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Equivalences of switching linear systems with inequality constraints

In Chapter 7 we studied switching linear systems as a special class of hybrid systems for which the discrete part of the hybrid dynamics is independent of the continuous one. In this chapter we consider switching linear systems with location invariants and guard conditions as a generalization towards hybrid automata [77]. The location invariants are described as polyhedral constraints while the guards correspond to the boundaries of the polyhedra. As a result of incorporating location invariants the discrete dynamics are influenced by the continuous ones since they can trigger transitions to other discrete states. Synchronous fulfillment of location invariants therefore is a requirement for bisimulation equivalence for this subclass of hybrid systems. To investigate which extra conditions are needed we start this chapter by studying bisimulation relations for linear systems with inequality constraints. We obtain a characterization for bisimulation relations using results from convex geometry, in particular the Farkas lemma. Section 8.1 can be seen as an extension of the theory for linear systems presented in Chapter 2. In the second part of this chapter we outline how this result can be used to define structural hybrid bisimulation relations for switching linear systems with inequality constraints. This is a first step towards a comprehensive bisimulation theory for general hybrid automata.

8.1. Bisimulation theory for linear systems with inequality constraints

Constraints on variables are imposed in many real-life control applications, e.g. due to saturation of sensors and actuators. Expressing the constraints by affine inequalities restricts the state space to polyhedral domains.

Definition 8.1. A polyhedron $\mathcal{P} \subset \mathbb{R}^n$ is described as

$$\mathcal{P} := \{x \in \mathbb{R}^n \mid Kx \leq k\}, K \in \mathbb{R}^{q \times n}, k \in \mathbb{R}^q, \quad (8.1)$$

For the special case $k = 0$, (8.1) describes a *polyhedral cone*.

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In the literature, various methods have been proposed how to analyze polyhedra, see e. g. the survey paper [7]. Invariant polyhedra for linear systems have been studied by [12] using the theory of essentially nonnegative matrices and a version of the Farkas lemma. In the following, we want to study bisimulation relations for linear systems with inequality constraints. The underlying motivation is our interest in hybrid systems given as switching linear systems with inequality constraints. In this context, the inequality constraints represent location invariants while the generating hyperplanes correspond to the guards of switching linear systems. The continuous dynamics of this type of hybrid system at every discrete location are given by a linear system with inequality constraints Σ_i^{con} ,

$$\Sigma_i^{\text{con}} : \begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i + L_i d_i \\ y_i &= C_i x_i \\ K_i x_i &\leq k_i \end{aligned} \quad (8.2)$$

with $u_i \in \mathcal{U}_i, y_i \in \mathcal{Y}_i$ and $d_i \in \mathcal{D}_i$ taken from appropriately dimensioned vector spaces. Hence, the state space of (8.2) is the polyhedron

$$\mathcal{P}_i = \{x_i \in \mathbb{R}^{n_i} \mid K_i x_i \leq k_i, K_i \in \mathbb{R}^{q_i \times n_i}, k_i \in \mathbb{R}^{q_i}\} \quad (8.3)$$

Associated to every constrained system Σ_i^{con} is the corresponding unconstrained system Σ_i ,

$$\Sigma_i : \begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i + L_i d_i \\ y_i &= C_i x_i \end{aligned} \quad (8.4)$$

where $x_i \in \mathcal{X}_i = \mathbb{R}^{n_i}$, and u_i, y_i, d_i as before. Before defining and characterizing bisimulations for constrained systems Σ_i^{con} we first recall some important facts about polyhedra.

8.1.1. Facts about polyhedra

Polyhedra have been studied in great detail, see e.g. [26] or [82] for an overview. In the following, some of the basic notations and definitions are summarized. In Definition 8.1 polyhedra are described as the intersection of finitely many half-spaces. Equivalently, a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ can be represented as the convex hull of a finite set of points v^1, \dots, v^N ,

$$\mathcal{P} = \text{conv}(V) = \text{conv} \left(\begin{bmatrix} v^1 & v^2 & \dots & v^N \end{bmatrix} \right) \quad (8.5)$$

Converting one representation into the other is a non-trivial task, but can be achieved using Fourier-Motzkin elimination, see e.g. [82]. The interior points of \mathcal{P} are characterized by the following

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Lemma 8.2. *A point $x \in \mathcal{P}$ is an interior point, $x \in \text{int}\mathcal{P}$, if and only if it can be written as a positive combination of the N vertices generating the polyhedron,*

$$x = \sum_{i=1}^N \lambda_i v_i, \quad \lambda_i > 0, \quad \sum_{i=1}^N \lambda_i = 1$$

Faces describe the boundaries of polyhedral domains. In particular, for a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$, all faces of dimension $n - 1$, the so-called facets, will be important.

Definition 8.3. Let $\mathcal{P} = \{x \in \mathbb{R}^n \mid Kx \leq k\} \subset \mathbb{R}^n$ be a polyhedron. Denote by $(K)^j$ and $(k)^j$ the j -th rows of K and k , respectively. A linear inequality $(K)^j x \leq (k)^j$ is *valid* if it is satisfied for all points $x \in \mathcal{P}$. A *face* F of \mathcal{P} is a set of the form

$$F = \mathcal{P} \cap \{x \in \mathbb{R}^n \mid (K)^j x = (k)^j\}. \quad (8.6)$$

Faces of dimension 0, 1, $\dim(\mathcal{P}) - 2$ and $\dim(\mathcal{P}) - 1$ are called *vertices*, *edges*, *ridges* and *facets*, respectively. We denote the set of facets of a polyhedron \mathcal{P} by $\mathcal{F}(\mathcal{P})$.

8.1.2. A solution approach using the Farkas lemma

We want to describe equivalences of constrained linear systems based on bisimulation relations. The definition of bisimulation relations for the constrained case will require invariance of the polyhedra and include a facet condition.

