(t) z_1(t) + e_2^T(t) z_2(t), \quad \forall t \geq 0$$

Thus, Ξ and $\Xi_1 \|_o \Xi_2$ are indeed bisimilar. (5.11) then follows from transitivity of simulation. Here, the conditions of Proposition 4.15 are fulfilled since

$$\mathrm{rank}\begin{bmatrix} \frac{\partial f(\xi,\xi_1,\xi_2)}{\partial \xi_1} & \frac{\partial f(\xi,\xi_1,x_1)}{\partial \xi_2} \\ \frac{\partial g(x_1,x_2,\xi_1,x_2)}{\partial \xi_1} & \frac{\partial g(x_1,x_2,\xi_1,x_2)}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \text{const.}$$

for all (ξ_1, ξ_2) and

$$\begin{aligned} \mathrm{rank}\tilde{S} &= \mathrm{rank}\left[\mathrm{Jac}(\xi - \frac{1}{2}x_1^T Q_1 x_2 + \frac{1}{2}x_2^T Q_2 x_2)\right] = \\ &= \mathrm{rank}\begin{bmatrix} 1 & -x_1^T Q_1 & -x_2^T Q_2 \end{bmatrix} = 1 \end{aligned}$$

for all $(\xi, x_1, x_2) \in \tilde{S}$. □

Note that although we considered *positive* feedback interconnections in Chapters 3 and 4 the results of these chapters can easily be seen to hold for negative feedback as well. Theorem 5.5 therefore establishes the link between passivity theory and compositional analysis techniques by showing that passivity is compositional under open feedback. Thus, a classical result of passivity theory is reinterpreted by means of compositional reasoning techniques. This evokes the question whether the converse implication also holds true. Recall that in Section 3.2.2 we showed that compositionality is complete for open feedback interconnections of linear systems. The same result also holds for nonlinear systems as shown in Chapter 4. We will now show that a similar result holds for negative feedback interconnections of passive (lossless) systems. We know that for such interconnections there exists a quadratic storage function

$$V(x_1, x_2) = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.16}$$

such that $\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ is symmetric and positive semi-definite.

Proposition 5.6. *Consider two linear systems $\Sigma_i, i = 1, 2,$ and systems Ξ_i of the form (5.4). If the open feedback interconnection $\Sigma_1 \|_o \Sigma_2$ is passive (lossless), then also the components $\Sigma_i, i = 1, 2,$ are passive (lossless). In other words, if there exists a simulation relation S of $\Sigma_1 \|_o \Sigma_2$ by Ξ, then there also exist simulation relations $S_i, i = 1, 2,$ for the components*

$$\begin{aligned} S_1 : \quad \Sigma_1 &\preccurlyeq \Xi_1 \\ S_1 : \quad \Sigma_2 &\preccurlyeq \Xi_2 \end{aligned} \tag{5.17}$$

Proof. Again we will only prove the passive case. Assume that $\Sigma_1 \|_o \Sigma_2$ is passive. Then there exists a quadratic storage function V as in (5.16) with $\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ symmetric and positive semi-definite. This implies that

$$\begin{aligned} Q_{11} = Q_{11}^T \geq 0, Q_{22} = Q_{22}^T \geq 0, \\ Q_{12}^T = Q_{21}, Q_{22} - Q_{21} Q_{11}^{-1} Q_{12} \geq 0 \end{aligned} \tag{5.18}$$

The simulation relation S of $\Sigma_1 \|_o \Sigma_2$ by Ξ is then given by

$$S = \{((x_1, x_2), \xi) \mid \xi = V(x_1, x_2)\}$$

Since $\Sigma_1 \|_o \Sigma_2$ is passive,

$$\dot{\xi} \leq e_1^T z_1 + e_2^T z_2 \tag{5.19}$$

This leads to

$$\begin{aligned}
\dot{\xi} &= \frac{1}{2}\frac{d}{dt}\left(x_1^T Q_{11} x_1 + x_2^T Q_{21} x_1 + x_1^T Q_{12} x_2 + x_2^T Q_{22} x_2\right) = \\
&= \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} A_1^T Q_{11} + Q_{11} A_1 + 2Q_{12} B_2 C_1 & -Q_{11} B_1 C_2 + C_1^T B_2^T Q_{22} + A_1^T Q_{12} + Q_{12} A_2 \\ -C_2^T B_1^T Q_{11} + Q_{22} B_2 C_1 + A_2^T Q_{21} + Q_{21} A_1 & -2Q_{12}^T B_1 C_2 + A_2^T Q_{22} + Q_{22} A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^T \begin{bmatrix} B_1^T Q_{11} & B_1^T Q_{12} \\ B_2^T Q_{12}^T & B_2^T Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&\leq e_1^T z_1 + e_2^T z_2 = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^T \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned} \tag{5.20}$$

(5.20) is equivalent to

$$\begin{bmatrix} A_1^T Q_{11} + Q_{11} A_1 + 2Q_{12} B_2 C_1 & -Q_{11} B_1 C_2 + C_1^T B_2^T Q_{22} + A_1^T Q_{12} + Q_{12} A_2 \\ -C_2^T B_1^T Q_{11} + Q_{22} B_2 C_1 + A_2^T Q_{21} + Q_{21} A_1 & -2Q_{12}^T B_1 C_2 + A_2^T Q_{22} + Q_{22} A_2 \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.21}$$

and

$$\begin{bmatrix} B_1^T Q_{11} & B_1^T Q_{12} \\ B_2^T Q_{12}^T & B_2^T Q_{22} \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \tag{5.22}$$

which in turn implies

$$\begin{aligned} B_1^T Q_{11} &= C_1, A_1^T Q_{11} + Q_{11} A_1 \leq 0 \\ B_2^T Q_{22} &= C_2, A_2^T Q_{22} + Q_{22} A_2 \leq 0 \end{aligned} \tag{5.23}$$

Now consider the systems

$$\Xi_i : \quad \dot{\xi}_i \leq u_i^T y_i, \quad i = 1, 2$$

and the relations

$$S_i := \{(x_i, \xi_i) \mid \xi = \frac{1}{2} x_i^T Q_{ii} x_i\}, \ i = 1, 2,$$

where Q_{ii} is symmetric and positive semi-definite by (5.18). $S_i \subset \mathcal{X}_i \times \mathcal{X}_{\xi_I}$ clearly fulfills the submanifold property in the sense of Theorem 4.6 with $\operatorname{rank}\begin{bmatrix} \frac{\partial S_i}{\partial x_i} & \frac{\partial S_i}{\partial \xi_i} \end{bmatrix}(x_i, \xi_i) = 1$ for all $(x_i, \xi_i) \in S_i$. Moreover, S_i is also invariant since

$$\dot{\xi}_i = \frac{d}{dt}\left(\tfrac{1}{2} x_i^T Q_{ii} x_i\right) = \tfrac{1}{2} x_i^T \left(A_i^T Q_{ii} + Q_{ii} A_i\right) x_i + u_i^T B_i^T Q_{ii} x_i \leq u_i^T y_i$$

due to (5.23). Hence, S_i defines a simulation relation of Σ_i by Ξ_i, $i = 1, 2$. Moreover, S_i is also full since for every x_i there exists a $\xi_i = \frac{1}{2}x_i^T Q_{ii} x_i$ such that $(\xi_i, x_i) \in S_i$. By Corollary 5.4 this means that Σ_i are passive. □

Since $V_i(x_i) = \frac{1}{2}x_i^T Q_{ii} x_i, i = 1, 2$, are storage functions for the passive components Σ_i, their sum

$$V(x_1, x_2) = V_1(x_1) + V_2(x_2) = \frac{1}{2}x_1^T Q_{11} x_1 + \frac{1}{2}x_2^T Q_{22} x_2 \tag{5.24}$$

by Theorem 5.5 is also a storage function of the interconnection $\Sigma_1 \|_o \Sigma_2$. Phrased differently, for every passive interconnection $\Sigma_1 \|_o \Sigma_2$ there always exists a decoupled storage function $V(x_1, x_2)$ of the form (5.24). For lossless systems we can give an even stronger result.

Proposition 5.7. *Consider two linear systems $\Sigma_i, i = 1, 2$, and assume that their interconnection $\Sigma_1 \|_o \Sigma_2$ is lossless. If Σ_i are controllable the storage function $V(x_1, x_2)$ of $\Sigma_1 \|_o \Sigma_2$ is* uniquely *determined as the sum of the storage functions of the components* (5.24) *(modulo a constant).*

Proof. Recall the arguments of the previous proof where inequality in (5.19) and subsequently is replaced by equality. Then (5.21) is equivalent to

$$\begin{aligned} B_1^T Q_{12} = 0, \quad B_1^T Q_{11} = C_1, \quad & A_1^T Q_{11} + Q_{11} A_1 = 0 \\ B_2^T Q_{12}^T = 0, \quad B_2^T Q_{22} = C_2, \quad & A_2^T Q_{22} + Q_{22} A_2 = 0 \\ A_1^T Q_{12} + Q_{12} A_2 = 0, \quad & A_2^T Q_{21} + Q_{21} A_2 = 0 \end{aligned} \tag{5.25}$$

Premultiplying (5.25) with $\begin{bmatrix} B_1^T \\ B_1^T A_1^T \\ \vdots \\ B_1^T (A_1^n)^T \end{bmatrix}$ and $\begin{bmatrix} B_2^T \\ B_2^T A_2^T \\ \vdots \\ (A_1^n)^T \end{bmatrix}$ yields

$$\begin{bmatrix} B_1^T \\ B_1^T A_1^T \\ \vdots \\ B_1^T (A_1^n)^T \end{bmatrix} Q_{12} = 0 \qquad \begin{bmatrix} B_2^T \\ B_2^T A_2^T \\ \vdots \\ B_2^T (A_2^n)^T \end{bmatrix} Q_{21} = 0$$

Thus, if (A_1, B_1) and (A_2, B_2) are controllable, the off-diagonal terms Q_{12}, $Q_{21} = Q_{12}^T$ in (5.16) necessarily vanish which proves that the storage function of $\Sigma_1 \|_o \Sigma_2$ is indeed of the form (5.24). □

Remark 5.8. As stated in Proposition 5.3 the storage function of a lossless and controllable linear system Σ is always unique up to a constant. Hence, if $\Sigma_1 \|_o \Sigma_2$ is controllable and lossless, then it has a unique storage function $V(x_1, x_2)$ of the form (5.24). Observe that the assumption of Proposition 5.7, namely that the components $\Sigma_i, i = 1, 2$, are controllable, is equivalent to the assumption that their interconnection $\Sigma_1 \|_o \Sigma_2$ is controllable, which is needed to prove the same result using passivity theory.

In the previous chapters we observed differences between open and closed interconnections with respect to compositionality. This discrepancy also becomes apparent here as the following proposition shows.

Proposition 5.9. *Consider two passive linear systems $\Sigma_i, i = 1, 2$, and their closed negative feedback interconnection $\Sigma_1 \|_{cl} \Sigma_2$. Then it holds that*

$$\left.\begin{array}{lccc} S_1 : & \Sigma_1 & \preccurlyeq & \Xi_1 \\ S_2 : & \Sigma_2 & \preccurlyeq & \Xi_2 \end{array}\right\} \implies S: \ \Sigma_1 \|_{cl} \Sigma_2 \preccurlyeq \Xi_1 \|_{cl} \Xi_2 \tag{5.26}$$

The converse of (5.26) does not hold for closed negative feedback interconnection.

Proof. Analogous to Theorem 5.5, construct the relation S from S_1 and S_2 as in (5.12). Since $\Sigma_i, i = 1, 2$ are passive, it holds that $\dot{\xi}_i \leq u_i^T y_i$. For the interconnection $\Xi_1 \|_{\mathrm{cl}} \Xi_2$, we obtain

$$\Xi_1 \|_{\mathrm{cl}} \Xi_2 : \quad \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} \leq \begin{bmatrix} -y_2^T y_1 \\ y_1^T y_2 \end{bmatrix} \tag{5.27}$$

and thus,

$$\dot{\xi}_1 + \dot{\xi}_2 \leq 0 \tag{5.28}$$

Since the inputs of $\Sigma_1 \|_{\mathrm{cl}} \Sigma_2$ are closed by the feedback interconnection, there is no energy flow between the system and its environment, $u^T y = 0$.
For the converse, consider the example of two damped mass-spring systems $\Sigma_i, i = 1, 2$,

$$\Sigma_i : \begin{array}{rcl} \begin{bmatrix} \dot{x}_i^1 \\ \dot{x}_i^2 \end{bmatrix} & = & \begin{bmatrix} 0 & 1 \\ \frac{k_i}{m_i} & \frac{d_i}{m_i} \end{bmatrix} \begin{bmatrix} x_i^1 \\ x_i^2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_i} \end{bmatrix} u_i \\ y_i & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_i^1 \\ x_i^2 \end{bmatrix} \end{array} \quad , i = 1, 2 \tag{5.29}$$

with damping coefficients $d_1 < 0$ in Σ_1 and $d_2 > 0$ in Σ_2 such that $d_1 + d_2 > 0$. Although Σ_1 is not passive, the interconnection $\Sigma_1 \|_{\mathrm{cl}} \Sigma_2$ is passive. Setting $d_1 = -1$, $d_2 = 2$ and $m_1 = m_2 = k_1 = k_2 = 1$, the quadratic storage function $V(x_1, x_2) = \frac{1}{2} \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is given by the numerically determined matrices

$$Q_{11} = \begin{bmatrix} 3.2856 & -5.3163 \\ -5.3163 & 8.6019 \end{bmatrix}, Q_{12} = \begin{bmatrix} -2.0306 & 0 \\ 3.2856 & 0 \end{bmatrix}, Q_{22} = \begin{bmatrix} 1.2250 & 0 \\ 0 & 0 \end{bmatrix}$$

□

Remark 5.10. Lyapunov stability is closely related to passivity. Indeed, a storage function $V(x)$ satisfying the dissipation inequality can serve as a Lyapunov function under certain conditions. In particular, if Σ is passive with storage function $V(x)$ and $V(x)$ has a strict (local) minimum at the equilibrium point $x^\star$, $x^\star$ is stable with Lyapunov function $L(x) = V(x) - V(x^\star)$, cf. [72]. Under this condition stability can then be specified in the same way as passivity, namely as the existence of a full nonlinear simulation relation between the system under consideration and the nonlinear system given by the Lyapunov condition for stability $\frac{d}{dt}V(x) \leq 0$. This holds in particular for closed interconnections where $u^T y = 0$. Thus, Proposition 5.9 can be translated into a result for *compositional stability analysis*. This reasoning is also valid for nonlinear systems as we will show in Corollary 5.13.

In the next step, we generalize the previous results to nonlinear input-affine systems Σ_i of the form (4.1). As an additional assumption, let $x_i = 0$ be an equilibrium of Σ_i, i.e.

$$f_i(0) = 0 \; . \tag{5.30}$$

Derived from the differential dissipation inequality, the Hill-Moylan conditions [33] will be used in the remainder as a characterization of passivity.

Proposition 5.11. *Let Σ be a nonlinear system of the form* (4.1) *and let $V(x)$ be a $\mathcal{C}^1$ storage function of Σ. Then Σ is passive (lossless) if and only if*

$$\begin{aligned} \frac{dV}{dx}(x)f(x) &\leq 0 (= 0) \\ \frac{dV}{dx}(x)g(x) &= h^T(x) \end{aligned} \tag{5.31}$$

We restate the classical result that negative feedback interconnections of passive systems are again passive using compositional reasoning.

Theorem 5.12. *For any two passive (lossless) nonlinear systems $\Sigma_i, i = 1, 2$, passivity (losslessness) is compositional under open negative feedback interconnection.*

Proof. We use again the differential dissipation inequality to prove compositionality of passivity. Assume the two systems $\Sigma_i, i = 1, 2$ are passive, then there exists a simulation relation $\mathcal{S}_1$ of Σ_i by Ξ_i,

$$\mathcal{S}_i := \{(x_i, \xi_i) \mid \xi_i = V(x_i)\} \tag{5.32}$$

where $V(x_i)$ is an unspecified storage function for system Σ_i such that

$$\dot{V}(x_1) \leq u_i^T y_i = u_i h_i^T(x_i) \tag{5.33}$$

The open feedback interconnection $\Sigma_1\|_o\Sigma_2$ is given by

$$\Sigma_1\|_o\Sigma_2: \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1) - g_1(x_1)h_2(x_2) \\ f_2(x_2) + g_2(x_2)h_1(x_1) \end{bmatrix} + \begin{bmatrix} g_1(x_1)e_1 \\ g_2(x_2)e_2 \end{bmatrix} \tag{5.34}$$
$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} h_1(x_1) \\ h_2(x_2) \end{bmatrix}$$

The simulation relation $\mathcal{S}$ of $\Sigma_1\|_o\Sigma_2$ by $\Xi_1\|_o\Xi_2$ is again constructed as

$$\mathcal{S} = \{((x_1, x_2), (\xi_1, x_2)) \mid \xi_1 + \xi_2 = V(x_1) + V(x_2)\} \tag{5.35}$$

since

$$\dot{\xi}_1 + \dot{\xi}_2 = \dot{V}_1(x_1) + \dot{V}_2(x_2) \leq (e_1 - y_2)^T y_1 + (e_2 + y_1)^T y_2 = e_1^T z_1 + e_2^T z_2 \tag{5.36}$$

which proves that $\Sigma_1\|_o\Sigma_2$ is passive. □

We can also formulate compositional reasoning for closed feedback interconnections as.

Corollary 5.13. *For any two passive nonlinear systems $\Sigma_i, i = 1, 2$, it holds that*

$$\left.\begin{array}{lll} S_1: & \Sigma_1 & \preccurlyeq & \Xi_1 \\ S_2: & \Sigma_2 & \preccurlyeq & \Xi_2 \end{array}\right\} \implies S: \quad \Sigma_1\|_{cl}\Sigma_2 \preccurlyeq \Xi_1\|_{cl}\Xi_2 \tag{5.37}$$

If we assume that the storage functions $V_i(x_i)$ of the nonlinear systems $\Sigma_i, i = 1, 2$, have a strict (local) minimum at $x_i^\star$, we can associate with the systems Ξ_i one-dimensional systems

$$\tilde{\Xi}_i: \quad \dot{\tilde{\xi}}_i \leq 0$$

Using the full nonlinear simulation relations

$$\tilde{S}_i := \{(x_i, \tilde{\xi}_i) \mid \tilde{\xi}_i = V(x_1) - V(x_1^\star)\}$$

Corollary 5.13 can then be interpreted as a result for compositional stability, namely that if the subsystems Σ_i are stable with Lyapunov functions $L(x_i) = V(x_1) - V(x_1^\star)$ then also their closed negative feedback interconnection is stable with Lyapnuov function $L(x_1, x_2) = L(x_1) + L(x_2)$. The converse, however, does not hold.
Theorem 5.12 states that feedback interconnections of passive systems are themselves passive. The converse implication has not yet been investigated in the literature but is equally valid. As shown for the linear case in Proposition 5.6, passivity as a compositional reasoning technique is complete. For the nonlinear case, we obtain the same result.

Theorem 5.14. *Let two nonlinear systems $\Sigma_i, i = 1, 2$, be given such that $\Sigma_1\|_o\Sigma_2$ is passive (lossless). Then also the systems Σ_i are passive (lossless).*

Proof. We will only prove the passive case, the same arguments hold for the lossless case. By Proposition 5.2, the interconnection $\Sigma_1\|_o\Sigma_2$ being passive is equivalent to the existence of a full simulation relation $\mathcal{S}$ of $\Sigma_1\|_o\Sigma_2$ by Ξ. Here, Ξ is given by

$$\Xi: \quad \dot{\xi} \leq e_1^T z_1 + e_2^T z_2 \tag{5.38}$$

and the simulation relation S is defined as

$$S = \{((x_1, x_2), \xi) \mid \xi = V(x_1, x_2)\} \tag{5.39}$$

with $V(x_1, x_2)$ an unspecified storage function of $\Sigma_1\|_o\Sigma_2$. Define

$$V_1(x_1) = V(x_1, 0) \quad , \quad V_2(x_2) = V(0, x_2) \tag{5.40}$$

as candidate storage functions for the component systems $\Sigma_i, i = 1, 2$. From (5.39) we obtain

$$\begin{aligned} \dot{\xi} \quad = \quad & \frac{\partial V}{\partial x_1}(x_1, x_2)\left(f_1(x_1) - g_1(x_1)h_2(x_2) + g_1(x_1)e_1\right) + \\ & \frac{\partial V}{\partial x_2}(x_1, x_2)\left(f_2(x_2) + g_2(x_2)h_1(x_1) + g_2(x_2)e_2\right) \leq \\ \leq \quad & e_1^T z_1 + e_2^T z_2 = e_1^T h_1(x_1) + e_2^T h_2(x_2) \end{aligned} \tag{5.41}$$

which is equivalent to

$$\begin{aligned} \frac{\partial V}{\partial x_1}(x_1, x_2)\left(f_1(x_1) - g_1(x_1)h_2(x_2)\right) + & \\ \frac{\partial V}{\partial x_2}(x_1, x_2)\left(f_2(x_2) + g_2(x_2)h_1(x_1)\right) \quad & \leq \quad 0 \end{aligned} \tag{5.42}$$

$$\frac{\partial V}{\partial x_1}(x_1, x_2)g_1(x_1) \quad = \quad h_1^T(x_1) \tag{5.43}$$

$$\frac{\partial V}{\partial x_2}(x_1, x_2)g_2(x_2) \quad = \quad h_2^T(x_2) \tag{5.44}$$

Plugging (5.43) and (5.44) in (5.42) results in

$$\begin{aligned} & \frac{\partial V}{\partial x_1}(x_1, x_2)f_1(x_1) - \underbrace{\frac{\partial V}{\partial x_1}(x_1, x_2)g_1(x_1)}_{=h_1^T(x_1)} h_2(x_2) + \frac{\partial V}{\partial x_2}(x_1, x_2)f_2(x_2) + \\ & \underbrace{\frac{\partial V}{\partial x_2}(x_1, x_2)g_2(x_2)}_{=h_2^T(x_2)} h_1(x_1) = \frac{\partial V}{\partial x_1}(x_1, x_2)f_1(x_1) + \frac{\partial V}{\partial x_2}(x_1, x_2)f_2(x_2) \leq 0 \end{aligned} \tag{5.45}$$

For $x_2 = 0$, (5.45) then becomes

$$\frac{\partial V}{\partial x_1}(x_1, 0)f_1(x_1) + \frac{\partial V}{\partial x_2}(x_1, 0)f_2(0) = \frac{\partial V}{\partial x_1}(x_1, 0)f_1(x_1) = \frac{dV_1}{dx_1}(x_1)f_1(x_1) \leq 0$$

since $f_2(0) = 0$ because of (5.30) while (5.43) becomes

$$\frac{\partial V}{\partial x_1}(x_1, 0)g_1(x_1) = \frac{dV_1}{dx_1}(x_1)g_1(x_1) = h_1^T(x_1) \tag{5.46}$$

Hence, $V_1(x_1) = V(x_1, 0)$ is a storage function for Σ_1 fulfilling the Hill-Moylan conditions (5.31). Similar arguments lead to Σ_2 being passive with storage function $V_2(x_2) = V(0, x_2)$. Consider now the relations $S_i, i = 1, 2$, of Σ_i by Ξ_i,

$$\begin{aligned} S_1 &:= \{(x_1, \xi_1) \mid \xi_1 = V_1(x_1)\} \\ S_2 &:= \{(x_2, \xi_2) \mid \xi_2 = V_2(x_2)\} \end{aligned} \tag{5.47}$$

with Ξ_i as in (5.4). By Proposition 4.6, $S_i, i = 1, 2$, are submanifolds since $\operatorname{rank}\begin{bmatrix} \frac{\partial S_i}{\partial x_i} & \frac{\partial S_i}{\partial \xi_i} \end{bmatrix} = 1$ for all $(x_i, \xi_i) \in S_i$. Since $V_i(x_i)$ is a storage function of Σ_i, invariance of the submanifolds S_i is guaranteed. Fullness follows directly since there exists for every x_i a $\xi_i = V_i(x_i)$ such that $(x_i, \xi_i) \in S_i$. Hence, S_i defines a full simulation relation of Σ_i by Ξ_i which is equivalent to Σ_i being passive. $\square$

Like in the linear case, an important implication of Theorem 5.12 is that whenever the open interconnection of two nonlinear systems is passive there exists a decoupled storage function

$$V(x_1, x_2) = V_1(x_1) + V_2(x_2) + c \tag{5.48}$$

where $V_i(x_i)$ are the storage functions of the components $\Sigma_i, i = 1, 2$ and $c \in \mathbb{R}$ a constant. For interconnections involving lossless components, we obtain even stronger results under accessibility assumptions.

Definition 5.15. Consider a nonlinear system Σ of the form (4.1). Then the *accessibility algebra* $\mathcal{C}$ is the smallest subalgebra of the Lie algebra of vector fields on $\mathcal{X}$ that contains f and all input vector fields $[g_1, \ldots, g_p] =: g$. Define $\mathcal{C}_0$ as the smallest subalgebra containing g and satisfying $[f, X] \in \mathcal{C}_0$ for all $X \in \mathcal{C}_0$.
Σ is *locally strongly accessible* if the reachable set $R_T^V(x_0) = \cup_{\tau \leq T} R^V(x_0, \tau)$,

$$\begin{aligned} R^V(x_0, T) &= \{(x \in \mathcal{X} \mid \exists u : [0, T] \to \mathcal{U} \text{ s. t. } x(t) \in V, 0 \leq t \leq T, \\ &\qquad x(0) = x_0, x(T) = x\} \end{aligned}$$

for all $x_0 \in \mathcal{X}$ contains a non-empty open set of $\mathcal{X}$ for all neighborhoods V of x_0 and any sufficiently small $T > 0$.
Σ is *reachable* from x_0 if $R_T^V(x_0) = \mathcal{X}$ for some $T \geq 0$.

As shown in [54], every element of the subalgebra $\mathcal{C}_0$ is a linear combination of repeated Lie brackets $[X_k, [X_{k-1}, [\ldots, [X_1, g] \ldots]]], k = 0, 1, \ldots$.

Proposition 5.16. *Let* Σ *be a nonlinear system of the form* (4.1). *If* Σ *is locally strongly accessible then* $\dim(\text{span}\{X(x_0) \mid X \in \mathcal{C}_0\}) = n = \dim\mathcal{X}$ *for* x_0 *in an open and dense subset of* $\mathcal{X}$.

We are now able to state the first result concerning the negative feedback interconnection of a passive and a lossless component.

Proposition 5.17. *Consider two nonlinear systems* $\Sigma_i, i = 1, 2$ *of the form* (4.1) *and let* Σ_1 *be passive and* Σ_2 *lossless. Assume that* Σ_1 *is locally strongly accessible. Then all storage functions* $V(x_1, x_2)$ *of the interconnection* $\Sigma_1\|_o\Sigma_2$ *are of the form* $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ *where* $V_1(x_1)$ *is a storage function of* Σ_1 *and* $V_2(x_2)$ *the unique storage function of* Σ_2. *The same also holds when interchanging* Σ_1 *and* Σ_2.

