Compositional analysis and control of dynamical systems

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6
Decentralized control

6.1. Introduction

In this chapter we want to use the compositional reasoning techniques developed in the previous chapters to derive decentralized control schemes. As a first step, we specialize compositional and assume-guarantee reasoning to a decentralized setting. Provided the local controllers used in this set-up are such that the locally controlled subsystems satisfy certain specifications themselves the network of locally controlled plants is then guaranteed to satisfy a given global specification. In the second step, we combine compositional analysis techniques with conditions under which one can find controllers that render the closed loop system satisfy a given specification. In particular, we focus on the so-called sandwich conditions which have been derived as necessary and sufficient conditions for achievable simulation. We present two bottom-up schemes starting from conditions on the locally controlled plants and one top-down scheme based on a global sandwich condition. An important consequence of the latter result is that whenever there exists a global controller satisfying a global specification it can be replaced by local ones due to completeness of circular assume-guarantee reasoning.

6.2. Problem setting

In our decentralized control setting we distinguish between the following types of linear continuous-time systems. *Plant systems* are of the form

$$\Sigma_i : \quad \begin{array}{rcl} \dot{x}_i &=& A_i x_i + B_i u_i + G_i e_i \\ y_i &=& C_i x_i \\ z_i &=& H_i x_i \end{array} \tag{6.1}$$

where u_i, y_i are the pair of *interconnection* variables and e_i, z_i the pair of external *specification* variables. All variables are taken from vector spaces of appropriate dimensions, $x_i \in \mathcal{X}_i, u_i \in \mathcal{U}_i, e_i \in \mathcal{E}_i, y_i \in \mathcal{Y}_i, z_i \in \mathcal{Z}_i$.
A *controller system* Σ_C is a linear system without external variables,

$$\Sigma_C : \quad \begin{array}{rcl} \dot{x}_C &=& A_C x_C + B_C u_C \\ y_C &=& C_C x_C \end{array} . \tag{6.2}$$

A *specification* Σ_Q defines the desired external behavior. Hence, Σ_Q does not have any interconnection variables and is given as

$$\Sigma_Q : \quad \begin{array}{rcl} \dot{x}_Q & = & A_Q x_Q + G_Q e_Q \\ z_Q & = & H_Q x_Q \end{array} . \tag{6.3}$$

6.3. Interconnections in control networks

The control systems defined in Section 6.2 can be interconnected in different ways. First, we discuss plant-controller interconnections for which the interconnection variables u_i, y_i are related by means of a permutation matrix.

Definition 6.1. Consider a plant system Σ_P of the form (6.1) and a controller system Σ_C. Then $\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C$ denotes the *plant-controller interconnection* with respect to the interconnection variables u, y and a permutation matrix Π,

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \quad , \quad \begin{bmatrix} u_P \\ y_P \end{bmatrix} = \Pi \begin{bmatrix} u_C \\ y_C \end{bmatrix} \tag{6.4}$$

The dynamics of the interconnected system $\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C$ are thus given by

$$\begin{array}{rcl} \begin{bmatrix} \dot{x}_P \\ \dot{x}_C \end{bmatrix} & = & \begin{bmatrix} A_P & B_P \Pi_{12} C_C \\ 0 & A_C \end{bmatrix} \begin{bmatrix} x_P \\ x_C \end{bmatrix} + \begin{bmatrix} B_P \Pi_{11} \\ B_C \end{bmatrix} u_c + \begin{bmatrix} G_P \\ 0 \end{bmatrix} e_p \\ & & \begin{bmatrix} C_P & -\Pi_{22} C_C \end{bmatrix} \begin{bmatrix} x_P \\ x_C \end{bmatrix} = \Pi_{21} u_c \\ z_P & = & \begin{bmatrix} H_P & 0 \end{bmatrix} \begin{bmatrix} x_P \\ x_C \end{bmatrix} \end{array} \tag{6.5}$$

In particular, $\|_{u,y}$ denotes the special case where Π is the identity matrix.

Remark 6.2. Allowing for a permutation matrix Π in the definition of plant-controller interconnections gives more freedom for controller design. Standard feedback interconnection is included in this framework by taking $\Pi = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. By contrast, the special case $\Pi = I$ entails algebraic constraints on the state variables. In this case (6.5) can be written as DAE system in pencil form,

$$\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C : \quad \begin{array}{rl} E_{PC} \dot{x}_{PC} & = A_{PC} x_{PC} \ , x_{PC} \in \mathcal{V}^\star_{PC}, w_{PC} \in \mathcal{W}^\star_{PC} \\ w_{PC} & = H_{PC} x_{PC} \ . \end{array} \tag{6.6}$$

where $\mathcal{V}^{\star}_{PC}$ denotes the consistent subspace and $\mathcal{W}^{\star}_{PC}$ the admissible inputs as defined in Definition 3.33 and

$$E_{PC} = \text{diag}\{G_P^{\perp}, G_P^{\perp}, 0\},\ x_{PC} = \begin{bmatrix} x_P \\ x_C \\ u_C \end{bmatrix},\ w_{PC} = \begin{bmatrix} z_P \\ u_C \end{bmatrix} \tag{6.7}$$

$$A_{PC} = \begin{bmatrix} G_P^{\perp}A_P & G_P^{\perp}B_P\Pi_{12}C_C & B_P\Pi_{11} \\ 0 & A_C & B_C \\ C_P & -\Pi_{22}C_C & -\Pi_{21} \end{bmatrix},\ H_{PC} = \begin{bmatrix} H_P & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

In a decentralized control setting the overall plant is usually given as an interconnection of subsystems. The topology of the global system model is determined by the type of interconnection between the individual components. In the remainder of this chapter, we consider *series* of feedback interconnections with respect to the external variables e_i, z_i as the standard interconnection between plant systems and specifications.