Definition 8.4. Given two constrained linear systems Σ_i^{con} , $i = 1, 2$, as defined in (8.2). A bisimulation relation R^{con} between Σ_1^{con} and Σ_2^{con} is a subset $R^{\text{con}} \subset \mathcal{P}_1 \times \mathcal{P}_2$ with the following properties:

Take any $(x_{10}, x_{20}) \in \text{int}\mathcal{P}_1 \times \text{int}\mathcal{P}_2$. Then for every state trajectory $x_1(\cdot)$ with $x_1(0) = x_{10}$, every joint input $u_1(\cdot) = u_2(\cdot) = u(\cdot)$ and every disturbance $d_1(\cdot)$ there should exist a trajectory $x_2(\cdot)$ with $x_2(0) = x_{20}$, a disturbance $d_2(\cdot)$ and a strictly positive time $T > 0 \in \mathbb{R}^+ \cup \{\infty\}$ such that

$$\begin{aligned} (i) : & \quad (x_1(t), x_2(t)) \in R^{\text{con}} \quad \forall t \in [0, T] \\ (ii) : & \quad C_1 x_1(t) = C_2 x_2(t) \quad \forall t \in [0, T] \\ (iii) : & \quad x_1(T) \in \mathcal{F}(\mathcal{P}_1) \iff x_2(T) \in \mathcal{F}(\mathcal{P}_2) \end{aligned} \quad (8.7)$$

Conversely, for any trajectory $x_2(\cdot)$ with $x_2(0) = x_{20}$ and any disturbance $d_2(\cdot)$ there should exist a trajectory $x_1(\cdot)$, $x_1(0) = x_{10}$, a disturbance $d_1(\cdot)$ and a time $T > 0 \in \mathbb{R}^+ \cup \{\infty\}$ such that (8.7) holds. Moreover, Σ_1 and Σ_2 are bisimilar, denoted $\Sigma_1^{\text{con}} \approx \Sigma_2^{\text{con}}$, if there exists a bisimulation relation R^{con} fulfilling $\Pi_i R^{\text{con}} = \mathcal{P}_i$, $i = 1, 2$, with $\Pi_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_i}$ the canonical projection from $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ to \mathbb{R}^{n_i} . In this case, R^{con} is called a *full* bisimulation relation.

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In order to obtain a linear-algebraic characterization of bisimulations for linear systems with inequality constraints, some results from the theory of nonnegative matrices (see e. g. [6]) will be needed.

Definition 8.5. A matrix $H = (h_{ij}) \in \mathbb{R}^{q \times r}$ is nonnegative, denoted by $H \geq 0$, if $h_{ij} \geq 0$ for all $i = 1, \dots, q, j = 1, \dots, r$.

Theorem 8.6. The set of all non-negative $n \times n$ -matrices forms a semi-group under matrix multiplication, denoted by \mathcal{N}_n .

Definition 8.7. An element $A \in \mathcal{N}_n$ is called regular if $A = ABA$ for some element $B \in \mathcal{N}_n$. If in addition $B = BAB$, then A and B are called *semi-inverses* of each other.

Definition 8.8. An $n \times n$ -matrix of rank $r \leq n$ is called r -monomial if each of its columns contains at most one nonzero entry. If $r = n$, then the matrix is called a monomial matrix or a generalized permutation matrix.

Theorem 8.9. Let $A \in \mathcal{N}_n$ be a matrix of rank r . Then the following statements are equivalent:

1. A is regular in \mathcal{N}_n .
2. A has a semi-inverse in \mathcal{N}_n which is r -monomial.
3. A has a monomial submatrix of order r .

The Farkas lemma is a very important lemma for the study of polyhedra. It characterizes the solvability of systems of linear inequalities and can occur in a lot of different variants. We first present a version formulated as a separation theorem, that is, a point either lies in a polyhedron or is separated from it by a hyperplane.

Lemma 8.10. Given a matrix $A \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^{n \times 1}$, the following two statements are equivalent:

$$\begin{aligned} (i) : \quad & \exists x \geq 0 \text{ such that } Ax = b \\ (ii) : \quad & \forall y : y^T A \geq 0 \implies y^T b \geq 0 \end{aligned} \tag{8.8}$$

A matrix version of the Farkas lemma first proved by [28] will be used in the following to transform linear inequality constraints.

Lemma 8.11. The set $\{x \in \mathbb{R}^n \mid K_1 x \leq k_1, K_1 \in \mathbb{R}^{q_1 \times n}, k_1 \in \mathbb{R}^{q_1}\}$ is contained in $\{x \in \mathbb{R}^n \mid K_2 x \leq k_2, K_2 \in \mathbb{R}^{q_2 \times n}, k_2 \in \mathbb{R}^{q_2}\}$ if and only if there exists a nonnegative matrix $M \in \mathbb{R}^{q_2 \times q_1}$ such that

$$MK_1 = K_2 \quad \text{and} \quad Mk_1 \leq k_2$$

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To characterize bisimulation relations for linear systems with inequality constraints we restrict the geometry of the polyhedral domains.

Assumption 8.12. *In the remainder, we assume that the polyhedral constraints*

$$K_i x_i \leq q_i, \quad K_i \in \mathbb{R}^{q_i \times n_i} \quad i = 1, 2,$$

satisfy

$$\text{rank}K_1 = q_1 = \text{rank}K_2 = q_2 =: q, \quad q \leq n_i \quad (8.9)$$

The question of how restrictive Assumption 8.12 is will be discussed in Remark 8.19. We now give a sufficient condition for the existence of bisimulation relation between two linear systems with inequality constraints.

