Compositional analysis and control of dynamical systems

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7

Compositional analysis of switching linear systems

7.1. Introduction

Hybrid systems combine elements of discrete and continuous dynamics. Many real-life applications are characterized by hybrid behavior, e.g. air traffic management [66, 70], multi-agent systems [10, 20] and genetic regulatory networks [18, 13]. Due to the interaction of continuous and discrete phenomena, the study of hybrid systems (depending on the area also called embedded [19] or cyber-physical systems) is an interdisciplinary field approached from both computer science and systems and control theory. As a result, there exists a large variety of subclasses of hybrid systems. In this work, we want to consider hybrid systems given as switching linear systems which combine discrete elements of labeled transition systems with linear continuous-time dynamics, see e.g. [60]. Exploiting their particular structure, we develop a theory of compositional and assume-guarantee reasoning for switching linear systems based on the theory of Chapter 3 and results from [21] for labeled transition systems. Like in previous chapters, we investigate the following proof rules:

- Compositionality:

$$\left.\begin{array}{rrcl} S_1: & \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_2: & \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_2} \end{array}\right\} \implies S: \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{7.1}$$

- Non-circular assume-guarantee reasoning:

$$\left.\begin{array}{rrcl} S_1: & \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_{II}: & \Sigma_{Q_1} \| \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array}\right\} \implies S: \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{7.2}$$

- Circular assume-guarantee reasoning:

$$\left.\begin{array}{rrcl} S_I: & \Sigma_{P_1} \| \Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \\ S_{II}: & \Sigma_{Q_1} \| \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \end{array}\right\} \implies S: \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{7.3}$$

In computer science, one usually describes the continuous part of the hybrid system by considering solutions of differential equations, see e.g. [2] or [30] and [21], thus treating the hybrid system as a generalized transition system. Our approach is different since it is based on the differential equations themselves. Hence, it does not depend upon actual system trajectories which are often, especially in the case of nonlinear dynamics, not available. Furthermore, it turns out that hybrid simulation relations can be checked efficiently working with differential equations. Structural hybrid simulation relations play a key role in our analysis since they separate the influence of the discrete from the continuous part of the hybrid dynamics. Thus, we can make use of well-established results for compositional analysis of labeled transition systems on the one hand and of linear continuous-time systems on the other hand.

7.2. Preliminaries and basic definitions

In our definition of switching linear systems, we will combine the structure of labeled transition systems on the level of the discrete dynamics and linear continuous-time systems on the level of the continuous dynamics.

Definition 7.1. A switching linear system Σ^{SLS} is a tuple $\Sigma^{\mathrm{SLS}} = (Q, \mathcal{X}, V, \mathcal{W}, \Sigma, E, M)$ with

- $Q = \{q_1, \ldots, q_N\}, N \in \mathbb{N}$, the set of discrete locations (or discrete states),
- $\mathcal{X}(q) \subset \mathbb{R}^{\dim(q)}, \dim : Q \to \mathbb{N}$ a linear vector space representing the continuous part of the hybrid dynamics at every location $q \in Q$,
- V the set of discrete transition labels,
- $\mathcal{W} \subset \mathcal{U} \times \mathcal{D} \times \mathcal{Y}$ the set of continuous communication variables with $u \in \mathcal{U} = \mathbb{R}^{\mathrm{m}}$ the inputs, $d \in \mathcal{D} = \mathbb{R}^{d}$ the disturbances, and $y \in \mathcal{Y} = \mathbb{R}^{\mathrm{p}}$ the outputs, respectively,
- Σ a function associating to every discrete state q the deterministic linear input-state-output system

$$\Sigma(q) : \begin{cases} \dot{x}(t) = A(q)x(t) + B(q)u(t) + G(q)d(t), \\ y(t) = C(q)x(t) \end{cases} \tag{7.4}$$

  with $x \in \mathcal{X}(q), u \in \mathcal{U}, d \in \mathcal{D}$ and $y \in \mathcal{Y}$,
- $E \subset Q \times V \times Q$ the discrete transition relation,
- for every $e = (q, v, q') \in E$, $M(e) : \mathcal{X}(q) \to \mathcal{X}(q')$ the linear reset map associated with every discrete transition $e = (q, v, q') \in E$ to reset the continuous state.

The hybrid state space of Σ^{SLS}, denoted by Δ, is given as the set $\Delta = \cup_{q \in Q}\{q\} \times \mathcal{X}(q)$. Changes of discrete states are triggered by (or accompanied by) discrete events $v \in V$. Whenever an event v occurs at an event time τ, the discrete state switches from $q(\tau^-) = \lim_{t \uparrow \tau} q(t)$ to $q(\tau^+) = \lim_{t \downarrow \tau} q(t)$ according to the transition relation E, while the continuous state is reset to $x(\tau^+) = M(e)x(\tau^-)$ where $x(\tau^-)$ and $x(\tau^+)$ represent the values of x just before and after the occurrence of transition $e = (q(\tau^-), v, q(\tau^+)) \in E$ at time τ, i.e. $x(\tau^-) = \lim_{t \uparrow \tau} x(t)$ and $x(\tau^+) = \lim_{t \downarrow \tau} x(t)$. In between event times, the associated continuous dynamics are governed by the linear system dynamics $\Sigma(q)$ in every discrete state. The discrete part of the hybrid dynamics has the structure of a labeled transition system as introduced in Section 2.1.

