Compositional analysis and control of dynamical systems

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Passivity theory and compositional analysis

The classical passivity theorem states that the negative feedback interconnection of two passive systems is again passive with respect to the same supply rate, cf. [63]. Clearly, this can be regarded as compositional reasoning for passive systems. In this chapter, a new approach towards passivity is developed using nonlinear simulation relations. Associating the storage function with the state variable and the inputs and outputs of the underlying dynamical system with the external variables, the differential dissipation inequality can be interpreted as a one-dimensional state space control system. Passivity is then equivalent to the existence of a nonlinear simulation relation between the original control system and the one-dimensional system given by the dissipation inequality. Formulating passivity properties by means of simulation relations allows us to apply some of the compositional reasoning techniques developed in the previous chapters. The passivity theorem mentioned above can thus be reinterpreted as compositionality of passivity. This is verified for both open and closed feedback interconnections of linear systems. We then show that passivity is also complete under open feedback, i.e. if the open feedback interconnection of two control systems is passive then the individual components must be passive as well. As a result, the structure of the storage function of the interconnected system can be specified in terms of the storage functions of the components. These results are then generalized to nonlinear systems. Thus, a strong relation between passivity and simulation theory has been established demonstrating that properties of interconnected control systems – in this case passivity – can be verified efficiently using compositional reasoning techniques. Due to its link with Lyapunov stability theory some of the results can, under mild assumptions on the storage functions, be reformulated as compositional stability analysis, in particular for closed feedback interconnections. This motivates further research on the topic of passivity based controller design methods, see e.g. [72].

5.1. A brief introduction to passivity theory

Passivity as a system theoretic concept can be defined in great generality. We consider continuous-time state-space systems with state variables x taking values in an $n-$ dimensional manifold $\mathcal{X} \subset \mathbb{R}^n$ and external variables u and y related by a set of differential and algebraic equations or inequalities

$$\Sigma : \quad F(x, \dot{x}, u, y) \ \leq \ 0 \ , \tag{5.1}$$

The input and output spaces $\mathcal{U}$ and $\mathcal{Y}$ are assumed to be dual spaces, i.e. $\mathcal{Y} = \mathcal{U}^\star$, in order to define the product $u^T y$.

Definition 5.1. A state space system Σ is *passive* if there exists a function $V : \mathcal{X} \to \mathbb{R}^+$, called the storage function, such that for all $x_0 \in \mathcal{X}$, all $t_1 \geq t_0$, and all input functions u

$$V(x(t_1)) \leq V(x(t_0)) + \int_{t_0}^{t_1} u^T(t) y(t) dt \tag{5.2}$$

where $x_0 = x(t_0)$.
If (5.2) holds with equality, then Σ is *lossless*.

If V is differentiable, the differential version of the dissipation inequality (5.2) is given by

$$V_x(x)\dot{x} \leq u^T y \tag{5.3}$$

for all $(x, \dot{x}, u, y)$ satisfying (5.1). Here $V_x(x)$ denotes the row vector of partial derivatives

$$V_x(x) = \begin{pmatrix} \frac{\partial V}{\partial x_1}(x) & \ldots & \frac{\partial V}{\partial x_n}(x) \end{pmatrix}$$

The next proposition expresses passivity properties of a state space system in terms of the existence of a nonlinear simulation relation.

Proposition 5.2. *For a continuous-time state space system Σ of the form* (4.1)*, the following two statements are equivalent:*

- *Σ is passive (lossless)*
- *Σ is simulated by the one-dimensional system*

$$\Xi : \quad \dot{\xi} \quad \leq (=) \quad u^T y, \quad \xi \in \mathcal{X}_\xi = \mathbb{R}^+ \tag{5.4}$$