Theorem 8.13. *Given two constrained linear systems $\Sigma_i^{\text{con}}, i = 1, 2$, of the form (8.2) and assume that the constraints satisfy Assumption 8.12. Then a relation $R^{\text{con}} \subset \mathcal{P}_1 \times \mathcal{P}_2$ is a bisimulation relation between Σ_1^{con} and Σ_2^{con} if*

1. *the linear closure $\mathcal{L}(R^{\text{con}})$ is a full bisimulation relation of the corresponding unconstrained systems Σ_i , and*
2. *there exist nonnegative generalized permutation matrices $M, N \in \mathcal{N}_q$ such that for all $(x_1, x_2) \in R^{\text{con}}$*

$$\begin{aligned} MK_1 x_1 &= K_2 x_2, & K_1 x_1 &= NK_2 x_2 \\ Mk_1 &\leq k_2, & Nk_2 &\leq k_1 \end{aligned} \quad (8.10)$$

Proof. Suppose we are given a relation R^{con} such that its linear closure $\mathcal{L}(R^{\text{con}})$ is a full bisimulation relation between the unconstrained systems $\Sigma_i, i = 1, 2$. By Theorem 2.6, $\mathcal{L}(R^{\text{con}})$ is a linear subspace contained in $\mathcal{X}_1 \times \mathcal{X}_2$ and can therefore be written in image representation,

$$\mathcal{L}(R^{\text{con}}) = \text{im} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.$$

We denote the dimension of $\mathcal{L}(R^{\text{con}})$ by r . Thus, the state variables are related,

$$\forall (x_1, x_2) \in \mathcal{L}(R^{\text{con}}) : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \lambda, \quad \lambda \in \mathbb{R}^r. \quad (8.11)$$

Based on this we reformulate condition (8.10) as

$$\exists M, N \in \mathcal{N}_q : \begin{aligned} MK_1 R_1 \lambda &= K_2 R_2 \lambda, & K_1 R_1 \lambda &= NK_2 R_2 \lambda \\ Mk_1 &\leq k_2, & Nk_2 &\leq k_1 \end{aligned} \quad (8.12)$$

for some $\lambda \in \mathbb{R}^r$. By the Farkas lemma, (8.12) is equivalent to

$$\begin{aligned} K_1 R_1 \lambda \leq k_1 &\implies K_2 R_2 \lambda \leq k_2 \\ K_1 R_1 \lambda \leq k_1 &\iff K_2 R_2 \lambda \leq k_2 \end{aligned} \quad (8.13)$$

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being satisfied for any $\lambda \in \mathbb{R}^r$ or, equivalently,

$$K_1 x_1 \leq k_1 \iff K_2 x_2 \leq k_2. \quad (8.14)$$

for all $(x_1, x_2) \in \mathcal{L}(R^{\text{con}})$. Consider now a pair of initial states $(x_{10}, x_{20}) \in \text{int}\mathcal{P}_1 \times \text{int}\mathcal{P}_2 \subset R^{\text{con}}$ and corresponding trajectories $x_i(t), i = 1, 2$, starting at $x_{i0} = x_i(0)$. Then condition (i) in Definition 8.4 is fulfilled because $(x_1(\cdot), x_2(\cdot))$ satisfies (8.14). The bisimulation subspace $\mathcal{L}(R^{\text{con}})$ is a subspace of $\ker \begin{bmatrix} C_1 & -C_2 \end{bmatrix}$, hence condition (ii) is fulfilled too. Moreover, since M is a generalized permutation matrix, if $x_1(\cdot)$ reaches a facet of \mathcal{P}_1 at time T , i.e. $(K_1)^j x_1(T) = (k_1)^j$ for some $j \in \{1, \dots, q\}$, then at the same time T the trajectory $x_2(\cdot)$ reaches a facet of \mathcal{P}_2 , $M(K_1)^j x_1(T) = (K_2)^l x_2(T) = M(k_1)^j = (k_2)^l$ for some $l \in \{1, \dots, q\}$ and vice versa. Thus, condition (iii) is also fulfilled. \square

Conditions 1 and 2 of Theorem 8.13 are very close to being necessary as well.

Proposition 8.14. *Given two linear constrained systems $\Sigma_i^{\text{con}}, i = 1, 2$, of the form (8.2) satisfying Assumption 8.12. Suppose there exists a bisimulation relation R^{con} between Σ_1^{con} and Σ_2^{con} such that*

$$\Pi_{\mathcal{X}_i} \mathcal{L}(R^{\text{con}}) = \mathcal{X}_i, i = 1, 2. \quad (8.15)$$

Then conditions 1 and 2 of Theorem 8.13 hold true.

Proof. For two given linear systems with inequality constraints $\Sigma_i^{\text{con}}, i = 1, 2$, such that Assumption 8.12 holds, let R^{con} define a bisimulation relation between Σ_1^{con} and Σ_2^{con} satisfying (8.15).

We first show that due to linearity the linear closure $\mathcal{L}(R^{\text{con}})$ has to be a full bisimulation relation for the corresponding unconstrained systems Σ_i . Consider a pair of initial states $(x_{10}, x_{20}) \in \text{int}\mathcal{P}_1 \times \text{int}\mathcal{P}_2$ and any pair of trajectories $x_i(\cdot)$ starting from $x_{i0} = x_i(0)$. If (x_{10}, x_{20}) is an element of a bisimulation relation R^{con} between the two constrained systems Σ_i^{con} then there exists a $T > 0$ such that $(x_1(t), x_2(t)) \in R^{\text{con}}$ and $y_1(t) = y_2(t)$ for all $0 \leq t \leq T$. Thus, for $0 \leq t \leq T$, R^{con} fulfills conditions 1 – 4 of Theorem 2.7. But then the linear closure of R^{con} , $\mathcal{L}(R^{\text{con}})$ should also satisfy conditions 1 – 4 of Theorem 2.7. Note that due to linearity we only need a time interval of positive length to satisfy the invariance conditions therein. This implies that $\mathcal{L}(R^{\text{con}})$ is a bisimulation relation between the corresponding unconstrained systems. Fullness is guaranteed by (8.15).