Definition 6.3. Consider k systems $\Sigma_i, i = 1, \ldots, k$ of the form (6.1) with external variables e_i, z_i and interconnection variables u_i, y_i. Then define the *series interconnection* $\Sigma_1 \| \ldots \| \Sigma_k$ with respect to the external variables e, z using feedback interconnections as follows:

$$\begin{array}{cccccccc} z_i^- & = & e_{i-1}^+ & , & z_i^+ & = & e_{i+1}^- & \\ e_i^- & = & z_{i-1}^+ & , & e_i^+ & = & z_{i+1}^- & \end{array} ,\quad i = 2, \ldots, k-1 \tag{6.8}$$

$$\begin{array}{cccccc} z_1^- & = & z_1 & , & z_1^+ & = & e_2^- \\ e_1^- & = & e_1 & , & e_1^+ & = & z_2^- \end{array} ,\quad \begin{array}{cccccc} z_k^- & = & e_{k-1}^+ & , & z_k^+ & = & z_k \\ e_k^- & = & z_{k-1}^+ & , & e_k^+ & = & e_k \end{array}$$

The matrices G_i and H_i corresponding to the external inputs are partitioned accordingly into submatrices

$$G_i = \begin{bmatrix} G_i^+ \\ G_i^- \end{bmatrix},\quad H_i = \begin{bmatrix} H_i^+ \\ H_i^- \end{bmatrix}, i = 1, \ldots, k\ . \tag{6.9}$$

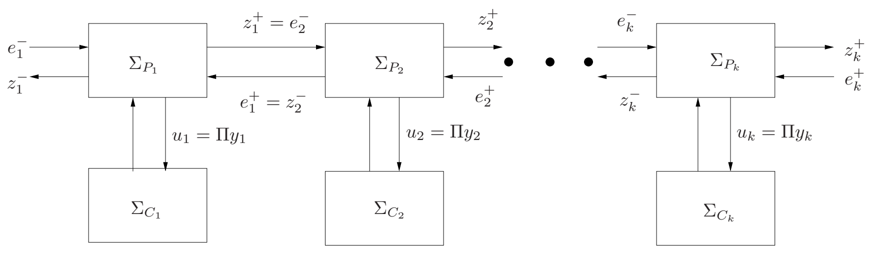

Figure 6.1.: Series interconnection $\left(\Sigma_{P_1} \|^{\Pi}_{u,y} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|^{\Pi}_{u,y} \Sigma_{C_1}\right) k$.

Remark 6.4. Considering series of feedback interconnections allows us to specialize the results of Chapter 3 to the decentralized setting. However, these results are valid in more generality to treat networks with other topologies, see also Remark 3.23.

6.4. Compositional analysis in the decentralized setting

In this section we will lay the foundation to analyze decentralized control problems using compositional analysis techniques based on simulation relations.

6.4.1. Simulation theory in the decentralized setting

In the following the notation in the definition of (bi)simulation relations is adjusted to be applicable in a decentralized setting. To that end, we require in the definition of a (bi)simulation relation that the external variables e_i, z_i remain equal whereas the interconnection variables u_i, y_i – if existent – are treated like disturbances.

Definition 6.5. A linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2, $\Sigma_i, i = 1, 2$, of the form (6.1), if it satisfies the following properties: Take any $(x_{10}, x_{20}) \in S$ and any joint external input function $e(\cdot) = e_1(\cdot) = e_2(\cdot)$. Then for any input function $u_1(\cdot)$ there exists an input function $u_2(\cdot)$ such that the resulting state trajectories $x_1(\cdot)$ and $x_2(\cdot)$, starting at $x_i(0) = x_{i0}$, satisfy

$$\begin{aligned} (i):& \qquad (x_1(t), x_2(t)) \in S \ \forall t \geq 0 \\ (ii):& \qquad z_1(t) = z_2(t) \ \forall t \geq 0 \end{aligned} \tag{6.10}$$

A simulation relation S is called full and denoted by $\Sigma_1 \preccurlyeq \Sigma_2$ if the projection on the first state component covers the whole state space, $\Pi_{\mathcal{X}_1} S = \mathcal{X}_1$.
A bisimulation relation R between Σ_1 and Σ_2, $\Sigma_i, i = 1, 2$, of the form (6.1), is a linear subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ with the following property: R defines a simulation relation of Σ_1 by Σ_2 and $R^{-1} := \{(x_2, x_1) \mid (x_1, x_2) \in R\}$ defines a simulation relation of Σ_2 by Σ_1. Moreover, R is full if $\Pi_{\mathcal{X}_i} R = \mathcal{X}_i, i = 1, 2$, which will be denoted by $\Sigma_1 \approx \Sigma_2$.

Algebraic characterizations and algorithms to compute (bi)simulation relations are immediately translated from the results in Chapter 2.

6.4.2. Decentralized control using compositional analysis techniques

We want to investigate control networks consisting of interconnections of arbitrarily (yet finitely) many systems. In our setting, the global plant system Σ_P is considered to be a series interconnection of component systems $\Sigma_{P_i}, i = 1, \ldots, k$, of the form (6.1),

$$\Sigma_P := \Sigma_{P_1} \| \ldots \| \Sigma_{P_k} . \tag{6.11}$$

The global specification, denoted by Σ_Q, is assumed to be decomposable into local subspecifications $\Sigma_{Q_i}, i = 1, \ldots, k$, of the form (6.3) corresponding to the plant subsystems,

$$\Sigma_Q := \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k} . \tag{6.12}$$

Making available the results obtained in Section 3.2.4 for compositional analysis of k systems will allow us to formulate decentralized control schemes. Note that in contrast to Section 3.2.4 the variables e_i, z_i will be used for feedback interconnections of plant systems and/or specifications. The only other plant variables u_i, y_i are exclusively used for plant-controller interconnections and are therefore not available for compositions between plants and specifications. Under these circumstances the results of Section 3.2.4 can be specialized immediately to a decentralized setting. We first state as a corollary of Theorem 3.24 that series of plant-controller interconnections are compositional.

Corollary 6.6. *Consider k plant-controller interconnections $\Sigma_{P_i} \|_{u,y}^{\Pi} \Sigma_{C_i}, i = 1, \ldots,$ k, of the form* (6.5) *and k specifications Σ_{Q_i} of the form* (3.1). *Then compositional reasoning is sound for series interconnections of k control systems, i.e.*

$$\begin{array}{rcl} \forall\, i = 1, \ldots, k: \ \Sigma_{P_i} \|_{u,y}^{\Pi} \Sigma_{C_i} & \preccurlyeq & \Sigma_{Q_i} \\ & \Longrightarrow & \\ \left(\Sigma_{P_1} \|_{u,y}^{\Pi} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|_{u,y}^{\Pi} \Sigma_{C_k}\right) & \preccurlyeq & \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k} \end{array} \tag{6.13}$$

Corollary 6.6 represents our first decentralized control scheme: Given local controllers $\Sigma_{C_i}, i = 1, 2, \ldots,$ that satisfy the local specifications Σ_{Q_i}, the global control network consisting of series interconnections of locally controlled plants is guaranteed to fulfill the global specification given itself by a series interconnection of local specifications.
A similar scheme can be derived on the basis of circular assume-guarantee reasoning. We therefore specialize Theorem 3.26 to the decentralized setting.

Corollary 6.7. *Consider $k \geq 2$ plant-controller interconnections $\Sigma_{P_i} \|_{u,y}^{\Pi} \Sigma_{C_i}, i =$ $1, \ldots, k$, of the form* (6.5) *and k corresponding specifications Σ_{Q_i} of the form* (6.3).