Proof. Since Σ_1 is passive and Σ_2 lossless, the interconnection $\Sigma_1\|_o\Sigma_2$ by Theorem 5.12 is also passive with storage function $V(x_1, x_2)$. Rewriting the dissipation inequality, this implies

$$L_{f_1}V + L_{g_1}V(e_1 - y_2) + L_{f_2}V + L_{g_2}V(e_2 + y_1) = e_1^T y_1 + e_2^T y_2 - W$$

with $W = W(x_1)$ a nonnegative function of x_1 or, equivalently,

$$\begin{aligned} L_{f_1}V + L_{f_2}V + W(x_1) &= 0 \\ L_{g_1}V &= h_1^T \\ L_{g_2}V &= h_2^T \end{aligned}$$

We claim that $L_X V$ is a function of x_1 only for all $X \in \mathcal{C}_0^1$. Clearly, $L_{g_1}V = h_1^T$ is a function of x_1. Moreover, $L_{[f_1,g_1]}V = L_{f_1}L_{g_1}V - L_{g_1}L_{f_1}V = L_{f_1}h_1^T + L_{g_1}L_{f_2}V + L_{g_1}W$ is a function of x_1 only since $L_{g_1}L_{f_2}V = L_{f_2}L_{g_1}V = L_{f_2}h_1^T = 0$. In fact, $L_{X_i}L_{X_j}V = L_{X_j}L_{X_i}V, (i,j) \in \{(1,2),(2,1)\}$ due to $[f_i, g_j] = 0$. Assume now that $L_{[X_k,[X_{k-1},[\ldots,[X_1,g]\ldots]]]}V$ is a function of x_1 only denoted by $R(x_1)$. To complete the induction step, consider first the case $X_{K+1} = g_1$. Then

$$\begin{aligned} L_{[g_1,[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]]}V = L_{g_1}L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}V - \\ L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}L_{g_1}V = L_{g_1}R(x_1) - L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}L_{g_1}V \\ = L_{g_1}R(x_1) - L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}h_1^T \end{aligned}$$

is a function of x_1 only since all $X_i, i = 1, 2, \ldots,$ depend on x_1 only. If $X_{k+1} = f_1$, then

$$\begin{aligned} L_{[f_1,[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]]}V = L_{f_1}L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}V - \\ L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}L_{f_1}V = L_{f_1}R(x_1) - L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}L_{f_1}V \\ = L_{f_1}R(x_1) + L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}(L_{f_2}V + W(x_1)) = \\ L_{f_1}R(x_1) + L_{f_2}L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}V + L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}W(x_1) \\ = L_{f_1}(x_1) + L_{f_2}R_{x_1} + L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}W(x_1) \end{aligned}$$

is also a function of x_1 only. Thus, $L_{\mathcal{C}_0^1}V$ is indeed a function of x_1 only, i.e.

$$\frac{\partial}{\partial x_2}\left\{L_{g_1}V, L_{[f_1,g_1]}V, \ldots, L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}V\right\} = 0 \tag{5.49}$$

Since Σ_1 is locally strongly accessible, $\dim\left(\text{span}\{X_1(x_1) \mid X_1 \in \mathcal{C}_0^1\}\right) = n_1$ for x_1 in an open and dense subset of $\mathcal{M}_1$. By continuity of $V(x_1,x_2)$, (5.49) thus implies that $\frac{\partial^2}{\partial x_1 \partial x_2}V(x_1,x_2) = 0$ and thus the storage function $V(x_1,x_2)$ of $\Sigma_1\|_{\text{o}}\Sigma_2$ is of the form $V(x_1,x_2) = V_1(x_1) + V_2(x_2)$ up to a constant. As a consequence of Theorem 5.14, $V_i(x_i)$ are storage functions of $\Sigma_i, i = 1,2$. Since Σ_2 is lossless, the storage function $V_2(x_2)$ is unique up to a constant. □

Next, we consider interconnections of lossless systems to obtain an even stronger result.

Proposition 5.18. *Consider two nonlinear systems $\Sigma_i, i = 1,2$ of the form* (4.1)*. Let $\Sigma_i, i = 1,2$, be lossless and locally strongly accessible. Then every storage function $V(x_1,x_2)$ of the interconnection $\Sigma_1\|_{\text{o}}\Sigma_2$ is of the form* (5.48)*.*

Proof. Observe first that by Theorems 5.12 and 5.14, $\Sigma_1\|_{\text{o}}\Sigma_2$ being lossless with storage function $V(x_1,x_2)$ is equivalent to both $\Sigma_i, i = 1,2$ being lossless with storage functions $V_i(x_i)$. Furthermore, $\Sigma_1\|_{\text{o}}\Sigma_2$ being lossless implies by (5.42) – (5.44) that

$$\begin{aligned} &L_{f_1}V(x_1,x_2) + L_{f_2}V(x_1,x_2) = 0, \\ &\qquad L_{g_1}V(x_1,x_2) = h_1^T(x_1), \quad L_{g_2}V(x_1,x_2) = h_2^T(x_2) \end{aligned} \tag{5.50}$$

We want to show that

$$L_{g_i}V, L_{[f_i,g_i]}V, \ldots, L_{[X_1^i,[X_2^i,\ldots,[X_k^i,X_{K+1}^i],\ldots]]}V, \quad i = 1,2, \tag{5.51}$$

are functions of x_i only for $i = 1,2$ and all $X_j^i, j \in k$ from the set $\{f_i, g_i\}$, $k \geq 1$. Clearly, $L_{g_i}V = h_i^T$ is a function of x_i only. The proof that also $L_{[X_1^i,[X_2^i,\ldots,[X_k^i,X_{K+1}^i],\ldots]]}V$ is a function of x_i only relies on the same arguments as used in the proof of Proposition 5.17. Hence, differentiation of (5.51) with respect to x_j yields

$$\begin{aligned} &\frac{\partial}{\partial x_j}\left\{L_{g_i}V, L_{[f_i,g_i]}V, \ldots, L_{[X_k,[X_{k-1},[\ldots,[X_1,g_i]\ldots]]]}V\right\} = 0, \\ &\qquad\qquad (i,j) \in \{(1,2),(2,1)\}\,. \end{aligned} \tag{5.52}$$

Since Σ_i are locally strongly accessible, $\mathcal{C}_0^i(x)$ has full rank for x in an open and dense subset of $\mathcal{M}_i$. Hence, 5.52 implies by continuity of $V(x_1,x_2)$ that $\frac{\partial^2 V}{\partial x_i \partial x_j} = 0, (i,j) \in \{(1,2),(2,1)\}$. Hence, any storage function $V(x_1,x_2)$ of $\Sigma_1\|_{\text{o}}\Sigma_2$ is of the form $V(x_1,x_2) = V_1(x_1) + V_2(x_2)$ where $V_i(x_i)$ by Theorem 5.14 is a storage function of Σ_i. □

A result similar to Proposition 5.18 can be obtained using arguments from passivity theory. In fact, it is a well-known fact that if $\Sigma_i, i = 1, 2$, is reachable from at least one point $x_i^\star$ it has a unique storage function $V_i(x_i)$. But then the open feedback interconnection $\Sigma_1\|_o\Sigma_2$ is reachable from $(x_1^\star, x_2^\star)$ using the inputs $e_1 = u_1 + h_2(x_2), e_2 = u_2 - h_1(x_1)$ and thus $\Sigma_1\|_o\Sigma_2$ has a unique storage function $V(x_1, x_2)$ as well. Theorem 5.12 tells us that $V(x_1, x_2)$ is given as the sum of the unique storage functions $V_i(x_i)$.

Finally, we present a passivity result for interconnections of an arbitrary system with a passive one.

Corollary 5.19. *Consider a nonlinear system Σ of the form* (4.1) *and two systems Ξ_2, Ξ of the form* (5.4). *Then the following statements hold:*

1. *If Ξ_2 is lossless then*

$$\Sigma\|_o\Xi_2 \preccurlyeq \Xi \quad \Longrightarrow \quad \exists\, \Xi_1 : \Sigma \preccurlyeq \Xi_1 \tag{5.53}$$

with Ξ_1 of the form (5.4).

2. *If Ξ_2 is passive and the simulation relation of $\Sigma\|_o\Xi_2$ by Ξ is such that*

$$\frac{\partial \xi}{\partial \xi_2} \geq 0, \xi = \xi(x, \xi_2), \tag{5.54}$$

then (5.53) *holds.*

Proof. Let Ξ be a system with state variable ξ such that $\dot{\xi} \leq e_1^T y_1 + e_2^T y_2$. According to (5.53) there exists a full simulation relation $\mathcal{S}$ of $\Sigma\|_o\Xi_2$ by Ξ. By Theorem 4.6, $\mathcal{S}$ can be written as

$$\mathcal{S} = \{(x, \xi_2, \xi) \mid \xi = \xi(x, \xi_2)\}\ . \tag{5.55}$$

Since (5.55) defines a nonlinear simulation relation, it has to hold for all $(x, \xi_2, \xi) \in \mathcal{S}$ and all e_1, e_2 that

$$\dot{\xi} = L_f\xi + L_g\xi(e_1 - y_2) + \frac{\partial \xi}{\partial \xi_2}\dot{\xi}_2 y_2 \leq e_1^T y_1 + e_2^T y_2 \tag{5.56}$$

Consider first the case that Ξ_2 is lossless. Then the system $\Sigma\|_o\Xi_2$ is given by

$$\Sigma\|_o\Xi_2 : \begin{array}{rcl} \dot{x} & = & f(x) - g(x)y_2 + g(x)e_1 \\ \dot{\xi}_2 & = & y_1^T y_2 + e_2^T y_2 \end{array} \tag{5.57}$$

Hence, (5.56) can be rewritten as

$$\dot{\xi} = L_f\xi + L_g\xi(e_1 - y_2) + \frac{\partial \xi}{\partial \xi_2}(y_1^T + e_2^T)y_2 \leq e_1^T y_1 + e_2^T y_2 \tag{5.58}$$

Since (5.56) has to hold for all $e_i, i = 1, 2,$, it follows that

$$\begin{aligned} L_f\xi - L_g\xi y_2 + \frac{\partial \xi}{\partial \xi_2} y_1^T y_2 &\leq 0 \\ L_g\xi &= h_1^T \\ \frac{\partial \xi}{\partial \xi_2} y_2 &= y_2 \end{aligned} \tag{5.59}$$

Conditions (5.59) are also obtained if Ξ_2 is passive and (5.54) holds. Indeed, (5.58) in this aces can be rewritten as

$$\begin{aligned} \dot{\xi} = L_f\xi + L_g\xi(e_1 - y_2) + \frac{\partial \xi}{\partial \xi_2}\dot{\xi}_2 y_2 \leq L_f\xi + L_g\xi(e_1 - y_2) + \frac{\partial \xi}{\partial \xi_2}(y_1^T + e_2^T)y_2 \\ \leq e_1^T y_1 + e_2^T y_2 \end{aligned}$$

The last line of (5.59) yields $\frac{\partial \xi}{\partial \xi_2} = 1$. Note that this is consistent with assumption (5.54) in the second case. Hence, the storage function ξ can be written as $\xi(x, \xi_2) = V(x) + \xi_2$. Substituting the last line of (5.59) into the first one results in

$$L_f V \leq 0, L_g V = h_1^T$$

which by Proposition 5.11 means that Σ is passive with storage function $V(x)$. Thus, Ξ_1 is given by

$$\Xi_1 : \xi_1 = V(x), \ \dot{\xi}_1 \leq u_1^T y_1$$

□

Remark 5.20. Corollary 5.19 states that if the open feedback interconnection of an arbitrary passive system with another arbitrary system Σ is passive, then Σ has to be passive as well. A similar result was presented in [15], namely if the *closed* negative feedback interconnection of an arbitrary passive system and Σ is stable then Σ must be passive.

5.3. Outlook

We showed in this chapter that passivity can be expressed as a nonlinear simulation relation between the system under consideration and the one-dimensional system given by the dissipation inequality. Based on this, we applied compositional analysis techniques to verify passivity properties of linear and nonlinear systems.
The close link between passivity and Lyapunov stability theory hints at further investigations to obtain similar result especially for closed feedback interconnections. In particular, assume-guarantee schemes could be applied to

check stability properties of nonlinear systems. Consider as an example two nonlinear systems $\Sigma_i, i = 1, 2$, and two systems Ξ_i of the form (5.4). Assume that both systems Σ_i can be stabilized by interconnection with $\Xi_j, (i, j) \in \{(1, 2), (2, 1)\}$, i.e.

$$\begin{aligned} \Sigma_1 \|_{\text{cl}} \Xi_2 &\preccurlyeq \Xi_1 \|_{\text{cl}} \Xi_2 \\ \Xi_1 \|_{\text{cl}} \Sigma_2 &\preccurlyeq \Xi_1 \|_{\text{cl}} \Xi_2 \end{aligned} \tag{5.60}$$

Under which conditions is the closed negative feedback interconnection of $\Sigma_1 \|_{\text{cl}} \Sigma_2$ stable,

$$\Sigma_1 \|_{\text{cl}} \Sigma_2 \overset{?}{\preccurlyeq} \Xi_1 \|_{\text{cl}} \Xi_2 \tag{5.61}$$

A similar example is mentioned in Remark 5.20, which shows that compositional reasoning for stability properties could have many practical applications, for example in robotics.

6
Decentralized control

6.1. Introduction

In this chapter we want to use the compositional reasoning techniques developed in the previous chapters to derive decentralized control schemes. As a first step, we specialize compositional and assume-guarantee reasoning to a decentralized setting. Provided the local controllers used in this set-up are such that the locally controlled subsystems satisfy certain specifications themselves the network of locally controlled plants is then guaranteed to satisfy a given global specification. In the second step, we combine compositional analysis techniques with conditions under which one can find controllers that render the closed loop system satisfy a given specification. In particular, we focus on the so-called sandwich conditions which have been derived as necessary and sufficient conditions for achievable simulation. We present two bottom-up schemes starting from conditions on the locally controlled plants and one top-down scheme based on a global sandwich condition. An important consequence of the latter result is that whenever there exists a global controller satisfying a global specification it can be replaced by local ones due to completeness of circular assume-guarantee reasoning.

6.2. Problem setting

In our decentralized control setting we distinguish between the following types of linear continuous-time systems. *Plant systems* are of the form

$$\Sigma_i : \quad \begin{array}{rcl} \dot{x}_i &=& A_i x_i + B_i u_i + G_i e_i \\ y_i &=& C_i x_i \\ z_i &=& H_i x_i \end{array} \tag{6.1}$$

where u_i, y_i are the pair of *interconnection* variables and e_i, z_i the pair of external *specification* variables. All variables are taken from vector spaces of appropriate dimensions, $x_i \in \mathcal{X}_i, u_i \in \mathcal{U}_i, e_i \in \mathcal{E}_i, y_i \in \mathcal{Y}_i, z_i \in \mathcal{Z}_i$.
A *controller system* Σ_C is a linear system without external variables,

$$\Sigma_C : \quad \begin{array}{rcl} \dot{x}_C &=& A_C x_C + B_C u_C \\ y_C &=& C_C x_C \end{array} \quad . \tag{6.2}$$

A *specification* Σ_Q defines the desired external behavior. Hence, Σ_Q does not have any interconnection variables and is given as

$$\Sigma_Q : \quad \begin{array}{rcl} \dot{x}_Q & = & A_Q x_Q + G_Q e_Q \\ z_Q & = & H_Q x_Q \end{array} . \tag{6.3}$$

6.3. Interconnections in control networks

The control systems defined in Section 6.2 can be interconnected in different ways. First, we discuss plant-controller interconnections for which the interconnection variables u_i, y_i are related by means of a permutation matrix.

Definition 6.1. Consider a plant system Σ_P of the form (6.1) and a controller system Σ_C. Then $\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C$ denotes the *plant-controller interconnection* with respect to the interconnection variables u, y and a permutation matrix Π,

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \quad , \quad \begin{bmatrix} u_P \\ y_P \end{bmatrix} = \Pi \begin{bmatrix} u_C \\ y_C \end{bmatrix} \tag{6.4}$$

The dynamics of the interconnected system $\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C$ are thus given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_P \\ \dot{x}_C \end{bmatrix} &= \begin{bmatrix} A_P & B_P \Pi_{12} C_C \\ 0 & A_C \end{bmatrix} \begin{bmatrix} x_P \\ x_C \end{bmatrix} + \begin{bmatrix} B_P \Pi_{11} \\ B_C \end{bmatrix} u_c + \begin{bmatrix} G_P \\ 0 \end{bmatrix} e_p \\ & \begin{bmatrix} C_P & -\Pi_{22} C_C \end{bmatrix} \begin{bmatrix} x_P \\ x_C \end{bmatrix} = \Pi_{21} u_c \\ z_P &= \begin{bmatrix} H_P & 0 \end{bmatrix} \begin{bmatrix} x_P \\ x_C \end{bmatrix} \end{aligned} \tag{6.5}$$

In particular, $\|_{u,y}$ denotes the special case where Π is the identity matrix.

Remark 6.2. Allowing for a permutation matrix Π in the definition of plant-controller interconnections gives more freedom for controller design. Standard feedback interconnection is included in this framework by taking $\Pi = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. By contrast, the special case $\Pi = I$ entails algebraic constraints on the state variables. In this case (6.5) can be written as DAE system in pencil form,

$$\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C : \quad \begin{aligned} E_{PC} \dot{x}_{PC} &= A_{PC} x_{PC} , x_{PC} \in \mathcal{V}^\star_{PC}, w_{PC} \in \mathcal{W}^\star_{PC} \\ w_{PC} &= H_{PC} x_{PC} . \end{aligned} \tag{6.6}$$

where $\mathcal{V}^{\star}_{PC}$ denotes the consistent subspace and $\mathcal{W}^{\star}_{PC}$ the admissible inputs as defined in Definition 3.33 and

$$E_{PC} = \text{diag}\{G_P^{\perp}, G_P^{\perp}, 0\},\; x_{PC} = \begin{bmatrix} x_P \\ x_C \\ u_C \end{bmatrix},\; w_{PC} = \begin{bmatrix} z_P \\ u_C \end{bmatrix} \tag{6.7}$$

$$A_{PC} = \begin{bmatrix} G_P^{\perp}A_P & G_P^{\perp}B_P\Pi_{12}C_C & B_P\Pi_{11} \\ 0 & A_C & B_C \\ C_P & -\Pi_{22}C_C & -\Pi_{21} \end{bmatrix},\; H_{PC} = \begin{bmatrix} H_P & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

In a decentralized control setting the overall plant is usually given as an interconnection of subsystems. The topology of the global system model is determined by the type of interconnection between the individual components. In the remainder of this chapter, we consider *series* of feedback interconnections with respect to the external variables e_i, z_i as the standard interconnection between plant systems and specifications.

Definition 6.3. Consider k systems $\Sigma_i, i = 1, \ldots, k$ of the form (6.1) with external variables e_i, z_i and interconnection variables u_i, y_i. Then define the *series interconnection* $\Sigma_1 \| \ldots \| \Sigma_k$ with respect to the external variables e, z using feedback interconnections as follows:

$$\begin{array}{llll} z_i^- = e_{i-1}^+ , & z_i^+ = e_{i+1}^- & , & i = 2, \ldots, k-1 \\ e_i^- = z_{i-1}^+ , & e_i^+ = z_{i+1}^- & & \end{array} \tag{6.8}$$

$$\begin{array}{llllll} z_1^- = z_1 , & z_1^+ = e_2^- & , & z_k^- = e_{k-1}^+ , & z_k^+ = z_k \\ e_1^- = e_1 , & e_1^+ = z_2^- & & e_k^- = z_{k-1}^+ , & e_k^+ = e_k \end{array}$$

The matrices G_i and H_i corresponding to the external inputs are partitioned accordingly into submatrices

$$G_i = \begin{bmatrix} G_i^+ \\ G_i^- \end{bmatrix} , \quad H_i = \begin{bmatrix} H_i^+ \\ H_i^- \end{bmatrix} , i = 1, \ldots, k \,. \tag{6.9}$$

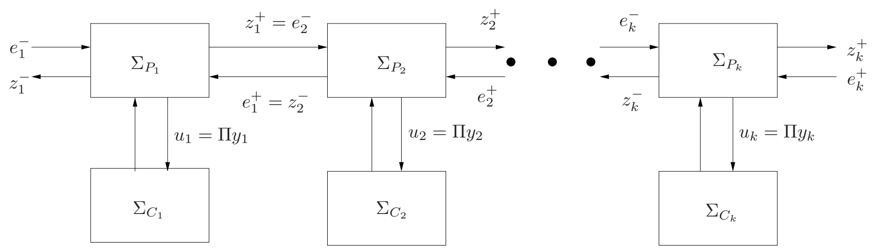

Figure 6.1.: Series interconnection $\left(\Sigma_{P_1} \|^{\Pi}_{u,y} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|^{\Pi}_{u,y} \Sigma_{C_1}\right) k$.

Remark 6.4. Considering series of feedback interconnections allows us to specialize the results of Chapter 3 to the decentralized setting. However, these results are valid in more generality to treat networks with other topologies, see also Remark 3.23.

6.4. Compositional analysis in the decentralized setting

In this section we will lay the foundation to analyze decentralized control problems using compositional analysis techniques based on simulation relations.

6.4.1. Simulation theory in the decentralized setting

In the following the notation in the definition of (bi)simulation relations is adjusted to be applicable in a decentralized setting. To that end, we require in the definition of a (bi)simulation relation that the external variables e_i, z_i remain equal whereas the interconnection variables u_i, y_i – if existent – are treated like disturbances.

Definition 6.5. A linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2, $\Sigma_i, i = 1, 2$, of the form (6.1), if it satisfies the following properties: Take any $(x_{10}, x_{20}) \in S$ and any joint external input function $e(\cdot) = e_1(\cdot) = e_2(\cdot)$. Then for any input function $u_1(\cdot)$ there exists an input function $u_2(\cdot)$ such that the resulting state trajectories $x_1(\cdot)$ and $x_2(\cdot)$, starting at $x_i(0) = x_{i0}$, satisfy

$$\begin{aligned} (i):& \qquad (x_1(t), x_2(t)) \in S \ \forall t \geq 0 \\ (ii):& \qquad z_1(t) = z_2(t) \ \forall t \geq 0 \end{aligned} \tag{6.10}$$

A simulation relation S is called full and denoted by $\Sigma_1 \preccurlyeq \Sigma_2$ if the projection on the first state component covers the whole state space, $\Pi_{\mathcal{X}_1} S = \mathcal{X}_1$.
A bisimulation relation R between Σ_1 and Σ_2, $\Sigma_i, i = 1, 2$, of the form (6.1), is a linear subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ with the following property: R defines a simulation relation of Σ_1 by Σ_2 and $R^{-1} := \{(x_2, x_1) \mid (x_1, x_2) \in R\}$ defines a simulation relation of Σ_2 by Σ_1. Moreover, R is full if $\Pi_{\mathcal{X}_i} R = \mathcal{X}_i, i = 1, 2$, which will be denoted by $\Sigma_1 \approx \Sigma_2$.

Algebraic characterizations and algorithms to compute (bi)simulation relations are immediately translated from the results in Chapter 2.

6.4.2. Decentralized control using compositional analysis techniques

We want to investigate control networks consisting of interconnections of arbitrarily (yet finitely) many systems. In our setting, the global plant system Σ_P is considered to be a series interconnection of component systems $\Sigma_{P_i}, i = 1, \ldots, k$, of the form (6.1),

$$\Sigma_P := \Sigma_{P_1} \| \ldots \| \Sigma_{P_k} \; . \tag{6.11}$$

The global specification, denoted by Σ_Q, is assumed to be decomposable into local subspecifications $\Sigma_{Q_i}, i = 1, \ldots, k$, of the form (6.3) corresponding to the plant subsystems,

$$\Sigma_Q := \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k} \; . \tag{6.12}$$

Making available the results obtained in Section 3.2.4 for compositional analysis of k systems will allow us to formulate decentralized control schemes. Note that in contrast to Section 3.2.4 the variables e_i, z_i will be used for feedback interconnections of plant systems and/or specifications. The only other plant variables u_i, y_i are exclusively used for plant-controller interconnections and are therefore not available for compositions between plants and specifications. Under these circumstances the results of Section 3.2.4 can be specialized immediately to a decentralized setting. We first state as a corollary of Theorem 3.24 that series of plant-controller interconnections are compositional.

Corollary 6.6. *Consider k plant-controller interconnections $\Sigma_{P_i} \|^{\Pi}_{u,y} \Sigma_{C_i}, i = 1, \ldots, k$, of the form* (6.5) *and k specifications Σ_{Q_i} of the form* (3.1). *Then compositional reasoning is sound for series interconnections of k control systems, i.e.*

$$\begin{array}{rcl} \forall\, i = 1, \ldots, k :\; \Sigma_{P_i} \|^{\Pi}_{u,y} \Sigma_{C_i} & \preccurlyeq & \Sigma_{Q_i} \\ & \Longrightarrow & \\ \left(\Sigma_{P_1} \|^{\Pi}_{u,y} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|^{\Pi}_{u,y} \Sigma_{C_k}\right) & \preccurlyeq & \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k} \end{array} \tag{6.13}$$

Corollary 6.6 represents our first decentralized control scheme: Given local controllers $\Sigma_{C_i}, i = 1, 2, \ldots$, that satisfy the local specifications Σ_{Q_i}, the global control network consisting of series interconnections of locally controlled plants is guaranteed to fulfill the global specification given itself by a series interconnection of local specifications.
A similar scheme can be derived on the basis of circular assume-guarantee reasoning. We therefore specialize Theorem 3.26 to the decentralized setting.

Corollary 6.7. *Consider $k \geq 2$ plant-controller interconnections $\Sigma_{P_i} \|^{\Pi}_{u,y} \Sigma_{C_i}, i = 1, \ldots, k$, of the form* (6.5) *and k corresponding specifications Σ_{Q_i} of the form* (6.3).

Let k circularly dependent conditions

$$
\begin{aligned}
S_I :&\quad \left(\Sigma_{P_1}\|^{\Pi}_{u,y}\Sigma_{C_1}\right)\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k} &\preccurlyeq&\quad \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k}\\
S_{II} :&\quad \Sigma_{Q_1}\|\left(\Sigma_{P_2}\|^{\Pi}_{u,y}\Sigma_{C_2}\right)\|\ldots\|\Sigma_{Q_k} &\preccurlyeq&\quad \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k}\\
\vdots &\quad\quad \vdots & \vdots &\quad\quad \vdots\\
S_k :&\quad \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\left(\Sigma_{P_k}\|^{\Pi}_{u,y}\Sigma_{C_k}\right) &\preccurlyeq&\quad \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k}
\end{aligned}
\tag{6.14}
$$

be satisfied. Then the global interconnected plant

$$\Sigma_P := \left(\Sigma_{P_1}\|^{\Pi}_{u,y}\Sigma_{C_1}\right)\|\ldots\|\left(\Sigma_{P_k}\|^{\Pi}_{u,y}\Sigma_{C_k}\right)$$

fulfills the global specification $\Sigma_Q := \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k}$, that is

$$S:\quad \Sigma_P \quad\preccurlyeq\quad \Sigma_Q \tag{6.15}$$

Moreover, if (6.14) *holds with bisimilarity then* (6.15) *also holds with bisimilarity.*

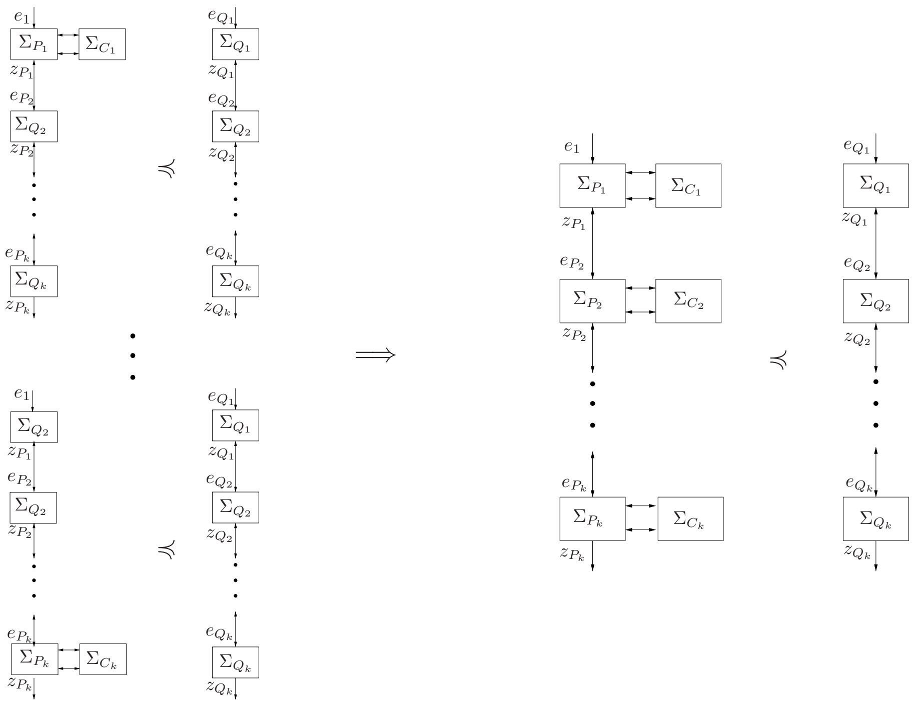

Figure 6.2.: Decentralized control scheme based on circular assume-guarantee reasoning.

Figure 6.4.2 depicts the decentralized control scheme based on Corollary 6.7. The conditions $S_i, i = I, II, \ldots, k$, require that the global specification Σ_Q is satisfied in each case for a network including the locally controlled plant $\Sigma_{P_i} \|_{u,y}^{\Pi} \Sigma_{C_i}$ assuming that the other locally controlled plants satisfy their individual specifications $\Sigma_{Q_j}, j = I, II, \ldots, k, j \neq i$. The k conditions S_i are therefore circularly dependent.
It is also possible to combine conditions of the form (6.13) and (6.14) in a triangular proof rule to obtain a decentralized control scheme based on non-circular assume-guarantee reasoning. Soundness is always ensured due to compositionality of series interconnections and transitivity of simulation. Not stating this formally, we provide a simple example instead to illustrate this point.