Next, we show that (8.10) is also a necessary condition for the existence of a bisimulation relation R^{con} between the two constrained systems Σ_i^{con} . By Definition 8.4, all elements $(x_1, x_2) \in R^{\text{con}}$ have to lie within the product of the two polyhedra,

$$K_1 x_1 \leq k_1 \iff K_2 x_2 \leq k_2 \quad (8.16)$$

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Moreover, since $\mathcal{L}(R^{\text{con}})$ is a bisimulation subspace, (8.11) holds so that (8.16) becomes

$$K_1 R_1 \lambda \leq k_1 \iff K_2 R_2 \lambda \leq k_2 \quad (8.17)$$

Splitting (8.17) into two implications and applying the Farkas lemma to both, (8.17) turns out to be equivalent to (8.12). Due to fullness of $\mathcal{L}(R^{\text{con}})$ and Assumption 8.12, it follows from $NMK_1 R_1 = NK_2 R_2 = K_1 R_1$ and $MNK_2 R_2 = MK_1 R_1 = K_2 R_2$ that $NM = I_q = MN$ and thus $MNM = M, NMN = N$. Hence, M and N are regular in \mathcal{N}_q . Moreover, due to (8.9), $\text{rank} M = \text{rank} N = q$. Theorem 8.9 then assures that M and N are generalized permutation matrices. Thus, we recovered condition (8.10). \square

Proposition 8.14 and Theorem 8.13 almost fully characterize bisimulation relations for linear systems with inequality constraints. However, not much is known about when conditions 1 and 2 are *not* necessary. Moreover, these conditions are not easy to check. From an algorithmic perspective, Theorem 8.16 suggests the following procedure to construct a bisimulation relation between two constrained linear systems $\Sigma_i^{\text{con}}, i = 1, 2$, satisfying Assumption 8.12:

1. Compute the maximal bisimulation relation R^* between the corresponding unconstrained systems Σ_i following Algorithm 2.8. If $R^* = \emptyset$ there does not exist any bisimulation relation between Σ_1^{con} and Σ_2^{con} . If R^* is not full, the procedure stops without result.
2. Check whether there exists a nonnegative generalized permutation matrix M such that

$$R^* \subset \ker \begin{bmatrix} MK_1 & -K_2 \end{bmatrix}. \quad (8.18)$$

Otherwise, check whether a lower dimensional bisimulation subspace $R \subset R^*, \Pi_{\mathcal{X}_i} R = \mathcal{X}_i$, satisfies (8.18). If (8.18) is not fulfilled for any R the procedure stops without result.

3. Construct R^{con} as $R^{\text{con}} = R \cap (\mathcal{P}_1 \times \mathcal{P}_2)$ with R the result of step 2. In this case, R^{con} is also maximal, i.e. for any other bisimulation relation \tilde{R}^{con} between Σ_1^{con} and $\Sigma_2^{\text{con}}, \tilde{R}^{\text{con}} \subset R^{\text{con}}$.

Next, we consider the special case that the state spaces of the corresponding unconstrained systems are related a priori by a surjective map. These so-called H -related systems were defined in [56] to obtain abstractions of systems.

Definition 8.15. Consider two linear systems of the form (8.4) and a surjective linear mapping $H : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}$. Then Σ_2 is H -related to Σ_1 if and only if

$$A_2 H = H A_1, B_2 = H B_1, G_2 = [H G_1, H A_1 v_1, \dots, H A_1 v_k] \quad C_2 H = C_1, \quad (8.19)$$

where $\ker H = \text{span}\{v_1, \dots, v_k\}$.

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The existence of a state space mapping ensures the existence of a bisimulation relation between Σ_1 and Σ_2 .

Proposition 8.16. *Given two H -related systems $\Sigma_i, i = 1, 2$. Then $\Sigma_1 \approx \Sigma_2$ with a full bisimulation relation R between Σ_1 and Σ_2 given by the graph of H ,*

$$R = \{(x_1, x_2) \mid x_2 = Hx_1\}. \quad (8.20)$$

Proof. The relation R in (8.20) clearly fulfills the conditions of Theorem 2.6. Moreover, $\Pi_1 R = \mathcal{X}_1$ and since the mapping H is surjective, also $\Pi_2 R = \mathcal{R}_2$. \square

Proposition 8.16 allows us to give a full characterization for bisimulation relations for constrained H -related systems.

Corollary 8.17. *Consider two constrained systems $\Sigma_i^{\text{con}}, i = 1, 2$, of the form (8.2). Assume that the polyhedral constraints satisfy Assumption 8.12 and let the corresponding unconstrained dynamics be H -related. Then a subset $R^{\text{con}} = \mathcal{P}_1 \times \mathcal{P}_2$ is a bisimulation relation between Σ_1^{con} and Σ_2^{con} if and only if*

1. *the linear closure $\mathcal{L}(R^{\text{con}})$ is a full bisimulation relation between the corresponding unconstrained systems $\Sigma_i, i = 1, 2$, and*
2. *there exist nonnegative generalized permutation matrices $M, N \in \mathcal{N}_q$ such that (8.10) holds for all $(x_1, x_2) \in R^{\text{con}}$.*