Definition 7.2. Given a switching linear system $\Sigma^{\text{SLS}} = (Q, \mathcal{X}, V, \mathcal{W}, \Sigma, E, M)$, the *associated labeled transition system* D^{SLS} is given by the triple $D^{\text{SLS}} = (Q, V, E)$ with $Q = \{q_1, \ldots, q_N\}$ the set of discrete states, V the set of transition labels and $E \subset Q \times V \times Q$ the discrete transition relation as in Definition 7.1.

The trajectories of a switching linear system will be specified with respect to a time interval $[0, T]$ including discrete event times.

Definition 7.3. An *execution* ρ of a switching linear system Σ^{SLS} on a time interval $[0, T]$ is a collection $\rho = (\mathcal{T}, q, x, v, w)$ where

- $\mathcal{T} \subset [0, T]$ is a finite set $\{\tau_1, \tau_2, \ldots\}$ of ordered discrete event times,
- $q : [0, T] \to Q$ is a function from the time interval $[0, T]$ to the set of locations describing the evolution of the discrete states, which is constant on every interval $[\tau_i, \tau_{i+1}), i = 1, 2, \ldots$,
- $x : [0, T] \to \mathcal{X}(q)$ is a time function satisfying for every $t \neq \mathcal{T}$ the differential equations $\Sigma(q(t))$ with $w = (u, d, y)$ and for every $\tau \in \mathcal{T}$, $x(\tau^+) = M(e)x(\tau^-)$,
- $v : \mathcal{T} \to V$ a function for the event labels satisfying for every $\tau \in \mathcal{T}$ the discrete transition relation $(q(\tau^-), v, q(\tau^+)) \in E$,
- $w = (u, d, y)$ are time functions $u : [0, T] \to \mathcal{U}$, $d : [0, T] \to \mathcal{D}$, $y : [0, T] \to \mathcal{Y}$ of the continuous inputs, disturbances and outputs valid for every $t \neq \mathcal{T}$.

For a more detailed discussion of the semantics of hybrid systems, allowing for more general descriptions, we refer to [77].

7.3. Compositions of hybrid systems

Figure 7.1 illustrates how switching linear systems are structured. The continuous dynamics are influenced by the discrete ones while, by contrast, the discrete dynamics are independent of the continuous ones. Changes of the continuous evolution are caused by discrete events. The occurrence of discrete events is assumed not to be controllable. Due to this particular structure, one

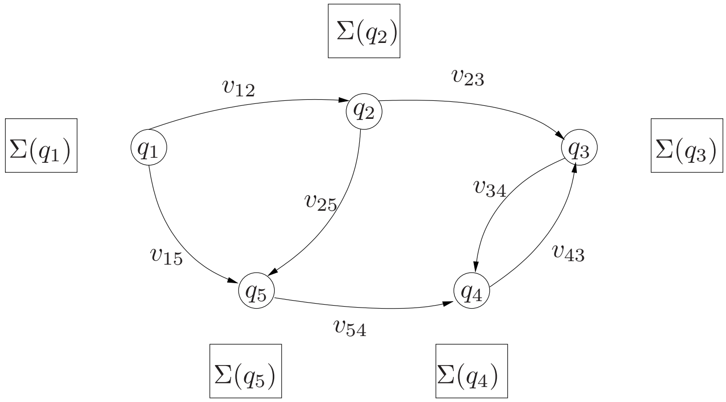

Figure 7.1.: Discrete and continuous layer of switching linear systems.

can define interconnections of switching linear systems by first determining the parallel composition of the associated labeled transition systems and then interconnecting the respective continuous systems for every discrete state of the parallel composition. Additionally, the reset maps have to be adjusted for non-shared transition labels $v \in V_i \setminus V_j, (i, j) \in \{(1, 2), (2, 1)\}$.

7.3.1. Interconnections of labeled transition systems

The interconnection of the discrete layer of two switching linear systems Σ_i^{SLS}, $i = 1, 2$, involves the labeled transition systems D_i^{SLS} associated to Σ_i^{SLS}. The treatment of these labeled transition systems follows the standard definitions found in the computing science literature.

Definition 7.4. Consider two labeled transition systems $D_i = (Q_i, V_i, E_i), i = 1, 2$, like in Definition 2.1. The parallel composition $D = D_1 \| D_2$ is again a labeled transition system $D = (Q, V, E_{12})$ with

- $Q = Q_1 \times Q_2$
- $V = V_1 \cup V_2$
- $E_{12} = \begin{cases} ((q_1, q_2), v, (q_1', q_2')) & , v \in V_1 \cap V_2, \\ & (q_i, v, q_i') \in E_i, i = 1, 2 \\ ((q_1, q_2), v, (q_1', q_2)) & , v \in V_1 \setminus V_2, \\ & (q_1, v, q_1') \in E_1 \\ ((q_1, q_2), v, (q_1, q_2')) & , v \in V_2 \setminus V_1, \\ & (q_2, v, q_2') \in E_2 \end{cases}$

The notion of simulation relations is instrumental for compositional analysis. We follow Definition 2.3 to relate labeled transition systems with the same transition structure by means of simulation relations. To prepare for the analysis of switching linear systems later, we recall from the literature, in particular [21], the most important results for compositional and assume-guarantee reasoning for labeled transition systems.