The non-linear simulation relation S of Σ by Ξ is given by

$$S = \{(x, \xi) \,|\, \xi = V(x)\} \tag{5.5}$$

Proof. $\mathcal{S}$ as given by (5.5) is defined on the product manifold $\mathcal{X} \times \mathcal{X}_\xi$ and has the submanifold property. Indeed, it is given as the zero level set of a smooth function $f(x, \xi) := \xi - V(x)$ such that the Jacobian $\mathrm{Jac}(f) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial \xi} \end{bmatrix} = \begin{bmatrix} V_x(x) & 1 \end{bmatrix}$ has full rank for all $(x, \xi) \in S$. Hence, by Theorem 4.6, S is a submanifold. Moreover, the external behavior of Σ and Ξ is identical. Thus, for any pair of states $(x_0, \xi_0) \in S$, the state trajectories $x(t), \xi(t) = V(x(t))$ remain within the submanifold $S \subset \mathcal{X} \times \mathcal{X}_\xi$ for all times $t \geq 0$ while the external variables $u(t)$ and $y(t)$ are the same. Moreover, S is full since for every x there exists a $\xi = V(x)$. $\square$

5.2. Compositional reasoning for passive systems

The motivation for using simulation theory to characterize passivity properties of control systems is to analyze interconnections of such systems. We first want to study *linear* state space systems before generalizing the results to nonlinear input-affine systems.
To simplify the exposition, we consider linear systems of the form

$$\Sigma_i : \begin{array}{rcll} \dot{x}_i & = & A_i x_i + B_i u_i, & x_i \in \mathcal{X}_i \\ y_i & = & C_i x_i & \end{array}, \; i = 1, 2, \tag{5.6}$$

without feedthrough terms.
Based on the differential dissipation inequality (5.3), the Kalman-Yakubovic conditions [80] characterize passivity in terms of the state space representation of linear systems.

Proposition 5.3. *A linear system Σ is passive (lossless) if there exists a quadratic storage function $V(x) = \frac{1}{2}x^T Qx, Q = Q^T \geq 0$ such that*

$$\begin{array}{rcl} A^T Q + QA & \leq (=) & 0 \\ B^T Q & = & C \end{array} \tag{5.7}$$

Moreover, if the system Σ is passive with a continuous and differentiable storage function V, then there exists a quadratic storage function $V = \frac{1}{2}x^T Qx$ with $Q = Q^T \geq 0$ such that Q fulfills (5.7).
If the system Σ is lossless and controllable, the storage function is unique (up to a constant) and given by a quadratic function $V(x) = \frac{1}{2}x^T Qx$.

The specialization of Proposition 5.2 to the linear case leads to the following

Corollary 5.4. *A linear system Σ of the form (5.6) is passive (lossless) if and only if there exists a full simulation relation S of Σ by Ξ, Ξ as in (5.4), where*

$$S = \{(x, \xi) \mid \xi = \frac{1}{2}x^T Qx, Q = Q^T \geq 0,\ Q \text{ fulfills } (5.7)\} \tag{5.8}$$

It is a well-known fact in electrical network and systems theory that the negative feedback interconnection of two passive systems is again passive [63]. Capturing passivity as nonlinear simulation this result can be reformulated as compositional reasoning.

Theorem 5.5. *Given two linear systems $\Sigma_i, i = 1, 2$ of the form (5.6) and two systems Ξ_i of the form (5.4). Then passivity (losslessness) is compositional under open negative feedback interconnection $\|_o$, i.e.*

$$\left.\begin{array}{lccc} S_1: & \Sigma_1 & \preccurlyeq & \Xi_1 \\ S_2: & \Sigma_2 & \preccurlyeq & \Xi_2 \end{array}\right\} \implies S: \ \Sigma_1\|_o\Sigma_2 \ \preccurlyeq \ \Xi_1\|_o\Xi_2 \tag{5.9}$$

Moreover, $\Xi_1\|_o\Xi_2 \approx \Xi$ where

$$\Xi: \quad \dot{\xi} \ \leq \ e_1^T y_1 + e_2^T y_2 \tag{5.10}$$

and thus

$$\tilde{S}: \quad \Sigma_1\|_o\Sigma_2 \preccurlyeq \Xi \tag{5.11}$$

Proof. We will prove the result for passive systems, the same arguments also hold for lossless systems. Construct the simulation relation S by setting

$$S := \left\{ ((x_1, x_2), (\xi_1, \xi_2)) \mid \xi_1 + \xi_2 = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\} \tag{5.12}$$