Example 6.8. Consider three plant systems $\Sigma_{P_i}, i = 1, 2, 3$, and three specifications Σ_{Q_i}. Let local controllers $\Sigma_{C_i}, i = 1, 2, 3$, be given such that the following conditions hold:

$$\begin{array}{rrcl} S_1 : & \Sigma_{P_1} \|_{u,y}^{\Pi_1} \Sigma_{C_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_{II} : & \Sigma_{Q_1} \| \left(\Sigma_{P_2} \|_{u,y}^{\Pi_2} \Sigma_{C_2} \right) & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \\ S_{III} : & \Sigma_{Q_1} \| \Sigma_{Q_2} \| \left(\Sigma_{P_3} \|_{u,y}^{\Pi_3} \Sigma_{C_3} \right) & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \end{array} \tag{6.16}$$

Combining S_1 and S_{II} by interconnecting the systems involved in S_1 with Σ_{S_2} yields

$$S_{1,II} : \quad \left(\Sigma_{P_1} \|_{u,y}^{\Pi_1} \Sigma_{C_1} \right) \| \left(\Sigma_{P_2} \|_{u,y}^{\Pi_2} \Sigma_{C_2} \right) \quad \preccurlyeq \quad \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{6.17}$$

while by the same reasoning, $S_{1,II}$ and S_{III} result in

$$S : \left(\Sigma_{P_1} \|_{u,y}^{\Pi_1} \Sigma_{C_1} \right) \| \left(\Sigma_{P_2} \|_{u,y}^{\Pi_2} \Sigma_{C_2} \right) \| \left(\Sigma_{P_3} \|_{u,y}^{\Pi_3} \Sigma_{C_3} \right) \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \tag{6.18}$$

Finally, as a special case of Theorem 3.31, circular assume-guarantee reasoning is also complete in the decentralized setting.

Corollary 6.9. *Consider k linear systems $\Sigma_{P_i}, i = 1, \ldots, k$, and k specifications Σ_{Q_i}, each of the form* (6.1), (6.3), (6.5) *or* (6.7). *Assume that*

$$\Sigma_{P_1} \| \ldots \| \Sigma_{P_k} \quad \preccurlyeq \quad \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k} \tag{6.19}$$

Then there also exist full simulation relations $S_i, i = 1, \ldots, k$, of

$$\Sigma_{Q_1} \| \ldots \| \Sigma_{Q_{i-1}} \| \Sigma_{P_i} \| \Sigma_{Q_{i+1}} \| \ldots \| \Sigma_{Q_k}$$

by

$$\Sigma_Q = \Sigma_{Q_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{Q_k}$$

6.5. Achievable simulations

In this section, we will discuss achievable simulations, i.e., given a plant Σ_P and a specification Σ_Q, under which conditions does there exist a controller Σ_C and a permutation matrix Π such that the plant-controller interconnection $\Sigma_P \|^{\Pi}_{u,y} \Sigma_C$ fulfills the desired specification Σ_Q. To obtain these conditions, we collect some basic facts about plant-interconnections of systems.
For every plant system Σ_i, define the *associated system* Σ_{N_i} by setting the interconnection variables to zero, $u_i = y_i \equiv 0$:

$$\Sigma_{N_i}: \quad \begin{aligned} \dot{x}_{N_i} &= A_i x_{N_i} + G_i e_{N_i} \\ z_{N_i} &= H_i x_{N_i} \\ C_i x_{N_i} &= 0 \end{aligned} \tag{6.20}$$

This entails algebraic constraints on the state variables since $x_{N_i} \in \ker C_i$.
The *null system* Σ_0 has all variables set to zero,

$$\Sigma_0: \; x_0 = 0, y_0 = u_0 = e_0 = z_0 = 0 \tag{6.21}$$

Proposition 6.10. *The system Σ_{N_P} associated with Σ_P is bisimilar to the plant-controller interconnection of Σ_P with the null system Σ_0.*

$$\Sigma_{N_P} \approx \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_0 \tag{6.22}$$

Proof. The interconnection $\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_0$ is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_P \\ \dot{x}_0 \end{bmatrix} &= \begin{bmatrix} A_P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_P \\ x_0 \end{bmatrix} + \begin{bmatrix} G_P \\ 0 \end{bmatrix} e_P \\ 0 &= \begin{bmatrix} C_P & 0 \end{bmatrix} \begin{bmatrix} x_P \\ x_0 \end{bmatrix} \\ z_P &= \begin{bmatrix} H_P & 0 \end{bmatrix} \begin{bmatrix} x_P \\ x_0 \end{bmatrix} \end{aligned} \tag{6.23}$$

which is an equivalent representation of Σ_{N_P} as in (6.20). Hence, the relation

$$S := \{(x_N, (x_P, x_0)) \mid x_N = x_P\} \tag{6.24}$$

defines a full bisimulation relation between Σ_{N_P} and $\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_0$. □

Proposition 6.11. *The null system is simulated by any other system, i.e. for any Σ_P*

$$\Sigma_0 \preccurlyeq \Sigma_P \tag{6.25}$$

Proof. The simulation relation S of Σ_0 by Σ_P is given by setting $x_Q = 0$,

$$S = \{(x_0, x_P) \mid x_P = 0\} \tag{6.26}$$

□

Proposition 6.12. *For any given linear system Σ_P of the form (6.1) and any controller system Σ_C it holds that*

$$\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C \preccurlyeq \Sigma_P \tag{6.27}$$

Proof. The plant-controller interconnection introduces a constraint on the state variables of Σ_P. Therefore,

$$S = \{(x_P, x_C), \bar{x}_P) \,|\, (x_P, x_C) \in \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C, x_P = \bar{x}_P\} \tag{6.28}$$

defines a simulation relation of $\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C$ by Σ_P. Indeed, setting $x_P = \bar{x}_P$ and taking $\bar{u} = \Pi_{12} C_C x_C + \Pi_{11} u$ yields

$$\dot{x}_P = A_P x_P + B_P \left(\Pi_{12} C_C x_C + \Pi_{11} u\right) + G_P e = \dot{\bar{x}}_P = A_P \bar{x}_P + B_P \bar{u} + G_P e$$

as well as $H_P x_P = H_P \bar{x}_P$. □

As a corollary of Theorem 3.38, compositional reasoning also holds for plant-controller interconnections.

Corollary 6.13. *Given two plants $\Sigma_{P_i}, i = 1, 2,$ of the form* (6.1) *and two controllers Σ_{C_i} of the form* (6.2), *plant-controller interconnection is compositional,*

$$\left.\begin{array}{ccc} \Sigma_{P_1} & \preccurlyeq & \Sigma_{P_2} \\ \Sigma_{C_1} & \preccurlyeq & \Sigma_{C_2} \end{array}\right\} \implies \Sigma_{P_1} \|_{u,y} \Sigma_{C_1} \preccurlyeq \Sigma_{P_2} \|_{u,y} \Sigma_{C_2} \tag{6.29}$$

6.5.1. The canonical controller

The canonical controller Σ_{can} has been introduced by van der Schaft in a behavioral setting [73]. An analogous definition for input-state-output systems was given in [78] interconnecting the plant with its specification through the external variables e and z.

Definition 6.14. The canonical controller for a plant system Σ_P and a specification Σ_Q is defined as

$$\Sigma_{\text{can}} := \Sigma_P \overset{I}{\underset{e,z}{\|}} \Sigma_Q \ , \tag{6.30}$$

i.e., by setting

$$e_P = e_Q, \qquad z_P = z_Q \tag{6.31}$$

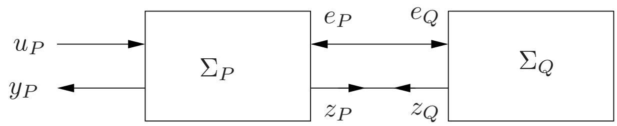

Figure 6.3.: The canonical controller $\Sigma_{\text{can}} = \Sigma_P \|_{\text{can}} \Sigma_Q$.

6.5.2. Sandwich conditions for achievable simulations

The decentralized control scheme presented in the following relies on checkable conditions for the existence of a controller Σ_C for a given plant Σ_P and a specification Σ_Q. The result presented here is formulated in terms of simulation relations and was first shown in [78].

Theorem 6.15. *For a given plant system Σ_P and a specification Σ_Q, the following statements hold*

(i): $\Sigma_Q \preccurlyeq \Sigma_P \implies \exists\, \Sigma_C,\, \Pi:\ \Sigma_Q \preccurlyeq \Sigma_P \|_{u,y}^{\Pi} \Sigma_C$

(ii): $\Sigma_{N_P} \preccurlyeq \Sigma_Q \implies \exists\, \Sigma_C,\, \Pi:\ \Sigma_P \|_{u,y}^{\Pi} \Sigma_C \preccurlyeq \Sigma_Q$

(iii): $\Sigma_{N_P} \preccurlyeq \Sigma_Q \preccurlyeq \Sigma_P \implies \exists\, \Sigma_C,\, \Pi:\ \Sigma_P \|_{u,y}^{\Pi} \Sigma_C \approx \Sigma_Q$

(iv): $\forall\, \Sigma_C\ ,\Pi\ :\ \Sigma_P \underset{u,y}{\overset{\Pi}{\|}} \Sigma_C \approx \Sigma_Q \implies \Sigma_{N_P} \preccurlyeq \Sigma_Q \preccurlyeq \Sigma_P$

Proof. The proof of (i) – (iii) follows the lines of [78].
(i): Consider the canonical controller $\Sigma_C = \Sigma_{\text{can}}$. Since there exists a full simulation relation S_{QP} of Σ_Q by Σ_P, we know that for every (x_Q, x_P) there exists a joint input $e = e_Q = e_P$ such that $z_s = z_P$. This ensures also that that the canonical controller has at least one state $(x_P, \bar{x}_P, x_s) \in \Sigma_P \|_{u,y}^{\Pi} \Sigma_{\text{can}}$ as it contains as states all the pairs $(x_P, x_Q) \in S_{QP}$. Take now any state $x_Q \in \Sigma_Q$. Due to S_{QP}, there exists a x_P such that $(x_P, x_P, x_Q) \in \Sigma_P \|_{u,y} \Sigma_{\text{can}} = \Sigma_P \|_{u,y} \left(\Sigma_P \underset{e,z}{\overset{I}{\|}} \Sigma_P \right)$ such that for every joint $e = e_p = e_{\Sigma_P \|_{u,y} \Sigma_{\text{can}}}$ the outputs are equal, that is $z_Q = H_Q x_Q = H_P x_P = z_{\Sigma_P \|_{u,y} \Sigma_{\text{can}}}$.
(ii): We want to show that by using the canonical controller there indeed exists a full simulation relation of $\Sigma_P \|_{u,y}^{\Pi} \Sigma_{\text{can}}$ by Σ_Q, i.e. for any $(x_P, \bar{x}_P, x_Q) \in \Sigma_P \|_{u,y}^{\Pi} \Sigma_{\text{can}}$ there exists a state $\bar{x}_Q \in \Sigma_Q$ such that $z_{\Sigma_P \|_{u,y}^{\Pi} \Sigma_{\text{can}}} = H_P x_P = H_x \bar{x}_Q = z_Q$. Observe first that for any state $(x_P, \bar{x}_P, x_Q) \in \Sigma_P \|_{u,y}^{\Pi} \Sigma_{\text{can}}$ it holds that $x_P - \bar{x}_P \in \Sigma_{N_P}$ since the plant-controller interconnection forces $C_P x_P = C_P \bar{x}_P$. Since all simulation relations considered here are linear subspaces, we can rewrite

$$(x_P, \bar{x}_P, x_s) = (\bar{x}_P + x_N, \bar{x}_P, x_Q) \in \Sigma_P \|_{u,y}^{\Pi} \Sigma_{\text{can}},\ x_N \in \Sigma_{N_P} \tag{6.32}$$

Moreover, since there exists a full simulation relation S_{NS} of Σ_{N_P} by Σ_Q, for every $x_N \in \Sigma_{N_P}$ there exists a $\bar{x}_Q \in \Sigma_Q$ such that $H_{N_P} x_N = H_Q \bar{x}_Q$. Consider now the state $x_Q + \bar{x}_Q \in \Sigma_Q$. Then the pair of states $(x_P, \bar{x}_P, x_s, x_Q + \bar{x}_Q)$ can be written as

$$(x_P, \bar{x}_P, x_s, x_Q + \bar{x}_Q) = (\bar{x}_P, \bar{x}_P, x_Q, x_Q) + (x_N, 0, 0, \bar{x}_Q) \tag{6.33}$$

where $(x_N, \bar{x}_Q) \in S_{NS}$ and $(\bar{x}_P, x_Q) \in \Sigma_{\text{can}}$. Thus, $H_{N_P} x_N = H_Q \bar{x}_Q$ and $H_P \bar{x}_P = H_Q x_s$ and therefore

$$H_P x_P = H_Q(x_Q + \bar{x}_Q) \ , \tag{6.34}$$

which proves the claim.
(iii): Combining the statements (i) and (ii) for the same Σ_C and Π yields $\Sigma_Q \preccurlyeq \Sigma_P \|_{u,y}^{\Pi} \Sigma_C \preccurlyeq \Sigma_Q$ and thus $\Sigma_Q \approx \Sigma_P \|_{u,y}^{\Pi} \Sigma_C$.
(iv): By Proposition 6.11, Σ_0 is simulated by any other system, so $\Sigma_0 \preccurlyeq_{e,z} \Sigma_C$. Moreover, Proposition 6.10 states that $\Sigma_{N_P} \preccurlyeq_{e,z} \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_0$. Since simulation is reflexive and plant-controller interconnection is compositional, we therefore conclude

$$\Sigma_{N_P} \preccurlyeq \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_0 \preccurlyeq \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C \preccurlyeq \Sigma_Q \tag{6.35}$$

and hence, $\Sigma_{N_P} \preccurlyeq \Sigma_Q$. Moreover, since $\Sigma_Q \approx \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C$, Proposition 6.12 yields

$$\Sigma_Q \preccurlyeq \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C \preccurlyeq \Sigma_P \tag{6.36}$$

□

6.6. Decentralized control and achievable simulation

Theorem 6.15 gives conditions for the existence of a controller for a given plant and specification, and a constructive procedure to compute such a controller. Combining sandwich conditions with compositional analysis techniques from Section 6.4 yields decentralized control schemes that include existence conditions for controllers guaranteed to satisfy the specification requirements. Like in (6.11), the overall plant Σ_P is given as a series of feedback interconnections of k subsystems Σ_{P_i}. Accordingly, the global specification Σ_Q is assumed to be given as a series of feedback interconnections of k subspecifications as in (6.12). We present two approaches to solve the control problem defined as follows:

Find necessary and sufficient conditions under which there exists a control strategy such that the global closed loop system satisfies the global specification.

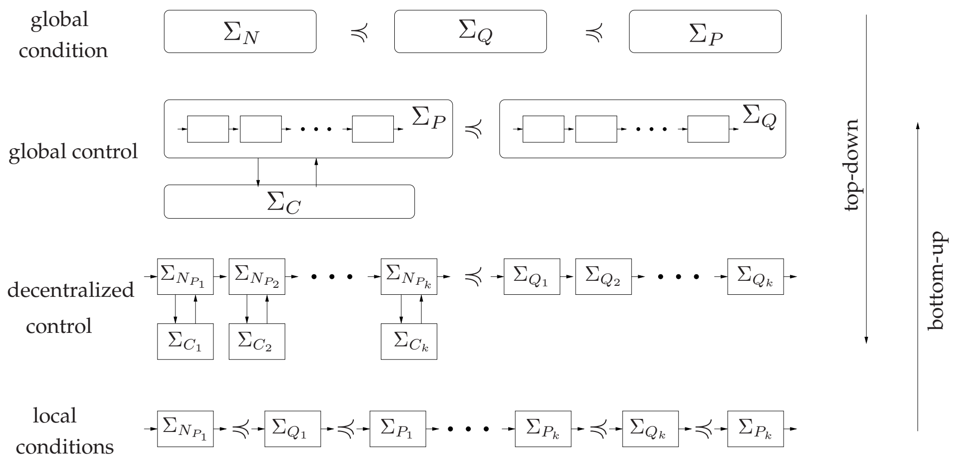

Figure 6.4.: Bottom-up and top-down schemes.

6.6.1. Bottom-up schemes using local sandwich conditions

As depicted in Figure 6.4, a *bottom-up* scheme uses local conditions for achievable simulation. These conditions ensure the existence of local controllers Σ_{C_i} for each component Σ_{P_i} of the global plant such that the overall decentralized control network fulfills the global specification Σ_Q. The first bottom-up scheme we present here uses soundness of compositional reasoning for k systems as stated in Corollary 6.6.

Theorem 6.16. *Consider a global plant system Σ_P of the form* (6.11). *Let $\Sigma_{N_{P_i}}, i = 1, \ldots, k$, be associated to the plant components Σ_{P_i}. Consider a corresponding specification Σ_Q be of the form* (6.12).

1. *If the local conditions*

$$\Sigma_{N_{P_i}} \preccurlyeq \Sigma_{Q_i} \; \forall i = 1, \ldots, k \tag{6.37}$$

are fulfilled, there exist local controllers Σ_{C_i} and permutation matrices Π_i such that the series interconnections fulfills the global specification, i.e.

$$\begin{gathered} \exists \; \Sigma_{C_i}, \Pi_i, \; i = 1, \ldots, k : \\ \left(\Sigma_{P_1} \|_{u,y}^{\Pi_1} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|_{u,y}^{\Pi_k} \Sigma_{C_k}\right) \preccurlyeq \Sigma_Q \end{gathered} \tag{6.38}$$

2. *If*

$$\Sigma_{N_{P_i}} \preccurlyeq \Sigma_{Q_i} \preccurlyeq \Sigma_{P_i} \; \forall i = 1, \ldots, k \tag{6.39}$$

then there exist local controllers Σ_{C_i} and permutation matrices Π_i such that

$$\begin{gathered} \exists \; \Sigma_{C_i}, \Pi_i, \; i = 1, \ldots, k : \\ \left(\Sigma_{P_1} \|_{u,y}^{\Pi_1} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|_{u,y}^{\Pi_k} \Sigma_{C_k}\right) \approx \Sigma_Q \end{gathered} \tag{6.40}$$

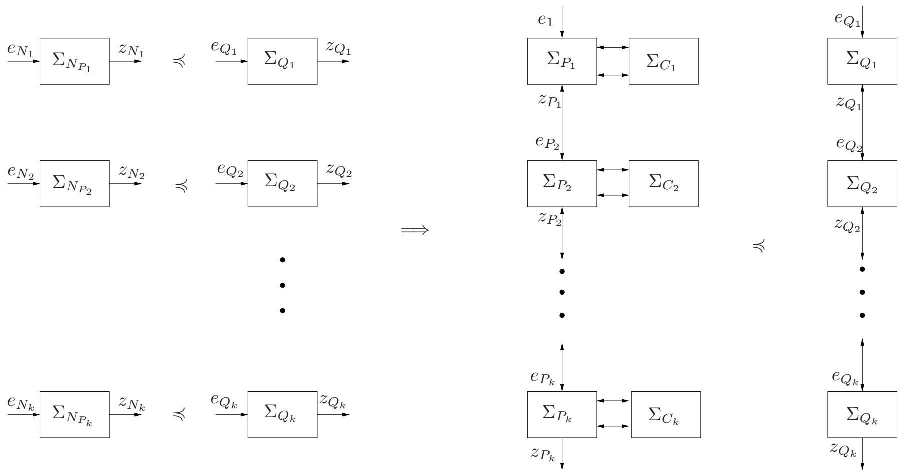

Figure 6.5.: Bottom-up decentralized control scheme combining local sandwich conditions and compositionality.

Proof. By Theorem 6.15 (i), the local conditions (6.37) guarantee the existence of local controllers such that

$$\Sigma_{P_i}\|_{u,y}^{\Pi_i}\Sigma_{C_i} \preccurlyeq \Sigma_{Q_i} \ \forall i = 1, \ldots, k \tag{6.41}$$

Compositionality for k plant-controller interconnections as stated in Corollary 6.6 then yields the desired result (6.38) for series interconnections.
If instead (6.39) holds, then by Theorem 6.15 (ii) the global controlled plant $\left(\Sigma_{P_1}\|_{u,y}^{\Pi_1}\Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k}\|_{u,y}^{\Pi_k}\Sigma_{C_k}\right)$ is bisimilar to the global specification Σ_Q. □

Next, we introduce a bottom-up scheme relying on soundness of circular assume guarantee reasoning, cf. Corollary 6.7. Figure 6.6.1 illustrates that the conditions for achievable simulation involve interconnections of plant subsystem $\Sigma_{P_i}, i = 1, \ldots, k$ with subspecifications $\Sigma_{Q_j}, j = 1, \ldots, k, j \neq i$, denoted by Σ_P^i and given as

$$\Sigma_P^i \quad := \quad \Sigma_{Q_1}\| \ldots \|\Sigma_{Q_{i-1}}\|\Sigma_{P_i}\|\Sigma_{Q_{i+1}} \ldots \|\Sigma_{Q_k} \ i = 1, \ldots, k \tag{6.42}$$

Associated with Σ_P^i are the systems

$$\Sigma_{N_P}^i \quad := \quad \Sigma_{Q_1}\| \ldots \|\Sigma_{Q_{i-1}}\|\Sigma_{N_{P_i}}\|\Sigma_{Q_{i+1}} \ldots \|\Sigma_{Q_k} \ i = 1, \ldots, k \tag{6.43}$$

Theorem 6.17. *Let the plant system Σ_P be of the form* (6.11) *and the corresponding specification Σ_Q be given as in* (6.12). *Consider k global systems $\Sigma_P^i, i = 1, \ldots, k$, and their associated systems $\Sigma_{N_P}^i$ as defined in* (6.42) *and* (6.43), *respectively. Then the following holds:*

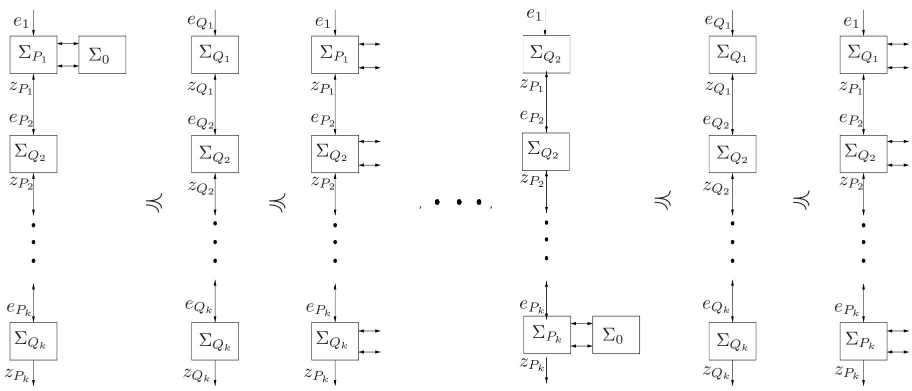

Figure 6.6.: Sandwich conditions 2 for bottom-up scheme based on circular assume-guarantee reasoning

1. *If*

$$\Sigma^i_{N_P} \quad \preccurlyeq \quad \Sigma_Q \ , \quad i = 1, \dots, k \ , \tag{6.44}$$

then there exist local controllers $\Sigma_{C_i}, i = 1, \dots, k$, *and permutation matrices* Π_i *such that*

$$\left(\Sigma_{P_1} \|^{\Pi_1}_{u,y} \Sigma_{C_1}\right) \| \dots \| \left(\Sigma_{P_k} \|^{\Pi_k}_{u,y} \Sigma_{C_k}\right) \quad \preccurlyeq \quad \Sigma_Q \tag{6.45}$$

2. *If*

$$\Sigma^i_{N_P} \quad \preccurlyeq \Sigma_Q \quad \preccurlyeq \Sigma^i_P \ , \quad i = 1, \dots, k \ , \tag{6.46}$$

then there exist local controllers Σ_{C_i} *and permutation matrices* Π_i *such that*

$$\left(\Sigma_{P_1} \|^{\Pi_1}_{u,y} \Sigma_{C_1}\right) \| \dots \| \left(\Sigma_{P_k} \|^{\Pi_k}_{u,y} \Sigma_{C_k}\right) \quad \approx \quad \Sigma_Q \tag{6.47}$$

Proof. By Theorem 6.15 (i), the local conditions (6.44) guarantee the existence of local controllers Σ_{C_i} and permutation matrices Π_i such that

$$\Sigma^i_P \|^{\Pi_i}_{u,y} \Sigma_{C_i} \preccurlyeq \Sigma_Q \ , \quad i = 1, \dots, k \ .$$

Soundness of assume-guarantee reasoning for k control systems as stated in Corollary 6.7 then yields the desired result for feedback interconnections.
If instead the sandwich condition holds, then by Theorem 6.15 (ii) the global controlled plant $\left(\Sigma_{P_1} \|^{\Pi_1}_{u,y} \Sigma_{C_1}\right) \| \dots \| \left(\Sigma_{P_k} \|^{\Pi_k}_{u,y} \Sigma_{C_k}\right)$ is bisimilar to the global specification Σ_Q. Indeed, it follows from (6.46) that there exists controllers Σ_{C_i} and permutation matrices Π_i such that

$$\Sigma^i_P \|^{\Pi_i}_{u,y} \Sigma_{C_i} \approx \Sigma_Q \ , \ i = 1, \dots, k \tag{6.48}$$

We then have to use that circular assume-guarantee reasoning for k systems is also sound using bisimulation relations (compare with Corollary 6.7) to conclude from (6.48) that (6.47) holds. □

So far we have shown that local conditions for achievable simulation are necessary and sufficient for the existence of local controller $\Sigma_{C_i}, i = 1, \ldots, k$, such that the global specification is satisfied. This holds for both bottom-up schemes relying on compositional and circular assume guarantee reasoning, respectively. Obviously, one can always construct a diagonally decoupled global controller Σ_C based on Σ_{C_i}, i.e. Σ_C consists of k subsystems running in parallel without interference,

$$\Sigma_C := (\Sigma_{C_1} \| \ldots \| \Sigma_{C_k}) \ . \tag{6.49}$$

The construction (6.49) is also consistent with our definition of feedback interconnection $\|$. According to Definition 6.3, the feedback interconnection $\|$ involves the external specification variables $e_i^{\pm}$ and $z_i^{\pm}$ which are absent in controller systems. Hence, it trivially holds that

$$\begin{array}{ccc} \left(\Sigma_{P_1} \|_{u,y}^{\Pi} \Sigma_{C_1}\right) \| & \ldots & \| \left(\Sigma_{P_N} \|_{u,y}^{\Pi} \Sigma_{C_k}\right) \\ & \approx & \\ (\Sigma_{P_1} \| \ldots \| \Sigma_{P_k}) & \|_{u,y}^{\Pi} & (\Sigma_{C_1} \| \ldots \| \Sigma_{C_k}) \end{array} \tag{6.50}$$

Thus, if (6.38) and (6.40) respectively (6.45) and (6.47) hold for the same permutation matrix Π, e.g. for $\Pi = I$, the decentralized control schemes of Theorems 6.16 and 6.17 can be interpreted as global feedback control strategies but based on *local* conditions for achievable simulation.

6.6.2. Top-down decentralized control scheme using global sandwich conditions

The *top-down* scheme for decentralized control starts from the perspective of the overall system, see Figure 6.4. Based on a global sandwich condition, we want to investigate whether the existence of a global controller Σ_C implies the existence of local controllers such that the overall controlled system satisfies the same specification. From the previous chapters it is known that compositional reasoning is not complete for closed interconnections. The result presented here therefore relies on completeness of circular assume-guarantee reasoning in the decentralized setting. Like before, we consider a global plant Σ_P composed of component systems $\Sigma_{P_i}, i = 1, \ldots, k$, interconnected in series by feedback as in (6.11) and a global specification Σ_Q assembled as in (6.12). The system Σ_{N_P} associated with the global plant Σ_P is given by

$$\begin{aligned} \Sigma_{N_P} = \Sigma_P \|_{u,y}^{\Pi} (\Sigma_0 \| \ldots \| \Sigma_0) \approx \left(\Sigma_{P_1} \|_{u,y}^{\Pi} \Sigma_0\right) \| \ldots \| \left(\Sigma_{P_k} \|_{u,y}^{\Pi} \Sigma_0\right) \\ = \Sigma_{N_{P_1}} \| \ldots \| \Sigma_{N_{P_k}} \end{aligned} \tag{6.51}$$

making use of Proposition 6.10.