Theorem 7.5. *Given four labeled transition systems D_i, $i \in \{P_1, P_2, Q_1, Q_2\}$ such that $V_{P_1} = V_{Q_1}, V_{P_2} = V_{Q_2}$ and define parallel composition $\|$ as in Definition 7.4. Then parallel composition of labeled transition systems is compositional with respect to simulation.*

Theorem 7.5 is proved indirectly in the literature. In fact, [21] shows that simulation for labeled transition systems is invariant under composition, i.e. if $D_{P_1} \preccurlyeq D_{Q_1}$ then for every D_{P_2}, $D_{P_1}\|D_{P_2} \preccurlyeq D_{Q_1}\|D_{P_2}$. Since parallel composition is symmetric, $D_{P_1}\|D_{P_2} \approx D_{P_2}\|D_{P_1}$, invariance under composition is equivalent to compositionality. Indeed, assuming that $D_{P_i} \preccurlyeq D_{Q_i}, i = 1, 2$, invariance under composition yields full simulation relations S_I and S_{II} such that

$$\begin{aligned} S_I : \quad & D_{P_1}\|D_{P_2} & \preccurlyeq \quad & D_{Q_1}\|D_{P_2} \\ S_{II} : \quad & D_{P_2}\|D_{Q_1} & \preccurlyeq \quad & D_{Q_2}\|D_{Q_1} \end{aligned}$$

Exploiting symmetry of parallel composition and transitivity of simulation, one obtains the desired result,

$$D_{P_1}\|D_{P_2} \preccurlyeq D_{Q_1}\|D_{P_2} \preccurlyeq D_{P_2}\|D_{Q_1} \preccurlyeq D_{Q_2}\|D_{Q_1} \preccurlyeq D_{Q_1}\|D_{Q_2}$$

Non-circular assume-guarantee reasoning is also sound since parallel composition of labeled transition systems is compositional and simulation is transitive (compare with Theorem 3.9 for the continuous case).

Theorem 7.6. *Given four labeled transition systems $D_i, i \in \{P_1, P_2, Q_1, Q_2\}$ such that $V_{P_1} = V_{Q_1}, V_{P_2} = V_{Q_2}$ and define parallel composition $\|$ as in Definition 7.4. Then non-circular assume-guarantee reasoning is sound.*

For circular assume-guarantee reasoning, only a slightly weaker result can be proved.

Theorem 7.7 (compare with [21])**.** *Given four labeled transition systems $D_i, i \in \{P_1, P_2, Q_1, Q_2\}$ such that $V_{P_1} = V_{Q_1}, V_{P_2} = V_{Q_2}$ and define parallel composition $\|$ as in Definition 7.4. Let full simulation relations S_I and S_{II} of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, respectively, be given as in the circular assume-guarantee reasoning rule* (7.3) *. Then there exists a simulation relation S of $\Sigma_{P_1}\|\Sigma_{P_2}$ by*

$\Sigma_{Q_1} \| \Sigma_{Q_2}$ if and only if $\forall((p_1, p_2), (q_1, q_2)) \in S, v \in V_{P_1} \cap V_{P_2}$, the following condition holds:

$$\begin{array}{rcl} ((p_1, p_2), v, (p_1', p_2')) & \in & E_{P_1 P_2} \\ & \Longrightarrow & \\ \exists q_1' : (q_1, v, q_1') \in E_{Q_1} & \vee & \exists q_2' : (q_2, v, q_2') \in E_{Q_2} \end{array} \tag{7.5}$$

Intuitively, condition (7.5) ensures that the evolution of the approximation $D_{Q_1} \| D_{Q_2}$ will not reach a deadlock while there are still executable transitions in $D_{P_1} \| D_{P_2}$. Note that even if (7.5) is satisfied, it is not ensured by Theorem 7.7 that $\Pi_{Q_i} S = Q_i, i = 1, 2$. Hence, circular assume-guarantee reasoning for switching linear systems will not be sound in general, see Theorem 7.18.

7.3.2. Interconnections of continuous-time dynamical systems

On the continuous layer, we are dealing with interconnections of linear systems. Recall the definitions of open and closed feedback interconnections of linear systems in Chapter 3.

Definition 7.8. For two linear systems of the form (7.4) such that $\dim \mathcal{U}_i = \dim \mathcal{Y}_j, (i, j) \in \{(1, 2), (2, 1)\}$, define the *open feedback interconnection* $\Sigma_1 \|_{\text{o}} \Sigma_2$ by

$$u_1 = y_2 + v_1 \quad , \quad u_2 = y_1 + v_2 \tag{7.6}$$

and, accordingly, the *closed feedback interconnection* $\Sigma_1 \|_{\text{cl}} \Sigma_2$ by

$$u_1 = y_2 \quad , \quad u_2 = y_1 \tag{7.7}$$

In Chapter 2 we defined simulation for linear systems and gave a linear-algebraic characterization which will be used in the following. With respect to compositional reasoning, we showed in Section 3.2.2 that there is an important difference between open and closed feedback interconnections: For open feedback there is no need for assume-guarantee reasoning since proof obligations for interconnections are equivalent to obligations for the individual components. I.e., one can always replace a simulation between systems interconnected by open feedback by the respective simulation relations of the components involved to simplify the verification task. For closed feedback interconnections, however, assume-guarantee reasoning *can* reduce the complexity of proof obligations. In the remainder, we will therefore only treat closed interconnections of linear systems. For completeness we recall Theorem 3.14 from Section 3.2.1.