Since $S_i, i = 1, 2$ is a simulation relation of Σ_i by Ξ_i, we know that there exists a quadratic storage function $\xi_i = \frac{1}{2} x_i^T Q_i x_i, Q_i = Q_i^T \geq 0$, such that

$$\dot{\xi}_i = \frac{1}{2} x_i^T (A_i^T Q_i + Q_i A_i) x_i + u_i^T B_i^T Q_i x_i \leq u_i^T y_i = u_i^T C_i x_i$$

The open negative feedback interconnection $u_1 = e_1 - y_2, u_2 = e_2 + y_1$ yields the closed loop systems

$$\Sigma_1\|_o\Sigma_2: \begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} A_1 & -B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} & = & \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array} \tag{5.13}$$

and

$$\Xi_1\|_o\Xi_2: \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} u_1^T y_1 \\ u_2^T y_2 \end{bmatrix} = \begin{bmatrix} (e_1 - y_2)^T y_1 \\ (e_2 + y_1)^T y_2 \end{bmatrix} \tag{5.14}$$

From (5.12), we compute

$$\begin{aligned}\dot{\xi}_1 + \dot{\xi}_2 &= \frac{1}{2}\frac{d}{dt}\left(x_1^T Q_1 x_1 + x_2^T Q_2 x_2\right) \leq \\ &\leq (e_1 - y_2)^T y_1 + (e_2 + y_1)^T y_2 = e_1^T y_1 + e_2^T y_2\end{aligned} \tag{5.15}$$

and thus, $\Sigma_1\|_o\Sigma_2$ is also passive with respect to the supply rate $s(e_1, e_2, y_1, y_2) = e_1^T y_1 + e_2^T y_2$.
Next, consider the system

$$\Xi: \quad \dot{\xi} = e^T z$$

with $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ and $z = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and a relation $\tilde{\mathcal{S}}$ given by

$$\tilde{\mathcal{S}} := \{((\xi_1, \xi_2), \xi) \mid \xi = \xi_1 + \xi_2\}$$

Again, $\tilde{S} \subset \mathcal{X}_\xi \times \mathcal{X}_{\xi_1} \times \mathcal{X}_{\xi_2}$ is defined as the zero level set of the smooth function $g(\xi, \xi_1, \xi_2) := \xi - \xi_1 - \xi_2$ such that the Jacobian $\mathrm{Jac}(g(\xi, \xi_1, \xi_2)) = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$ has full rank for all $(\xi, \xi_1, \xi_2) \in \tilde{S}$. Hence, $\tilde{S}$ is indeed a submanifold. Take now a set of initial states $((\xi_1(0), \xi_2(0)), \xi(0)) \in \tilde{S}$. Starting from $\xi_1(0) + \xi_2(0) = \xi(0)$, the trajectories by definition satisfy

$$\xi_1(t) + \xi_2(t) = \xi(t), \quad \forall t \geq 0$$

while the corresponding external variables are the same,

$$e^T(t)z(t) = e_1^T(t)z_1(t) + e_2^T(t)z_2(t), \quad \forall t \geq 0$$

Thus, Ξ and $\Xi_1\|_o\Xi_2$ are indeed bisimilar. (5.11) then follows from transitivity of simulation. Here, the conditions of Proposition 4.15 are fulfilled since

$$\mathrm{rank}\begin{bmatrix} \frac{\partial f(\xi,\xi_1,\xi_2)}{\partial \xi_1} & \frac{\partial f(\xi,\xi_1,x_1)}{\partial \xi_2} \\ \frac{\partial g(x_1,x_2,\xi_1,x_2)}{\partial \xi_1} & \frac{\partial g(x_1,x_2,\xi_1,x_2)}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \text{const.}$$

for all (ξ_1, ξ_2) and

$$\begin{aligned}\mathrm{rank}\tilde{S} &= \mathrm{rank}\left[\mathrm{Jac}(\xi - \frac{1}{2}x_1^T Q_1 x_2 + \frac{1}{2}x_2^T Q_2 x_2)\right] = \\ &= \mathrm{rank}\begin{bmatrix} 1 & -x_1^T Q_1 & -x_2^T Q_2 \end{bmatrix} = 1\end{aligned}$$