Let k circularly dependent conditions

$$
\begin{array}{llcl}
S_I: & \left(\Sigma_{P_1}\|^{\Pi}_{u,y}\Sigma_{C_1}\right)\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k} & \preccurlyeq & \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k} \\
S_{II}: & \Sigma_{Q_1}\|\left(\Sigma_{P_2}\|^{\Pi}_{u,y}\Sigma_{C_2}\right)\|\ldots\|\Sigma_{Q_k} & \preccurlyeq & \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k} \\
\vdots & \vdots & \vdots & \vdots \\
S_k: & \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\left(\Sigma_{P_k}\|^{\Pi}_{u,y}\Sigma_{C_k}\right) & \preccurlyeq & \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k}
\end{array}
\tag{6.14}
$$

be satisfied. Then the global interconnected plant

$$\Sigma_P := \left(\Sigma_{P_1}\|^{\Pi}_{u,y}\Sigma_{C_1}\right)\|\ldots\|\left(\Sigma_{P_k}\|^{\Pi}_{u,y}\Sigma_{C_k}\right)$$

fulfills the global specification $\Sigma_Q := \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k}$, that is

$$S: \quad \Sigma_P \preccurlyeq \Sigma_Q \tag{6.15}$$

Moreover, if (6.14) *holds with bisimilarity then* (6.15) *also holds with bisimilarity.*

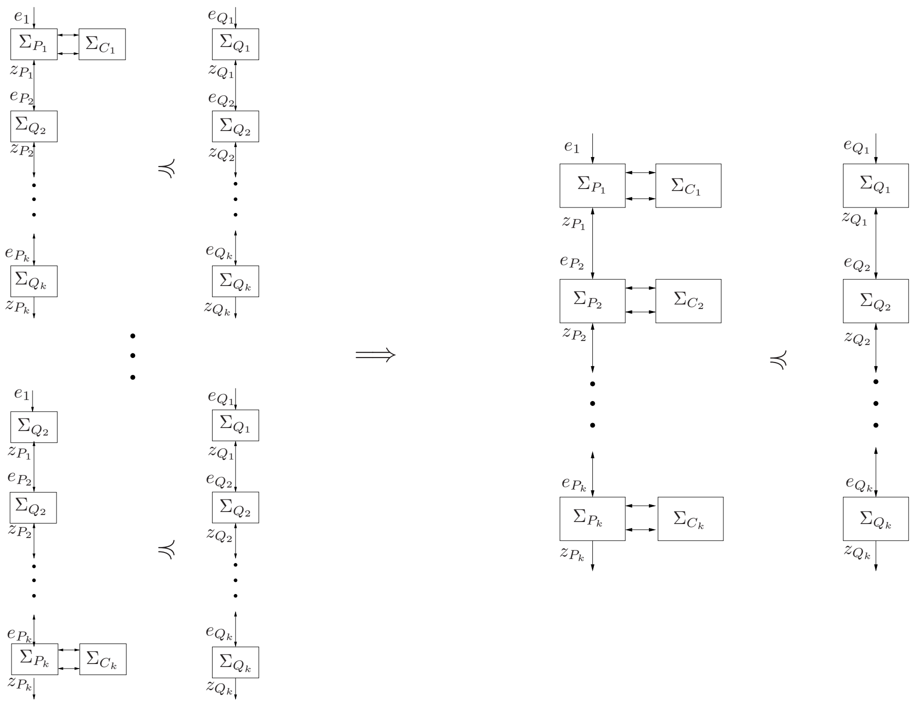

Figure 6.2.: Decentralized control scheme based on circular assume-guarantee reasoning.

Figure 6.4.2 depicts the decentralized control scheme based on Corollary 6.7. The conditions $S_i, i = I, II, \ldots, k$, require that the global specification Σ_Q is satisfied in each case for a network including the locally controlled plant $\Sigma_{P_i} \|_{u,y}^{\Pi} \Sigma_{C_i}$ assuming that the other locally controlled plants satisfy their individual specifications $\Sigma_{Q_j}, j = I, II, \ldots, k, j \neq i$. The k conditions S_i are therefore circularly dependent.
It is also possible to combine conditions of the form (6.13) and (6.14) in a triangular proof rule to obtain a decentralized control scheme based on non-circular assume-guarantee reasoning. Soundness is always ensured due to compositionality of series interconnections and transitivity of simulation. Not stating this formally, we provide a simple example instead to illustrate this point.

Example 6.8. Consider three plant systems $\Sigma_{P_i}, i = 1, 2, 3$, and three specifications Σ_{Q_i}. Let local controllers $\Sigma_{C_i}, i = 1, 2, 3$, be given such that the following conditions hold:

$$\begin{array}{rrcl} S_1 : & \Sigma_{P_1} \|_{u,y}^{\Pi_1} \Sigma_{C_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_{II} : & \Sigma_{Q_1} \| \left(\Sigma_{P_2} \|_{u,y}^{\Pi_2} \Sigma_{C_2} \right) & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \\ S_{III} : & \Sigma_{Q_1} \| \Sigma_{Q_2} \| \left(\Sigma_{P_3} \|_{u,y}^{\Pi_3} \Sigma_{C_3} \right) & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \end{array} \tag{6.16}$$

Combining S_1 and S_{II} by interconnecting the systems involved in S_1 with Σ_{S_2} yields

$$S_{1,II} : \quad \left(\Sigma_{P_1} \|_{u,y}^{\Pi_1} \Sigma_{C_1} \right) \| \left(\Sigma_{P_2} \|_{u,y}^{\Pi_2} \Sigma_{C_2} \right) \quad \preccurlyeq \quad \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{6.17}$$

while by the same reasoning, $S_{1,II}$ and S_{III} result in

$$S : \left(\Sigma_{P_1} \|_{u,y}^{\Pi_1} \Sigma_{C_1} \right) \| \left(\Sigma_{P_2} \|_{u,y}^{\Pi_2} \Sigma_{C_2} \right) \| \left(\Sigma_{P_3} \|_{u,y}^{\Pi_3} \Sigma_{C_3} \right) \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \tag{6.18}$$

Finally, as a special case of Theorem 3.31, circular assume-guarantee reasoning is also complete in the decentralized setting.