Theorem 7.9. *Given four linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ of the form* (7.4). *Then simulation is compositional with respect to closed feedback interconnection. Moreover, both non-circular and circular assume-guarantee reasoning is also sound.*

7.3.3. Interconnections of switching linear systems

Combining parallel composition of labeled transition systems and closed feedback interconnection of linear systems we define interconnections of switching linear systems as follows:

Definition 7.10. Consider two switching linear systems $\Sigma_i^{\text{SLS}} = (Q_i, \mathcal{X}_i, V_i, \mathcal{W}_i, \Sigma_i, E_i, M_i), i = 1, 2$. The interconnection $\Sigma_1^{\text{SLS}} \| \Sigma_2^{\text{SLS}}$ is a switching linear system $\Sigma^{\text{SLS}} = (Q, \mathcal{X}, V, \mathcal{W}, \Sigma, E_{12}, M)$ with

- $Q = Q_1 \times Q_2$ the set of discrete locations resulting from the parallel composition $D_1^{\text{SLS}} \| D_2^{\text{SLS}}$ of the associated labeled transition systems D_i^{SLS},
- $\mathcal{X}(q_1, q_2) = \mathcal{X}_1(q_1) \times \mathcal{X}_2(q_2)$ the continuous state space at every location $(q_1, q_2) \in Q$,
- $\mathcal{W} = (\mathcal{U}, \mathcal{D}, \mathcal{Y})$ where $\mathcal{U}$ is void since we consider closed feedback interconnections $\Sigma_1(q_1) \|_{\text{cl}} \Sigma_2(q_2)$ in every location $(q_1, q_2) \in Q$, $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$,
- V and E_{12} as determined by the parallel composition of the respective associated labeled transition systems D_i^{SLS},
- $M(e) = \text{diag}\{\bar{M}_1, \bar{M}_2\}$ with

$$\bar{M}_i(e_i) = \begin{cases} M_i(e_i), & \text{if } q_i' \neq q_i, e_i = (q_i, v, q_i') \in E_i \\ I, & \text{if } q_i = q_i' \end{cases}$$

Note that the adjusted reset map $\bar{M} : \mathcal{X}(q_1, q_2) \to \mathcal{X}(q_1', q_2')$ works as a self-loop at q_i if the transition label is not shared, $v \in V_j \setminus V_i, (i, j) \in \{(1, 2), (2, 1)\}$. Event times of the interconnection $\Sigma_1^{\text{SLS}} \| \Sigma_2^{\text{SLS}}$ are synchronized for shared labels, while for non-shared labels $v \in V_j \setminus V_i, (i, j) \in \{(1, 2), (2, 1)\}$ there is an event in Σ_j^{SLS} and a self-loop in Σ_i^{SLS}.

7.4. Simulation relations for switching linear systems

Hybrid simulation theory has been treated by several authors, e.g. [75, 2]. We follow [60] and start with

Definition 7.11. Given two switching linear systems $\Sigma_i^{\text{SLS}} = (Q_i, \mathcal{X}_i, V, \mathcal{W}_i, \Sigma_i, E_i, M_i), i = 1, 2$, with the same continuous input and output spaces $\mathcal{U} = \mathcal{U}_1 = \mathcal{U}_2, \mathcal{Y} = \mathcal{Y}_1 = \mathcal{Y}_2$ and the same set of discrete labels $V = V_1 = V_2$. A hybrid simulation relation S of Σ_1^{SLS} by Σ_2^{SLS} is a subset of the product of the hybrid state spaces Δ_i, $S \subset \Delta_1 \times \Delta_2$, with the following properties. Take any initial

hybrid state $(\xi_1^0, \xi_2^0) \in S$ and any input function $u = u_1 = u_2$.
Then for any $d_1 \in \mathcal{D}_1$ and any hybrid execution $\rho_1 = (\mathcal{T}_1, q_1, x_1, v, w_1)$ there should exist a $d_2 \in \mathcal{D}_2$ and an execution $\rho_2 = (\mathcal{T}_2, q_2, x_2, v, w_2)$ such that for all times t for which the hybrid execution ρ_1 is defined,

$$\begin{aligned}
(i):&\quad \mathcal{T}_1 = \mathcal{T}_2 =: \mathcal{T} \\
(ii):&\quad \forall\, u_1(t) = u_2(t), \forall\, d_1(t)\ \exists\, d_2(t) : y_1(t) = y_2(t)\ \forall\, t \geq 0, t \notin \mathcal{T} \\
(iii):&\quad v_1(t) = v_2(t)\ \forall\, t \geq 0,\ t \in \mathcal{T} \\
(iv):&\quad (q_1(t), x_1(t), q_2(t), x_2(t)) \in S\ \forall\, t \geq 0, t \notin \mathcal{T}
\end{aligned} \tag{7.8}$$

Moreover, if there exists a hybrid simulation relation S of Σ_1^{SLS} by Σ_2^{SLS} such that $\Pi|_{\Delta_1} S = \Delta_1$, then Σ_1^{SLS} *simulates* Σ_2^{SLS}, denoted by $\Sigma_1^{\text{SLS}} \preccurlyeq \Sigma_2^{\text{SLS}}$. In this case, S is called a *full simulation relation*.

Hybrid simulation relations possess the same properties as their discrete and continuous counterparts.

Proposition 7.12. *Hybrid simulation as defined in Definition 7.11 is a preorder, i.e., it is reflexive and transitive.*

Proof. Consider switching linear system $\Sigma_1^{\text{SLS}}, \Sigma_2^{\text{SLS}}$ and Σ_3^{SLS}.
(*Reflexivity*) Define the relation $S = \{(q, x), (q, x))|(q, x) \in \Delta_1\}$. It is easy to check that S is a hybrid simulation relation of Σ_1^{SLS} by Σ_1^{SLS}.
(*Transitivity*) Assume there exist hybrid simulation relations S_{12} and S_{23} of Σ_1^{SLS} by Σ_2^{SLS} and of Σ_2^{SLS} by Σ_3^{SLS}, respectively. Then

$$\begin{aligned}
S_{13} \quad := \quad & \{((q_1, x_1), (q_3, x_3))|\exists (q_2, x_2) : ((q_1, x_1), (q_2, x_2)) \in S_{12}, \\
& ((q_2, x_2), (q_3, x_3)) \in S_{23}\}
\end{aligned}$$

is a hybrid simulation relation of Σ_1^{SLS} by Σ_3^{SLS}. □

The structure of switching linear systems allows to split a hybrid simulation subset S into a relation Q_S between the discrete states and, associated with every pair of discrete locations $(q_1, q_2) \in Q_S$, subsets of the continuous product space $W(q_1, q_2) \subset \mathcal{X}_1 \times \mathcal{X}_2$.