for all $(\xi, x_1, x_2) \in \tilde{S}$. □

Note that although we considered *positive* feedback interconnections in Chapters 3 and 4 the results of these chapters can easily be seen to hold for negative feedback as well. Theorem 5.5 therefore establishes the link between passivity theory and compositional analysis techniques by showing that passivity is compositional under open feedback. Thus, a classical result of passivity theory is reinterpreted by means of compositional reasoning techniques. This evokes the question whether the converse implication also holds true. Recall that in Section 3.2.2 we showed that compositionality is complete for open feedback interconnections of linear systems. The same result also holds for nonlinear systems as shown in Chapter 4. We will now show that a similar result holds for negative feedback interconnections of passive (lossless) systems. We know that for such interconnections there exists a quadratic storage function

$$V(x_1, x_2) = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.16}$$

such that $\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ is symmetric and positive semi-definite.

Proposition 5.6. *Consider two linear systems $\Sigma_i, i = 1, 2$, and systems Ξ_i of the form (5.4). If the open feedback interconnection $\Sigma_1 \|_o \Sigma_2$ is passive (lossless), then also the components $\Sigma_i, i = 1, 2$, are passive (lossless). In other words, if there exists a simulation relation S of $\Sigma_1 \|_o \Sigma_2$ by Ξ, then there also exist simulation relations $S_i, i = 1, 2$, for the components*

$$\begin{aligned} S_1 : \quad & \Sigma_1 \preccurlyeq \Xi_1 \\ S_1 : \quad & \Sigma_2 \preccurlyeq \Xi_2 \end{aligned} \tag{5.17}$$

Proof. Again we will only prove the passive case. Assume that $\Sigma_1 \|_o \Sigma_2$ is passive. Then there exists a quadratic storage function V as in (5.16) with $\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ symmetric and positive semi-definite. This implies that

$$\begin{aligned} Q_{11} = Q_{11}^T \geq 0, Q_{22} = Q_{22}^T \geq 0, \\ Q_{12}^T = Q_{21}, Q_{22} - Q_{21} Q_{11}^{-1} Q_{12} \geq 0 \end{aligned} \tag{5.18}$$

The simulation relation S of $\Sigma_1 \|_o \Sigma_2$ by Ξ is then given by

$$S = \{((x_1, x_2), \xi) \mid \xi = V(x_1, x_2)\}$$

Since $\Sigma_1 \|_o \Sigma_2$ is passive,

$$\dot{\xi} \leq e_1^T z_1 + e_2^T z_2 \tag{5.19}$$

This leads to

$$\begin{aligned}
\dot{\xi} &= \frac{1}{2}\frac{d}{dt}\left(x_1^T Q_{11} x_1 + x_2^T Q_{21} x_1 + x_1^T Q_{12} x_2 + x_2^T Q_{22} x_2\right) = \\
&= \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} A_1^T Q_{11} + Q_{11} A_1 + 2Q_{12} B_2 C_1 & -Q_{11} B_1 C_2 + C_1^T B_2^T Q_{22} + A_1^T Q_{12} + Q_{12} A_2 \\ -C_2^T B_1^T Q_{11} + Q_{22} B_2 C_1 + A_2^T Q_{21} + Q_{21} A_1 & -2Q_{12}^T B_1 C_2 + A_2^T Q_{22} + Q_{22} A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&\quad + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^T \begin{bmatrix} B_1^T Q_{11} & B_1^T Q_{12} \\ B_2^T Q_{12}^T & B_2^T Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&\leq e_1^T z_1 + e_2^T z_2 = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^T \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned} \tag{5.20}$$

(5.20) is equivalent to

$$\begin{bmatrix} A_1^T Q_{11} + Q_{11} A_1 + 2Q_{12} B_2 C_1 & -Q_{11} B_1 C_2 + C_1^T B_2^T Q_{22} + A_1^T Q_{12} + Q_{12} A_2 \\ -C_2^T B_1^T Q_{11} + Q_{22} B_2 C_1 + A_2^T Q_{21} + Q_{21} A_1 & -2Q_{12}^T B_1 C_2 + A_2^T Q_{22} + Q_{22} A_2 \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.21}$$

and

$$\begin{bmatrix} B_1^T Q_{11} & B_1^T Q_{12} \\ B_2^T Q_{12}^T & B_2^T Q_{22} \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \tag{5.22}$$

which in turn implies

$$\begin{aligned} B_1^T Q_{11} = C_1, A_1^T Q_{11} + Q_{11} A_1 \leq 0 \\ B_2^T Q_{22} = C_2, A_2^T Q_{22} + Q_{22} A_2 \leq 0 \end{aligned} \tag{5.23}$$