Corollary 6.9. *Consider k linear systems $\Sigma_{P_i}, i = 1, \ldots, k$, and k specifications Σ_{Q_i}, each of the form* (6.1), (6.3), (6.5) *or* (6.7). *Assume that*

$$\Sigma_{P_1} \| \ldots \| \Sigma_{P_k} \quad \preccurlyeq \quad \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k} \tag{6.19}$$

Then there also exist full simulation relations $S_i, i = 1, \ldots, k$, of

$$\Sigma_{Q_1} \| \ldots \| \Sigma_{Q_{i-1}} \| \Sigma_{P_i} \| \Sigma_{Q_{i+1}} \| \ldots \| \Sigma_{Q_k}$$

by

$$\Sigma_Q = \Sigma_{Q_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{Q_k}$$

6.5. Achievable simulations

In this section, we will discuss achievable simulations, i.e., given a plant Σ_P and a specification Σ_Q, under which conditions does there exist a controller Σ_C and a permutation matrix Π such that the plant-controller interconnection $\Sigma_P \|_{u,y}^{\Pi} \Sigma_C$ fulfills the desired specification Σ_Q. To obtain these conditions, we collect some basic facts about plant-interconnections of systems.
For every plant system Σ_i, define the *associated system* Σ_{N_i} by setting the interconnection variables to zero, $u_i = y_i \equiv 0$:

$$\Sigma_{N_i}: \quad \begin{aligned} \dot{x}_{N_i} &= A_i x_{N_i} + G_i e_{N_i} \\ z_{N_i} &= H_i x_{N_i} \\ C_i x_{N_i} &= 0 \end{aligned} \tag{6.20}$$

This entails algebraic constraints on the state variables since $x_{N_i} \in \ker C_i$.
The *null system* Σ_0 has all variables set to zero,

$$\Sigma_0: \; x_0 = 0, y_0 = u_0 = e_0 = z_0 = 0 \tag{6.21}$$

Proposition 6.10. *The system Σ_{N_P} associated with Σ_P is bisimilar to the plant-controller interconnection of Σ_P with the null system Σ_0.*

$$\Sigma_{N_P} \approx \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_0 \tag{6.22}$$

Proof. The interconnection $\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_0$ is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_P \\ \dot{x}_0 \end{bmatrix} &= \begin{bmatrix} A_P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_P \\ x_0 \end{bmatrix} + \begin{bmatrix} G_P \\ 0 \end{bmatrix} e_P \\ 0 &= \begin{bmatrix} C_P & 0 \end{bmatrix} \begin{bmatrix} x_P \\ x_0 \end{bmatrix} \\ z_P &= \begin{bmatrix} H_P & 0 \end{bmatrix} \begin{bmatrix} x_P \\ x_0 \end{bmatrix} \end{aligned} \tag{6.23}$$

which is an equivalent representation of Σ_{N_P} as in (6.20). Hence, the relation

$$S := \{(x_N, (x_P, x_0)) \mid x_N = x_P\} \tag{6.24}$$

defines a full bisimulation relation between Σ_{N_P} and $\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_0$. □

Proposition 6.11. *The null system is simulated by any other system, i.e. for any Σ_P*

$$\Sigma_0 \preccurlyeq \Sigma_P \tag{6.25}$$

Proof. The simulation relation S of Σ_0 by Σ_P is given by setting $x_Q = 0$,

$$S = \{(x_0, x_P) \mid x_P = 0\} \tag{6.26}$$

□

Proposition 6.12. *For any given linear system Σ_P of the form (6.1) and any controller system Σ_C it holds that*

$$\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C \preccurlyeq \Sigma_P \tag{6.27}$$

Proof. The plant-controller interconnection introduces a constraint on the state variables of Σ_P. Therefore,

$$S = \{(x_P, x_C), \bar{x}_P) \,|\, (x_P, x_C) \in \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C, x_P = \bar{x}_P\} \tag{6.28}$$

defines a simulation relation of $\Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C$ by Σ_P. Indeed, setting $x_P = \bar{x}_P$ and taking $\bar{u} = \Pi_{12} C_C x_C + \Pi_{11} u$ yields

$$\dot{x}_P = A_P x_P + B_P\left(\Pi_{12} C_C x_C + \Pi_{11} u\right) + G_P e = \dot{\bar{x}}_P = A_P \bar{x}_P + B_P \bar{u} + G_P e$$

as well as $H_P x_P = H_P \bar{x}_P$. □

As a corollary of Theorem 3.38, compositional reasoning also holds for plant-controller interconnections.

Corollary 6.13. *Given two plants $\Sigma_{P_i}, i = 1, 2,$ of the form* (6.1) *and two controllers Σ_{C_i} of the form* (6.2), *plant-controller interconnection is compositional,*

$$\left.\begin{array}{ccc} \Sigma_{P_1} & \preccurlyeq & \Sigma_{P_2} \\ \Sigma_{C_1} & \preccurlyeq & \Sigma_{C_2} \end{array}\right\} \implies \Sigma_{P_1} \|_{u,y} \Sigma_{C_1} \preccurlyeq \Sigma_{P_2} \|_{u,y} \Sigma_{C_2} \tag{6.29}$$

6.5.1. The canonical controller

The canonical controller Σ_{can} has been introduced by van der Schaft in a behavioral setting [73]. An analogous definition for input-state-output systems was given in [78] interconnecting the plant with its specification through the external variables e and z.

Definition 6.14. The canonical controller for a plant system Σ_P and a specification Σ_Q is defined as

$$\Sigma_{\text{can}} := \Sigma_P \overset{I}{\underset{e,z}{\|}} \Sigma_Q \ , \tag{6.30}$$

i.e., by setting

$$e_P = e_Q, \qquad z_P = z_Q \tag{6.31}$$

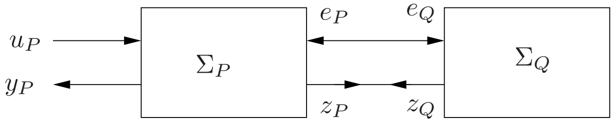

Figure 6.3.: The canonical controller $\Sigma_{\text{can}} = \Sigma_P \|_{\text{can}} \Sigma_Q$.

6.5.2. Sandwich conditions for achievable simulations

The decentralized control scheme presented in the following relies on checkable conditions for the existence of a controller Σ_C for a given plant Σ_P and a specification Σ_Q. The result presented here is formulated in terms of simulation relations and was first shown in [78].