Proposition 7.13. *Let S be a hybrid simulation relation of Σ_1^{SLS} by Σ_2^{SLS}. Then there exists a relation $Q_S \subset Q_1 \times Q_2$ and for any $(q_1, q_2) \in Q_S$ suitable sets $W(q_1, q_2) \subset \mathcal{X}_1(q_1) \times \mathcal{X}_2(q_2)$ such that*

$$((q_1, x_1), (q_2, x_2)) \in S \quad \Leftrightarrow \quad (q_1, q_2) \in Q_S, (x_1, x_2) \in W(q_1, q_2)$$

Remark 7.14. It has been shown in [60] that the sets $W(q_1, q_2)$ can be assumed to be linear subspaces. In fact, given a hybrid simulation relation S of Σ_1^{SLS} by Σ_2^{SLS}, its linear closure $\mathcal{L}(S)$ is also a hybrid simulation relation of Σ_1^{SLS} by Σ_2^{SLS}. Therefore, we will assume in the remainder that the subsets of the continuous variable spaces $W(q_1, q_2)$ are linear subspaces.

As a consequence of Proposition 7.13 a hybrid simulation relation should fulfill the following conditions:

- The set of discrete states Q_S is a simulation relation between the associated labeled transition systems
- The linear subspaces $W(q_1, q_2)$ at every location $(q_1, q_2) \in Q_S$ define a simulation relation between the underlying linear systems $\Sigma_1(q_1)$ and $\Sigma_2(q_2)$.

The coupling between discrete and continuous dynamics is imposed by the reset maps $M_i(e_i)$. This leads to a more checkable notion of hybrid simulation which is usually referred to as structural hybrid simulation relation in the literature [75].

Theorem 7.15. *Given two switching linear systems Σ_1^{SLS} and Σ_2^{SLS} with the same set of labels $V = V_1 = V_2$, a set*

$$S = \{((q_1, x_1), (q_2, x_2)) \in \Delta_1 \times \Delta_2 | (q_1, q_2) \in Q_S, (x_1, x_2) \in W(q_1, q_2)\} \quad (7.9)$$

is a hybrid simulation relation of Σ_1^{SLS} by Σ_2^{SLS} if and only if the following properties hold:

(i) Q_S is a simulation relation of D_1^{SLS} by D_2^{SLS} where D_i^{SLS} are the labeled transition systems associated to $\Sigma_i^{SLS}, i = 1, 2$, and for every $(q_1, q_2) \in Q_S$, $W(q_1, q_2)$ is a simulation relation of $\Sigma_1(q_1)$ by $\Sigma_2(q_2)$;

(ii) for every $(q_1, q_2) \in Q_S$ and every possible set of successor states $(q_1', q_2') \in Q_S$ such that $(q_1, v, q_1') \in E_1$ and $(q_2, v, q_2') \in E_2$,

$$diag\{M_1(e_1), M_2(e_2)\} W(q_1, q_2) \subset W(q_1', q_2')$$

Hybrid simulation relations of the form (7.9) *are called* structural hybrid simulation relations.

Proof. ($\Longrightarrow$):
Assume Q_S is not a simulation relation of D_1^{SLS} by D_2^{SLS} and there exists at least one pair $(q_1, q_2) \in Q_S$ such that there does not exist a simulation relation of $\Sigma_1(q_1)$ by $\Sigma_2(q_2)$. Then even if $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}$ one can find for some joint input function $u_1 = u_2 = u$ a disturbance d_1 and a hybrid execution $\rho_1 = (\mathcal{T}_1, q_i, x_1, v, w_1)$ such that for any disturbance d_2 and any execution

$\rho_2 = (\mathcal{T}_2, q_2, x_2, v, w_2)$ there exists an event time $T \in \mathcal{T}_1$ and a $v \in V$ for which $(q_1, v, q_1') \in E_1$ but there does not exist a q_2' such that $(q_2, v, q_2') \in E_2$. Similarly, one can show that if there does not exist any simulation relation between the underlying linear systems $\Sigma_1(q_1)$ and $\Sigma_2(q_2)$ one can find an execution ρ_1 with continuous output $y_1(t)$ such that any execution ρ_2 violates condition 2 in Definition 7.11, i.e. $y_2(t) \neq y_1(t)$ for some t for which ρ_1 is defined.
($\Longleftarrow$):
Due to Theorem 7.15 (i), the set Q_S is a simulation relation of D_1^{SLS} by D_2^{SLS} and for every $(q_1, q_2) \in Q_S$, $W(q_1, q_2)$ is a simulation relation of $\Sigma_1(q_1)$ by $\Sigma_2(q_2)$. By the respective definitions of simulation relations for labeled transition and linear systems, Theorem 7.15 (i) therefore guarantees that for any hybrid execution ρ_1 there exists a hybrid execution ρ_2 such that the set of event times are equal, $\mathcal{T}_1 = \mathcal{T}_2$, and moreover Definition 7.11 (ii) and (iii) hold. Furthermore, Theorem 7.15 (ii) ensures that just after every event time $t \in \mathcal{T}$ the reset of the continuous state remains within the simulation subspace $W(q_1', q_2')$ associated with the new location (q_1', q_2'). Thus, condition (iv) of Definition 7.11 is also fulfilled. □