Proof. Sufficiency is a direct consequence of Theorem 8.13. For necessity recall that since Σ_1 and Σ_2 are H -related, the graph of H defines a full bisimulation relation $R = \{(x_1, x_2) \mid x_2 = Hx_1\}$ between them. Suppose now there exists a bisimulation relation R^{con} between Σ_1^{con} and Σ_2^{con} . Condition (i) in Definition 8.4 can then be rewritten as

$$K_1 x_1 \leq k_1 \iff K_2 H x_1 \leq k_2 \quad (8.21)$$

Applying the Farkas lemma to (8.21), there have to exist nonnegative matrices $M, N \in \mathcal{N}_q$ such that

$$\begin{aligned} MK_1 &= K_2 H & , & & K_1 &= NK_2 H \\ Mk_1 &\leq k_2 & , & & k_1 &\leq Nk_2 \end{aligned} \quad (8.22)$$

The first line of (8.22) implies that $MK_1 x_1 = x_2$ for all $(x_1, x_2) \in R^{\text{con}}$. Since due to Proposition 8.14 $\mathcal{L}(R^{\text{con}})$ has to be a bisimulation relation between Σ_1 and Σ_2 , it follows that

$$\mathcal{L}(R^{\text{con}}) \subset \ker \begin{bmatrix} MK_1 & -K_2 \end{bmatrix} = \ker \begin{bmatrix} K_2 \begin{bmatrix} H & -I \end{bmatrix} \end{bmatrix} \quad (8.23)$$

However, (8.23) is always fulfilled by $R = \text{graph}H$. Hence, if there exists a bisimulation relation R^{con} , it can be constructed as

$$R^{\text{con}} := \text{graph}H \cap (\mathcal{P}_1 \times \mathcal{P}_2) \quad (8.24)$$

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so that $\mathcal{L}(R^{\text{con}}) = \text{graph}H$ is a full bisimulation relation between Σ_1 and Σ_2 . Then, repeating the same arguments as before, (8.22) implies that $MN = NM = I_q$ and hence M and N are generalized permutation matrices. \square

Corollary 8.17 also indicates how to compute the bisimulation relation between two constrained systems $\Sigma_i^{\text{con}}, i = 1, 2$, with H -related dynamics. First, check whether there exists a nonnegative generalized permutation matrix M such that $MK_1 = K_2H$. If so, R^{con} as defined in (8.23) is a bisimulation relation between Σ_1^{con} and Σ_2^{con} . Otherwise, there does not exist any such bisimulation relation.

Example 8.18. Consider the following H -related constrained systems

$$\begin{aligned} \Sigma_1^{\text{con}} : \quad & \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d_1 \\ & y_1 = [1 \ 0] x_1 \\ & \mathcal{P}_1 = \{(x^1, x^2) \mid x^1 \leq 0\} \end{aligned}$$

and

$$\Sigma_2^{\text{con}} : \quad \begin{aligned} \dot{z} &= z + u_2 \\ y_2 &= 1 \\ \mathcal{P}_2 &= \{z \mid z \leq 0\} \end{aligned}$$

and the surjective mapping $H : \mathbb{R}^2 \rightarrow \mathbb{R}^1$ defined as

$$z = [1 \ 0] \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

A quick computation reveals that condition (8.22) is fulfilled with $M = N = 1$. The graph of H is given by

$$R = \text{graph}H = \{(x^1, x^2), z \mid x^2 = 0, x^1 = z\}$$

Hence, a bisimulation relation R^{con} between the linear systems Σ_1^{con} and Σ_2^{con} is given by

$$R^{\text{con}} = \{(x^1, 0), z \mid x^1 \leq 0, z \leq 0, x^1 = z\}$$

Remark 8.19. Assumption (8.9) is clearly restrictive since it limits the number of facets of the polyhedra to be no greater than the dimension of the embedding space. An approach to relax this could be to find a facet mapping between the two polyhedra \mathcal{P}_1 and \mathcal{P}_2 : First, enlarge $K_i, k_i, i = 1, 2$, by multiples of existing rows to obtain \tilde{K}_i, \tilde{k}_i of dimensions $q_1 q_2 \times n_i, q_1 q_2 \times 1$. Then, the existence of generalized permutation matrices \tilde{M}, \tilde{N} such that

$$\begin{aligned} \tilde{M}\tilde{K}_1 &= \tilde{K}_2 & \text{and} & & \tilde{M}k_1 &\leq & \tilde{k}_2 \\ \tilde{K}_1 &= \tilde{N}\tilde{K}_2 & \text{and} & & k_1 &\leq & \tilde{N}\tilde{k}_2 \end{aligned} \quad (8.25)$$

would establish a relation between the respective facets of the two polyhedra and could thus guarantee condition (iii) of Definition 8.4.

8.2. Bisimulation relations for switching linear systems with inequality constraints

Adding inequality constraints to the definition of switching linear systems introduced in Chapter 7 is an important step towards general hybrid automata. In practice, many models of hybrid systems require inequality constraints, for example the simple model of a thermostat or complementarity systems [11]. However, these constraints increase the level of difficulty. Compared to Chapter 7, switching linear systems with inequality constraints exhibit mutual dependencies of the discrete and continuous dynamics. As before, the occurrence of a discrete event leads to a discrete transition accompanied by a change of the continuous dynamics with prior reset of the continuous state. Additionally, the continuous dynamics have to satisfy an invariance condition at every discrete location. If imminent to be violated the continuous evolution triggers a change of the discrete location. These transitions are controlled by guards which are related to the boundary of the constraints as illustrated in Figure 8.1.

8.2.1. Definition and semantics of switching linear systems with inequality constraints

Our definition of switching linear systems with location invariants is derived from the general definition of hybrid automata that can be found in [77].

Definition 8.20. A switching linear system with inequality constraints Σ^{SLScon} is described by a tuple $\Sigma^{\text{SLScon}} = (Q, \mathcal{P}, V, \mathcal{W}, \mathcal{F}, \mathcal{E})$ where the symbols have the following meaning:

- Q is a finite set of discrete locations (or discrete states).
- $\mathcal{P}(q)$ is a polyhedron, given as linear inequalities

$$\mathcal{P}(q) = \{x \in \mathbb{R}^{n(q)} \mid K(q)x \leq k(q)\}, \quad (8.26)$$

representing the constrained continuous part of the hybrid state space at every location $q \in Q$.