Theorem 6.15. *For a given plant system Σ_P and a specification Σ_Q, the following statements hold*

(i): $\Sigma_Q \preccurlyeq \Sigma_P \implies \exists\, \Sigma_C,\, \Pi :\ \Sigma_Q \preccurlyeq \Sigma_P \|^{\Pi}_{u,y} \Sigma_C$

(ii): $\Sigma_{N_P} \preccurlyeq \Sigma_Q \implies \exists\, \Sigma_C,\, \Pi :\ \Sigma_P \|^{\Pi}_{u,y} \Sigma_C \preccurlyeq \Sigma_Q$

(iii): $\Sigma_{N_P} \preccurlyeq \Sigma_Q \preccurlyeq \Sigma_P \implies \exists\, \Sigma_C,\, \Pi :\ \Sigma_P \|^{\Pi}_{u,y} \Sigma_C \approx \Sigma_Q$

(iv): $\forall\, \Sigma_C\ , \Pi\ :\ \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C \approx \Sigma_Q \implies \Sigma_{N_P} \preccurlyeq \Sigma_Q \preccurlyeq \Sigma_P$

Proof. The proof of (i) – (iii) follows the lines of [78].
(i): Consider the canonical controller $\Sigma_C = \Sigma_{\text{can}}$. Since there exists a full simulation relation S_{QP} of Σ_Q by Σ_P, we know that for every (x_Q, x_P) there exists a joint input $e = e_Q = e_P$ such that $z_s = z_P$. This ensures also that that the canonical controller has at least one state $(x_P, \bar{x}_P, x_s) \in \Sigma_P \|^{\Pi}_{u,y} \Sigma_{\text{can}}$ as it contains as states all the pairs $(x_P, x_Q) \in S_{QP}$. Take now any state $x_Q \in \Sigma_Q$. Due to S_{QP}, there exists a x_P such that $(x_P, x_P, x_Q) \in \Sigma_P \|_{u,y} \Sigma_{\text{can}} = \Sigma_P \|_{u,y} \left(\Sigma_P \overset{I}{\underset{e,z}{\|}} \Sigma_P \right)$ such that for every joint $e = e_p = e_{\Sigma_P \|_{u,y} \Sigma_{\text{can}}}$ the outputs are equal, that is $z_Q = H_Q x_Q = H_P x_P = z_{\Sigma_P \|_{u,y} \Sigma_{\text{can}}}$.
(ii): We want to show that by using the canonical controller there indeed exists a full simulation relation of $\Sigma_P \|^{\Pi}_{u,y} \Sigma_{\text{can}}$ by Σ_Q, i.e. for any $(x_P, \bar{x}_P, x_Q) \in \Sigma_P \|^{\Pi}_{u,y} \Sigma_{\text{can}}$ there exists a state $\bar{x}_Q \in \Sigma_Q$ such that $z_{\Sigma_P \|^{\Pi}_{u,y} \Sigma_{\text{can}}} = H_P x_P = H_x \bar{x}_Q = z_Q$. Observe first that for any state $(x_P, \bar{x}_P, x_Q) \in \Sigma_P \|^{\Pi}_{u,y} \Sigma_{\text{can}}$ it holds that $x_P - \bar{x}_P \in \Sigma_{N_P}$ since the plant-controller interconnection forces $C_P x_P = C_P \bar{x}_P$. Since all simulation relations considered here are linear subspaces, we can rewrite

$$(x_P, \bar{x}_P, x_s) = (\bar{x}_P + x_N, \bar{x}_P, x_Q) \in \Sigma_P \|^{\Pi}_{u,y} \Sigma_{\text{can}},\ x_N \in \Sigma_{N_P} \tag{6.32}$$

Moreover, since there exists a full simulation relation S_{NS} of Σ_{N_P} by Σ_Q, for every $x_N \in \Sigma_{N_P}$ there exists a $\bar{x}_Q \in \Sigma_Q$ such that $H_{N_P} x_N = H_Q \bar{x}_Q$. Consider now the state $x_Q + \bar{x}_Q \in \Sigma_Q$. Then the pair of states $(x_P, \bar{x}_P, x_s, x_Q + \bar{x}_Q)$ can be written as

$$(x_P, \bar{x}_P, x_s, x_Q + \bar{x}_Q) = (\bar{x}_P, \bar{x}_P, x_Q, x_Q) + (x_N, 0, 0, \bar{x}_Q) \tag{6.33}$$

where $(x_N, \bar{x}_Q) \in S_{NS}$ and $(\bar{x}_P, x_Q) \in \Sigma_{\text{can}}$. Thus, $H_{N_P} x_N = H_Q \bar{x}_Q$ and $H_P \bar{x}_P = H_Q x_s$ and therefore

$$H_P x_P = H_Q(x_Q + \bar{x}_Q)\ , \tag{6.34}$$

which proves the claim.
(iii): Combining the statements (i) and (ii) for the same Σ_C and Π yields $\Sigma_Q \preccurlyeq \Sigma_P \|_{u,y}^{\Pi} \Sigma_C \preccurlyeq \Sigma_Q$ and thus $\Sigma_Q \approx \Sigma_P \|_{u,y}^{\Pi} \Sigma_C$.
(iv): By Proposition 6.11, Σ_0 is simulated by any other system, so $\Sigma_0 \preccurlyeq_{e,z} \Sigma_C$. Moreover, Proposition 6.10 states that $\Sigma_{N_P} \preccurlyeq_{e,z} \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_0$. Since simulation is reflexive and plant-controller interconnection is compositional, we therefore conclude

$$\Sigma_{N_P} \preccurlyeq \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_0 \preccurlyeq \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C \preccurlyeq \Sigma_Q \tag{6.35}$$

and hence, $\Sigma_{N_P} \preccurlyeq \Sigma_Q$. Moreover, since $\Sigma_Q \approx \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C$, Proposition 6.12 yields

$$\Sigma_Q \preccurlyeq \Sigma_P \overset{\Pi}{\underset{u,y}{\|}} \Sigma_C \preccurlyeq \Sigma_P \tag{6.36}$$

□

6.6. Decentralized control and achievable simulation

Theorem 6.15 gives conditions for the existence of a controller for a given plant and specification, and a constructive procedure to compute such a controller. Combining sandwich conditions with compositional analysis techniques from Section 6.4 yields decentralized control schemes that include existence conditions for controllers guaranteed to satisfy the specification requirements. Like in (6.11), the overall plant Σ_P is given as a series of feedback interconnections of k subsystems Σ_{P_i}. Accordingly, the global specification Σ_Q is assumed to be given as a series of feedback interconnections of k subspecifications as in (6.12). We present two approaches to solve the control problem defined as follows:

Find necessary and sufficient conditions under which there exists a control strategy such that the global closed loop system satisfies the global specification.