Theorem 6.18. *Consider a global plant Σ_P as in* (6.11) *with an associated system Σ_{N_P} given by* (6.51) *and a global specification Σ_Q as in* (6.12)*. Then the following two statements are equivalent:*

1. *There exists a global controller Σ_C and a permutation matrix Π_1 such that*

$$\Sigma_P \|_{u,y}^{\Pi_1} \Sigma_C \approx \Sigma_Q \tag{6.52}$$

2. *There exist local controllers $\Sigma_{C_i}, i = 1, \ldots, k$, and a permutation matrix Π_2 such that*

$$\left(\Sigma_{P_1} \|_{u,y}^{\Pi_2} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|_{u,y}^{\Pi_2} \Sigma_{C_k}\right) \quad \approx \quad \Sigma_Q \tag{6.53}$$

Proof. "2 $\Longrightarrow$ 1": This is a consequence of (6.50), i.e. the global decoupled controller Σ_C is the series interconnection of the local controllers $\Sigma_{C_i}, i = 1, \ldots, k$ with $\Pi_1 = \Pi_2$.
"1 $\Longrightarrow$ 2": Assume there exists a global controller Σ_C and a permutation matrix Π such that (6.52) holds. Then by Theorem 6.15, iv, the global sandwich condition

$$\Sigma_{N_P} \preccurlyeq \Sigma_Q \preccurlyeq \Sigma_P \tag{6.54}$$

is fulfilled. Making use of (6.51), (6.54) can be rewritten as

$$\Sigma_{N_{P_1}} \| \ldots \| \Sigma_{N_{P_k}} \preccurlyeq \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k} \preccurlyeq \Sigma_{P_1} \| \ldots \| \Sigma_{P_k} \tag{6.55}$$

We now split (6.55) into two statements and use the fact that circular assume-guarantee reasoning is complete in the decentralized setting (Corollary 6.7). Hence, we obtain from the first statement k full simulation relations $S_i^l, i = I, II, \ldots, k$, of the form

$$S_i^l : \quad \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_{i-1}} \| \Sigma_{N_{P_i}} \| \Sigma_{Q_{i+1}} \| \ldots \| \Sigma_{Q_k} \preccurlyeq \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k}$$

which by (6.43) can be simplified to

$$S_i^l : \ \Sigma_{N_P}^i \preccurlyeq \Sigma_Q \ . \tag{6.56}$$

The second statement results in k full simulation relations $S_i^r, i = I, II, \ldots, k$, of the form

$$S_i^r : \quad \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k} \preccurlyeq \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_{i-1}} \| \Sigma_{P_i} \| \Sigma_{Q_{i+1}} \| \ldots \| \Sigma_{Q_k}$$

or, due to (6.42),

$$S_i^r : \quad \Sigma_Q \preccurlyeq \Sigma_P^i \tag{6.57}$$

Note that (6.57) implies that Σ_{Q_i} refines Σ_{P_i} as mentioned in Remark 3.22. Hence, in order to apply circular assume-guarantee reasoning, we have to construct S_i^r as

$$\begin{aligned} S_i^r \quad := \quad & \{(x_{Q_1}, \ldots, x_{Q_k}, x_{Q_1}, \ldots, x_{Q_{i-1}}, x_{P_i}, x_{Q_{i+1}}, \ldots, x_{Q_k}) \mid \exists\, x_{P_1}, \ldots, x_{P_{i-1}}, \\ & x_{P_{i+1}}, \ldots, x_{P_k} : (x_{Q_1}, \ldots, x_{Q_k}, x_{P_1}, \ldots, x_{P_k}) \in S^r\} \end{aligned}$$

where S^r is a full simulation relation of Σ_Q by Σ_P. Transitivity of simulation allows to combine (6.56) and (6.57) to obtain sandwich conditions

$$S_i : \quad \Sigma^i_{N_P} \preccurlyeq \Sigma_Q \preccurlyeq \Sigma^i_P \ , i = 1, \ldots, k \ . \tag{6.58}$$

Theorem 6.17, 2, then ensures that there exist local controllers Σ_{C_i} and permutation matrices Π_i such that

$$\left(\Sigma_{P_1} \|^{\Pi_1}_{u,y} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|^{\Pi_k}_{u,y} \Sigma_{C_k}\right) \quad \approx \quad \Sigma_Q$$

holds. Using canonical controllers Σ^i_{can}, we can choose $\Pi_1 = \ldots = \Pi_k = I$ and thus the claim is proved. □

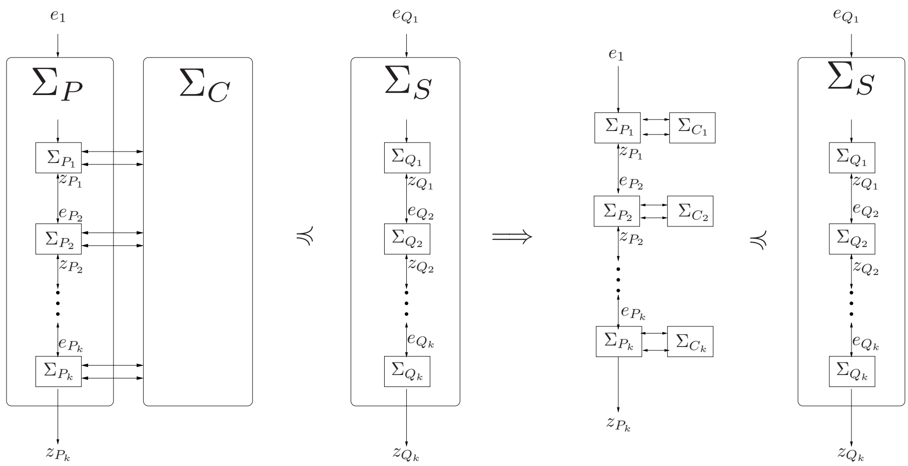

Figure 6.7.: Top-down decentralized control scheme.

Figure 6.7 illustrates an intriguing consequence of Theorem 6.18: Although nothing is known about the structure of the global controller Σ_C, there always exist local controllers Σ_{C_i} in our decentralized control setting that satisfy the same global control target Σ_Q. Thus, provided the conditions of Theorem 6.18 hold – in particular that the specification Σ_Q is given as $\Sigma_Q = \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k}$ – decentralized control can achieve the same control performance as a more complex global feedback controller.

7

Compositional analysis of switching linear systems

7.1. Introduction

Hybrid systems combine elements of discrete and continuous dynamics. Many real-life applications are characterized by hybrid behavior, e.g. air traffic management [66, 70], multi-agent systems [10, 20] and genetic regulatory networks [18, 13]. Due to the interaction of continuous and discrete phenomena, the study of hybrid systems (depending on the area also called embedded [19] or cyber-physical systems) is an interdisciplinary field approached from both computer science and systems and control theory. As a result, there exists a large variety of subclasses of hybrid systems. In this work, we want to consider hybrid systems given as switching linear systems which combine discrete elements of labeled transition systems with linear continuous-time dynamics, see e.g. [60]. Exploiting their particular structure, we develop a theory of compositional and assume-guarantee reasoning for switching linear systems based on the theory of Chapter 3 and results from [21] for labeled transition systems. Like in previous chapters, we investigate the following proof rules:

- Compositionality:

$$\left.\begin{array}{rrcl} S_1: & \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_2: & \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_2} \end{array}\right\} \implies S: \Sigma_{P_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2} \quad (7.1)$$

- Non-circular assume-guarantee reasoning:

$$\left.\begin{array}{rrcl} S_1: & \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_{II}: & \Sigma_{Q_1}\|\Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1}\|\Sigma_{Q_2} \end{array}\right\} \implies S: \Sigma_{P_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2} \quad (7.2)$$

- Circular assume-guarantee reasoning:

$$\left.\begin{array}{rrcl} S_I: & \Sigma_{P_1}\|\Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1}\|\Sigma_{Q_2} \\ S_{II}: & \Sigma_{Q_1}\|\Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1}\|\Sigma_{Q_2} \end{array}\right\} \implies S: \Sigma_{P_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2} \quad (7.3)$$

In computer science, one usually describes the continuous part of the hybrid system by considering solutions of differential equations, see e.g. [2] or [30] and [21], thus treating the hybrid system as a generalized transition system. Our approach is different since it is based on the differential equations themselves. Hence, it does not depend upon actual system trajectories which are often, especially in the case of nonlinear dynamics, not available. Furthermore, it turns out that hybrid simulation relations can be checked efficiently working with differential equations. Structural hybrid simulation relations play a key role in our analysis since they separate the influence of the discrete from the continuous part of the hybrid dynamics. Thus, we can make use of well-established results for compositional analysis of labeled transition systems on the one hand and of linear continuous-time systems on the other hand.

7.2. Preliminaries and basic definitions

In our definition of switching linear systems, we will combine the structure of labeled transition systems on the level of the discrete dynamics and linear continuous-time systems on the level of the continuous dynamics.

Definition 7.1. A switching linear system Σ^{SLS} is a tuple $\Sigma^{\mathrm{SLS}} = (Q, \mathcal{X}, V, \mathcal{W}, \Sigma, E, M)$ with

- $Q = \{q_1, \ldots, q_N\}, N \in \mathbb{N}$, the set of discrete locations (or discrete states),
- $\mathcal{X}(q) \subset \mathbb{R}^{\dim(q)}$, $\dim : Q \to \mathbb{N}$ a linear vector space representing the continuous part of the hybrid dynamics at every location $q \in Q$,
- V the set of discrete transition labels,
- $\mathcal{W} \subset \mathcal{U} \times \mathcal{D} \times \mathcal{Y}$ the set of continuous communication variables with $u \in \mathcal{U} = \mathbb{R}^{\mathrm{m}}$ the inputs, $d \in \mathcal{D} = \mathbb{R}^{d}$ the disturbances, and $y \in \mathcal{Y} = \mathbb{R}^{\mathrm{p}}$ the outputs, respectively,
- Σ a function associating to every discrete state q the deterministic linear input-state-output system

$$\Sigma(q) : \begin{cases} \dot{x}(t) = A(q)x(t) + B(q)u(t) + G(q)d(t), \\ y(t) = C(q)x(t) \end{cases} \tag{7.4}$$

  with $x \in \mathcal{X}(q), u \in \mathcal{U}, d \in \mathcal{D}$ and $y \in \mathcal{Y}$,
- $E \subset Q \times V \times Q$ the discrete transition relation,
- for every $e = (q, v, q') \in E$, $M(e) : \mathcal{X}(q) \to \mathcal{X}(q')$ the linear reset map associated with every discrete transition $e = (q, v, q') \in E$ to reset the continuous state.

The hybrid state space of Σ^{SLS}, denoted by Δ, is given as the set $\Delta = \cup_{q\in Q}\{q\} \times \mathcal{X}(q)$. Changes of discrete states are triggered by (or accompanied by) discrete events $v \in V$. Whenever an event v occurs at an event time τ, the discrete state switches from $q(\tau^-) = \lim_{t\uparrow\tau} q(t)$ to $q(\tau^+) = \lim_{t\downarrow\tau} q(t)$ according to the transition relation E, while the continuous state is reset to $x(\tau^+) = M(e)x(\tau^-)$ where $x(\tau^-)$ and $x(\tau^+)$ represent the values of x just before and after the occurrence of transition $e = (q(\tau^-), v, q(\tau^+)) \in E$ at time τ, i.e. $x(\tau^-) = \lim_{t\uparrow\tau} x(t)$ and $x(\tau^+) = \lim_{t\downarrow\tau} x(t)$. In between event times, the associated continuous dynamics are governed by the linear system dynamics $\Sigma(q)$ in every discrete state. The discrete part of the hybrid dynamics has the structure of a labeled transition system as introduced in Section 2.1.

Definition 7.2. Given a switching linear system $\Sigma^{\text{SLS}} = (Q, \mathcal{X}, V, \mathcal{W}, \Sigma, E, M)$, the *associated labeled transition system* D^{SLS} is given by the triple $D^{\text{SLS}} = (Q, V, E)$ with $Q = \{q_1, \ldots, q_N\}$ the set of discrete states, V the set of transition labels and $E \subset Q \times V \times Q$ the discrete transition relation as in Definition 7.1.

The trajectories of a switching linear system will be specified with respect to a time interval $[0, T]$ including discrete event times.

Definition 7.3. An *execution* ρ of a switching linear system Σ^{SLS} on a time interval $[0, T]$ is a collection $\rho = (\mathcal{T}, q, x, v, w)$ where

- $\mathcal{T} \subset [0, T]$ is a finite set $\{\tau_1, \tau_2, \ldots\}$ of ordered discrete event times,
- $q : [0, T] \to Q$ is a function from the time interval $[0, T]$ to the set of locations describing the evolution of the discrete states, which is constant on every interval $[\tau_i, \tau_{i+1}), i = 1, 2, \ldots$,
- $x : [0, T] \to \mathcal{X}(q)$ is a time function satisfying for every $t \neq \mathcal{T}$ the differential equations $\Sigma(q(t))$ with $w = (u, d, y)$ and for every $\tau \in \mathcal{T}$, $x(\tau^+) = M(e)x(\tau^-)$,
- $v : \mathcal{T} \to V$ a function for the event labels satisfying for every $\tau \in \mathcal{T}$ the discrete transition relation $(q(\tau^-), v, q(\tau^+)) \in E$,
- $w = (u, d, y)$ are time functions $u : [0, T] \to \mathcal{U}, d : [0, T] \to \mathcal{D}, y : [0, T] \to \mathcal{Y}$ of the continuous inputs, disturbances and outputs valid for every $t \neq \mathcal{T}$.

For a more detailed discussion of the semantics of hybrid systems, allowing for more general descriptions, we refer to [77].

7.3. Compositions of hybrid systems

Figure 7.1 illustrates how switching linear systems are structured. The continuous dynamics are influenced by the discrete ones while, by contrast, the discrete dynamics are independent of the continuous ones. Changes of the continuous evolution are caused by discrete events. The occurrence of discrete events is assumed not to be controllable. Due to this particular structure, one

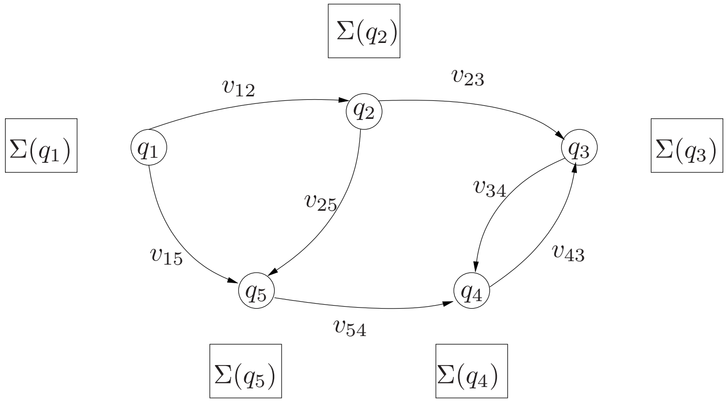

Figure 7.1.: Discrete and continuous layer of switching linear systems.

can define interconnections of switching linear systems by first determining the parallel composition of the associated labeled transition systems and then interconnecting the respective continuous systems for every discrete state of the parallel composition. Additionally, the reset maps have to be adjusted for non-shared transition labels $v \in V_i \setminus V_j, (i,j) \in \{(1,2),(2,1)\}$.

7.3.1. Interconnections of labeled transition systems

The interconnection of the discrete layer of two switching linear systems Σ_i^{SLS}, $i = 1,2$, involves the labeled transition systems D_i^{SLS} associated to Σ_i^{SLS}. The treatment of these labeled transition systems follows the standard definitions found in the computing science literature.

Definition 7.4. Consider two labeled transition systems $D_i = (Q_i, V_i, E_i), i = 1,2$, like in Definition 2.1. The parallel composition $D = D_1 \| D_2$ is again a labeled transition system $D = (Q, V, E_{12})$ with

- $Q = Q_1 \times Q_2$
- $V = V_1 \cup V_2$
- $$E_{12} = \begin{cases} ((q_1,q_2),v,(q_1',q_2')) & ,v \in V_1 \cap V_2, \\ & (q_i,v,q_i') \in E_i, i = 1,2 \\ ((q_1,q_2),v,(q_1',q_2)) & ,v \in V_1 \setminus V_2, \\ & (q_1,v,q_1') \in E_1 \\ ((q_1,q_2),v,(q_1,q_2')) & ,v \in V_2 \setminus V_1, \\ & (q_2,v,q_2') \in E_2 \end{cases}$$

The notion of simulation relations is instrumental for compositional analysis. We follow Definition 2.3 to relate labeled transition systems with the same transition structure by means of simulation relations. To prepare for the analysis of switching linear systems later, we recall from the literature, in particular [21], the most important results for compositional and assume-guarantee reasoning for labeled transition systems.

Theorem 7.5. *Given four labeled transition systems D_i, $i \in \{P_1, P_2, Q_1, Q_2\}$ such that $V_{P_1} = V_{Q_1}, V_{P_2} = V_{Q_2}$ and define parallel composition $\|$ as in Definition 7.4. Then parallel composition of labeled transition systems is compositional with respect to simulation.*

Theorem 7.5 is proved indirectly in the literature. In fact, [21] shows that simulation for labeled transition systems is invariant under composition, i.e. if $D_{P_1} \preccurlyeq D_{Q_1}$ then for every D_{P_2}, $D_{P_1}\|D_{P_2} \preccurlyeq D_{Q_1}\|D_{P_2}$. Since parallel composition is symmetric, $D_{P_1}\|D_{P_2} \approx D_{P_2}\|D_{P_1}$, invariance under composition is equivalent to compositionality. Indeed, assuming that $D_{P_i} \preccurlyeq D_{Q_i}, i = 1, 2$, invariance under composition yields full simulation relations S_I and S_{II} such that

$$
\begin{aligned}
S_I : \quad & D_{P_1}\|D_{P_2} && \preccurlyeq && D_{Q_1}\|D_{P_2} \\
S_{II} : \quad & D_{P_2}\|D_{Q_1} && \preccurlyeq && D_{Q_2}\|D_{Q_1}
\end{aligned}
$$

Exploiting symmetry of parallel composition and transitivity of simulation, one obtains the desired result,

$$
D_{P_1}\|D_{P_2} \preccurlyeq D_{Q_1}\|D_{P_2} \preccurlyeq D_{P_2}\|D_{Q_1} \preccurlyeq D_{Q_2}\|D_{Q_1} \preccurlyeq D_{Q_1}\|D_{Q_2}
$$

Non-circular assume-guarantee reasoning is also sound since parallel composition of labeled transition systems is compositional and simulation is transitive (compare with Theorem 3.9 for the continuous case).

Theorem 7.6. *Given four labeled transition systems $D_i, i \in \{P_1, P_2, Q_1, Q_2\}$ such that $V_{P_1} = V_{Q_1}, V_{P_2} = V_{Q_2}$ and define parallel composition $\|$ as in Definition 7.4. Then non-circular assume-guarantee reasoning is sound.*

For circular assume-guarantee reasoning, only a slightly weaker result can be proved.

Theorem 7.7 (compare with [21])**.** *Given four labeled transition systems $D_i, i \in \{P_1, P_2, Q_1, Q_2\}$ such that $V_{P_1} = V_{Q_1}, V_{P_2} = V_{Q_2}$ and define parallel composition $\|$ as in Definition 7.4. Let full simulation relations S_I and S_{II} of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, respectively, be given as in the circular assume-guarantee reasoning rule* (7.3) *. Then there exists a simulation relation S of $\Sigma_{P_1}\|\Sigma_{P_2}$ by*

$\Sigma_{Q_1} \| \Sigma_{Q_2}$ *if and only if* $\forall((p_1,p_2),(q_1,q_2)) \in S, v \in V_{P_1} \cap V_{P_2}$, *the following condition holds:*

$$\begin{array}{ccc} ((p_1,p_2),v,(p_1',p_2')) & \in & E_{P_1P_2} \\ & \Longrightarrow & \\ \exists q_1' : (q_1,v,q_1') \in E_{Q_1} & \vee & \exists q_2' : (q_2,v,q_2') \in E_{Q_2} \end{array} \tag{7.5}$$

Intuitively, condition (7.5) ensures that the evolution of the approximation $D_{Q_1} \| D_{Q_2}$ will not reach a deadlock while there are still executable transitions in $D_{P_1} \| D_{P_2}$. Note that even if (7.5) is satisfied, it is not ensured by Theorem 7.7 that $\Pi_{Q_i} S = Q_i, i = 1,2$. Hence, circular assume-guarantee reasoning for switching linear systems will not be sound in general, see Theorem 7.18.

7.3.2. Interconnections of continuous-time dynamical systems

On the continuous layer, we are dealing with interconnections of linear systems. Recall the definitions of open and closed feedback interconnections of linear systems in Chapter 3.

Definition 7.8. For two linear systems of the form (7.4) such that $\dim \mathcal{U}_i = \dim \mathcal{Y}_j, (i,j) \in \{(1,2),(2,1)\}$, define the *open feedback interconnection* $\Sigma_1 \|_{\text{o}} \Sigma_2$ by

$$u_1 = y_2 + v_1 \quad , \quad u_2 = y_1 + v_2 \tag{7.6}$$

and, accordingly, the *closed feedback interconnection* $\Sigma_1 \|_{\text{cl}} \Sigma_2$ by

$$u_1 = y_2 \quad , \quad u_2 = y_1 \tag{7.7}$$

In Chapter 2 we defined simulation for linear systems and gave a linear-algebraic characterization which will be used in the following. With respect to compositional reasoning, we showed in Section 3.2.2 that there is an important difference between open and closed feedback interconnections: For open feedback there is no need for assume-guarantee reasoning since proof obligations for interconnections are equivalent to obligations for the individual components. I.e., one can always replace a simulation between systems interconnected by open feedback by the respective simulation relations of the components involved to simplify the verification task. For closed feedback interconnections, however, assume-guarantee reasoning *can* reduce the complexity of proof obligations. In the remainder, we will therefore only treat closed interconnections of linear systems. For completeness we recall Theorem 3.14 from Section 3.2.1.

Theorem 7.9. *Given four linear systems* $\Sigma_i, i \in \{P_1,P_2,Q_1,Q_2\}$ *of the form* (7.4). *Then simulation is compositional with respect to closed feedback interconnection. Moreover, both non-circular and circular assume-guarantee reasoning is also sound.*

7.3.3. Interconnections of switching linear systems

Combining parallel composition of labeled transition systems and closed feedback interconnection of linear systems we define interconnections of switching linear systems as follows:

Definition 7.10. Consider two switching linear systems $\Sigma_i^{\text{SLS}} = (Q_i, \mathcal{X}_i, V_i, \mathcal{W}_i, \Sigma_i, E_i, M_i), i = 1, 2$. The interconnection $\Sigma_1^{\text{SLS}} \| \Sigma_2^{\text{SLS}}$ is a switching linear system $\Sigma^{\text{SLS}} = (Q, \mathcal{X}, V, \mathcal{W}, \Sigma, E_{12}, M)$ with

- $Q = Q_1 \times Q_2$ the set of discrete locations resulting from the parallel composition $D_1^{\text{SLS}} \| D_2^{\text{SLS}}$ of the associated labeled transition systems D_i^{SLS},
- $\mathcal{X}(q_1, q_2) = \mathcal{X}_1(q_1) \times \mathcal{X}_2(q_2)$ the continuous state space at every location $(q_1, q_2) \in Q$,
- $\mathcal{W} = (\mathcal{U}, \mathcal{D}, \mathcal{Y})$ where $\mathcal{U}$ is void since we consider closed feedback interconnections $\Sigma_1(q_1) \|_{\text{cl}} \Sigma_2(q_2)$ in every location $(q_1, q_2) \in Q$, $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$,
- V and E_{12} as determined by the parallel composition of the respective associated labeled transition systems D_i^{SLS},
- $M(e) = \text{diag}\{\bar{M}_1, \bar{M}_2\}$ with

$$\bar{M}_i(e_i) = \begin{cases} M_i(e_i), & \text{if } q_i' \neq q_i, e_i = (q_i, v, q_i') \in E_i \\ I, & \text{if } q_i = q_i' \end{cases}$$

Note that the adjusted reset map $\bar{M} : \mathcal{X}(q_1, q_2) \to \mathcal{X}(q_1', q_2')$ works as a self-loop at q_i if the transition label is not shared, $v \in V_j \setminus V_i, (i, j) \in \{(1, 2), (2, 1)\}$. Event times of the interconnection $\Sigma_1^{\text{SLS}} \| \Sigma_2^{\text{SLS}}$ are synchronized for shared labels, while for non-shared labels $v \in V_j \setminus V_i, (i, j) \in \{(1, 2), (2, 1)\}$ there is an event in Σ_j^{SLS} and a self-loop in Σ_i^{SLS}.

7.4. Simulation relations for switching linear systems

Hybrid simulation theory has been treated by several authors, e.g. [75, 2]. We follow [60] and start with

Definition 7.11. Given two switching linear systems $\Sigma_i^{\text{SLS}} = (Q_i, \mathcal{X}_i, V, \mathcal{W}_i, \Sigma_i, E_i, M_i), i = 1, 2$, with the same continuous input and output spaces $\mathcal{U} = \mathcal{U}_1 = \mathcal{U}_2, \mathcal{Y} = \mathcal{Y}_1 = \mathcal{Y}_2$ and the same set of discrete labels $V = V_1 = V_2$. A hybrid simulation relation S of Σ_1^{SLS} by Σ_2^{SLS} is a subset of the product of the hybrid state spaces Δ_i, $S \subset \Delta_1 \times \Delta_2$, with the following properties. Take any initial

hybrid state $(\xi_1^0, \xi_2^0) \in S$ and any input function $u = u_1 = u_2$.
Then for any $d_1 \in \mathcal{D}_1$ and any hybrid execution $\rho_1 = (\mathcal{T}_1, q_1, x_1, v, w_1)$ there should exist a $d_2 \in \mathcal{D}_2$ and an execution $\rho_2 = (\mathcal{T}_2, q_2, x_2, v, w_2)$ such that for all times t for which the hybrid execution ρ_1 is defined,

$$\begin{aligned}
(i): &\quad \mathcal{T}_1 = \mathcal{T}_2 =: \mathcal{T} \\
(ii): &\quad \forall\, u_1(t) = u_2(t), \forall\, d_1(t)\ \exists\, d_2(t) : y_1(t) = y_2(t)\ \forall\, t \geq 0, t \notin \mathcal{T} \\
(iii): &\quad v_1(t) = v_2(t)\ \forall\, t \geq 0,\ t \in \mathcal{T} \\
(iv): &\quad (q_1(t), x_1(t), q_2(t), x_2(t)) \in S\ \forall\, t \geq 0, t \notin \mathcal{T}
\end{aligned} \tag{7.8}$$

Moreover, if there exists a hybrid simulation relation S of Σ_1^{SLS} by Σ_2^{SLS} such that $\Pi|_{\Delta_1} S = \Delta_1$, then Σ_1^{SLS} *simulates* Σ_2^{SLS}, denoted by $\Sigma_1^{\mathrm{SLS}} \preccurlyeq \Sigma_2^{\mathrm{SLS}}$. In this case, S is called a *full simulation relation*.

Hybrid simulation relations possess the same properties as their discrete and continuous counterparts.

Proposition 7.12. *Hybrid simulation as defined in Definition 7.11 is a preorder, i.e., it is reflexive and transitive.*

Proof. Consider switching linear system $\Sigma_1^{\mathrm{SLS}}, \Sigma_2^{\mathrm{SLS}}$ and Σ_3^{SLS}.
(*Reflexivity*) Define the relation $S = \{(q,x),(q,x))|(q,x) \in \Delta_1\}$. It is easy to check that S is a hybrid simulation relation of Σ_1^{SLS} by Σ_1^{SLS}.
(*Transitivity*) Assume there exist hybrid simulation relations S_{12} and S_{23} of Σ_1^{SLS} by Σ_2^{SLS} and of Σ_2^{SLS} by Σ_3^{SLS}, respectively. Then

$$\begin{aligned}
S_{13} \quad := \quad & \{((q_1,x_1),(q_3,x_3))|\exists(q_2,x_2) : ((q_1,x_1),(q_2,x_2)) \in S_{12}, \\
& ((q_2,x_2),(q_3,x_3)) \in S_{23}\}
\end{aligned}$$

is a hybrid simulation relation of Σ_1^{SLS} by Σ_3^{SLS}. □

The structure of switching linear systems allows to split a hybrid simulation subset S into a relation Q_S between the discrete states and, associated with every pair of discrete locations $(q_1, q_2) \in Q_S$, subsets of the continuous product space $W(q_1, q_2) \subset \mathcal{X}_1 \times \mathcal{X}_2$.

Proposition 7.13. *Let S be a hybrid simulation relation of Σ_1^{SLS} by Σ_2^{SLS}. Then there exists a relation $Q_S \subset Q_1 \times Q_2$ and for any $(q_1, q_2) \in Q_S$ suitable sets $W(q_1, q_2) \subset \mathcal{X}_1(q_1) \times \mathcal{X}_2(q_2)$ such that*

$$((q_1,x_1),(q_2,x_2)) \in S \quad \Leftrightarrow \quad (q_1,q_2) \in Q_S, (x_1,x_2) \in W(q_1,q_2)$$

Remark 7.14. It has been shown in [60] that the sets $W(q_1, q_2)$ can be assumed to be linear subspaces. In fact, given a hybrid simulation relation S of Σ_1^{SLS} by Σ_2^{SLS}, its linear closure $\mathcal{L}(S)$ is also a hybrid simulation relation of Σ_1^{SLS} by Σ_2^{SLS}. Therefore, we will assume in the remainder that the subsets of the continuous variable spaces $W(q_1, q_2)$ are linear subspaces.

As a consequence of Proposition 7.13 a hybrid simulation relation should fulfill the following conditions:

- The set of discrete states Q_S is a simulation relation between the associated labeled transition systems
- The linear subspaces $W(q_1, q_2)$ at every location $(q_1, q_2) \in Q_S$ define a simulation relation between the underlying linear systems $\Sigma_1(q_1)$ and $\Sigma_2(q_2)$.