- V is a finite set of event labels.
- $\mathcal{W} = \mathcal{U} \times \mathcal{D} \times \mathcal{Y}$ denotes the continuous communication variables taken from linear spaces of appropriate dimensions, with $u \in \mathcal{U}$, $d \in \mathcal{D}$ and $y \in \mathcal{Y}$ the continuous inputs, disturbances and outputs, respectively.

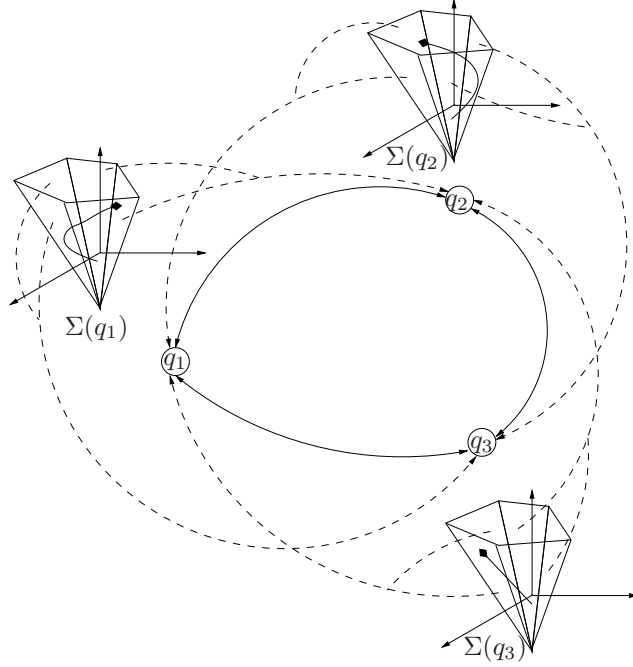


Figure 8.1.: Switching linear systems with location invariants and guards.

- The flow conditions \mathcal{F} are described by differential equations of the form (8.4).
- The event conditions \mathcal{E} are defined by a set of linear equations,

$$(q, x, v, q', x') \in \mathcal{E} \iff \begin{cases} (q, v, q') \in E \\ x' = M(q, q')x \\ (K(q))^j x = (k(q))^j \text{ for some } j \end{cases} \quad (8.27)$$

where $E \subset Q \times V \times Q$ is the discrete transition relation, x and x' describe the continuous variables just before and after a transition from q to q' and $(K(q))^j x = (k(q))^j$ represents the guard condition for this transition.

The hybrid state space Δ^{SLScn} of a switching linear system Σ^{SLScn} is given by the product $\Delta^{\text{SLScn}} = \bigcup_{q \in Q} \{q\} \times \mathcal{P}(q)$.

Remark 8.21. Note that if x lies in a lower-dimensional face of $\mathcal{P}(q)$, e.g. a vertex, several guard conditions $(K(q))^j x = (k(q))^j$ could be fulfilled simultaneously in (8.27). This non-determinism is characteristic for switching linear systems with inequality constraints.

8. Equivalences of switching linear systems with inequality constraints

From the above definition the structure of switching linear systems with inequality constraints Σ^{SLscon} is described as follows: On the level of the discrete dynamics, there exists a labeled transition system associated with Σ^{SLscon} and denoted by D^{SLscon} ,

$$D^{\text{SLscon}} = (Q, V, E) \quad (8.28)$$

The continuous dynamics in a discrete location are described by a linear system with inequality constraints $\Sigma^{\text{con}}(q)$ of the form (8.2). The location invariant is determined by a system of linear inequalities $K(q)x \leq k(q)$ while the guard conditions correspond to facets $\mathcal{F}(\mathcal{P}(q))$ of the polyhedron $\mathcal{P}(q)$. In the remainder, we assume that switching linear systems with inequality constraints are *deadlock-free*¹. In particular, every facet $\mathcal{F}(\mathcal{P}(q))$ guards a transition to another discrete location, i.e., as soon as a guard condition is satisfied a transition of the discrete state is triggered to continue the continuous evolution of the hybrid dynamics. The semantics of a switching linear systems with inequality constraints are described similarly to Definition 7.3 as follows.

Definition 8.22. A hybrid execution r of a switching linear system $\Sigma^{\text{SLscon}} = (Q, \mathcal{P}, V, \mathcal{W}, \mathcal{F}, \mathcal{E})$ on a time interval $[0, T]$ is a collection $(\mathcal{T}, q, x, v, w)$ where

- $\mathcal{T} \subset [0, T]$ is a finite set $\{\tau_1, \tau_2, \dots\}$ of ordered discrete event times,
- $q : [0, T] \rightarrow Q$ is a function from the time interval $[0, T]$ to the set of locations describing the evolution of the discrete states, which is constant on every interval $[\tau_i, \tau_{i+1})$, $i = 1, 2, \dots$,
- $x : [0, T] \rightarrow \mathcal{X}(q)$ and $u : [0, T] \rightarrow \mathcal{U}$, $d : [0, T] \rightarrow \mathcal{D}$, $y : [0, T] \rightarrow \mathcal{Y}$ are time functions satisfying for every $t \notin \mathcal{T}$ the flow conditions (8.4),
- $v : \mathcal{T} \rightarrow V$ a function for the event labels satisfying for every $\tau \in \mathcal{T}$ the event condition $(q(\tau^-), x(\tau^-), v, q(\tau^+), x(\tau^+)) \in \mathcal{E}$ where $q(\tau^-) = \lim_{t \uparrow \tau} q(t)$, $q(\tau^+) = \lim_{t \downarrow \tau} q(t)$ denote the discrete locations involved in the transition $(q(\tau^-), v, q(\tau^+)) \in E$ and $x(\tau^-) = \lim_{t \uparrow \tau} x(t)$, $x(\tau^+) = \lim_{t \downarrow \tau} x(t)$ the continuous states just before and after the event time τ ,
- $w = (u, d, y)$ are time functions $u : [0, T] \rightarrow \mathcal{U}$, $d : [0, T] \rightarrow \mathcal{D}$, $y : [0, T] \rightarrow \mathcal{Y}$ of the continuous inputs, disturbances and outputs valid for every $t \notin \mathcal{T}$.