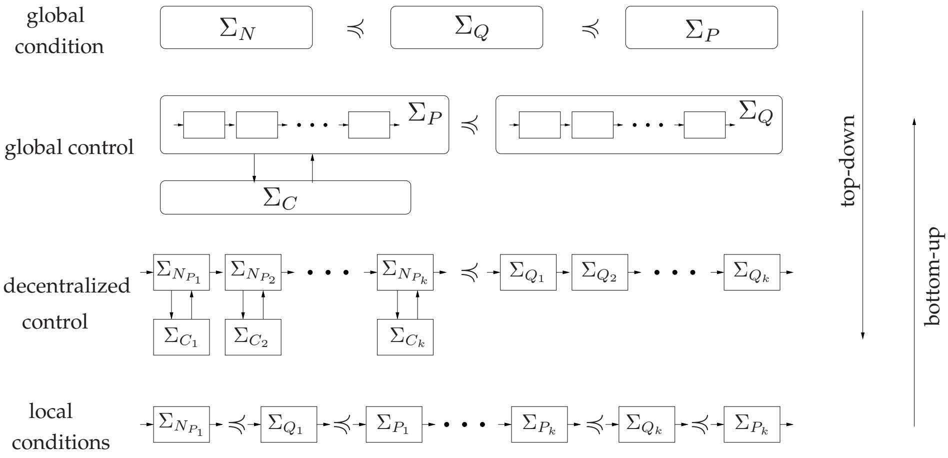

Figure 6.4.: Bottom-up and top-down schemes.

6.6.1. Bottom-up schemes using local sandwich conditions

As depicted in Figure 6.4, a *bottom-up* scheme uses local conditions for achievable simulation. These conditions ensure the existence of local controllers Σ_{C_i} for each component Σ_{P_i} of the global plant such that the overall decentralized control network fulfills the global specification Σ_Q. The first bottom-up scheme we present here uses soundness of compositional reasoning for k systems as stated in Corollary 6.6.

Theorem 6.16. *Consider a global plant system Σ_P of the form* (6.11)*. Let $\Sigma_{N_{P_i}}, i = 1, \ldots, k$, be associated to the plant components Σ_{P_i}. Consider a corresponding specification Σ_Q be of the form* (6.12)*.*

1. *If the local conditions*

$$\Sigma_{N_{P_i}} \preccurlyeq \Sigma_{Q_i} \ \forall i = 1, \ldots, k \tag{6.37}$$

are fulfilled, there exist local controllers Σ_{C_i} and permutation matrices Π_i such that the series interconnections fulfills the global specification, i.e.

$$\begin{gathered} \exists\, \Sigma_{C_i}, \Pi_i,\ i = 1, \ldots, k : \\ \left(\Sigma_{P_1} \|_{u,y}^{\Pi_1} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|_{u,y}^{\Pi_k} \Sigma_{C_k}\right) \preccurlyeq \Sigma_Q \end{gathered} \tag{6.38}$$

2. *If*

$$\Sigma_{N_{P_i}} \preccurlyeq \Sigma_{Q_i} \preccurlyeq \Sigma_{P_i} \ \forall i = 1, \ldots, k \tag{6.39}$$

then there exist local controllers Σ_{C_i} and permutation matrices Π_i such that

$$\begin{gathered} \exists\, \Sigma_{C_i}, \Pi_i,\ i = 1, \ldots, k : \\ \left(\Sigma_{P_1} \|_{u,y}^{\Pi_1} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|_{u,y}^{\Pi_k} \Sigma_{C_k}\right) \approx \Sigma_Q \end{gathered} \tag{6.40}$$

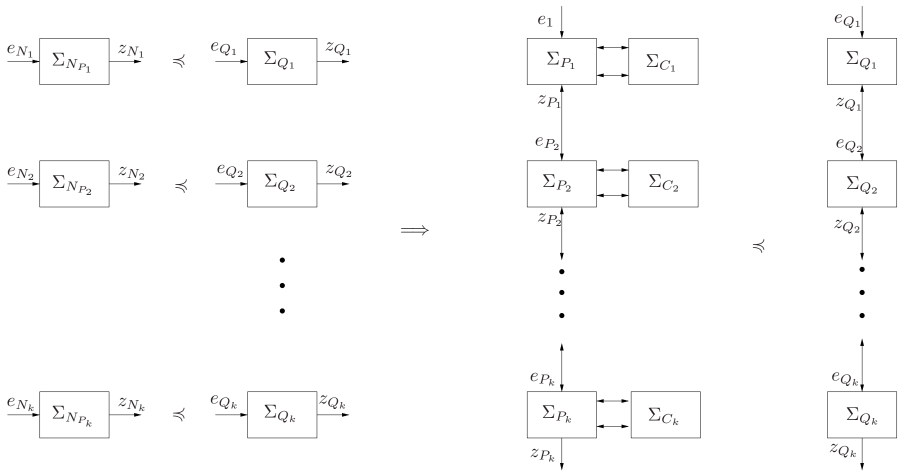

Figure 6.5.: Bottom-up decentralized control scheme combining local sandwich conditions and compositionality.

Proof. By Theorem 6.15 (i), the local conditions (6.37) guarantee the existence of local controllers such that

$$\Sigma_{P_i} \|_{u,y}^{\Pi_i} \Sigma_{C_i} \preccurlyeq \Sigma_{Q_i} \ \forall i = 1, \dots, k \tag{6.41}$$

Compositionality for k plant-controller interconnections as stated in Corollary 6.6 then yields the desired result (6.38) for series interconnections.
If instead (6.39) holds, then by Theorem 6.15 (ii) the global controlled plant $\left(\Sigma_{P_1} \|_{u,y}^{\Pi_1} \Sigma_{C_1}\right) \| \dots \| \left(\Sigma_{P_k} \|_{u,y}^{\Pi_k} \Sigma_{C_k}\right)$ is bisimilar to the global specification Σ_Q. $\square$

Next, we introduce a bottom-up scheme relying on soundness of circular assume guarantee reasoning, cf. Corollary 6.7. Figure 6.6.1 illustrates that the conditions for achievable simulation involve interconnections of plant subsystem $\Sigma_{P_i}, i = 1, \dots, k$ with subspecifications $\Sigma_{Q_j}, j = 1, \dots, k, j \neq i$, denoted by Σ_P^i and given as

$$\Sigma_P^i \quad := \quad \Sigma_{Q_1} \| \dots \| \Sigma_{Q_{i-1}} \| \Sigma_{P_i} \| \Sigma_{Q_{i+1}} \dots \| \Sigma_{Q_k} \ i = 1, \dots, k \tag{6.42}$$

Associated with Σ_P^i are the systems

$$\Sigma_{N_P}^i \quad := \quad \Sigma_{Q_1} \| \dots \| \Sigma_{Q_{i-1}} \| \Sigma_{N_{P_i}} \| \Sigma_{Q_{i+1}} \dots \| \Sigma_{Q_k} \ i = 1, \dots, k \tag{6.43}$$