The coupling between discrete and continuous dynamics is imposed by the reset maps $M_i(e_i)$. This leads to a more checkable notion of hybrid simulation which is usually referred to as structural hybrid simulation relation in the literature [75].

Theorem 7.15. *Given two switching linear systems Σ_1^{SLS} and Σ_2^{SLS} with the same set of labels $V = V_1 = V_2$, a set*

$$S = \{((q_1, x_1), (q_2, x_2)) \in \Delta_1 \times \Delta_2 | (q_1, q_2) \in Q_S, (x_1, x_2) \in W(q_1, q_2)\} \quad (7.9)$$

is a hybrid simulation relation of Σ_1^{SLS} by Σ_2^{SLS} if and only if the following properties hold:

(i) Q_S is a simulation relation of D_1^{SLS} by D_2^{SLS} where D_i^{SLS} are the labeled transition systems associated to $\Sigma_i^{SLS}, i = 1, 2$, and for every $(q_1, q_2) \in Q_S$, $W(q_1, q_2)$ is a simulation relation of $\Sigma_1(q_1)$ by $\Sigma_2(q_2)$;

(ii) for every $(q_1, q_2) \in Q_S$ and every possible set of successor states $(q_1', q_2') \in Q_S$ such that $(q_1, v, q_1') \in E_1$ and $(q_2, v, q_2') \in E_2$,

$$diag\{M_1(e_1), M_2(e_2)\} W(q_1, q_2) \subset W(q_1', q_2')$$

Hybrid simulation relations of the form (7.9) *are called* structural hybrid simulation relations.

Proof. ($\Longrightarrow$):
Assume Q_S is not a simulation relation of D_1^{SLS} by D_2^{SLS} and there exists at least one pair $(q_1, q_2) \in Q_S$ such that there does not exist a simulation relation of $\Sigma_1(q_1)$ by $\Sigma_2(q_2)$. Then even if $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}$ one can find for some joint input function $u_1 = u_2 = u$ a disturbance d_1 and a hybrid execution $\rho_1 = (\mathcal{T}_1, q_i, x_1, v, w_1)$ such that for any disturbance d_2 and any execution

$\rho_2 = (\mathcal{T}_2, q_2, x_2, v, w_2)$ there exists an event time $T \in \mathcal{T}_1$ and a $v \in V$ for which $(q_1, v, q_1') \in E_1$ but there does not exist a q_2' such that $(q_2, v, q_2') \in E_2$. Similarly, one can show that if there does not exist any simulation relation between the underlying linear systems $\Sigma_1(q_1)$ and $\Sigma_2(q_2)$ one can find an execution ρ_1 with continuous output $y_1(t)$ such that any execution ρ_2 violates condition 2 in Definition 7.11, i.e. $y_2(t) \neq y_1(t)$ for some t for which ρ_1 is defined.
($\Longleftarrow$):
Due to Theorem 7.15 (i), the set Q_S is a simulation relation of D_1^{SLS} by D_2^{SLS} and for every $(q_1, q_2) \in Q_S$, $W(q_1, q_2)$ is a simulation relation of $\Sigma_1(q_1)$ by $\Sigma_2(q_2)$. By the respective definitions of simulation relations for labeled transition and linear systems, Theorem 7.15 (i) therefore guarantees that for any hybrid execution ρ_1 there exists a hybrid execution ρ_2 such that the set of event times are equal, $\mathcal{T}_1 = \mathcal{T}_2$, and moreover Definition 7.11 (ii) and (iii) hold. Furthermore, Theorem 7.15 (ii) ensures that just after every event time $t \in \mathcal{T}$ the reset of the continuous state remains within the simulation subspace $W(q_1', q_2')$ associated with the new location (q_1', q_2'). Thus, condition (iv) of Definition 7.11 is also fulfilled. □

It was shown in [60] that for two similar switching linear systems $\Sigma_i^{\text{SLS}}, i = 1, 2$, the maximal hybrid simulation relation $S^\star$ is of the form (7.9) where $W(q_1, q_2)$ are linear subspaces, i.e. $S^\star$ is a structural hybrid simulation relation. As mentioned in Remark 7.14, given a hybrid simulation relation S, its linear closure $\mathcal{L}(S)$ is also a hybrid simulation relation. Moreover, $S \subset \mathcal{L}(S)$. For other classes of hybrid systems, in particular for hybrid systems with location invariants and guards, this no longer holds true since the continuous part of the hybrid state space is not a linear subspace nor is the linear closure $\mathcal{L}(S)$ in general a hybrid simulation relation. Thus, the maximal hybrid simulation relation might not be well-defined.

7.5. Compositional reasoning

Knowing that compositionality holds on the level of both the discrete and continuous dynamics, we establish a similar result for switching linear systems.

Theorem 7.16. *For any four given switching linear systems $\Sigma_i^{SLS}, i \in \{P_1, P_2, Q_1, Q_2\}$, such that $V_{P_1} = V_{Q_1}, V_{P_2} = V_{Q_2}$ and interconnections $\|$, hybrid simulation is compositional.*

Proof. Assume we are given hybrid simulation relations S_1 and S_2 of $\Sigma_{P_1}^{\text{SLS}}$ by $\Sigma_{Q_1}^{\text{SLS}}$ and $\Sigma_{P_2}^{\text{SLS}}$ by $\Sigma_{Q_2}^{\text{SLS}}$, respectively. Define the relation

$$S := \{((\xi_{P_1}, \xi_{P_2}), (\xi_{Q_1}, \xi_{Q_2})) \mid (\xi_{P_1}, \xi_{Q_1}) \in S_1, (\xi_{P_2}, \xi_{Q_2}) \in S_2\} \quad (7.10)$$

Reordering the components, one obtains

$$\tilde{S} \quad = \quad \{((\xi_{P_1}, \xi_{Q_1}), (\xi_{P_2}, \xi_{Q_2})) \mid ((\xi_{P_1}, \xi_{P_2}), (\xi_{Q_1}, \xi_{Q_2})) \in S\} = S_1 \times S_2$$

From Theorems 7.5 and 7.9, we know that the interconnections of the associated labeled transition systems $D_i^{\mathrm{SLS}}, i \in \{P_1, P_2, Q_1, Q_2\}$, as well as the interconnections of the respective linear systems $\Sigma_i(q_i)$ at every location $((p_1, p_2), (q_1, q_2)) \in Q_S$ are compositional so that condition (i) in Theorem 7.15 is fulfilled.
Since S_1, S_2 are hybrid simulation relations, it holds for every $(p_1, q_1) \in Q_{S_1}$ and every possible successor state $(p_1', q_1') \in Q_{S_1}$ that

$$\mathrm{diag}\{M_{P_1}(e_{P_1}), M_{Q_1}(e_{Q_1})\} W_1(p_1, q_1) \subset W_1(p_1', q_1') \tag{7.11}$$

with $e_{P_1} = (p_1, v, p_1') \in E_{P_1}$ and $e_{Q_1} = (q_1, v, q_1') \in E_{Q_1}$ for some $v \in V_{P_1}$. Similarly, for every $(p_2, q_2) \in Q_{S_2}$ and every successor state $(p_2', q_2') \in Q_{S_2}$ such that $e_{P_2} = (p_2, v, p_2') \in E_{P_2}$ and $e_{Q_2} = (q_2, v, q_2') \in E_{Q_2}$ for some $v \in V_{P_2}$ the following holds:

$$\mathrm{diag}\{M_{P_2}(e_{P_2}), M_{Q_2}(e_{Q_2})\} W_2(p_2, q_2) \subset W_2(p_2', q_2') \tag{7.12}$$

Depending on the discrete transition label $v \in V = V_{P_1} \cup V_{P_2} = V_{Q_1} \cup V_{Q_2}$, three cases have to be distinguished:

1. $v \in V_{P_1} \cap V_{P_2}$: $((p_1, p_2), v, (p_1', p_2')) \in E_{P_1 P_2}$ implies $((q_1, q_2), v, (q_1', q_2')) \in E_{Q_1 Q_2}$ and therefore $(p_1', q_1', p_2', q_2') \in Q_{\tilde{S}}$. From (7.11) and (7.12) it follows that

$$\mathrm{diag}\{M_{P_1}(e_{P_1 P_2}), M_{Q_1}(e_{Q_1 Q_2}), M_{P_2}(e_{P_1 P_2}), M_{Q_1 Q_2}(e_{Q_1 Q_2})\} \\ \tilde{W}(p_1, q_1, p_2, q_2) \subset \tilde{W}(p_1', q_1', p_2', q_2')$$

2. $v \in V_{P_1} \setminus V_{P_2}$: Here, $((p_1, p_2), v, (p_1', p_2)) \in E_{P_1 P_2}$ implies $((q_1, q_2), v, (q_1', q_2)) \in E_{Q_1 Q_2}$ and therefore $(p_1', q_1', p_2, q_2) \in Q_{\tilde{S}}$. Thus,

$$\mathrm{diag}\{M_{P_1}(e_{P_1 P_2}), M_{Q_1}(e_{Q_1 Q_2}), I, I)\} \tilde{W}(p_1, q_1, p_2, q_2) \subset \tilde{W}(p_1', q_1', p_2, q_2)$$

3. $v \in V_{P_1} \setminus V_{P_2}$: This case is symmetrical to (2).

We can therefore conclude that for every $(p_1, p_2, q_1, q_2) \in Q_{\tilde{S}}$ and every possible successor state $(p_1', q_1', p_2', q_2') \in Q_{\tilde{S}}$

$$\mathrm{diag}\{\bar{M}_{P_1}(e_{P_1 P_2}), \bar{M}_{Q_1}(e_{Q_1 Q_2}), \bar{M}_{P_2}(e_{P_1 P_2}), \bar{M}_{Q_1 Q_2}(e_{Q_2})\} \tilde{W}(p_1, q_1, p_2, q_2) \subset \\ \subset \tilde{W}(p_1', q_1', p_2', q_2')$$

where $e_{P_1 P_2} = ((p_1, p_2), v, (p_1', p_2')) \in E_{P_1 P_2}$ and $e_{Q_1 Q_2} = ((q_1, q_2), v, (q_1', q_2')) \in E_{Q_1 Q_2}$. Reordering the components, S as defined in (7.10) indeed defines a structural hybrid simulation relation of $\Sigma_{P_1}^{\mathrm{SLS}} \parallel \Sigma_{P_2}^{\mathrm{SLS}}$ by $\Sigma_{Q_1}^{\mathrm{SLS}} \parallel \Sigma_{Q_2}^{\mathrm{SLS}}$. □

7.6. Assume-guarantee reasoning

Due to its triangular structure the non-circular assume-guarantee reasoning rule (7.2) can immediately be shown to hold true.

Theorem 7.17. *Consider switching linear systems $\Sigma_i^{SLS}, i \in \{P_1, P_2, Q_1, Q_2\}$ such that $V_{P_1} = V_{Q_1}, V_{P_2} = V_{Q_2}$ and assume that there exist full hybrid simulation relations S_1 and S_{II} of $\Sigma_{P_1}^{SLS}$ by $\Sigma_{Q_1}^{SLS}$ and of $\Sigma_{Q_1}^{SLS} \parallel \Sigma_{P_2}^{SLS}$ by $\Sigma_{Q_1}^{SLS} \parallel \Sigma_{Q_2}^{SLS}$, respectively. Then there also exists a full simulation relation S of $\Sigma_{P_1}^{SLS} \parallel \Sigma_{P_2}^{SLS}$ by $\Sigma_{Q_1}^{SLS} \parallel \Sigma_{Q_2}^{SLS}$, i.e., non-circular assume-guarantee reasoning is sound with respect to hybrid simulation.*

Proof. Theorem 7.16 ensures that interconnections of switching linear systems are compositional. Together with the transitivity property of hybrid simulation (cf. Proposition 7.12) the claim follows. □

It cannot be expected, however, that circular assume-guarantee reasoning is unconditionally sound. From Theorem 7.7 it is known that already on the level of the discrete dynamics, an additional condition is needed. It turns out that this condition (7.5) also ensures that there exists a hybrid simulation relation S of $\Sigma_{P_1}^{\text{SLS}} \parallel \Sigma_{P_2}^{\text{SLS}}$ by $\Sigma_{Q_1}^{\text{SLS}} \parallel \Sigma_{Q_2}^{\text{SLS}}$ given S_I and S_{II} as in (7.3). However, fullness of S is not guaranteed. This is formalized in the following theorem.

Theorem 7.18. *Consider any given switching linear systems $\Sigma_i^{SLS}, i \in \{P_1, P_2, Q_1, Q_2\}$, such that $V_{P_1} = V_{Q_1}, V_{P_2} = V_{Q_2}$ and assume there exist full hybrid simulation relations S_I and S_{II} of $\Sigma_{P_1}^{SLS} \parallel \Sigma_{Q_2}^{SLS}$ by $\Sigma_{Q_1}^{SLS} \parallel \Sigma_{Q_2}^{SLS}$ and of $\Sigma_{Q_1}^{SLS} \parallel \Sigma_{P_2}^{SLS}$ by $S_{Q_1} \|_{SLS} S_{Q_2}$, respectively. Then there exists a hybrid simulation relation S of $\Sigma_{P_1}^{SLS} \parallel \Sigma_{P_2}^{SLS}$ by $\Sigma_{P_1}^{SLS} \parallel \Sigma_{P_2}^{SLS}$ if and only if condition* (7.5) *for the associated labeled transition systems is fulfilled.*

Proof. First, if (7.5) does not hold, one can construct a counterexample in the same spirit as in [21], Example 4.2, where all components P_1, P_2, Q_1, Q_2 share the same set of of states Q, labels V and continuous dynamics $\Sigma(q)$. While $P_i, i = 1, 2$ allow for any transition $v \in V$ between any two states, i.e. $E_{P_i} = Q \times V \times Q$, the abstractions Q_i do not allow for any transition between any two states, $E_{Q_i} = \emptyset$.
Conversely, assume we are given hybrid simulation relations S_I and S_{II} of $\Sigma_{P_1}^{\text{SLS}} \parallel \Sigma_{Q_2}^{\text{SLS}}$ and $\Sigma_{Q_1}^{\text{SLS}} \parallel \Sigma_{P_2}^{\text{SLS}}$ by $\Sigma_{Q_1}^{\text{SLS}} \parallel \Sigma_{Q_2}^{\text{SLS}}$, respectively. Construct the relation

$$S := \Big\{(\xi_{P_1}, \xi_{P_2}, \xi_{Q_1}, \xi_{Q_2}) \mid \exists \hat{\xi}_{Q_1}, \hat{\xi}_{Q_2} : (\xi_{P_1}, \xi_{Q_2}, \xi_{Q_1}, \hat{\xi}_{Q_2}) \in S_I, \; (\xi_{Q_1}, \xi_{P_2}, \hat{\xi}_{Q_1}, \xi_{Q_2}) \in S_{II}\Big\} \quad (7.13)$$

We claim that S is a hybrid simulation relation of $\Sigma_{P_1}^{\text{SLS}} \parallel \Sigma_{P_2}^{\text{SLS}}$ by $\Sigma_{Q_1}^{\text{SLS}} \parallel \Sigma_{Q_2}^{\text{SLS}}$. From Theorem (7.5), we know that the associated labeled transition systems fulfill the circular assume-guarantee reasoning rule 3 if and only if condition

(7.5) holds. Furthermore, for the respective linear systems at every location $(p_1, p_2, q_1, q_2) \in Q_S$, circular assume-guarantee reasoning is sound as stated in Theorem 7.9. Thus, condition (i) in Theorem 7.15 is readily fulfilled.
Now take for any $(p_1, p_2, q_1, q_2) \in Q_S$ an arbitrary element $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in W_S(p_1, p_2, q_1, q_2)$. Define first for every location $(p_1, p_2, q_1, q_2) \in Q_S$ the subspaces

$$\begin{aligned}
W_{Q_1Q_2}(q_1, q_2) &= \{(x_{Q_1}, x_{Q_2}) \mid \exists x_{P_1}, \bar{x}_{Q_2}, x_{P_2}, \bar{x}_{Q_1} : (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in \\
&\quad W_I(p_1, q_2, q_1, \bar{q}_2), (x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in W_{II}(q_1, p_2, \bar{q}_1, q_2)\} \\
W_{P_1}(p_1) &= \{x_{P_1} \mid \exists x_{P_2}, x_{Q_1}, x_{Q_2} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in W(p_1, p_2, q_1, q_2)\} \\
W_{P_2}(p_2) &= \{x_{P_2} \mid \exists x_{P_1}, x_{Q_1}, x_{Q_2} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in W(p_1, p_2, q_1, q_2)\}
\end{aligned}$$

Since S_I and S_{II} are simulation relations, they fulfill condition (ii) in Theorem 7.15 so that for every $(p_1, q_2, q_1, \hat{q}_2) \in Q_{S_I}$ and every possible successor state $(p_1', q_2', q_1', \hat{q}_2') \in Q_{S_I}$

$$\begin{aligned}
&\mathrm{diag}\{\bar{M}_{P_1}(e_{P_1Q_2}), \bar{M}_{Q_2}(e_{P_1Q_2}), \bar{M}_{Q_1}(e_{Q_1Q_2}), \bar{M}_{Q_2}(e_{Q_1Q_2})\} \\
&\qquad W_I(p_1, p_2, q_1, \hat{q}_2) \subset W_I(p_1', q_2', q_1', \hat{q}_2')
\end{aligned}$$

and similarly for every $(q_1, p_2, \hat{q}_1, q_2) \in Q_{S_{II}}$ and every possible successor state $(q_1', p_2', \hat{q}_1', q_2') \in Q_{S_{II}}$

$$\begin{aligned}
&\mathrm{diag}\{\bar{M}_{Q_1}(e_{Q_1P_2}), \bar{M}_{P_2}(e_{Q_1P_2}), \bar{M}_{Q_1}(e_{Q_1Q_2}), \bar{M}_{Q_2}(e_{Q_1Q_2})\} \\
&\qquad W_{II}(q_1, p_2, \hat{q}_1, q_2) \subset W_{II}(q_1', p_2', \hat{q}_1', q_2')
\end{aligned}$$

It therefore holds for every $(p_1, p_2, q_1, q_2) \in Q_S$ and every possible discrete successor state $(p_1', p_2', q_1', q_2') \in Q_S$ that

$$\mathrm{diag}\{\bar{M}_{Q_1}(e_{Q_1Q_2}), \bar{M}_{Q_2}(e_{Q_1Q_2})\}Q_T(q_1, q_2) \subset W_{Q_1Q_2}(q_1', q_2')$$

and similarly

$$\bar{M}_{P_1}(e_{P_1Q_2})W_{P_1}(p_1) \subset W_{P_1}(p_1'), \bar{M}_{P_2}(e_{Q_1P_2})W_{P_2}(p_2) \subset W_{P_2}(p_2'),$$

Thus, condition (ii) is fulfilled which proves that S as in (7.13) is indeed a hybrid simulation relation of $\Sigma_{P_1}^{\mathrm{SLS}} \parallel \Sigma_{P_2}^{\mathrm{SLS}}$ by $\Sigma_{Q_1}^{\mathrm{SLS}} \parallel \Sigma_{Q_2}^{\mathrm{SLS}}$. □

7.7. Conclusions

We developed a framework for compositional analysis of switching linear systems as a special class of hybrid systems. Their particular structure, i.e. the fact that the discrete dynamics is independent of the continuous one, allows to define structural hybrid simulation relations that are easy to check and to apply. We proved that compositional and non-circular assume-guarantee reasoning are sound and gave a necessary and sufficient condition for the existence of a simulation relation S for the circular reasoning scheme (7.3). In the next chapter, we consider switching linear systems with inequality constraints as a more general class of hybrid systems.

8

Equivalences of switching linear systems with inequality constraints

In Chapter 7 we studied switching linear systems as a special class of hybrid systems for which the discrete part of the hybrid dynamics is independent of the continuous one. In this chapter we consider switching linear systems with location invariants and guard conditions as a generalization towards hybrid automata [77]. The location invariants are described as polyhedral constraints while the guards correspond to the boundaries of the polyhedra. As a result of incorporating location invariants the discrete dynamics are influenced by the continuous ones since they can trigger transitions to other discrete states. Synchronous fulfillment of location invariants therefore is a requirement for bisimulation equivalence for this subclass of hybrid systems. To investigate which extra conditions are needed we start this chapter by studying bisimulation relations for linear systems with inequality constraints. We obtain a characterization for bisimulation relations using results from convex geometry, in particular the Farkas lemma. Section 8.1 can be seen as an extension of the theory for linear systems presented in Chapter 2. In the second part of this chapter we outline how this result can be used to define structural hybrid bisimulation relations for switching linear systems with inequality constraints. This is a first step towards a comprehensive bisimulation theory for general hybrid automata.

8.1. Bisimulation theory for linear systems with inequality constraints

Constraints on variables are imposed in many real-life control applications, e.g. due to saturation of sensors and actuators. Expressing the constraints by affine inequalities restricts the state space to polyhedral domains.

Definition 8.1. A *polyhedron* $\mathcal{P} \subset \mathbb{R}^n$ is described as

$$\mathcal{P} := \{x \in \mathbb{R}^n \mid Kx \leq k\}, K \in \mathbb{R}^{q\times n}, k \in \mathbb{R}^q , \tag{8.1}$$

For the special case $k = 0$, (8.1) describes a *polyhedral cone*.

In the literature, various methods have been proposed how to analyze polyhedra, see e. g. the survey paper [7]. Invariant polyhedra for linear systems have been studied by [12] using the theory of essentially nonnegative matrices and a version of the Farkas lemma. In the following, we want to study bisimulation relations for linear systems with inequality constraints. The underlying motivation is our interest in hybrid systems given as switching linear systems with inequality constraints. In this context, the inequality constraints represent location invariants while the generating hyperplanes correspond to the guards of switching linear systems. The continuous dynamics of this type of hybrid system at every discrete location are given by a linear system with inequality constraints Σ_i^{con},

$$\Sigma_i^{\text{con}}: \quad \begin{array}{rcl} \dot{x}_i &=& A_i x_i + B_i u_i + L_i d_i \\ y_i &=& C_i x_i \\ K_i x_i &\leq& k_i \end{array} \tag{8.2}$$

with $u_i \in \mathcal{U}_i, y_i \in \mathcal{Y}_i$ and $d_i \in \mathcal{D}_i$ taken from appropriately dimensioned vector spaces. Hence, the state space of (8.2) is the polyhedron

$$\mathcal{P}_i = \left\{ x_i \in \mathbb{R}^{n_i} \mid K_i x_i \leq k_i, K_i \in \mathbb{R}^{q_i \times n_i}, k_i \in \mathbb{R}^{q_i} \right\} \tag{8.3}$$

Associated to every constrained system Σ_i^{con} is the corresponding unconstrained system Σ_i,

$$\Sigma_i: \quad \begin{array}{rcl} \dot{x}_i &=& A_i x_i + B_i u_i + L_i d_i \\ y_i &=& C_i x_i \end{array} \tag{8.4}$$

where $x_i \in \mathcal{X}_i = \mathbb{R}^{n_i}$, and u_i, y_i, d_i as before. Before defining and characterizing bisimulations for constrained systems Σ_i^{con} we first recall some important facts about polyhedra.

8.1.1. Facts about polyhedra

Polyhedra have been studied in great detail, see e.g. [26] or [82] for an overview. In the following, some of the basic notations and definitions are summarized. In Definition 8.1 polyhedra are described as the intersection of finitely many half-spaces. Equivalently, a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ can be represented as the convex hull of a finite set of points $v^1, \dots, v^N$,

$$\mathcal{P} = \text{conv}(V) = \text{conv}\left(\left[\begin{array}{cccc} v^1 & v^2 & \dots & v^N \end{array}\right]\right) \tag{8.5}$$

Converting one representation into the other is a non-trivial task, but can be achieved using Fourier-Motzkin elimination, see e.g. [82]. The interior points of $\mathcal{P}$ are characterized by the following

Lemma 8.2. *A point $x \in \mathcal{P}$ is an interior point, $x \in \text{int}\mathcal{P}$, if and only if it can be written as a positive combination of the N vertices generating the polyhedron,*

$$x = \Sigma_{i=1}^{N} \lambda_i v_i,\ \lambda_i > 0,\ \Sigma_{i=1}^{N} \lambda_i = 1$$

Faces describe the boundaries of polyhedral domains. In particular, for a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$, all faces of dimension $n-1$, the so-called facets, will be important.

Definition 8.3. Let $\mathcal{P} = \{x \in \mathbb{R}^n \mid Kx \leq k\} \subset \mathbb{R}^n$ be a polyhedron. Denote by $(K)^j$ and $(k)^j$ the $j-$th rows of K and k, respectively. A linear inequality $(K)^j x \leq (k)^j$ is *valid* if it is satisfied for all points $x \in \mathcal{P}$.
A *face* F of $\mathcal{P}$ is a set of the form

$$F = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid (K)^j x = (k)^j\}\ . \tag{8.6}$$

Faces of dimension 0, 1, $\dim(\mathcal{P}) - 2$ and $\dim(\mathcal{P}) - 1$ are called *vertices*, *edges*, *ridges* and *facets*, respectively. We denote the set of facets of a polyhedron $\mathcal{P}$ by $\mathcal{F}(\mathcal{P})$.

8.1.2. A solution approach using the Farkas lemma

We want to describe equivalences of constrained linear systems based on bisimulation relations. The definition of bisimulation relations for the constrained case will require invariance of the polyhedra and include a facet condition.

Definition 8.4. Given two constrained linear systems $\Sigma_i^{\text{con}}, i = 1, 2$, as defined in (8.2). A bisimulation relation R^{con} between Σ_1^{con} and Σ_2^{con} is a subset $R^{\text{con}} \subset \mathcal{P}_1 \times \mathcal{P}_2$ with the following properties:
Take any $(x_{10}, x_{20}) \in \text{int}\mathcal{P}_1 \times \text{int}\mathcal{P}_2$. Then for every state trajectory $x_1(\cdot)$ with $x_1(0) = x_{10}$, every joint input $u_1(\cdot) = u_2(\cdot) = u(\cdot)$ and every disturbance $d_1(\cdot)$ there should exist a trajectory $x_2(\cdot)$ with $x_2(0) = x_{20}$, a disturbance $d_2(\cdot)$ and a strictly positive time $T > 0 \in \mathbb{R}^+ \cup \{\infty\}$ such that

$$\begin{array}{rl} (i): & (x_1(t), x_2(t)) \in R^{\text{con}} \quad \forall t \in [0, T] \\ (ii): & C_1 x_1(t) = C_2 x_2(t) \quad \forall t \in [0, T] \\ (iii): & x_1(T) \in \mathcal{F}(\mathcal{P}_1) \iff x_2(T) \in \mathcal{F}(\mathcal{P}_2) \end{array} \tag{8.7}$$

Conversely, for any trajectory $x_2(\cdot)$ with $x_2(0) = x_{20}$ and any disturbance $d_2(\cdot)$ there should exist a trajectory $x_1(\cdot)$, $x_1(0) = x_{10}$, a disturbance $d_2(\cdot)$ and a time $T > 0 \in \mathbb{R}^+ \cup \{\infty\}$ such that (8.7) holds. Moreover, Σ_1 and Σ_2 are bisimilar, denoted $\Sigma_1^{\text{con}} \approx \Sigma_2^{\text{con}}$, if there exists a bisimulation relation R^{con} fulfilling $\Pi_i R^{\text{con}} = \mathcal{P}_i, i = 1, 2$, with $\Pi_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_i}$ the canonical projection from $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ to $\mathbb{R}^{n_i}$. In this case, R^{con} is called a *full* bisimulation relation.

In order to obtain a linear-algebraic characterization of bisimulations for linear systems with inequality constraints, some results from the theory of nonnegative matrices (see e. g. [6]) will be needed.

Definition 8.5. A matrix $H = (h_{ij}) \in \mathbb{R}^{q\times r}$ is nonnegative, denoted by $H \geq 0$, if $h_{ij} \geq 0$ for all $i = 1, \ldots, q, j = 1, \ldots, r$.

Theorem 8.6. *The set of all non-negative $n \times n$-matrices forms a semi-group under matrix multiplication, denoted by $\mathcal{N}_n$.*

Definition 8.7. An element $A \in \mathcal{N}_n$ is called regular if $A = ABA$ for some element $B \in \mathcal{N}_n$. If in addition $B = BAB$, then A and B are called *semi-inverses* of each other.

Definition 8.8. An $n \times n$-matrix of rank $r \leq n$ is called $r-$monomial if each of its columns contains at most one nonzero entry. If $r = n$, then the matrix is called a monomial matrix or a generalized permutation matrix.

Theorem 8.9. *Let $A \in \mathcal{N}_n$ be a matrix of rank r. Then the following statements are equivalent:*

1. *A is regular in $\mathcal{N}_n$.*
2. *A has a semi-inverse in $\mathcal{N}_n$ which is $r-$ monomial.*
3. *A has a monomial submatrix of order r.*

The Farkas lemma is a very important lemma for the study of polyhedra. It characterizes the solvability of systems of linear inequalities and can occur in a lot of different variants. We first present a version formulated as a separation theorem, that is, a point either lies in a polyhedron or is separated from it by a hyperplane.