8.2.2. Structural bisimulation relations for switching linear systems with inequality constraints

A standard way of defining bisimulation relations for hybrid systems is based on matching executions. Definition 7.11 is also valid for switching linear systems with inequality constraints. We now want to develop a structural notion

¹A switching linear system Σ^{SLscon} is deadlock-free if every hybrid execution can be extended to an execution defined on an infinite time interval

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of hybrid bisimulation. To do so, we recall Proposition 7.13 to exploit the particular structure of the bisimulation subsets defined on the hybrid state space.

Proposition 8.23. *Let the set $R \subset \Delta_1 \times \Delta_2$ be a hybrid bisimulation relation between the switching linear systems Σ_i^{SLScn} , $i = 1, 2$. Then there exists $Q_R \subset Q_1 \times Q_2$ and for every $(q_1, q_2) \in Q_R$ suitable sets $W(q_1, q_2) \subset \mathcal{P}_1(q_1) \times \mathcal{P}_2(q_2)$ such that*

$$(q_1, x_1, q_2, x_2) \in R \iff (q_1, q_2) \in Q_R, (x_1, x_2) \in W(q_1, q_2) \quad (8.29)$$

Note that in contrast to Chapter 7 $W(q_1, q_2) \subset \mathcal{P}_1(q_1) \times \mathcal{P}_2(q_2)$ cannot be assumed to have any nice properties. In fact, $W(q_1, q_2)$ need not be a polyhedron itself. Nevertheless, we can define structural hybrid bisimulations for switching linear systems with inequality constraints as subsets of the form (8.29).

Definition 8.24. Consider two switching linear systems $\Sigma_i^{\text{SLScn}} = (Q_i, \mathcal{P}_i, V_i, \mathcal{W}_i, \mathcal{F}_i, \mathcal{E}_i)$, $i = 1, 2$. A structural hybrid bisimulation relation R between Σ_1^{SLScn} and Σ_2^{SLScn} is a subset $R \subset \Delta_1 \times \Delta_2$ with the following property. Take any $(q_1, x_1, q_2, x_2) \in R$. Then for every q'_1, x'_1 for which $(q_1, x_1, v, q'_1, x'_1) \in \mathcal{E}_1$ there should exist q'_2, x'_2 such that $(q_2, x_2, v, q'_2, x'_2) \in \mathcal{E}_2$ while $(q'_1, x'_1, q'_2, x'_2) \in R$, and conversely. Furthermore, take any $(q_1, x_1, q_2, x_2) \in R$. Then for every joint continuous input $u_1 = u_2$ and every disturbance d_1 there should exist a disturbance d_2 such that $y_1 = y_2$ while $(A_1 x_1 + B_1 u + G_1 d_1, A_2 x_2 + B_2 u + G_2 d_2) \in W(q_1, q_2)$ and conversely.

Definition 8.24 facilitates a characterization of bisimulation relation for switching linear systems with inequality constraints.

Theorem 8.25. *Given two switching linear systems with inequality constraints Σ_i^{SLScn} , $i = 1, 2$. A relation*

$$R = \{(q_1, x_1, q_2, x_2) \mid (q_1, q_2) \in Q_R, (x_1, x_2) \in W(q_1, q_2)\} \quad (8.30)$$

is a hybrid bisimulation relation between Σ_1^{SLScn} and Σ_2^{SLScn} if and only if the following holds:

1. $Q_R(q_1, q_2)$ is a bisimulation relation of the associated labeled transition systems D_i^{SLScn} , $i = 1, 2$
2. for every $(q_1, q_2) \in Q_R$, $W(q_1, q_2)$ is a bisimulation relation between the corresponding linear systems with inequality constraints $\Sigma_1^{\text{con}}(q_1)$ and $\Sigma_2^{\text{con}}(q_2)$
3. for every $(q_1, x_1, v, q'_1, x'_1) \in \mathcal{E}_1$ there should exist q'_2, x'_2 such that $(q_2, x_2, v, q'_2, x'_2) \in \mathcal{E}_2$ and

$$(x'_1, x'_2) \in W(q'_1, q'_2) \quad (8.31)$$

and vice versa, for every $(q_2, x_2, v, q'_2, x'_2) \in \mathcal{E}_2$ there should exist q'_1, x'_1 such that $(q_1, x_1, v, q'_1, x'_1) \in \mathcal{E}_1$ and (8.31) holds.