Theorem 6.17. *Let the plant system Σ_P be of the form* (6.11) *and the corresponding specification Σ_Q be given as in* (6.12)*. Consider k global systems $\Sigma_P^i, i = 1, \dots, k$, and their associated systems $\Sigma_{N_P}^i$ as defined in* (6.42) *and* (6.43)*, respectively. Then the following holds:*

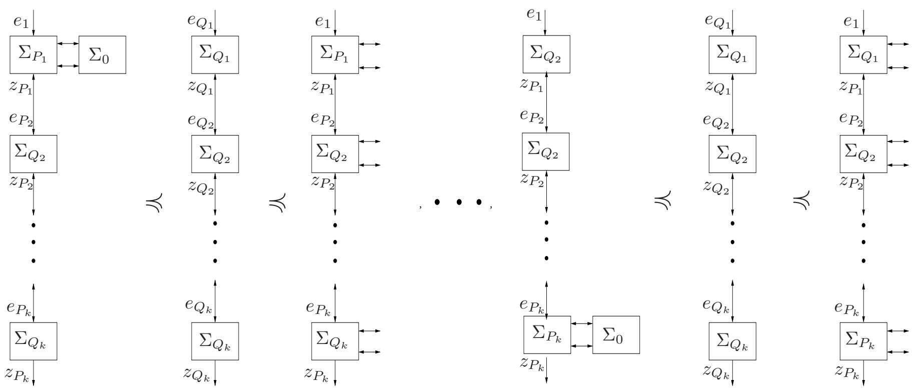

Figure 6.6.: Sandwich conditions 2 for bottom-up scheme based on circular assume-guarantee reasoning

1. *If*

$$\Sigma^i_{N_P} \quad \preccurlyeq \quad \Sigma_Q \ , \quad i = 1, \ldots, k \ , \tag{6.44}$$

then there exist local controllers $\Sigma_{C_i}, i = 1, \ldots, k$*, and permutation matrices* Π_i *such that*

$$\left(\Sigma_{P_1} \|^{\Pi_1}_{u,y} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|^{\Pi_k}_{u,y} \Sigma_{C_k}\right) \quad \preccurlyeq \quad \Sigma_Q \tag{6.45}$$

2. *If*

$$\Sigma^i_{N_P} \quad \preccurlyeq \Sigma_Q \quad \preccurlyeq \Sigma^i_P \ , \quad i = 1, \ldots, k \ , \tag{6.46}$$

then there exist local controllers Σ_{C_i} *and permutation matrices* Π_i *such that*

$$\left(\Sigma_{P_1} \|^{\Pi_1}_{u,y} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|^{\Pi_k}_{u,y} \Sigma_{C_k}\right) \quad \approx \quad \Sigma_Q \tag{6.47}$$

Proof. By Theorem 6.15 (i), the local conditions (6.44) guarantee the existence of local controllers Σ_{C_i} and permutation matrices Π_i such that

$$\Sigma^i_P \|^{\Pi_i}_{u,y} \Sigma_{C_i} \preccurlyeq \Sigma_Q \ , \quad i = 1, \ldots, k \ .$$

Soundness of assume-guarantee reasoning for k control systems as stated in Corollary 6.7 then yields the desired result for feedback interconnections.
If instead the sandwich condition holds, then by Theorem 6.15 (ii) the global controlled plant $\left(\Sigma_{P_1} \|^{\Pi_1}_{u,y} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|^{\Pi_k}_{u,y} \Sigma_{C_k}\right)$ is bisimilar to the global specification Σ_Q. Indeed, it follows from (6.46) that there exists controllers Σ_{C_i} and permutation matrices Π_i such that

$$\Sigma^i_P \|^{\Pi_i}_{u,y} \Sigma_{C_i} \approx \Sigma_Q \ , \ i = 1, \ldots, k \tag{6.48}$$

We then have to use that circular assume-guarantee reasoning for k systems is also sound using bisimulation relations (compare with Corollary 6.7) to conclude from (6.48) that (6.47) holds. □

So far we have shown that local conditions for achievable simulation are necessary and sufficient for the existence of local controller $\Sigma_{C_i}, i = 1, \ldots, k$, such that the global specification is satisfied. This holds for both bottom-up schemes relying on compositional and circular assume guarantee reasoning, respectively. Obviously, one can always construct a diagonally decoupled global controller Σ_C based on Σ_{C_i}, i.e. Σ_C consists of k subsystems running in parallel without interference,

$$\Sigma_C := (\Sigma_{C_1} \| \ldots \| \Sigma_{C_k}) \ . \tag{6.49}$$

The construction (6.49) is also consistent with our definition of feedback interconnection $\|$. According to Definition 6.3, the feedback interconnection $\|$ involves the external specification variables $e_i^\pm$ and $z_i^\pm$ which are absent in controller systems. Hence, it trivially holds that

$$\begin{array}{ccc} \left(\Sigma_{P_1} \|_{u,y}^{\Pi} \Sigma_{C_1}\right) \| & \ldots & \| \left(\Sigma_{P_N} \|_{u,y}^{\Pi} \Sigma_{C_k}\right) \\ & \approx & \\ (\Sigma_{P_1} \| \ldots \| \Sigma_{P_k}) & \|_{u,y}^{\Pi} & (\Sigma_{C_1} \| \ldots \| \Sigma_{C_k}) \end{array} \tag{6.50}$$

Thus, if (6.38) and (6.40) respectively (6.45) and (6.47) hold for the same permutation matrix Π, e.g. for $\Pi = I$, the decentralized control schemes of Theorems 6.16 and 6.17 can be interpreted as global feedback control strategies but based on *local* conditions for achievable simulation.

6.6.2. Top-down decentralized control scheme using global sandwich conditions

The *top-down* scheme for decentralized control starts from the perspective of the overall system, see Figure 6.4. Based on a global sandwich condition, we want to investigate whether the existence of a global controller Σ_C implies the existence of local controllers such that the overall controlled system satisfies the same specification. From the previous chapters it is known that compositional reasoning is not complete for closed interconnections. The result presented here therefore relies on completeness of circular assume-guarantee reasoning in the decentralized setting. Like before, we consider a global plant Σ_P composed of component systems $\Sigma_{P_i}, i = 1, \ldots, k$, interconnected in series by feedback as in (6.11) and a global specification Σ_Q assembled as in (6.12). The system Σ_{N_P} associated with the global plant Σ_P is given by

$$\begin{aligned} \Sigma_{N_P} = \Sigma_P \|_{u,y}^{\Pi} (\Sigma_0 \| \ldots \| \Sigma_0) \approx \left(\Sigma_{P_1} \|_{u,y}^{\Pi} \Sigma_0\right) \| \ldots \| \left(\Sigma_{P_k} \|_{u,y}^{\Pi} \Sigma_0\right) \\ = \Sigma_{N_{P_1}} \| \ldots \| \Sigma_{N_{P_k}} \end{aligned} \tag{6.51}$$

making use of Proposition 6.10.