It was shown in [60] that for two similar switching linear systems $\Sigma_i^{\mathrm{SLS}}, i = 1, 2$, the maximal hybrid simulation relation $S^\star$ is of the form (7.9) where $W(q_1, q_2)$ are linear subspaces, i.e. $S^\star$ is a structural hybrid simulation relation. As mentioned in Remark 7.14, given a hybrid simulation relation S, its linear closure $\mathcal{L}(S)$ is also a hybrid simulation relation. Moreover, $S \subset \mathcal{L}(S)$. For other classes of hybrid systems, in particular for hybrid systems with location invariants and guards, this no longer holds true since the continuous part of the hybrid state space is not a linear subspace nor is the linear closure $\mathcal{L}(S)$ in general a hybrid simulation relation. Thus, the maximal hybrid simulation relation might not be well-defined.

7.5. Compositional reasoning

Knowing that compositionality holds on the level of both the discrete and continuous dynamics, we establish a similar result for switching linear systems.

Theorem 7.16. *For any four given switching linear systems $\Sigma_i^{SLS}, i \in \{P_1, P_2, Q_1, Q_2\}$, such that $V_{P_1} = V_{Q_1}, V_{P_2} = V_{Q_2}$ and interconnections $\|$, hybrid simulation is compositional.*

Proof. Assume we are given hybrid simulation relations S_1 and S_2 of $\Sigma_{P_1}^{\mathrm{SLS}}$ by $\Sigma_{Q_1}^{\mathrm{SLS}}$ and $\Sigma_{P_2}^{\mathrm{SLS}}$ by $\Sigma_{Q_2}^{\mathrm{SLS}}$, respectively. Define the relation

$$S := \{((\xi_{P_1}, \xi_{P_2}), (\xi_{Q_1}, \xi_{Q_2})) \mid (\xi_{P_1}, \xi_{Q_1}) \in S_1, (\xi_{P_2}, \xi_{Q_2}) \in S_2\} \quad (7.10)$$

Reordering the components, one obtains

$$\tilde{S} \;=\; \{((\xi_{P_1}, \xi_{Q_1}), (\xi_{P_2}, \xi_{Q_2})) \mid ((\xi_{P_1}, \xi_{P_2}), (\xi_{Q_1}, \xi_{Q_2})) \in S\} = S_1 \times S_2$$

From Theorems 7.5 and 7.9, we know that the interconnections of the associated labeled transition systems $D_i^{\mathsf{SLS}}, i \in \{P_1, P_2, Q_1, Q_2\}$, as well as the interconnections of the respective linear systems $\Sigma_i(q_i)$ at every location $((p_1, p_2), (q_1, q_2)) \in Q_S$ are compositional so that condition (i) in Theorem 7.15 is fulfilled.
Since S_1, S_2 are hybrid simulation relations, it holds for every $(p_1, q_1) \in Q_{S_1}$ and every possible successor state $(p_1', q_1') \in Q_{S_1}$ that

$$\operatorname{diag}\{M_{P_1}(e_{P_1}), M_{Q_1}(e_{Q_1})\} W_1(p_1, q_1) \subset W_1(p_1', q_1') \tag{7.11}$$

with $e_{P_1} = (p_1, v, p_1') \in E_{P_1}$ and $e_{Q_1} = (q_1, v, q_1') \in E_{Q_1}$ for some $v \in V_{P_1}$. Similarly, for every $(p_2, q_2) \in Q_{S_2}$ and every successor state $(p_2', q_2') \in Q_{S_2}$ such that $e_{P_2} = (p_2, v, p_2') \in E_{P_2}$ and $e_{Q_2} = (q_2, v, q_2') \in E_{Q_2}$ for some $v \in V_{P_2}$ the following holds:

$$\operatorname{diag}\{M_{P_2}(e_{P_2}), M_{Q_2}(e_{Q_2})\} W_2(p_2, q_2) \subset W_2(p_2', q_2') \tag{7.12}$$

Depending on the discrete transition label $v \in V = V_{P_1} \cup V_{P_2} = V_{Q_1} \cup V_{Q_2}$, three cases have to be distinguished:

1. $v \in V_{P_1} \cap V_{P_2}$: $((p_1, p_2), v, (p_1', p_2')) \in E_{P_1 P_2}$ implies $((q_1, q_2), v, (q_1', q_2')) \in E_{Q_1 Q_2}$ and therefore $(p_1', q_1', p_2', q_2') \in Q_{\tilde{S}}$. From (7.11) and (7.12) it follows that

$$\begin{aligned}\operatorname{diag}\{M_{P_1}(e_{P_1P_2}), M_{Q_1}(e_{Q_1Q_2}), M_{P_2}(e_{P_1P_2}), M_{Q_1Q_2}(e_{Q_1Q_2})\}\\ \tilde{W}(p_1, q_1, p_2, q_2) \subset \tilde{W}(p_1', q_1', p_2', q_2')\end{aligned}$$

2. $v \in V_{P_1} \setminus V_{P_2}$: Here, $((p_1, p_2), v, (p_1', p_2)) \in E_{P_1 P_2}$ implies $((q_1, q_2), v, (q_1', q_2)) \in E_{Q_1 Q_2}$ and therefore $(p_1', q_1', p_2, q_2) \in Q_{\tilde{S}}$. Thus,

$$\operatorname{diag}\{M_{P_1}(e_{P_1P_2}), M_{Q_1}(e_{Q_1Q_2}), I, I\}\tilde{W}(p_1, q_1, p_2, q_2) \subset \tilde{W}(p_1', q_1', p_2, q_2)$$

3. $v \in V_{P_1} \setminus V_{P_2}$: This case is symmetrical to (2).