Lemma 8.10. *Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^{n\times 1}$, the following two statements are equivalent:*

$$\begin{aligned} (i): &\quad \exists x \geq 0 \text{ such that } Ax = b \\ (ii): &\quad \forall y: \; y^T A \geq 0 \Longrightarrow y^T b \geq 0 \end{aligned} \tag{8.8}$$

A matrix version of the Farkas lemma first proved by [28] will be used in the following to transform linear inequality constraints.

Lemma 8.11. *The set $\{x \in \mathbb{R}^n \,|\, K_1 x \leq k_1, K_1 \in \mathbb{R}^{q_1\times n}, k_1 \in \mathbb{R}^{q_1}\}$ is contained in $\{x \in \mathbb{R}^n \,|\, K_2 x \leq k_2, K_2 \in \mathbb{R}^{q_2\times n}, k_2 \in \mathbb{R}^{q_2}\}$ if and only if there exists a nonnegative matrix $M \in \mathbb{R}^{q_2\times q_1}$ such that*

$$MK_1 = K_2 \quad \textit{and} \quad Mk_1 \leq k_2$$

To characterize bisimulation relations for linear systems with inequality constraints we restrict the geometry of the polyhedral domains.

Assumption 8.12. *In the remainder, we assume that the polyhedral constraints*

$$K_i x_i \leq q_i, \; K_i \in \mathbb{R}^{q_i \times n_i} \; i = 1, 2,$$

satisfy

$$\text{rank} K_1 = q_1 = \text{rank} K_2 = q_2 =: q, \; q \leq n_i \tag{8.9}$$

The question of how restrictive Assumption 8.12 is will be discussed in Remark 8.19. We now give a sufficient condition for the existence of bisimulation relation between two linear systems with inequality constraints.

Theorem 8.13. *Given two constrained linear systems $\Sigma_i^{con}, i = 1, 2$, of the form (8.2) and assume that the constraints satisfy Assumption 8.12. Then a relation $R^{con} \subset \mathcal{P}_1 \times \mathcal{P}_2$ is a bisimulation relation between Σ_1^{con} and Σ_2^{con} if*

1. *the linear closure $\mathcal{L}(R^{con})$ is a full bisimulation relation of the corresponding unconstrained systems Σ_i, and*
2. *there exist nonnegative generalized permutation matrices $M, N \in \mathcal{N}_q$ such that for all $(x_1, x_2) \in R^{con}$*

$$\begin{array}{ccccccc} MK_1x_1 & = & K_2x_2 & , & K_1x_1 & = & NK_2x_2 \\ Mk_1 & \leq & k_2 & , & Nk_2 & \leq & k_1 \end{array} \tag{8.10}$$

Proof. Suppose we are given a relation R^{con} such that its linear closure $\mathcal{L}(R^{\text{con}})$ is a full bisimulation relation between the unconstrained systems $\Sigma_i, i = 1, 2$. By Theorem 2.6, $\mathcal{L}(R^{\text{con}})$ is a linear subspace contained in $\mathcal{X}_1 \times \mathcal{X}_2$ and can therefore be written in image representation,

$$\mathcal{L}(R^{\text{con}}) = \text{im} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} .$$

We denote the dimension of $\mathcal{L}(R^{\text{con}})$ by r. Thus, the state variables are related,

$$\forall \, (x_1, x_2) \in \mathcal{L}(R^{\text{con}}) : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \lambda, \; \lambda \in \mathbb{R}^r . \tag{8.11}$$

Based on this we reformulate condition (8.10) as

$$\exists M, N \in \mathcal{N}_q : \begin{array}{ccccccc} MK_1R_1\lambda & = & K_2R_2\lambda & , & K_1R_1\lambda & = & NK_2R_2\lambda \\ Mk_1 & \leq & k_2 & , & Nk_2 & \leq & k_1 \end{array} \tag{8.12}$$

for some $\lambda \in \mathbb{R}^r$. By the Farkas lemma, (8.12) is equivalent to

$$\begin{array}{ccc} K_1R_1\lambda \leq k_1 & \Longrightarrow & K_2R_2\lambda \leq k_2 \\ K_1R_1\lambda \leq k_1 & \Longleftarrow & K_2R_2\lambda \leq k_2 \end{array} \tag{8.13}$$

being satisfied for any $\lambda \in \mathbb{R}^r$ or, equivalently,

$$K_1 x_1 \leq k_1 \quad \Longleftrightarrow \quad K_2 x_2 \leq k_2 \ . \tag{8.14}$$

for all $(x_1, x_2) \in \mathcal{L}(R^{\text{con}})$. Consider now a pair of initial states $(x_{10}, x_{20}) \in \text{int}\mathcal{P}_1 \times \text{int}\mathcal{P}_2 \subset R^{\text{con}}$ and corresponding trajectories $x_i(t), i = 1, 2$, starting at $x_{i0} = x_i(0)$. Then condition (i) in Definition 8.4 is fulfilled because $(x_1(\cdot), x_2(\cdot))$ satisfies (8.14). The bisimulation subspace $\mathcal{L}(R^{\text{con}})$ is a subspace of $\ker \begin{bmatrix} C_1 & -C_2 \end{bmatrix}$, hence condition (ii) is fulfilled too. Moreover, since M is a generalized permutation matrix, if $x_1(\cdot)$ reaches a facet of $\mathcal{P}_1$ at time T, i.e. $(K_1)^j x_1(T) = (k_1)^j$ for some $j \in \{1, \ldots, q\}$, then at the same time T the trajectory $x_2(\cdot)$ reaches a facet of $\mathcal{P}_2$, $M(K_1)^j x_1(T) = (K_2)^l x_2(T) = M(k_1)^j = (k_2)^l$ for some $l \in \{1, \ldots, q\}$ and vice versa. Thus, condition (iii) is also fulfilled. □

Conditions 1 and 2 of Theorem 8.13 are very close to being necessary as well.

Proposition 8.14. *Given two linear constrained systems $\Sigma_i^{con}, i = 1, 2$, of the form* (8.2) *satisfying Assumption 8.12. Suppose there exists a bisimulation relation R^{con} between Σ_1^{con} and Σ_2^{con} such that*

$$\Pi_{\mathcal{X}_i} \mathcal{L}(R^{con}) = \mathcal{X}_i, i = 1, 2 \ . \tag{8.15}$$

Then conditions 1 and 2 of Theorem 8.13 hold true.

Proof. For two given linear systems with inequality constraints $\Sigma_i^{\text{con}}, i = 1, 2$, such that Assumption 8.12 holds, let R^{con} define a bisimulation relation between Σ_1^{con} and Σ_2^{con} satisfying (8.15).
We first show that due to linearity the linear closure $\mathcal{L}(R^{\text{con}})$ has to be a full bisimulation relation for the corresponding unconstrained systems Σ_i. Consider a pair of initial states $(x_{10}, x_{20}) \in \text{int}\mathcal{P}_1 \times \text{int}\mathcal{P}_2$ and any pair of trajectories $x_i(\cdot)$ starting from $x_{i0} = x_i(0)$. If (x_{10}, x_{20}) is an element of a bisimulation relation R^{con} between the two constrained systems Σ_i^{con} then there exists a $T > 0$ such that $(x_1(t), x_2(t) \in R^{\text{con}}$ and $y_1(t) = y_2(t)$ for all $0 \leq t \leq T$. Thus, for $0 \leq t \leq T$, R^{con} fulfills conditions 1 – 4 of Theorem 2.7. But then the linear closure of R^{con}, $\mathcal{L}(\mathcal{R}^{\text{con}})$ should also satisfy conditions 1 – 4 of Theorem 2.7. Note that due to linearity we only need a time interval of positive length to satisfy the invariance conditions therein. This implies that $\mathcal{L}(\mathcal{R}^{\text{con}})$ is a bisimulation relation between the corresponding unconstrained systems. Fullness is guaranteed by (8.15).
Next, we show that (8.10) is also a necessary condition for the existence of a bisimulation relation R^{con} between the two constrained systems Σ_i^{con}. By Definition 8.4, all elements $(x_1, x_2) \in R^{\text{con}}$ have to lie within the product of the two polyhedra,

$$K_1 x_1 \leq k_1 \quad \Longleftrightarrow \quad K_2 x_2 \leq k_2 \tag{8.16}$$

Moreover, since $\mathcal{L}(R^{\text{con}})$ is a bisimulation subspace, (8.11) holds so that (8.16) becomes

$$K_1R_1\lambda \leq k_1 \quad \Longleftrightarrow \quad K_2R_2\lambda \leq k_2 \tag{8.17}$$

Splitting (8.17) into two implications and applying the Farkas lemma to both, (8.17) turns out to be equivalent to (8.12). Due to fullness of $\mathcal{L}(R^{\text{con}})$ and Assumption 8.12, it follows from $NMK_1R_1 = NK_2R_2 = K_1R_1$ and $MNK_2R_2 = MK_1R_1 = K_2R_2$ that $NM = I_q = MN$ and thus $MNM = M, NMN = N$. Hence, M and N are regular in $\mathcal{N}_q$. Moreover, due to (8.9), $\text{rank}M = \text{rank}N = q$. Theorem 8.9 then assures that M and N are generalized permutation matrices. Thus, we recovered condition (8.10). □

Proposition 8.14 and Theorem 8.13 almost fully characterize bisimulation relations for linear systems with inequality constraints. However, not much is known about when conditions 1 and 2 are *not* necessary. Moreover, these conditions are not easy to check. From an algorithmic perspective, Theorem 8.16 suggests the following procedure to construct a bisimulation relation between two constrained linear systems $\Sigma_i^{\text{con}}, i = 1, 2$, satisfying Assumption 8.12:

1. Compute the maximal bisimulation relation $R^\star$ between the corresponding unconstrained systems Σ_i following Algorithm 2.8. If $R^\star = \emptyset$ there does not exist any bisimulation relation between Σ_1^{con} and Σ_2^{con}. If $R^\star$ is not full, the procedure stops without result.

2. Check whether there exists a nonnegative generalized permutation matrix M such that

$$R^\star \subset \ker \begin{bmatrix} MK_1 & -K_2 \end{bmatrix} . \tag{8.18}$$

   Otherwise, check whether a lower dimensional bisimulation subspace $R \subset R^\star, \Pi_{\mathcal{X}_i} R = \mathcal{X}_i$, satisfies (8.18). If (8.18) is not fulfilled for any R the procedure stops without result.

3. Construct R^{con} as $R^{\text{con}} = R \cap (\mathcal{P}_1 \times \mathcal{P}_2)$ with R the result of step 2. In this case, R^{con} is also maximal, i.e. for any other bisimulation relation $\tilde{R}^{\text{con}}$ between Σ_1^{con} and Σ_2^{con}, $\tilde{R}^{\text{con}} \subset R^{\text{con}}$.

Next, we consider the special case that the state spaces of the corresponding unconstrained systems are related a priori by a surjective map. These so-called $H-$related systems were defined in [56] to obtain abstractions of systems.

Definition 8.15. Consider two linear systems of the form (8.4) and a surjective linear mapping $H : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ Then Σ_2 is H-related to Σ_1 if and only if

$$A_2H = HA_1, B_2 = HB_1, G_2 = [HG_1, HA_1v_1, \ldots, HA_1v_k] \quad C_2H = C_1, \tag{8.19}$$

where $\ker H = \text{span}\{v_1, \ldots, v_k\}$.

The existence of a state space mapping ensures the existence of a bisimulation relation between Σ_1 and Σ_2.

Proposition 8.16. *Given two $H-$related systems $\Sigma_i, i = 1, 2$. Then $\Sigma_1 \approx \Sigma_2$ with a full bisimulation relation R between Σ_1 and Σ_2 given by the graph of H,*

$$R = \{(x_1, x_2) \mid x_2 = Hx_1\} . \tag{8.20}$$

Proof. The relation R in (8.20) clearly fulfills the conditions of Theorem 2.6. Moreover, $\Pi_1 R = \mathcal{X}_1$ and since the mapping H is surjective, also $\Pi_2 R = \mathcal{R}_2$. ☐

Proposition 8.16 allows us to give a full characterization for bisimulation relations for constrained $H-$ related systems.

Corollary 8.17. *Consider two constrained systems $\Sigma_i^{con}, i = 1, 2$, of the form (8.2). Assume that the polyhedral constraints satisfy Assumption 8.12 and let the corresponding unconstrained dynamics be $H-$related. Then a subset $R^{con} = \mathcal{P}_1 \times \mathcal{P}_2$ is a bisimulation relation between Σ_1^{con} and Σ_2^{con} if and only if*

1. *the linear closure $\mathcal{L}(R^{con})$ is a full bisimulation relation between the corresponding unconstrained systems $\Sigma_i, i = 1, 2$, and*
2. *there exist nonnegative generalized permutation matrices $M, N \in \mathcal{N}_q$ such that* (8.10) *holds for all $(x_1, x_2) \in R^{con}$.*

Proof. Sufficiency is a direct consequence of Theorem 8.13. For necessity recall that since Σ_1 and Σ_2 are $H-$related, the graph of H defines a full bisimulation relation $R = \{(x_1, x_2) \mid x_2 = Hx_1\}$ between them. Suppose now there exists a bisimulation relation R^{con} between Σ_1^{con} and Σ_2^{con}. Condition (i) in Definition 8.4 can then be rewritten as

$$K_1 x_1 \le k_1 \quad \Longleftrightarrow \quad K_2 H x_1 \le k_2 \tag{8.21}$$

Applying the Farkas lemma to (8.21), there have to exist nonnegative matrices $M, N \in \mathcal{N}_q$ such that

$$\begin{array}{rclcrcl} MK_1 & = & K_2 H & , & K_1 & = & NK_2 H \\ Mk_1 & \le & k_2 & , & k_1 & \le & Nk_2 \end{array} \tag{8.22}$$

The first line of (8.22) implies that $MK_1 x_1 = x_2$ for all $(x_1, x_2) \in R^{\mathrm{con}}$. Since due to Proposition 8.14 $\mathcal{L}(R^{\mathrm{con}})$ has to be a bisimulation relation between Σ_1 and Σ_2, it follows that

$$\mathcal{L}(R^{\mathrm{con}}) \subset \ker \begin{bmatrix} MK_1 & -K_2 \end{bmatrix} = \ker \left[K_2 \begin{bmatrix} H & -I \end{bmatrix} \right] \tag{8.23}$$

However, (8.23) is always fulfilled by $R = \mathrm{graph} H$. Hence, if there exists a bisimulation relation R^{con}, it can be constructed as

$$R^{\mathrm{con}} := \mathrm{graph} H \cap (\mathcal{P}_1 \times \mathcal{P}_2) \tag{8.24}$$

so that $\mathcal{L}(R^{\text{con}}) = \text{graph}H$ is a full bisimulation relation between Σ_1 and Σ_2. Then, repeating the same arguments as before, (8.22) implies that $MN = NM = I_q$ and hence M and N are generalized permutation matrices. □

Corollary 8.17 also indicates how to compute the bisimulation relation between two constrained systems $\Sigma_i^{\text{con}}, i = 1, 2$, with $H-$related dynamics. First, check whether there exists a nonnegative generalized permutation matrix M such that $MK_1 = K_2 H$. If so, R^{con} as defined in (8.23) is a bisimulation relation between Σ_1^{con} and Σ_2^{con}. Otherwise, there does not exist any such bisimulation relation.

Example 8.18. Consider the following $H-$related constrained systems

$$\Sigma_1^{\text{con}}: \quad \begin{aligned} \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d_1 \\ y_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_1 \\ \mathcal{P}_1 &= \{(x^1, x^2) \mid x^1 \leq 0\} \end{aligned}$$

and

$$\Sigma_2^{\text{con}}: \quad \begin{aligned} \dot{z} &= z + u_2 \\ y_2 &= 1 \\ \mathcal{P}_2 &= \{z \mid z \leq 0\} \end{aligned}$$

and the surjective mapping $H : \mathbb{R}^2 \to \mathbb{R}^1$ defined as

$$z = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

A quick computation reveals that condition (8.22) is fulfilled with $M = N = 1$. The graph of H is given by

$$R = \text{graph}H = \{((x^1, x^2), z) \mid x^2 = 0, x^1 = z\}$$

Hence, a bisimulation relation R^{con} between the linear systems Σ_1^{con} and Σ_2^{con} is given by

$$R^{\text{con}} = \{(x^1, 0), z) \mid x^1 \leq 0, z \leq 0, x^1 = z\}$$

Remark 8.19. Assumption (8.9) is clearly restrictive since it limits the number of facets of the polyhedra to be no greater than the dimension of the embedding space. An approach to relax this could be to find a facet mapping between the two polyhedra $\mathcal{P}_1$ and $\mathcal{P}_2$: First, enlarge $K_i, k_i, i = 1, 2$, by multiples of existing rows to obtain $\tilde{K}_i, \tilde{k}_i$ of dimensions $q_1 q_2 \times n_i, q_1 q_2 \times 1$. Then, the existence of generalized permutation matrices $\tilde{M}, \tilde{N}$ such that

$$\begin{aligned} \tilde{M}\tilde{K}_1 &= \tilde{K}_2 \quad \text{and} \quad \tilde{M}k_1 \leq \tilde{k}_2 \\ \tilde{K}_1 &= \tilde{N}\tilde{K}_2 \quad \text{and} \quad k_1 \leq \tilde{N}\tilde{k}_2 \end{aligned} \tag{8.25}$$

would establish a relation between the respective facets of the two polyhedra and could thus guarantee condition (iii) of Definition 8.4.

8.2. Bisimulation relations for switching linear systems with inequality constraints

Adding inequality constraints to the definition of switching linear systems introduced in Chapter 7 is an important step towards general hybrid automata. In practice, many models of hybrid systems require inequality constraints, for example the simple model of a thermostat or complementarity systems [11]. However, these constraints increase the level of difficulty. Compared to Chapter 7, switching linear systems with inequality constraints exhibit mutual dependencies of the discrete and continuous dynamics. As before, the occurrence of a discrete event leads to a discrete transition accompanied by a change of the continuous dynamics with prior reset of the continuous state. Additionally, the continuous dynamics have to satisfy an invariance condition at every discrete location. If imminent to be violated the continuous evolution triggers a change of the discrete location. These transitions are controlled by guards which are related to the boundary of the constraints as illustrated in Figure 8.1.

8.2.1. Definition and semantics of switching linear systems with inequality constraints

Our definition of switching linear systems with location invariants is derived from the general definition of hybrid automata that can be found in [77].

Definition 8.20. A switching linear system with inequality constraints Σ^{SLScon} is described by a tuple $\Sigma^{\mathrm{SLScon}} = (Q, \mathcal{P}, V, \mathcal{W}, \mathcal{F}, \mathcal{E})$ where the symbols have the following meaning:

- Q is a finite set of discrete locations (or discrete states).
- $\mathcal{P}(q)$ is a polyhedron, given as linear inequalities

$$\mathcal{P}(q) = \{x \in \mathbb{R}^{n(q)} \mid K(q)x \leq k(q)\} \, , \tag{8.26}$$

  representing the constrained continuous part of the hybrid state space at every location $q \in Q$.
- V is a finite set of event labels.
- $\mathcal{W} = \mathcal{U} \times \mathcal{D} \times \mathcal{Y}$ denotes the continuous communication variables taken from linear spaces of appropriate dimensions, with $u \in \mathcal{U}, d \in \mathcal{D}$ and $y \in \mathcal{Y}$ the continuous inputs, disturbances and outputs, respectively.

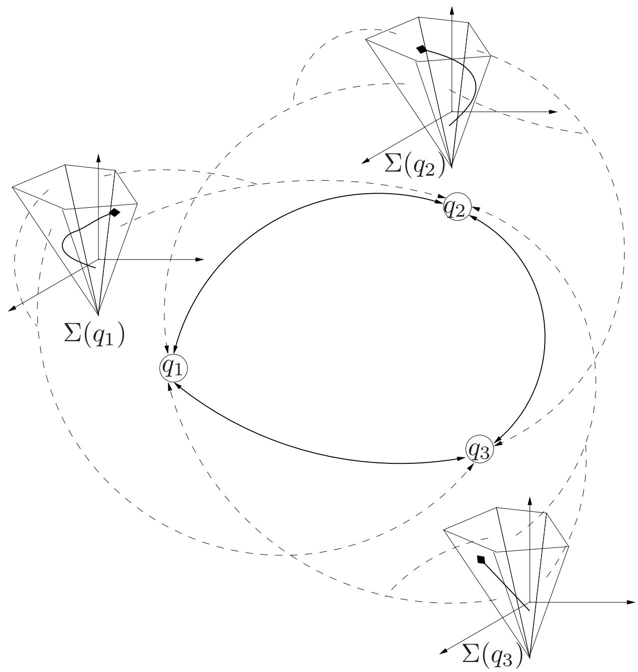

Figure 8.1.: Switching linear systems with location invariants and guards.

- The flow conditions $\mathcal{F}$ are described by differential equations of the form (8.4).
- The event conditions $\mathcal{E}$ are defined by a set of linear equations,

$$(q, x, v, q', x') \in \mathcal{E} \Longleftrightarrow \begin{array}{rcl} (q, v, q') & \in & E \\ x' & = & M(q, q')x \\ (K(q))^j x & = & (k(q))^j \text{ for some } j \end{array} \tag{8.27}$$

  where $E \subset Q \times V \times Q$ is the discrete transition relation, x and x' describe the continuous variables just before and after a transition from q to q' and $(K(q))^j x = (k(q))^j$ represents the guard condition for this transition.

The hybrid state space Δ^{SLScon} of a switching linear system Σ^{SLScon} is given by the product $\Delta^{\text{SLScon}} = \bigcup_{q \in Q} \{q\} \times \mathcal{P}(q)$.

Remark 8.21. Note that if x lies in a lower-dimensional face of $\mathcal{P}(q)$, e.g. a vertex, several guard conditions $(K(q))^j x- = (k(q))^j$ could be fulfilled simultaneously in (8.27). This non-determinism is characteristic for switching linear systems with inequality constraints.

From the above definition the structure of switching linear systems with inequality constraints Σ^{SLScon} is described as follows: On the level of the discrete dynamics, there exists a labeled transition system associated with Σ^{SLScon} and denoted by D^{SLScon},

$$D^{\text{SLScon}} = (Q, V, E) \tag{8.28}$$

The continuous dynamics in a discrete location are described by a linear system with inequality constraints $\Sigma^{\text{con}}(q)$ of the form (8.2). The location invariant is determined by a system of linear inequalities $K(q)x \leq k(q)$ while the guard conditions correspond to facets $\mathcal{F}(\mathcal{P}(q))$ of the polyhedron $\mathcal{P}(q)$. In the remainder, we assume that switching linear systems with inequality constraints are *deadlock-free*[1]. In particular, every facet $\mathcal{F}(\mathcal{P}(q))$ guards a transition to another discrete location, i.e., as soon as a guard condition is satisfied a transition of the discrete state is triggered to continue the continuous evolution of the hybrid dynamics. The semantics of a switching linear systems with inequality constraints are described similarly to Definition 7.3 as follows.

Definition 8.22. A hybrid execution r of a switching linear system $\Sigma^{\text{SLScon}} = (Q, \mathcal{P}, V, \mathcal{W}, \mathcal{F}, \mathcal{E})$ on a time interval $[0, T]$ is a collection $(\mathcal{T}, q, x, v, w)$ where

- $\mathcal{T} \subset [0, T]$ is a finite set $\{\tau_1, \tau_2, \ldots\}$ of ordered discrete event times,
- $q : [0, T] \to Q$ is a function from the time interval $[0, T)$ to the set of locations describing the evolution of the discrete states, which is constant on every interval $[\tau_i, \tau_{i+1}), i = 1, 2, \ldots,$
- $x : [0, T] \to \mathcal{X}(q)$ and $u : [0, T] \to \mathcal{U}$, $d : [0, T] \to \mathcal{D}$, $y : [0, T] \to \mathcal{Y}$ are time functions satisfying for every $t \notin \mathcal{T}$ the flow conditions (8.4),
- $v : \mathcal{T} \to V$ a function for the event labels satisfying for every $\tau \in \mathcal{T}$ the event condition $(q(\tau^-), x(\tau^-), v, q(\tau^+), x(\tau^+)) \in \mathcal{E}$ where $q(\tau^-) = \lim_{t\uparrow\tau} q(t), q(\tau^+) = \lim_{t\downarrow\tau} q(t)$ denote the discrete locations involved in the transition $(q(\tau^-), v, q(\tau^+)) \in E$ and $x(\tau^-) = \lim_{t\uparrow\tau} x(t), x(\tau^+) = \lim_{t\downarrow\tau} x(t)$ the continuous states just before and after the event time τ,
- $w = (u, d, y)$ are time functions $u : [0, T] \to \mathcal{U}, d : [0, T] \to \mathcal{D}, y : [0, T] \to \mathcal{Y}$ of the continuous inputs, disturbances and outputs valid for every $t \neq \mathcal{T}$.

8.2.2. Structural bisimulation relations for switching linear systems with inequality constraints

A standard way of defining bisimulation relations for hybrid systems is based on matching executions. Definition 7.11 is also valid for switching linear systems with inequality constraints. We now want to develop a structural notion

[1] A switching linear system Σ^{SLScon} is deadlock-free if every hybrid execution can be extended to an execution defined on an infinite time interval

of hybrid bisimulation. To do so, we recall Proposition 7.13 to exploit the particular structure of the bisimulation subsets defined on the hybrid state space.

Proposition 8.23. *Let the set $R \subset \Delta_1 \times \Delta_2$ be a hybrid bisimulation relation between the switching linear systems $\Sigma_i^{SLScon}, i = 1, 2$. Then there exists $Q_R \subset Q_1 \times Q_2$ and for every $(q_1, q_2) \in Q_\mathcal{R}$ suitable sets $W(q_1, q_2) \subset \mathcal{P}_1(q_1) \times \mathcal{P}_2(q_2)$ such that*

$$(q_1, x_1, q_2, x_2) \in R \Longleftrightarrow (q_1, q_2) \in Q_R, (x_1, x_2) \in W(q_1, q_2) \tag{8.29}$$

Note that in contrast to Chapter 7 $W(q_1, q_2) \subset \mathcal{P}_1(q_1) \times \mathcal{P}_2(q_2)$ cannot be assumed to have any nice properties. In fact, $W(q_1, q_2)$ need not be a polyhedron itself. Nevertheless, we can define structural hybrid bisimulations for switching linear systems with inequality constraints as subsets of the form (8.29).

Definition 8.24. Consider two switching linear systems $\Sigma_i^{\text{SLScon}} = (Q_i, \mathcal{P}_i, V_i, \mathcal{W}_i, \mathcal{F}_i, \mathcal{E}_i), i = 1, 2$. A *structural hybrid bisimulation relation* R between Σ_1^{SLScon} and Σ_2^{SLScon} is a subset $R \subset \Delta_1 \times \Delta_2$ with the following property. Take any $(q_1, x_1, q_2, x_2) \in R$. Then for every q_1', x_1' for which $(q_1, x_1, v, q_1', x_1') \in \mathcal{E}_1$ there should exist q_2', x_2' such that $(q_2, x_2, v, q_2', x_2') \in \mathcal{E}_2$ while $(q_1', x_1', q_2', x_2') \in R$, and conversely. Furthermore, take any $(q_1, x_1, q_2, x_2) \in R$. Then for every joint continuous input $u_1 = u_2$ and every disturbance d_1 there should exist a disturbance d_2 such that $y_1 = y_2$ while $(A_1x_1 + B_1u + G_1d_1, A_2x_2 + B_2u + G_2d_2) \in W(q_1, q_2)$ and conversely.

Definition 8.24 facilitates a characterization of bisimulation relation for switching linear systems with inequality constraints.

Theorem 8.25. *Given two switching linear systems with inequality constraints $\Sigma_i^{SLScon}, i = 1, 2$. A relation*

$$R = \{(q_1, x_1, q_2, x_2) \mid (q_1, q_2) \in Q_\mathcal{R}, (x_1, x_2) \in W(q_1, q_2)\} \tag{8.30}$$

is a hybrid bisimulation relation between Σ_1^{SLScon} and Σ_2^{SLScon} if and only if the following holds:

1. *$Q_R(q_1, q_2)$ is a bisimulation relation of the associated labeled transition systems $D_i^{SLScon}, i = 1, 2$*
2. *for every $(q_1, q_2) \in Q_R$, $W(q_1, q_2)$ is a bisimulation relation between the corresponding linear systems with inequality constraints $\Sigma_1^{con}(q_1)$ and $\Sigma_2^{con}(q_2)$*
3. *for every $(q_1, x_1, v, q_1', x_1') \in \mathcal{E}_1$ there should exist q_2', x_2' such that $(q_2, x_2, v, q_2', x_2') \in \mathcal{E}_2$ and*

$$(x_1', x_2') \in W(q_1', q_2') \tag{8.31}$$

and vice versa, for every $(q_2, x_2, v, q_2', x_2') \in \mathcal{E}_2$ there should exist q_1', x_1' such that $(q_1, x_1, v, q_1', x_1') \in \mathcal{E}_1$ and (8.31) holds.