Theorem 6.18. *Consider a global plant Σ_P as in* (6.11) *with an associated system Σ_{N_P} given by* (6.51) *and a global specification Σ_Q as in* (6.12). *Then the following two statements are equivalent:*

1. *There exists a global controller Σ_C and a permutation matrix Π_1 such that*

$$\Sigma_P \|_{u,y}^{\Pi_1} \Sigma_C \approx \Sigma_Q \tag{6.52}$$

2. *There exist local controllers $\Sigma_{C_i}, i = 1, \ldots, k$, and a permutation matrix Π_2 such that*

$$\left(\Sigma_{P_1} \|_{u,y}^{\Pi_2} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|_{u,y}^{\Pi_2} \Sigma_{C_k}\right) \quad \approx \quad \Sigma_Q \tag{6.53}$$

Proof. "$2 \implies 1$": This is a consequence of (6.50), i.e. the global decoupled controller Σ_C is the series interconnection of the local controllers $\Sigma_{C_i}, i = 1, \ldots, k$ with $\Pi_1 = \Pi_2$.
"$1 \implies 2$": Assume there exists a global controller Σ_C and a permutation matrix Π such that (6.52) holds. Then by Theorem 6.15, iv, the global sandwich condition

$$\Sigma_{N_P} \preccurlyeq \Sigma_Q \preccurlyeq \Sigma_P \tag{6.54}$$

is fulfilled. Making use of (6.51), (6.54) can be rewritten as

$$\Sigma_{N_{P_1}} \| \ldots \| \Sigma_{N_{P_k}} \preccurlyeq \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k} \preccurlyeq \Sigma_{P_1} \| \ldots \| \Sigma_{P_k} \tag{6.55}$$

We now split (6.55) into two statements and use the fact that circular assume-guarantee reasoning is complete in the decentralized setting (Corollary 6.7). Hence, we obtain from the first statement k full simulation relations $S_i^l, i = I, II, \ldots, k$, of the form

$$S_i^l : \quad \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_{i-1}} \| \Sigma_{N_{P_i}} \| \Sigma_{Q_{i+1}} \| \ldots \| \Sigma_{Q_k} \preccurlyeq \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k}$$

which by (6.43) can be simplified to

$$S_i^l : \; \Sigma_{N_P}^i \preccurlyeq \Sigma_Q \; . \tag{6.56}$$

The second statement results in k full simulation relations $S_i^r, i = I, II, \ldots, k$, of the form

$$S_i^r : \quad \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k} \preccurlyeq \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_{i-1}} \| \Sigma_{P_i} \| \Sigma_{Q_{i+1}} \| \ldots \| \Sigma_{Q_k}$$

or, due to (6.42),

$$S_i^r : \quad \Sigma_Q \preccurlyeq \Sigma_P^i \tag{6.57}$$

Note that (6.57) implies that Σ_{Q_i} refines Σ_{P_i} as mentioned in Remark 3.22. Hence, in order to apply circular assume-guarantee reasoning, we have to construct S_i^r as

$$\begin{aligned} S_i^r \quad := \quad & \{(x_{Q_1}, \ldots, x_{Q_k}, x_{Q_1}, \ldots, x_{Q_{i-1}}, x_{P_i}, x_{Q_{i+1}}, \ldots, x_{Q_k}) \mid \exists\, x_{P_1}, \ldots, x_{P_{i-1}}, \\ & x_{P_{i+1}}, \ldots, x_{P_k} : (x_{Q_1}, \ldots, x_{Q_k}, x_{P_1}, \ldots, x_{P_k}) \in S^r\} \end{aligned}$$

where S^r is a full simulation relation of Σ_Q by Σ_P. Transitivity of simulation allows to combine (6.56) and (6.57) to obtain sandwich conditions

$$S_i : \quad \Sigma^i_{N_P} \preccurlyeq \Sigma_Q \preccurlyeq \Sigma^i_P \ , i = 1, \ldots, k \ . \tag{6.58}$$

Theorem 6.17, 2, then ensures that there exist local controllers Σ_{C_i} and permutation matrices Π_i such that

$$\left(\Sigma_{P_1} \|^{\Pi_1}_{u,y} \Sigma_{C_1}\right) \| \ldots \| \left(\Sigma_{P_k} \|^{\Pi_k}_{u,y} \Sigma_{C_k}\right) \quad \approx \quad \Sigma_Q$$

holds. Using canonical controllers Σ^i_{can}, we can choose $\Pi_1 = \ldots = \Pi_k = I$ and thus the claim is proved. □

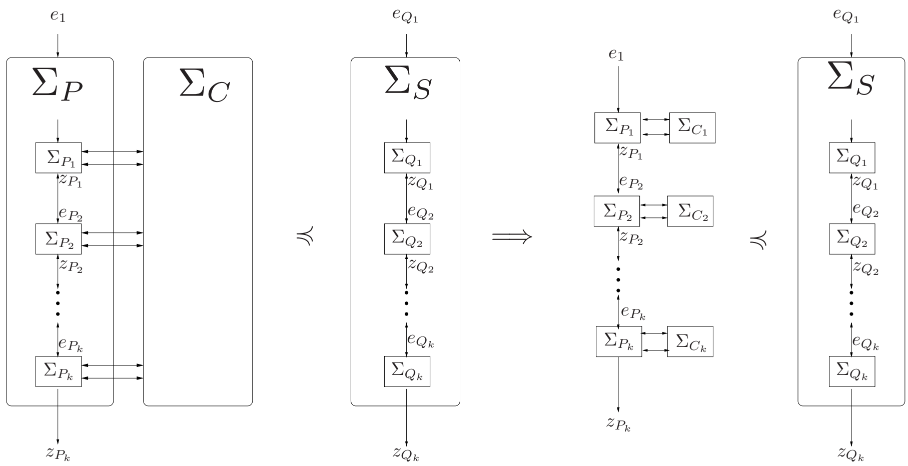

Figure 6.7.: Top-down decentralized control scheme.

Figure 6.7 illustrates an intriguing consequence of Theorem 6.18: Although nothing is known about the structure of the global controller Σ_C, there always exist local controllers Σ_{C_i} in our decentralized control setting that satisfy the same global control target Σ_Q. Thus, provided the conditions of Theorem 6.18 hold – in particular that the specification Σ_Q is given as $\Sigma_Q = \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k}$ – decentralized control can achieve the same control performance as a more complex global feedback controller.