We can therefore conclude that for every $(p_1, p_2, q_1, q_2) \in Q_{\tilde{S}}$ and every possible successor state $(p_1', q_1', p_2', q_2') \in Q_{\tilde{S}}$

$$\begin{aligned}\operatorname{diag}\{\bar{M}_{P_1}(e_{P_1P_2}), \bar{M}_{Q_1}(e_{Q_1Q_2}), \bar{M}_{P_2}(e_{P_1P_2}), \bar{M}_{Q_1Q_2}(e_{Q_2})\}\tilde{W}(p_1, q_1, p_2, q_2) \subset\\ \subset \tilde{W}(p_1', q_1', p_2', q_2')\end{aligned}$$

where $e_{P_1P_2} = ((p_1, p_2), v, (p_1', p_2')) \in E_{P_1P_2}$ and $e_{Q_1Q_2} = ((q_1, q_2), v, (q_1', q_2')) \in E_{Q_1Q_2}$. Reordering the components, S as defined in (7.10) indeed defines a structural hybrid simulation relation of $\Sigma_{P_1}^{\mathsf{SLS}} \parallel \Sigma_{P_2}^{\mathsf{SLS}}$ by $\Sigma_{Q_1}^{\mathsf{SLS}} \parallel \Sigma_{Q_2}^{\mathsf{SLS}}$. □

7.6. Assume-guarantee reasoning

Due to its triangular structure the non-circular assume-guarantee reasoning rule (7.2) can immediately be shown to hold true.

Theorem 7.17. *Consider switching linear systems $\Sigma_i^{SLS}, i \in \{P_1, P_2, Q_1, Q_2\}$ such that $V_{P_1} = V_{Q_1}, V_{P_2} = V_{Q_2}$ and assume that there exist full hybrid simulation relations S_1 and S_{II} of $\Sigma_{P_1}^{SLS}$ by $\Sigma_{Q_1}^{SLS}$ and of $\Sigma_{Q_1}^{SLS} \parallel \Sigma_{P_2}^{SLS}$ by $\Sigma_{Q_1}^{SLS} \parallel \Sigma_{Q_2}^{SLS}$, respectively. Then there also exists a full simulation relation S of $\Sigma_{P_1}^{SLS} \parallel \Sigma_{P_2}^{SLS}$ by $\Sigma_{Q_1}^{SLS} \parallel \Sigma_{Q_2}^{SLS}$, i.e., non-circular assume-guarantee reasoning is sound with respect to hybrid simulation.*

Proof. Theorem 7.16 ensures that interconnections of switching linear systems are compositional. Together with the transitivity property of hybrid simulation (cf. Proposition 7.12) the claim follows. □

It cannot be expected, however, that circular assume-guarantee reasoning is unconditionally sound. From Theorem 7.7 it is known that already on the level of the discrete dynamics, an additional condition is needed. It turns out that this condition (7.5) also ensures that there exists a hybrid simulation relation S of $\Sigma_{P_1}^{\mathrm{SLS}} \parallel \Sigma_{P_2}^{\mathrm{SLS}}$ by $\Sigma_{Q_1}^{\mathrm{SLS}} \parallel \Sigma_{Q_2}^{\mathrm{SLS}}$ given S_I and S_{II} as in (7.3). However, fullness of S is not guaranteed. This is formalized in the following theorem.

Theorem 7.18. *Consider any given switching linear systems $\Sigma_i^{SLS}, i \in \{P_1, P_2, Q_1, Q_2\}$, such that $V_{P_1} = V_{Q_1}, V_{P_2} = V_{Q_2}$ and assume there exist full hybrid simulation relations S_I and S_{II} of $\Sigma_{P_1}^{SLS} \parallel \Sigma_{Q_2}^{SLS}$ by $\Sigma_{Q_1}^{SLS} \parallel \Sigma_{Q_2}^{SLS}$ and of $\Sigma_{Q_1}^{SLS} \parallel \Sigma_{P_2}^{SLS}$ by $S_{Q_1} \|_{SLS} S_{Q_2}$, respectively. Then there exists a hybrid simulation relation S of $\Sigma_{P_1}^{SLS} \| \Sigma_{P_2}^{SLS}$ by $\Sigma_{P_1}^{SLS} \parallel \Sigma_{P_2}^{SLS}$ if and only if condition* (7.5) *for the associated labeled transition systems is fulfilled.*

Proof. First, if (7.5) does not hold, one can construct a counterexample in the same spirit as in [21], Example 4.2, where all components P_1, P_2, Q_1, Q_2 share the same set of of states Q, labels V and continuous dynamics $\Sigma(q)$. While $P_i, i = 1, 2$ allow for any transition $v \in V$ between any two states, i.e. $E_{P_i} = Q \times V \times Q$, the abstractions Q_i do not allow for any transition between any two states, $E_{Q_i} = \emptyset$.
Conversely, assume we are given hybrid simulation relations S_I and S_{II} of $\Sigma_{P_1}^{\mathrm{SLS}} \parallel \Sigma_{Q_2}^{\mathrm{SLS}}$ and $\Sigma_{Q_1}^{\mathrm{SLS}} \parallel \Sigma_{P_2}^{\mathrm{SLS}}$ by $\Sigma_{Q_1}^{\mathrm{SLS}} \parallel \Sigma_{Q_2}^{\mathrm{SLS}}$, respectively. Construct the relation

$$S := \Big\{ (\xi_{P_1}, \xi_{P_2}, \xi_{Q_1}, \xi_{Q_2}) \mid \exists \hat{\xi}_{Q_1}, \hat{\xi}_{Q_2} : (\xi_{P_1}, \xi_{Q_2}, \xi_{Q_1}, \hat{\xi}_{Q_2}) \in S_I, \quad (\xi_{Q_1}, \xi_{P_2}, \hat{\xi}_{Q_1}, \xi_{Q_2}) \in S_{II} \Big\} \tag{7.13}$$