Proof. Suppose that R given by (8.30) is a bisimulation relation between Σ_i^{SLScon}, $i = 1, 2$ and consider an execution $r_1 = (\mathcal{T}_1, q_1, x_1, v_1, w_1)$ of Σ_1^{SLScon} with $\mathcal{T}_1 = \{\tau_1^1, \tau_1^2, \ldots\}$. Start with the time interval $[0, \tau_1^1]$ during which $\dot{x}_1(t) = A_1x_1(t)+B_1u_1(t)+G_1d_1(t), y_1(t) = C_1x_1(t)\ , t \in [0, \tau_1^1)$, $(q_1(\tau_1^{1,-}), v, q_1(\tau_1^{1,+})) \in D_1^{\text{SLScon}}$ and

$$(q_1(\tau_1^{1,-}), x_1(\tau_1^{1,-}), v_1, q_1(\tau_1^{1,+}), x_1(\tau_1^{1,+})) \in \mathcal{E}_1$$

where $\tau_1^{1,-}, \tau_1^{1,+}$ denote the time instants just before and after τ_1^1. By condition 2 and since $\Sigma_i^{\text{SLS}}, i = 1, 2$, are deadlock-free, there exists a d_2 such that $(A_1x_1 + B_1u_1 + G_1d_1, A_2x_2 + B_2u_2 + G_2d_2)_{|_{\tau_1^{1,-}}} \in W(q_1(\tau_1^{1,-}), q_2(\tau_2^{1,-}))$ and $y_1(t) = y_2(t)$ for $t \geq 0$. Moreover, since there exists a bisimulation relation between $\Sigma_1^{\text{con}}(q_1(\tau_1^{1,-}))$ and $\Sigma_2^{\text{con}}(q_2(\tau_2^{1,-}))$ and at time $\tau_1^{1,-}$, $x_1(\tau_1^{1,-}) \in \mathcal{F}(\mathcal{I}(q_1(\tau_1^{1,-}))$, it follows from (8.7), (iii) that also $x_2(\tau_1^{1,-}) \in \mathcal{F}(\mathcal{I}(q_2(\tau_1^{1,-}))$. The first discrete switch in execution r_1 occurs at τ_1^1, and since $\Sigma_2^{\text{con}}(q_2)$ fulfills its guard condition $x_2(\tau_1^{1,-}) \in \mathcal{F}(\mathcal{I}(q_2(\tau_1^{1,-}))$, it makes a transition at the same time $\tau_2^1 = \tau_1^1 =: \tau^1$. The existence of a bisimulation relation between D_1^{SLScon} and D_2^{SLScon} ensures that there exist $q_2(\tau^{1,-}), q_2(\tau^{1,+})$ such that $(q_2(\tau^{1,-}), v, q_2(\tau^{1,+})) \in D_2^{\text{SLScon}}$ such that $(q_1(\tau^{1,-}), q_2(\tau^{1,-})) \in Q_R$ as well as $(q_1(\tau^{1,+}), q_2(\tau^{1,+})) \in Q_R$. Condition 3 of Theorem 8.25 then ensures that $(x_1(\tau^{1,+}), x_2(\tau^{1,+})) \in W(q_1(\tau^{1,+}), q_2(\tau^{1,+}))$. Repeating the same arguments starting with an execution r_2 of Σ_2^{SLScon} and inductively proceeding in time considering intervals $[\tau^i, \tau^{i+1}], i = 1, 2, \ldots$ completes this part of the proof.
For the converse, suppose there exists a hybrid bisimulation relation $R \subset Q_1 \times \mathcal{P}_1 \times Q_2 \times \mathcal{P}_2$ in the sense of Definition 7.11. Assume there does not exist any bisimulation relation between the associated labeled transition systems D_i^{SLScon}. Then for any v such that $r_1 = (\mathcal{T}_1, q_1, x_1, v_1, w_1)$ is an execution of Σ_1^{SLScon} there does not exist any execution r_2 of Σ_2^{SLScon} such that (7.8), (iii) holds. Similarly, one can prove by contradiction that condition 2 of Theorem 8.25 is necessary. Next, consider any execution r_1 of Σ_1^{SLScon} such that there exists an execution r_2 of Σ_2^{SLScon} satisfying conditions (7.8). For any $\tau \in \mathcal{T}$, denote by τ^- and τ^+ the time instants just before and after the switching event τ. Then by (7.8), $y_1(\tau^\star) = y_2(\tau^\star)$, $v_1 = v_2$ and $(q_1(\tau^\star), x_1(\tau^\star), q_2(\tau^\star), x_2(\tau^\star)) \in R$ for $\star \in \{-, +\}$. Since $W(q_1(\tau^\star), q_2(\tau^\star)$ is a bisimulation relation of the continuous-time systems $\Sigma_i^{\text{con}}(q_i(\tau^\star)), i = 1, 2$, it follows by definition of $\mathcal{E}_i$ that

$$(q_i(\tau^-), x_i(\tau^-), v, q_i(\tau^+), x(\tau^+)) \in \mathcal{E}_i$$

and $(x_1(\tau^\star), x_2(\tau^\star)) \in W(q_1(\tau^\star), q_2(\tau^\star))$. Repeating the same arguments for any execution r_2 it follows that condition 3 is indeed satisfied. ☐

Theorem 8.25 combines the well-established results for bisimulation relations of labeled transition systems (see Chapter 2) with the notion of bisimulation relations for linear systems with inequality constraints previously given in Definition 8.4. Resets and guards coupling the continuous and discrete dynamics are synchronized due to condition 3.

Since our definition of switching linear systems with location invariants is based on hybrid automata, it should be possible to characterize bisimulation relations for other classes of hybrid systems in a similar way as in Theorem 8.25, compare also with [76]. However, in the most general case, the continuous-time dynamics are described by nonlinear flow conditions. From Chapter 4 we know that bisimulation relations of nonlinear systems are defined on submanifolds. One important question would therefore be whether the sets $\mathcal{P}_R(q_1, q_2)$ can be proved to be submanifolds. Another important issue is the existence of a maximal hybrid bisimulation relation. This necessarily depends on whether there exists a maximal bisimulation relation between the constrained continuous-time systems at every discrete location. As an example, Corollary 8.17 contains necessary and sufficient conditions for the existence of a bisimulation relation between constrained H-related linear systems, which allows to compute the maximal relation satisfying these conditions. In this case, the algorithm presented in [60] for switching linear systems in the sense of Definition 7.1 can be modified to compute maximal bisimulation relations between switching linear systems with inequality constraints.

9

Conclusions

9.1. Summary

Compositional techniques are widely used in formal verification to analyze complex computer programs. The difficulty lies in the structure of these programs which often consist of large numbers of interacting concurrent processes. As a result, the state dimension is huge since it grows exponentially with the number of components. Models of physical systems are often equally complex as they are characterized by networks of interacting subsystems as well. In this thesis we show that compositional analysis methods developed for the verification of concurrent programs can be used to simplify analysis and synthesis problems in systems theory and control.
In Chapter 3 we adopt reasoning schemes based on simulation theory to analyze properties of interconnected linear systems. As one of the main results of this thesis we prove that circular assume-guarantee reasoning is sound and complete for linear systems. We extend this result to other types of interconnections and arbitrary numbers of components involved. Moreover, compositional and assume-guarantee reasoning schemes are also valid when using bisimulation instead of simulation relations. Altogether, this yields a comprehensive theory of compositional analysis methods for linear systems. We illustrate these findings with an example from circuit theory using the fact that simulation relations give rise to abstractions of linear systems. This makes it possible to replace a large interconnected system by a lower-dimensional one consisting of abstractions of the original components. Since abstractions computed by simulation can replace the original systems the verification task thus becomes less complex. Most of the results for linear systems can be generalized to nonlinear systems. However, as shown in Chapter 4, additional constant rank assumptions are needed to prove soundness of compositional and assume-guarantee reasoning for nonlinear simulation. In Chapter 5 we discuss passivity properties of both linear and nonlinear systems using compositional analysis techniques. The pivotal observation is that passivity can be captured as a nonlinear simulation relation between the system to be checked and the corresponding one-dimensional system given by the differential dissipation inequality. Compositionality of nonlinear simulation then allows to

infer passivity properties of interconnections from passivity properties of the components involved. In the case of open feedback, also the converse holds true. Since passivity is related to other concepts such as Lyapunov stability, the results of Chapter 5 can serve as a paradigm how to verify properties of complex interconnected systems using compositional analysis techniques.
Complementary to compositional analysis is the notion of decentralized control. The basic principle is to delegate a global control task to local controllers interacting with subsystems of the overall plant. In Chapter 6 we use compositional and assume-guarantee reasoning to derive decentralized control schemes. Combined with conditions for achievable simulation we obtain two bottom-up and one top-down scheme. The latter is particularly interesting since it proves that the existence of a global diagonally decoupled controller is equivalent to the existence of local feedback controllers with respect to the same global control target provided the global specification can be decomposed.
The last two chapters are dedicated to the study of switching linear systems as a special class of hybrid systems. Combining elements of discrete and continuous dynamics, the study of hybrid systems is an intrinsically interdisciplinary field. Like in previous chapters, our approach unites a representation stemming from systems theory – we describe the continuous part of the hybrid dynamics by means of differential equations rather than trajectories – with analysis methods borrowed from formal verification. This allows us to use structural hybrid (bi)simulation relations to prove soundness of compositional reasoning schemes for switching linear systems. Before these results can be generalized to switching linear systems with location invariants a bisimulation theory for these systems has to be developed. In particular, the influence of the continuous on the discrete dynamics due to polyhedral constraints on the state spaces requires extra conditions for bisimulation equivalence. As a first step, we characterize bisimulation relations for constrained linear systems. We then derive a structural notion of bisimulation relations for switching linear systems with location invariants which allows to directly apply the previous results to hybrid case.

9.2. Recommendations for future work

To extend the results obtained in this thesis for compositional analysis and control of dynamical systems, we recommend to investigate the following problems in the future:

- In Chapter 5 we mainly used compositional reasoning to infer passivity properties of both linear and nonlinear control systems. A natural extension would be to consider deduction schemes based on assume-guarantee reasoning. E.g., consider two nonlinear systems $\Sigma_i, i = 1, 2$, and assume that Σ_i can be stabilized by arbitrary passive systems Ξ_i,

$$\begin{array}{llll} S_1: & \Sigma_1 \|_{\mathrm{cl}} \Xi_2 & \preccurlyeq & \Xi_1 \|_{\mathrm{cl}} \Xi_2 \\ S_2: & \Xi_1 \|_{\mathrm{cl}} \Sigma_2 & \preccurlyeq & \Xi_1 \|_{\mathrm{cl}} \Xi_2 \end{array} . \tag{9.1}$$

Does (9.1) then imply that the interconnection $\Sigma_1 \| \Sigma_2$ is also stable, in other words does there exist a full simulation relation S of $\Sigma_1 \| \Sigma_2$ by $\Xi_1 \|_{\mathrm{cl}} \Xi_2$?

- For most parts of this thesis we assumed that a global specification is given as the interconnection of subspecifications related to subsystems of the global plant. For some applications, however, it would be useful to have available methods to decompose global specifications, i.e., given a specification Σ_Q and a plant $\Sigma_P = \Sigma_{P_1} \| \Sigma_{P_2}$, under which conditions and how can Σ_Q be decomposed into two subspecifications $\Sigma_{Q_i}, i = 1, 2$, such that $\Sigma_Q \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}$?

- In computer science, implementations and specifications can usually be expressed in the same language, e.g. as labeled transition systems. The proof rules for compositional and assume-guarantee reasoning are based on this idea. In this thesis, we assumed throughout that the specification (the property to be checked) can be expressed as an input-state-output system. Chapter 5 showed how to this is done for passivity and, at least partially, for stability as well. It would be interesting to further investigate whether system theoretic properties such as observability, control performance etc., can be formalized as control systems themselves.

- The area of hybrid systems remains a promising application for compositional techniques. In particular, deriving checkable conditions and algorithms to compute structural hybrid (bi)simulations for general systems with location invariants and guard conditions is an open problem.

A

Soundness of circular assume-guarantee reasoning for linear systems

A.1. Proof of Lemma 3.11

We give the proof with respect to $S_I + \bar{S}_I$, the result for $S_{II} + \bar{S}_{II}$ follows from symmetry.
Take any $(x_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, -x_{Q_2}) \in S_1$. Since all components fulfill

$$C_{P_1} x_{P_1} = C_{Q_1} x_{Q_1} = 0, C_{Q_2} x_{Q_2} = -C_{Q_2} \bar{x}_{Q_2} = 0 \tag{A.1}$$

and

$$H_{P_1} x_{P_1} = H_{Q_1} x_{Q_1} = 0, H_{Q_2} x_{Q_2} = -H_{Q_2} \bar{x}_{Q_2} = 0\,, \tag{A.2}$$

condition (iii) in Theorem 3.4 is fulfilled. By Definition 3.2 there exists a $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I$ and since S_I is a simulation relation, condition (ii) in Theorem 3.4 ensures that there exists a $(w_{P_1}, w_{Q_2}, w_{Q_1}, \bar{w}_{Q_2}) \in S_I$ such that

$$\begin{bmatrix} A_{P_1} x_{P_1} \\ A_{Q_2} x_{Q_2} \\ A_{Q_1} x_{Q_1} \\ A_{Q_2} \bar{x}_{Q_2} \end{bmatrix} = \begin{bmatrix} w_{P_1} \\ w_{Q_2} \\ w_{Q_1} \\ \bar{w}_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_{Q_1}\alpha \\ L_{Q_2}\beta \end{bmatrix} \tag{A.3}$$

Note that since $(w_{P_1}, w_{Q_2}, w_{Q_1}, \bar{w}_{Q_2}) \in S_I$, $(w_{P_1}, \bar{w}_{Q_2}, w_{Q_1}, -w_{Q_2}) \in \bar{S}_I$. Hence

$$\begin{bmatrix} A_{P_1} x_{P_1} \\ A_{Q_2} \bar{x}_{Q_2} \\ A_{Q_1} x_{Q_1} \\ -A_{Q_2} x_{Q_2} \end{bmatrix} = \begin{bmatrix} w_{P_1} \\ \bar{w}_{Q_2} \\ w_{Q_1} \\ -w_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ L_{Q_2}\beta \\ L_{Q_1}\alpha \\ 0 \end{bmatrix} \tag{A.4}$$

Since S_I is a simulation relation, there exists for every $x \in \mathrm{im} G_{Q_2}$ an element $(0, x, \tilde{x}_{Q_1}, \tilde{x}_{Q_2}) \in S_I$ such that

$$\begin{bmatrix} 0 \\ x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ \tilde{x}_{Q_1} \\ \tilde{x}_{Q_2} \end{bmatrix} + \mathrm{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{A.5}$$

Therefore, (A.4) can be rewritten as

$$\begin{bmatrix} A_{P_1}x_{P_1} \\ A_{Q_2}\bar{x}_{Q_2} \\ A_{Q_1}x_{Q_1} \\ -A_{Q_2}x_{Q_2} \end{bmatrix} = \underbrace{\begin{bmatrix} w_{P_1} \\ \bar{w}_{Q_2} \\ w_{Q_1} \\ -w_{Q_2} \end{bmatrix}}_{\in \bar{S}_I} + \underbrace{\begin{bmatrix} 0 \\ L_{Q_2}\beta \\ \tilde{x}_{Q_1} \\ \tilde{x}_{Q_2} \end{bmatrix}}_{\in S_I} + \mathrm{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \quad \text{(A.6)}$$

$$\in S_I + \bar{S}_I + \mathrm{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}$$

which proves that condition (ii) in Theorem 3.4 is also fulfilled. Condition (i) is also fulfilled due to S_I being a simulation relation. Indeed,

$$\mathrm{im} \begin{bmatrix} G_{P_1} & 0 \\ 0 & G_{Q_2} \\ G_{Q_1} & 0 \\ o & G_{Q_2} \end{bmatrix} + \mathrm{im} \begin{bmatrix} L_{P_1} & 0 \\ 0 & L_{Q_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \subset S_I + \mathrm{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \quad \text{(A.7)}$$

$$\subset S_I + \bar{S}_I + \mathrm{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}$$

Moreover, since S_I is a full simulation relation, $\Pi_{X_{P_1} X_{Q_2}} S_I = \Pi_{X_{P_1} X_{Q_2}} (S_I + \bar{S}_I) = \mathcal{X}_{P_1} \times \mathcal{X}_{Q_2}$ and thus $S_I + \bar{S}_I$ is a full simulation relation of $\Sigma_{P_1} \| \Sigma_{Q_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$.

A.2. Proof of Lemma 3.12

Again, the statement will be proved only for S_I^{sym}. Condition (i) and fullness of S_I^{sym} follow from fullness of S_I. Condition (iii) holds true since by interchanging the components, still $C_{Q_2}x_{Q_2} = C_{Q_2}\bar{x}_{Q_2}$ as well as $H_{Q_2}x_{Q_2} = H_{Q_2}\bar{x}_{Q_2}$. Finally, condition (ii) is proved analogously to (A.6) observing that for every $(x_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, x_{Q_2}) \in \tilde{S}_I$, there exists a $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I$ for which

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P-1}C_{Q_2}x_{Q_2} \\ A_{Q_2}x_{Q_2} + B_{Q_2}C_{P_1}x_{P_1} \\ A_{Q_1}x_{Q_1} + B_{Q_1}C_{Q_2}\bar{x}_{Q_2} \\ A_{Q_2}\bar{x}_{Q_2} + B_{Q_2}C_{Q_1}x_{Q_1} \end{bmatrix} = \underbrace{\begin{bmatrix} w_{P_1} \\ w_{Q_2} \\ w_{Q_1} \\ \bar{w}_{Q_2} \end{bmatrix}}_{\in S_I} + \mathrm{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \quad \text{(A.8)}$$

Furthermore, since S_I is a simulation relation,

$$\mathrm{im}\begin{bmatrix} 0 \\ L_{Q_2} \\ 0 \\ 0 \end{bmatrix} \subset S_I + \mathrm{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & {}_{Q_2} \end{bmatrix} \tag{A.9}$$

and therefore

$$\begin{bmatrix} A_{P_1} & B_{P_1}C_{Q_2} & 0 & 0 \\ B_{Q_2}C_{P_1} & A_{Q_2} & 0 & 0 \\ 0 & 0 & A_{Q_1} & B_{Q_1}C_{Q_2} \\ 0 & 0 & B_{Q_2}C_{Q_1} & A_{Q_2} \end{bmatrix} \tilde{S}_I \subset \tag{A.10}$$

$$S_I + \tilde{S}_I + \mathrm{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}$$

A.3. Proof of Lemma 3.13

Again, we will only prove the first half of the lemma. Since S_I is a full simulation relation, it holds that for every $(0, x)$ there exists x_{Q_1}, x_{Q_2} such that $(0, x, x_{Q_1}, x_{Q_2}) \in S_I$ with $x_{Q_1} \in \ker C_{Q_1} \cap \ker H_{Q_1}$. If we take $x \in \ker C_{Q_2} \cap \ker H_{Q_2}$ then also $x_{Q_2} \in \ker C_{Q_2} \cap \ker H_{Q_2}$. Then $(0, x_{Q_2}, x_{Q_1}, -x) \in \bar{S}_I$ and therefore

$$\begin{bmatrix} 0 \\ x \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} - \begin{bmatrix} 0 \\ x_{Q_2} \\ x_{Q_1} \\ -x \end{bmatrix} = \begin{bmatrix} 0 \\ x - x_{Q_2} \\ 0 \\ x + x_{Q_2} \end{bmatrix} \in S_I + \bar{S}_I \tag{A.11}$$

Moreover, $(0, x + x_{Q_2}, 0, x - x_{Q_2}) \in (S_I + \bar{S}_I)^{\mathrm{sym}}$ and by the subspace property also

$$\begin{bmatrix} 0 \\ x - x_{Q_2} \\ 0 \\ x + x_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ x + x_{Q_2} \\ 0 \\ x - x_{Q_2} \end{bmatrix} = 2\begin{bmatrix} 0 \\ x \\ 0 \\ x \end{bmatrix} \in (S_I + \bar{S}_I)^{\mathrm{sym}} \tag{A.12}$$

A.4. Proof of Theorem 3.14

Firstly, it is easy to see that S indeed defines a linear subspace.
Secondly, we have to show that it defines a simulation relation of $\Sigma_{P_1} \| \Sigma_{P_2}$ by

$\Sigma_{Q_1}\|\Sigma_{Q_2}$. Take any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. Then there exist $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in (S_I + \bar{S}_I)^{\text{sym}}$ and $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in (S_{II} + \bar{S}_{II})^{\text{sym}}$ with the property that

$$C_{P_1}x_{P_1} = C_{Q_1}x_{Q_1} = C_{Q_1}\bar{x}_{Q_1}, C_{P_2}x_{P_2} = C_{Q_2}x_{Q_2} = C_{Q_2}\bar{x}_{Q_2} \tag{A.13}$$

and

$$H_{P_1}x_{P_1} = H_{Q_1}x_{Q_1} = H_{Q_1}\bar{x}_{Q_1}, H_{P_2}x_{P_2} = H_{Q_2}x_{Q_2} = H_{Q_2}\bar{x}_{Q_2} \tag{A.14}$$

so that condition (iii) of Theorem 3.4 is already fulfilled. To show that condition (i) also holds, take first any $d_{P_1} \in \operatorname{im} L_{P_1}$. Since $(S_I + \bar{S}_I)^{\text{sym}}$ is a simulation relation, there exist $x_{Q_i} \in \operatorname{im} L_{Q_i}, i = 1, 2$ such that

$$\begin{bmatrix} d_{P_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_{P_1} \\ 0 \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{A.15}$$

with $(d_{P_1}, 0, x_{Q_1}, x_{Q_2}) \in (S_I + \bar{S}_I)^{\text{sym}}$. Since $x_{Q_1} \in \operatorname{im} G_{Q_1}$ and $(S_{II} + \bar{S}_{II})^{\text{sym}}$ is also a full simulation relation, there exist $\bar{x}_{Q_1} \in \operatorname{im} L_{Q_1}$ and $\bar{x}_{Q_2} \in \operatorname{im} L_{Q_2} \cap \ker C_{Q_2} \cap \ker H_{Q_1}$ such that $(x_{Q_1}, 0, \bar{x}_{Q_1}, \bar{x}_{Q_2}) \in (S_{II} + \bar{S}_{II})^{\text{sym}}$. By Lemma 3.13, there exists an element $(0, \bar{x}_{Q_2}, 0, \bar{x}_{Q_2}) \in (S_I + \bar{S}_I)^{\text{sym}}$ and therefore

$$\begin{aligned} \begin{bmatrix} d_{P_1} \\ 0 \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{x}_{Q_2} \\ 0 \\ \bar{x}_{Q_2} \end{bmatrix} = \begin{bmatrix} d_{P_1} \\ \bar{x}_{Q_2} \\ x_{Q_1} \\ x_{Q_2} + \bar{x}_{Q_2} \end{bmatrix} &\in (S_I + \bar{S}_I)^{\text{sym}}, \\ \begin{bmatrix} x_{Q_1} \\ 0 \\ \bar{x}_{Q_1} \\ \bar{x}_{Q_2} \end{bmatrix} &\in (S_{II} + \bar{S}_{II})^{\text{sym}} \\ &\Longrightarrow \begin{bmatrix} d_{P_1} \\ 0 \\ x_{Q_1} \\ \bar{x}_{Q_2} \end{bmatrix} \in S \end{aligned} \tag{A.16}$$

for any $d_{P_1} \in \operatorname{im} L_{P_1}$ with $x_{Q_1} \in \operatorname{im} L_{Q_1}$ and $\bar{x}_{Q_2} \in \operatorname{im} L_{Q_2} \cap \ker C_{Q_2} \cap \ker H_{Q_2}$. Hence,

$$\operatorname{im} \begin{bmatrix} L_{P_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \subset S + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{A.17}$$

By the same arguments one can also show that

$$\operatorname{im}\begin{bmatrix} 0 \\ L_{P_2} \\ 0 \\ 0 \end{bmatrix} \subset S + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{A.18}$$

Similarly, consider any

$$\begin{bmatrix} g_{P_1} \\ 0 \\ g_{Q_1} \\ 0 \end{bmatrix} \in \operatorname{im}\begin{bmatrix} G_{P_1} \\ 0 \\ G_{Q_1} \\ 0 \end{bmatrix} \subset (S_I + \bar{S}_I)^{\text{sym}} + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{A.19}$$

From (A.19) it follows that there exists an element $(x_{P_1}, 0, x_{Q_1}, x_{Q_2}) \in (S_I + \bar{S}_I)^{\text{sym}}$ such that

$$\begin{bmatrix} g_{P_1} \\ 0 \\ g_{Q_1} \\ 0 \end{bmatrix} = \begin{bmatrix} x_{P_1} \\ 0 \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{A.20}$$

with $x_{Q_2} \in \operatorname{im} L_{Q_2} \cap \ker C_{Q_2} \cap \ker H_{Q_2}$. Since $(S_{II} + \bar{S}_{II})^{\text{sym}}$ is full, there exists an element $(x_{Q_1}, 0, \bar{x}_{Q_1}, \bar{x}_{Q_2}) \in (S_{II} + \bar{S}_{II})^{\text{sym}}$ such that $\bar{x}_{Q_2} \in \operatorname{im} L_{Q_2} \cap \ker C_{Q_2} \cap \ker H_{Q_2}$. Lemma 3.13 ensures that there also exists an element $(0, \bar{x}_{Q_2}, 0, \bar{x}_{Q_2}) \in (S_I + \bar{S}_I)^{\text{sym}}$ since $x_{Q_2}, \bar{x}_{Q_2}$ and therefore $\bar{x}_{Q_2} \in \operatorname{im} L_{Q_2} \cap \ker C_{Q_2} \cap \ker H_{Q_2}$ as well as

$$\begin{bmatrix} x_{P_1} \\ 0 \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{x}_{Q_2} \\ 0 \\ \bar{x}_{Q_2} \end{bmatrix} = \begin{bmatrix} x_{P_1} \\ \bar{x}_{Q_2} \\ x_{Q_1} \\ \bar{x}_{Q_2} + x_{Q_2} \end{bmatrix} \in (S_I + \bar{S}_I)^{\text{sym}} \tag{A.21}$$

Therefore, there exists an element $(x_{P_1}, 0, x_{Q_1}, \bar{x}_{Q_2}) \in S$ with $\bar{x}_{Q_2} \in \operatorname{im} L_{Q_2}$ such that

$$\begin{bmatrix} g_{P_1} \\ 0 \\ g_{Q_1} \\ 0 \end{bmatrix} = \begin{bmatrix} x_{P_1} \\ 0 \\ x_{Q_1} \\ \bar{x}_{Q_2} \end{bmatrix} + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{A.22}$$

which proves that

$$\operatorname{im}\begin{bmatrix} G_{P_1} \\ 0 \\ G_{Q_1} \\ 0 \end{bmatrix} \subset S + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{A.23}$$

Similarly, one can show that

$$\mathrm{im}\begin{bmatrix} 0 \\ G_{P_2} \\ 0 \\ G_{Q_2} \end{bmatrix} \subset S + \mathrm{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix} \tag{A.24}$$

and therefore condition (i) in Theorem 3.4 is completely fulfilled.
As to condition (ii), take any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. Since $(S_i + \bar{S}_i)^{\mathrm{sym}}, i = 1, 2$ are simulation relations, there exist $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}), (v_{P_1}, v_{Q_2}, v_{Q_1}, \bar{v}_{Q_2}) \in (S_I + \bar{S}_I)^{\mathrm{sym}}$ and $a, b \in \mathbb{R}$ such that

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}C_{Q_2}x_{Q_2} \\ A_{Q_2}x_{Q_2} + B_{Q_2}C_{P_1}x_{P_1} \\ A_{Q_1}x_{Q_1} + B_{Q_1}C_{Q_2}\bar{x}_{Q_2} \\ A_{Q_2}\bar{x}_{Q_2} + B_{Q_1}C_{Q_1}x_{Q_1} \end{bmatrix} = \begin{bmatrix} v_{P_1} \\ v_{Q_2} \\ v_{Q_1} \\ \bar{v}_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_{Q_1}a \\ L_{Q_2}b \end{bmatrix} \tag{A.25}$$

as well as $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}), (w_{Q_1}, w_{P_2}, \bar{w}_{Q_1}, w_{Q_2}) \in (S_{II}+\bar{S}_{II})^{\mathrm{sym}}$ and $l, m \in \mathbb{R}$ such that

$$\begin{bmatrix} A_{Q_1}x_{Q_1} + B_{Q_1}C_{P_2}x_{P_2} \\ A_{P_2}x_{P_2} + B_{P_2}C_{Q_1}x_{Q_1} \\ A_{Q_1}\bar{x}_{Q_1} + B_{Q_1}C_{Q_2}x_{Q_2} \\ A_{Q_2}x_{Q_2} + B_{Q_1}C_{Q_1}\bar{x}_{Q_1} \end{bmatrix} = \begin{bmatrix} w_{Q_1} \\ w_{P_2} \\ \bar{w}_{Q_1} \\ w_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_{Q_1}l \\ L_{Q_2}m \end{bmatrix} \tag{A.26}$$