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Proof. Suppose that R given by (8.30) is a bisimulation relation between Σ_i^{SLScn} , $i = 1, 2$ and consider an execution $r_1 = (\mathcal{T}_1, q_1, x_1, v_1, w_1)$ of Σ_1^{SLScn} with $\mathcal{T}_1 = \{\tau_1^1, \tau_1^2, \dots\}$. Start with the time interval $[0, \tau_1^1]$ during which $\dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t) + G_1 d_1(t)$, $y_1(t) = C_1 x_1(t)$, $t \in [0, \tau_1^1)$, $(q_1(\tau_1^{1,-}), v, q_1(\tau_1^{1,+})) \in D_1^{\text{SLScn}}$ and

$$(q_1(\tau_1^{1,-}), x_1(\tau_1^{1,-}), v_1, q_1(\tau_1^{1,+}), x_1(\tau_1^{1,+})) \in \mathcal{E}_1$$

where $\tau_1^{1,-}, \tau_1^{1,+}$ denote the time instants just before and after τ_1^1 . By condition 2 and since Σ_i^{SLScn} , $i = 1, 2$, are deadlock-free, there exists a d_2 such that $(A_1 x_1 + B_1 u_1 + G_1 d_1, A_2 x_2 + B_2 u_2 + G_2 d_2)|_{\tau_1^{1,-}} \in W(q_1(\tau_1^{1,-}), q_2(\tau_2^{1,-}))$ and $y_1(t) = y_2(t)$ for $t \geq 0$. Moreover, since there exists a bisimulation relation between $\Sigma_1^{\text{con}}(q_1(\tau_1^{1,-}))$ and $\Sigma_2^{\text{con}}(q_2(\tau_2^{1,-}))$ and at time $\tau_1^{1,-}$, $x_1(\tau_1^{1,-}) \in \mathcal{F}(\mathcal{I}(q_1(\tau_1^{1,-})))$, it follows from (8.7), (iii) that also $x_2(\tau_1^{1,-}) \in \mathcal{F}(\mathcal{I}(q_2(\tau_1^{1,-})))$. The first discrete switch in execution r_1 occurs at τ_1^1 , and since $\Sigma_2^{\text{con}}(q_2)$ fulfills its guard condition $x_2(\tau_1^{1,-}) \in \mathcal{F}(\mathcal{I}(q_2(\tau_1^{1,-})))$, it makes a transition at the same time $\tau_2^1 = \tau_1^1 =: \tau^1$. The existence of a bisimulation relation between D_1^{SLScn} and D_2^{SLScn} ensures that there exist $q_2(\tau^{1,-}), q_2(\tau^{1,+})$ such that $(q_2(\tau^{1,-}), v, q_2(\tau^{1,+})) \in D_2^{\text{SLScn}}$ such that $(q_1(\tau^{1,-}), q_2(\tau^{1,-})) \in Q_R$ as well as $(q_1(\tau^{1,+}), q_2(\tau^{1,+})) \in Q_R$. Condition 3 of Theorem 8.25 then ensures that $(x_1(\tau^{1,+}), x_2(\tau^{1,+})) \in W(q_1(\tau^{1,+}), q_2(\tau^{1,+}))$. Repeating the same arguments starting with an execution r_2 of Σ_2^{SLScn} and inductively proceeding in time considering intervals $[\tau^i, \tau^{i+1}]$, $i = 1, 2, \dots$ completes this part of the proof.

For the converse, suppose there exists a hybrid bisimulation relation $R \subset Q_1 \times \mathcal{P}_1 \times Q_2 \times \mathcal{P}_2$ in the sense of Definition 7.11. Assume there does not exist any bisimulation relation between the associated labeled transition systems D_i^{SLScn} . Then for any v such that $r_1 = (\mathcal{T}_1, q_1, x_1, v_1, w_1)$ is an execution of Σ_1^{SLScn} there does not exist any execution r_2 of Σ_2^{SLScn} such that (7.8), (iii) holds. Similarly, one can prove by contradiction that condition 2 of Theorem 8.25 is necessary. Next, consider any execution r_1 of Σ_1^{SLScn} such that there exists an execution r_2 of Σ_2^{SLScn} satisfying conditions (7.8). For any $\tau \in \mathcal{T}$, denote by τ^- and τ^+ the time instants just before and after the switching event τ . Then by (7.8), $y_1(\tau^*) = y_2(\tau^*)$, $v_1 = v_2$ and $(q_1(\tau^*), x_1(\tau^*), q_2(\tau^*), x_2(\tau^*)) \in R$ for $\star \in \{-, +\}$. Since $W(q_1(\tau^*), q_2(\tau^*))$ is a bisimulation relation of the continuous-time systems $\Sigma_i^{\text{con}}(q_i(\tau^*))$, $i = 1, 2$, it follows by definition of \mathcal{E}_i that

$$(q_i(\tau^-), x_i(\tau^-), v, q_i(\tau^+), x(\tau^+)) \in \mathcal{E}_i$$

and $(x_1(\tau^*), x_2(\tau^*)) \in W(q_1(\tau^*), q_2(\tau^*))$. Repeating the same arguments for any execution r_2 it follows that condition 3 is indeed satisfied. \square

Theorem 8.25 combines the well-established results for bisimulation relations of labeled transition systems (see Chapter 2) with the notion of bisimulation relations for linear systems with inequality constraints previously given

in Definition 8.4. Resets and guards coupling the continuous and discrete dynamics are synchronized due to condition 3.

Since our definition of switching linear systems with location invariants is based on hybrid automata, it should be possible to characterize bisimulation relations for other classes of hybrid systems in a similar way as in Theorem 8.25, compare also with [76]. However, in the most general case, the continuous-time dynamics are described by nonlinear flow conditions. From Chapter 4 we know that bisimulation relations of nonlinear systems are defined on submanifolds. One important question would therefore be whether the sets $\mathcal{P}_R(q_1, q_2)$ can be proved to be submanifolds. Another important issue is the existence of a maximal hybrid bisimulation relation. This necessarily depends on whether there exists a maximal bisimulation relation between the constrained continuous-time systems at every discrete location. As an example, Corollary 8.17 contains necessary and sufficient conditions for the existence of a bisimulation relation between constrained H -related linear systems, which allows to compute the maximal relation satisfying these conditions. In this case, the algorithm presented in [60] for switching linear systems in the sense of Definition 7.1 can be modified to compute maximal bisimulation relations between switching linear systems with inequality constraints.