We claim that S is a hybrid simulation relation of $\Sigma_{P_1}^{\mathrm{SLS}} \parallel \Sigma_{P_2}^{\mathrm{SLS}}$ by $\Sigma_{Q_1}^{\mathrm{SLS}} \parallel \Sigma_{Q_2}^{\mathrm{SLS}}$. From Theorem (7.5), we know that the associated labeled transition systems fulfill the circular assume-guarantee reasoning rule 3 if and only if condition

(7.5) holds. Furthermore, for the respective linear systems at every location $(p_1, p_2, q_1, q_2) \in Q_S$, circular assume-guarantee reasoning is sound as stated in Theorem 7.9. Thus, condition (i) in Theorem 7.15 is readily fulfilled.
Now take for any $(p_1, p_2, q_1, q_2) \in Q_S$ an arbitrary element $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in W_S(p_1, p_2, q_1, q_2)$. Define first for every location $(p_1, p_2, q_1, q_2) \in Q_S$ the subspaces

$$\begin{aligned} W_{Q_1Q_2}(q_1, q_2) &= \{(x_{Q_1}, x_{Q_2}) \mid \exists x_{P_1}, \bar{x}_{Q_2}, x_{P_2}, \bar{x}_{Q_1} : (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in \\ &\quad W_I(p_1, q_2, q_1, \bar{q}_2), (x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in W_{II}(q_1, p_2, \bar{q}_1, q_2)\} \\ W_{P_1}(p_1) &= \{x_{P_1} \mid \exists x_{P_2}, x_{Q_1}, x_{Q_2} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in W(p_1, p_2, q_1, q_2)\} \\ W_{P_2}(p_2) &= \{x_{P_2} \mid \exists x_{P_1}, x_{Q_1}, x_{Q_2} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in W(p_1, p_2, q_1, q_2)\} \end{aligned}$$

Since S_I and S_{II} are simulation relations, they fulfill condition (ii) in Theorem 7.15 so that for every $(p_1, q_2, q_1, \hat{q}_2) \in Q_{S_I}$ and every possible successor state $(p_1', q_2', q_1', \hat{q}_2') \in Q_{S_I}$

$$\begin{aligned} &\text{diag}\{\bar{M}_{P_1}(e_{P_1Q_2}), \bar{M}_{Q_2}(e_{P_1Q_2}), \bar{M}_{Q_1}(e_{Q_1Q_2}), \bar{M}_{Q_2}(e_{Q_1Q_2})\} \\ &\qquad\qquad W_I(p_1, p_2, q_1, \hat{q}_2) \subset W_I(p_1', q_2', q_1', \hat{q}_2') \end{aligned}$$

and similarly for every $(q_1, p_2, \hat{q}_1, q_2) \in Q_{S_{II}}$ and every possible successor state $(q_1', p_2', \hat{q}_1', q_2') \in Q_{S_{II}}$

$$\begin{aligned} &\text{diag}\{\bar{M}_{Q_1}(e_{Q_1P_2}), \bar{M}_{P_2}(e_{Q_1P_2}), \bar{M}_{Q_1}(e_{Q_1Q_2}), \bar{M}_{Q_2}(e_{Q_1Q_2})\} \\ &\qquad\qquad W_{II}(q_1, p_2, \hat{q}_1, q_2) \subset W_{II}(q_1', p_2', \hat{q}_1', q_2') \end{aligned}$$

It therefore holds for every $(p_1, p_2, q_1, q_2) \in Q_S$ and every possible discrete successor state $(p_1', p_2', q_1', q_2') \in Q_S$ that

$$\text{diag}\{\bar{M}_{Q_1}(e_{Q_1Q_2}), \bar{M}_{Q_2}(e_{Q_1Q_2})\}Q_T(q_1, q_2) \subset W_{Q_1Q_2}(q_1', q_2')$$

and similarly

$$\bar{M}_{P_1}(e_{P_1Q_2})W_{P_1}(p_1) \subset W_{P_1}(p_1'), \bar{M}_{P_2}(e_{Q_1P_2})W_{P_2}(p_2) \subset W_{P_2}(p_2'),$$

Thus, condition (ii) is fulfilled which proves that S as in (7.13) is indeed a hybrid simulation relation of $\Sigma_{P_1}^{\text{SLS}} \parallel \Sigma_{P_2}^{\text{SLS}}$ by $\Sigma_{Q_1}^{\text{SLS}} \parallel \Sigma_{Q_2}^{\text{SLS}}$. □

7.7. Conclusions

We developed a framework for compositional analysis of switching linear systems as a special class of hybrid systems. Their particular structure, i.e. the fact that the discrete dynamics is independent of the continuous one, allows to define structural hybrid simulation relations that are easy to check and to apply. We proved that compositional and non-circular assume-guarantee reasoning are sound and gave a necessary and sufficient condition for the existence of a simulation relation S for the circular reasoning scheme (7.3). In the next chapter, we consider switching linear systems with inequality constraints as a more general class of hybrid systems.