Because of (A.13), observe that $v_{Q_2} = w_{Q_2} + L_{Q_2}m$ and $v_{Q_1} + L_{Q_1}a = w_{Q_1}$. Furthermore, we know that there exists an element $(0, L_{Q_2}m, L_{Q_1}c, L_{Q_2}d) \in (S_I+\bar{S}_I)^{\mathrm{sym}}$ with $L_{Q_1}c \in \ker C_{Q_1} \cap \ker H_{Q_1}$ and similarly, $(L_{Q_1}a, 0, L_{Q_1}n, L_{Q_2}p) \in (S_{II} + \bar{S}_{II})^{\mathrm{sym}}$ with $L_{Q_2}p \in \ker C_{Q_2} \cap \ker H_{Q_2}$. With Lemma 3.13, also $(0, L_{Q_2}p, 0, L_{Q_2}p) \in (S_I + \bar{S}_I)^{\mathrm{sym}}$ and $(L_{Q_1}c, 0, L_{Q_1}c, 0) \in (S_{II} + \bar{S}_{II})^{\mathrm{sym}}$. Hence,

$$\begin{aligned} \begin{bmatrix} v_{P_1} \\ v_{Q_2} \\ v_{Q_1} \\ \bar{v}_{Q_2} \end{bmatrix} &= \begin{bmatrix} v_{P_1} \\ v_{Q_2} - L_{Q_2}m - L_{Q_2}p \\ v_{Q_1} - L_{Q_1}c \\ \bar{v}_{Q_2} - L_{Q_2}d - L_{Q_2}p \end{bmatrix} + \begin{bmatrix} 0 \\ L_{Q_2}(m+p) \\ L_{Q_1}c \\ L_{Q_2}(d+p) \end{bmatrix} \\ &= \begin{bmatrix} v_{P_1} \\ w_{Q_2} - L_{Q_2}p \\ v_{Q_1} - L_{Q_1}c \\ \bar{v}_{Q_2} - L_{Q_2}(d+p) \end{bmatrix} + \begin{bmatrix} 0 \\ L_{Q_2}(m+p) \\ L_{Q_1}c \\ L_{Q_2}(d+p) \end{bmatrix} \end{aligned} \tag{A.27}$$

and

$$\begin{bmatrix} w_{Q_1} \\ w_{P_2} \\ \bar{w}_{Q_1} \\ w_{Q_2} \end{bmatrix} = \begin{bmatrix} w_{Q_1} - L_{Q_1}a - L_{Q_1}c \\ w_{P_2} \\ \bar{w}_{Q_1} - L_{Q_1}n - L_{Q_1}c \\ w_{Q_2} - L_{Q_2}p \end{bmatrix} + \begin{bmatrix} L_{Q_1}(a+c) \\ 0 \\ L_{Q_1}(n+c) \\ L_{Q_2}p \end{bmatrix} \tag{A.28}$$

$$= \begin{bmatrix} v_{Q_1} - L_{Q_1}c \\ w_{P_2} \\ \bar{w}_{Q_1} - L_{Q_1}(n+c) \\ w_{Q_2} - L_{Q_2}p \end{bmatrix} + \begin{bmatrix} L_{Q_1}(a+c) \\ 0 \\ L_{Q_1}(n+c) \\ L_{Q_2}p \end{bmatrix}$$

Thus, (A.25) can be rewritten as

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}C_{Q_2}x_{Q_2} \\ A_{Q_2}x_{Q_2} + B_{Q_2}C_{P_1}x_{P_1} \\ A_{Q_1}x_{Q_1} + B_{Q_1}C_{Q_2}\bar{x}_{Q_2} \\ A_{Q_2}\bar{x}_{Q_2} + B_{Q_1}C_{Q_1}x_{Q_1} \end{bmatrix} = \begin{bmatrix} v_{P_1} \\ w_{Q_2} - L_{Q_2}p \\ v_{Q_1} - L_{Q_1}c \\ \bar{v}_{Q_2} - L_{Q_2}(d+p) \end{bmatrix} + \begin{bmatrix} 0 \\ L_{Q_2}(m+p) \\ L_{Q_1}(c+a) \\ L_{Q_2}(d+p+b) \end{bmatrix}$$

and similarly, (A.26) becomes

$$\begin{bmatrix} A_{Q_1}x_{Q_1} + B_{Q_1}C_{P_2}x_{P_2} \\ A_{P_2}x_{P_2} + B_{P_2}C_{Q_1}x_{Q_1} \\ A_{Q_1}\bar{x}_{Q_1} + B_{Q_1}C_{Q_2}x_{Q_2} \\ A_{Q_2}x_{Q_2} + B_{Q_1}C_{Q_1}\bar{x}_{Q_1} \end{bmatrix} = \begin{bmatrix} v_{Q_1} - L_{Q_1}c \\ w_{P_2} \\ \bar{w}_{Q_1} - L_{Q_1}(n+c) \\ w_{Q_2} - L_{Q_2}p \end{bmatrix} + \begin{bmatrix} L_{Q_1}(a+c) \\ 0 \\ L_{Q_1}(n+c+l) \\ L_{Q_2}(p+m) \end{bmatrix}$$

Consequently, there exists an element $(v_{P_1}, w_{P_2}, v_{Q_1} - L_{Q_1}c, w_{Q_2} - L_{Q_2}p) \in S$ such that

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}C_{P_2}x_{P_2} \\ A_{P_2}x_{P_2} + B_{P_2}C_{P_1}x_{P_1} \\ A_{Q_1}x_{Q_1} + B_{Q_1}C_{Q_2}x_{Q_2} \\ A_{Q_2}x_{Q_2} + B_{Q_1}C_{Q_1}x_{Q_1} \end{bmatrix} = \begin{bmatrix} v_{P_1} \\ w_{P_2} \\ v_{Q_1} - L_{Q_1}c \\ w_{Q_2} - L_{Q_2}p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_{Q_1}(a+c) \\ L_{Q_2}(p+m) \end{bmatrix}$$

which concludes the proof for S being a simulation relation of $\Sigma_{P_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$.
Thirdly, it has to be shown that S as defined in (3.16) is full, i.e. for any (x_{P_1}, x_{P_2}) there has to exist a (x_{Q_1}, x_{Q_2}) such that $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. Since $(S_I + \bar{S}_I)^{\text{sym}}$ is a full simulation relation, there exists for every (x_{P_1}, x_{Q_2}) a $(\bar{x}_{Q_1}, \bar{x}_{Q_1})$ such that $(x_{P_1}, x_{Q_2}, \bar{x}_{Q_1}, \bar{x}_{Q_1}) \in (S_I + \bar{S}_I)^{\text{sym}}$. Moreover, since also $(S_{II} + \bar{S}_{II})^{\text{sym}}$ is full, there exists for an arbitrary x_{P_2} and the given $\bar{x}_{Q_1}$ a $(\hat{x}_{Q_1}, \hat{x}_{Q_2})$ such that $(\bar{x}_{Q_1}, x_{P_2}\hat{x}_{Q_1}, \hat{x}_{Q_2}) \in (S_{II} + \bar{S}_{II})^{\text{sym}}$. Fullness of $(S_I + \bar{S}_I)^{\text{sym}}$ also ensures that there exists an element $(0, \hat{x}_{Q_2}, \tilde{x}_{Q_1}, \tilde{x}_{Q_2}) \in (S_I + \bar{S}_I)^{\text{sym}}$ with $\tilde{x}_{Q_1} \in \ker C_{Q_1} \cap \ker H_{Q_1}$. By Lemma 3.13, however, an element

$(\tilde{x}_{Q_1}, 0, \tilde{x}_{Q_1}, 0)$ is contained in $(S_{II} + \bar{S}_{II})^{\text{sym}}$. Hence

$$\begin{bmatrix} x_{P_1} \\ x_{Q_2} \\ \bar{x}_{Q_1} \\ \bar{x}_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{x}_{Q_2} - x_{Q_2} \\ \tilde{x}_{Q_1} \\ \tilde{x}_{Q_2} \end{bmatrix} = \begin{bmatrix} x_{P_1} \\ \hat{x}_{Q_2} \\ \bar{x}_{Q_1} + \tilde{x}_{Q_1} \\ \bar{x}_{Q_2} + \tilde{x}_{Q_2} \end{bmatrix} \in (S_I + \bar{S}_I)^{\text{sym}}$$

$$\begin{bmatrix} \bar{x}_{Q_1} \\ x_{P_2} \\ \hat{x}_{Q_1} \\ \hat{x}_{Q_2} \end{bmatrix} + \begin{bmatrix} \tilde{x}_{Q_1} \\ 0 \\ \tilde{x}_{Q_1} \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{x}_{Q_1} + \tilde{x}_{Q_1} \\ x_{P_2} \\ \hat{x}_{Q_1} + \tilde{x}_{Q_1} \\ \hat{x}_{Q_2} \end{bmatrix} \in (S_{II} + \bar{S}_{II})^{\text{sym}}$$

from which the element

$$\begin{bmatrix} x_{P_1} \\ x_{P_2} \\ \hat{x}_{Q_1} + \tilde{x}_{Q_1} \\ \hat{x}_{Q_2} \end{bmatrix} \in S$$

can be constructed for any (x_{P_1}, x_{P_2}).

B

Completeness of circular assume-guarantee reasoning for linear systems

B.1. Proof of Theorem 3.20

We will only prove that S_I is a full simulation relation; the result for S_{II} follows by symmetry. Observe first that S_I indeed defines a linear subspace and is non-empty. To show that S_I is a simulation relation of $\Sigma_{P_1}\|\Sigma_{Q_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, note first that by construction

$$S_I \subset \ker \begin{bmatrix} C_{P_1} & 0 & -C_{Q_1} & 0 \\ 0 & C_{Q_2} & 0 & -C_{Q_2} \\ H_{P_1} & 0 & -H_{Q_1} & 0 \\ 0 & H_{Q_2} & 0 & -H_{Q_2} \end{bmatrix}$$

Furthermore, since for every $x_{Q_2} \in \mathrm{im} L_{Q_2}$ $(0, x_{Q_2}, 0, x_{Q_2}) \in S_I$ it holds that

$$\mathrm{im} \begin{bmatrix} 0 \\ L_{Q_2} \\ 0 \\ 0 \end{bmatrix} \subset S_I + \mathrm{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ L_{Q_2} \end{bmatrix}$$

Moreover, since S is a full simulation relation, for every $g_{P_1} \in \mathrm{im} L_{P_1}$ there exists a $(g_{P_1}, 0, x_{Q_1}, x_{Q_2}) \in S$ such that $x_{Q_i} \in \mathrm{im} L_{Q_i}, i = 1, 2$. But then there also exists an element $(g_{P_1}, x_{Q_2}, x_{Q_2}, x_{Q_2}) \in S_I$ such that for some α, λ

$$\begin{bmatrix} g_{P_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} g_{P_1} \\ x_{Q_2} \\ x_{Q_1} \\ x_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ L_{Q_2}\lambda \\ L_{Q_1}\alpha \\ L_{Q_2}\lambda \end{bmatrix} = \underbrace{\begin{bmatrix} g_{P_1} \\ 0 \\ x_{Q_1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_{Q_2} + L_{Q_2}\lambda \\ 0 \\ x_{Q_2} + L_{Q_2}\lambda \end{bmatrix}}_{\in S_I}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ L_{Q_1}\alpha \\ 0 \end{bmatrix} \in S_I + \mathrm{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}.$$

Clearly, also $\operatorname{im}\begin{bmatrix} 0 & 0 \\ B_{Q_2} & G_{Q_2} \\ 0 & 0 \\ B_{Q_2} & G_{Q_2} \end{bmatrix} \in S_I$. Since $\operatorname{im}\begin{bmatrix} B_{P_1} & G_{P_1} \\ 0 & 0 \\ B_{Q_1} & G_{Q_1} \\ 0 & 0 \end{bmatrix} \in S$, there exists for all β, δ an element $(w_{P_1}, w_{Q_2}, w_{Q_1}, w_{Q_2}) \in S_I$ such that

$$\begin{bmatrix} B_{P_1} & G_{P_1} \\ 0 & 0 \\ B_{Q_1} & G_{Q_1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \delta \end{bmatrix} = \begin{bmatrix} w_{P_1} \\ w_{Q_2} \\ w_{Q_1} \\ w_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ G_{Q_2}\rho \\ G_{Q_1}\tau \\ G_{Q_2}\rho \end{bmatrix} = \underbrace{\begin{bmatrix} w_{P_1} \\ 0 \\ w_{Q_1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ w_{Q_2} + G_{Q_2}\rho \\ 0 \\ w_{Q_2} + G_{Q_2}\rho \end{bmatrix}}_{\in S_I} + \begin{bmatrix} 0 \\ 0 \\ G_{Q_1}\tau \\ 0 \end{bmatrix}$$

for some ρ, τ. Finally, for any $(x_{P_1}, x_{Q_2}, x_{Q_1}, x_{Q_2}) \in S_I$ there exists a $(z_{P_1}, z_{Q_2}, z_{Q_1}, z_{Q_2}) \in S_I$ such that

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}C_{Q_2}x_{Q_2} \\ B_{Q_2}C_{P_1}x_{P_1} + A_{Q_2}x_{Q_2} \\ A_{Q_1}x_{Q_1} + B_{Q_1}C_{Q_2}x_{Q_2} \\ B_{Q_2}C_{Q_1}x_{Q_1} + A_{Q_2}x_{Q_2} \end{bmatrix} = \begin{bmatrix} z_{P_1} \\ z_{Q_2} \\ z_{Q_1} \\ z_{Q_2} \end{bmatrix} + \begin{bmatrix} 0 \\ G_{Q_2}\mu \\ G_{Q_1}\nu \\ G_{Q_2}\mu \end{bmatrix} = \underbrace{\begin{bmatrix} z_{P_1} \\ 0 \\ z_{Q_1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ z_{Q_2} + G_{Q_2}\mu \\ 0 \\ z_{Q_2} + G_{Q_2}\mu \end{bmatrix}}_{\in S_I} + \begin{bmatrix} 0 \\ 0 \\ G_{Q_1}\nu \\ 0 \end{bmatrix}.$$

In particular, for any $(0, x, 0, x) \in S_I$ it holds that

$$\begin{bmatrix} B_{P_1}C_{Q_2}x \\ A_{Q_2}x \\ B_{Q_1}C_{Q_2}x \\ A_{Q_2} \end{bmatrix} = \begin{bmatrix} B_{P_1}C_{Q_2}x \\ 0 \\ B_{Q_1}C_{Q_2}x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ A_{Q_2}x \\ 0 \\ A_{Q_2}x \end{bmatrix} \subset S_I + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ G_{Q_1} & 0 \\ 0 & G_{Q_2} \end{bmatrix}$$

since

$$\begin{bmatrix} B_{P_1}C_{Q_2}x \\ 0 \\ B_{Q_1}C_{Q_2}x \\ 0 \end{bmatrix} \in \operatorname{im}\begin{bmatrix} B_{P_1} \\ 0 \\ B_{Q_1} \\ 0 \end{bmatrix} \subset S_I + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ G_{Q_1} & 0 \\ 0 & G_{Q_2} \end{bmatrix}$$

In order to prove fullness of S_I, we first note that for any $x_{Q_2} \in \mathcal{X}_{Q_2}$ there is an element $(0, x_{Q_2}, 0, x_{Q_2}) \in S_I$. Moreover, for any $x_{P_1} \in \mathcal{X}_{P_1}$ there is a $(x_{P_1}, 0, w_{Q_1}, w_{Q_2}) \in S$. Hence there is a $(x_{P_1}, w_{Q_2}, w_{Q_1}, w_{Q_2}) \in S_I$ and therefore also

$$\begin{bmatrix} x_{P_1} \\ 0 \\ w_{Q_1} \\ 0 \end{bmatrix} = \begin{bmatrix} x_{P_1} \\ w_{Q_2} \\ w_{Q_1} \\ w_{Q_2} \end{bmatrix} - \begin{bmatrix} 0 \\ w_{Q_2} \\ 0 \\ w_{Q_2} \end{bmatrix} \in S_I$$

Combining these two results we conclude that for every (x_{P_1}, x_{Q_2}) there exists a (w_{Q_1}, x_{Q_2}) such that $(x_{P_1}, x_{Q_2}, w_{Q_1}, x_{Q_2}) \in S_I$ which proves fullness.

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Index

Samenvatting

Op het gebied van programmacorrectheid wordt vaak gebruik gemaakt van compositionele analyse-technieken. Het analyseren van individuele processen helpt om de complexiteit van het geheel veroorzaakt door de koppeling van concurrente processen tegen te gaan. In gelijke mate vertonen technische processen een hoge complexiteit. De structurering in deelsystemen laat een wiskundige modellering toe, maar heeft vaak toestandsexplosie tot gevolg. Het doel van dit proefschrift is om compositionele methodes afkomstig uit de theoretische informatica over te dragen op en bruikbaar te maken voor de analyse van dynamische systemen zoals bekend in de regel- en systeemtheorie.

In Hoofdstuk 3 worden redeneermethodes voor lineaire systemen op basis van simulaties bestudeerd. Een belangrijk resultaat van dit proefschrift is dat circulair "assume-guarantee" redeneren logisch correct en compleet is voor feedbackinterconnecties van lineaire tijd-continue systemen. Deze conclusie geldt ook zowel voor andere types van interconnecties zoals het parallele product, als voor feedback interconnecties van meer dan twee deelsystemen. Bovendien zijn de redeneermethodes ook correct als bisimulaties in plaats van simulaties worden gebruikt. Deze resultaten leiden tot een volledige theorie van compositionele analysemethodes voor lineaire systemen. De praktische relevantie wordt door een voorbeeld over elektrische circuits aangetoond, waarbij we regelsystemen middels simulatierelaties abstraheren. In Hoofdstuk 4 onderzoeken wij of deze resultaten tot niet-lineaire systemen kunnen worden veralgemeend. Omdat niet-lineaire simulaties op deelvarieteiten gedefineerd zijn dienen extra constante rang condities verfuld te worden. Deze rang condities zijn onder meer noodzakelijk voor het bewijs dat niet-lineare simulatie een preorde is. In Hoofdstuk 5 worden niet-lineaire simulaties gebruikt om passiviteitseigenschappen te onderzoeken. Deze op het principe van "model checking" lijkende toepassing is mogelijk omdat passiviteit als een simulatie tussen het bewuste systeem en één gegeneraliseerd een-dimensioneel dynamisch systeem afgeleid uit de differentiele dissipativiteitsongelijkheid kan worden beschreven. De klassieke stelling dat de feedbackinterconnectie van twee passieve systemen zelf weer passief is kan op deze manier als compositionaliteit van passiviteit worden hergeinterpreteerd. Compositioneel redeneren levert nog meer resultaten op; zo geldt bijvoorbeeld ook de omkering van de passiviteitsstelling en laat zich daaruit informatie over de structuur van de opslagfunctie afleiden. Ook kunnen

stabiliteitseigenschappen van lineaire en niet-lineaire systemen op deze manier bestudeerd worden omdat een passief systeem stabiel is indien de opslagfunctie een locaal minimum in het evenwichtspunt heeft. Hoofdstuk 5 is exemplarisch voor de manier waarop eigenschappen van complexe regelsystemen met behulp van compositioneel redeneren efficient kunnen worden bepaald.
Gedecentraliseerde regeling berust op het zelfde principe als compositionele analyse methodes, te weten de splitsing van een *globaal* probleem (voor het gehele systeem) in meerdere *locale* problemen (voor de deelsystemen). In Hoofdstuk 6 onderzoeken wij hoe een globaal regelprobleem door gedecentraliseerde regelaars kan worden opgelost. In de eerste stap passen wij compositioneel redeneren op het gedecentraliseerde scenario toe. Gecombineerd met condities voor het bestaan van een regelaar zodat het geregelde systeem aan de gegeven specificatie voldoet ontwerpen wij daarna twee bottom-up (van locale condities naar de globale specificatie) en een top-down (van een globale conditie tot gedecentraliseerde regelaars) procedure voor de gedecentraliseerde regeling van lineaire systemen.
De laatste twee hoofdstukken behandelen hybride systemen gegeven als schakelende lineaire systemen ("switching linear systems"). Hybride systemen vereinigen kenmerken van discrete en continue dynamica en kunnen daarom interdisciplinair worden onderzocht. Zoals in de voorafgaande hoofdstukken van dit proefschrift benaderen wij schakelende lineaire systemen door het toepassen van structurele technieken (ontleend aan de theoretische informatica) op differentiaalvergelijkingen (de gebruikelijke representatievorm voor dynamische systemen in de regel- en systeemtheorie in tegenstelling tot oplossingen van het system). Het voordeel van dit aanpak is dat structurele (bi)simulaties kunnen worden gebruikt. Daardoor vereenvoudigt het bewijs dat compositioneel redeneren ook voor schakelende lineaire systemen correct is. Tenslotte voegen we aan de definitie invarianten voor de continue dynamica toe. Dit heeft een wederzijdse invloed van continue en discrete dynamica tot gevolg, hetgeen de moelijkheidsgraad aanzienlijk verhoogt. In eerste instantie onderzoeken wij alleen de continue dynamica, die gegeven wordt door lineaire systemen met ongelijkheidsnevenvoorwaarden. Met behulp van het lemma van Farkas karakteriseren wij bisimulaties voor dit soort systemen. Daarop voortbouwend kan een structurele notie van (bi)simulaties voor schakelende lineaire systemen met invarianten worden geformuleerd. Deze is ook op gegeneraliseerde hybride systemen toepasbaar, en maakt daarmee een belangrijke stap in de ontwikkeling van compositionele methodes voor hybride systemen.

Zusammenfassung

Im Bereich der formalen Verifikation von Softwareprogrammen finden kompositionelle Analysemethoden häufig Anwendung. Parallele, d. h. nebenläufig ausgeführte Prozesse führen in ihrer Verknüpfung zu einem hochkomplexen Gesamtsystem, da die Dimension des Beschreibungsraumes exponentiell mit der Anzahl der Prozesse wächst. Auch Modelle technischer Prozesse weisen in der Praxis eine hohe Komplexität auf. Die Gründe hierfür sind gleichermaßen strukturell bedingt, da ein kompletter Prozess oft nur als Produkt gekoppelter Teilsysteme beschreibbar ist. Ziel dieser Arbeit ist es, Methoden und formale Konzepte, die in der theoretischen Informatik für die Analyse paraller Programme entwickelt wurden, auf dynamische Systeme, wie sie in der Systemtheorie und Regelungstechnik vorkommen, zu übertragen und dadurch regelungstechnische Probleme effizienter lösen zu können.
Kapitel 3 befasst sich mit Deduktionsverfahren für lineare zeitkontinuierliche Systeme auf der Basis von Simulationsrelationen. Als ein zentrales Ergebnis dieser Arbeit weisen wir in diesem Kapitel nach, dass zirkelschlüssiges "assume-guarantee reasoning" logisch korrekt und komplett ist für durch Rückkoppelung verbundene Systeme. Dieses Resultat lässt sich sowohl auf andere Verknüpfungstypen wie das parallele Produkt zweier Systeme als auch auf Verknüpfungen von mehr als zwei Systemen übertragen. Darüber hinaus gelten die Deduktionsverfahren auch, wenn Simulation durch Bisimulation ersetzt wird. Zusammengenommen ergeben diese Ergebnisse eine vollständige Theorie kompositioneller Analysemethoden für lineare Systeme. Anhand des Beispiels geregelter elektrischer Schaltkreise lässt sich die praktische Relevanz dieser Theorie illustrieren, wobei die Bedeutung von Simulationsrelationen für die Abstraktion von dynamischen Systemen ebenfalls veranschaulicht wird. Die Verallgemeinerung der Theorie hin zu nichtlinearen Systemen ist größtenteils möglich, wie Kapitel 4 zeigt. Jedoch sind zusätzliche Annahmen nötig, um zu garantieren, dass die nichtlinearen Simulationsrelationen die differentialgeometrischen Eigenschaften von Untermannigfaltigkeiten besitzen. Dies ist insbesondere für den Beweis erforderlich, dass nichtlineare Simulation eine Quasiordnung darstellt wie auch dafür, dass "assume-guarantee reasoning" logisch korrekt ist. Nichtlineare Simulationsrelationen als Grundlage kompositioneller Analyseverfahren finden in Kapitel 5 eine direkte Anwendung im Bereich der Passivitätstheorie. Voraussetzung dafür ist die Erkenntnis, dass die Eigenschaft eines Systems, passiv zu sein, gleichsam durch eine volle nichtlineare Simulationsrelation

zwischen dem System selbst und dem sich aus der differentiellen Dissipativitätsungleichung ergebenden eindimensionalen System beschrieben werden kann. Der klassiche Satz, dass die Verknüpfung zweier passiver Systeme durch negative Rückkopplung wiederum passiv ist, kann so neu interpretiert werden als Kompositionalität von Passivität. Weitere Ergebnisse lassen sich mit Hilfe der Theorie der vorangegangenen Kapitel direkt ableiten, z.B. gilt für offene negative Rückkopplungen auch die Umkehrung des klassischen Passivitätstheorems. Stabilitätseigenschaften linearer und nichtlinearer Systeme können auf diese Weise ebenfalls behandelt werden, da passive Systeme auch stabil sind, wenn die Speicherfunktion ein lokales Minimum im Gleichgewichtspunkt besitzt. Kapitel 5 zeigt somit auf, wie Eigenschaften komplexer Regelsysteme mit kompositionellen Methoden systematisch und effizient überprüft werden können.

Dezentrale Regelung und kompositionelle Analyse beruhen auf demselben Prinzip, dass eine *globale* Aufgabe (auf der Ebene des Gesamtsystems) in mehrere *lokale* Aufgaben (auf der Ebene von Teilsystemen) aufgeteilt wird. In Kapitel 6 nutzten wir dieses Vorgehen, um eine globale Reglerspezifikation mit einem dezentralen Regelungskonzept zu erfüllen. Zu diesem Zweck werden zuerst die kompositionellen Deduktionsverfahren an das dezentrale Regelungsszenario angepasst. Zusammen mit Bedingungen, die die Erreichbarkeit eines Regelungszieles für ein gegebenes Streckenmodell charakterisieren, entwerfen wir zwei *bottom-up* (von lokalen Bedingungen zur globalen Spezifikation) und ein *top-down* (von der globalen Bedingung zur dezentralen Regelung) Verfahren. Letzteres kann so interpretiert werden, dass in dem gegebenen Szenario die Existenz eines globalen Reglers immer die Existenz lokaler Regler impliziert, die dieselbe Spzifikation erfüllen können.

Die letzten beiden Kapitel befassen sich mit der Analyse hybrider Systeme, genauer gesagt konzentrieren wir uns auf die praxisrelevante Untergruppierung der sogenannten geschalteten linearen Systeme ("switching linear systems"). Hybride Systeme verbinden Elemente zeitkontinuierlicher und zeitdiskreter Dynamik miteinander, was sie zu einem interdisziplinären Forschungsgegenstand macht. Dem Motiv dieser Arbeit folgend vereint unser Ansatz eine in der Systemtheorie gängige Repräsentationsform – die Beschreibung der zeitkontinuierlichen Dynamik durch Differenzialgleichung statt durch Lösungskurven – mit Analysemethoden für Transitionssysteme, die traditionell in der formellen Verifikation Anwendung finden. Der Vorteil dieser Herangehensweise liegt darin, dass strukturelle (Bi)simulationsrelationen angewandt werden können. Dadurch lässt sich auf einfachere Weise zeigen, dass die in den vorangehenden Kapiteln gebrauchten Deduktionsverfahren auch für geschaltete lineare Systeme anwendbar sind. Zum Schluss betrachten wir geschaltete lineare Systeme mit Stelleninvarianten, welche einen zusätzlichen Schwierigkeitsgrad darstellen. Diese bewirken nämlich, dass neben dem Einfluss der zeitdiskreten auf die zeitkontinuierliche Dynamik auch ein umgekehrter Einfluss besteht, wenn nämlich eine Transition zu einer anderen Stelle durch das Erreichen bzw. bevorstehende Verletzen der momentanen Stelleninvarianten ausgelöst wird. In einem ersten Schritt beschränken wir uns deshalb auf den zeitkontinuierlichen Part der hybriden Dynamik von geschalteten linearen Systemen mit Selleninvarianten, um eine Bisimulationstheorie für lineare Systeme mit Ungleichungszwangsbedingungen zu entwerfen. Darauf aufbauend lässt sich die strukturelle Variante einer Bisimulationsrelation ableiten. Diese hat das Potential, auch bei anderen Klassen hybrider Systeme angewendbar zu sein und stellt damit einen wichtigen Schritt zur Entwicklung kompositioneller Analsyseverfahren für hybride Systeme dar.