Now consider the systems

$$\Xi_i : \quad \dot{\xi}_i \leq u_i^T y_i, \quad i = 1, 2$$

and the relations

$$S_i := \{(x_i, \xi_i) \mid \xi = \frac{1}{2} x_i^T Q_{ii} x_i\}, \ i = 1, 2,$$

where Q_{ii} is symmetric and positive semi-definite by (5.18). $S_i \subset \mathcal{X}_i \times \mathcal{X}_{\xi_I}$ clearly fulfills the submanifold property in the sense of Theorem 4.6 with rank $\begin{bmatrix} \frac{\partial S_i}{\partial x_i} & \frac{\partial S_i}{\partial \xi_i} \end{bmatrix}(x_i, \xi_i) = 1$ for all $(x_i, \xi_i) \in S_i$. Moreover, S_i is also invariant since

$$\dot{\xi}_i = \frac{d}{dt}\left(\tfrac{1}{2} x_i^T Q_{ii} x_i\right) = \tfrac{1}{2} x_i^T \left(A_i^T Q_{ii} + Q_{ii} A_i\right) x_i + u_i^T B_i^T Q_{ii} x_i \leq u_i^T y_i$$

due to (5.23). Hence, S_i defines a simulation relation of Σ_i by Ξ_i, $i = 1, 2$. Moreover, S_i is also full since for every x_i there exists a $\xi_i = \frac{1}{2}x_i^T Q_{ii} x_i$ such that $(\xi_i, x_i) \in S_i$. By Corollary 5.4 this means that Σ_i are passive. □

Since $V_i(x_i) = \frac{1}{2}x_i^T Q_{ii} x_i, i = 1, 2$, are storage functions for the passive components Σ_i, their sum

$$V(x_1, x_2) = V_1(x_1) + V_2(x_2) = \frac{1}{2}x_1^T Q_{11} x_1 + \frac{1}{2}x_2^T Q_{22} x_2 \tag{5.24}$$

by Theorem 5.5 is also a storage function of the interconnection $\Sigma_1 \|_o \Sigma_2$. Phrased differently, for every passive interconnection $\Sigma_1 \|_o \Sigma_2$ there always exists a decoupled storage function $V(x_1, x_2)$ of the form (5.24). For lossless systems we can give an even stronger result.

Proposition 5.7. *Consider two linear systems $\Sigma_i, i = 1, 2$, and assume that their interconnection $\Sigma_1 \|_o \Sigma_2$ is lossless. If Σ_i are controllable the storage function $V(x_1, x_2)$ of $\Sigma_1 \|_o \Sigma_2$ is* uniquely *determined as the sum of the storage functions of the components* (5.24) *(modulo a constant).*

Proof. Recall the arguments of the previous proof where inequality in (5.19) and subsequently is replaced by equality. Then (5.21) is equivalent to

$$\begin{aligned} B_1^T Q_{12} = 0, \quad B_1^T Q_{11} = C_1, \quad & A_1^T Q_{11} + Q_{11} A_1 = 0 \\ B_2^T Q_{12}^T = 0, \quad B_2^T Q_{22} = C_2, \quad & A_2^T Q_{22} + Q_{22} A_2 = 0 \\ A_1^T Q_{12} + Q_{12} A_2 = 0, \quad & A_2^T Q_{21} + Q_{21} A_2 = 0 \end{aligned} \tag{5.25}$$

Premultiplying (5.25) with $\begin{bmatrix} B_1^T \\ B_1^T A_1^T \\ \vdots \\ B_1^T (A_1^n)^T \end{bmatrix}$ and $\begin{bmatrix} B_2^T \\ B_2^T A_2^T \\ \vdots \\ (A_1^n)^T \end{bmatrix}$ yields

$$\begin{bmatrix} B_1^T \\ B_1^T A_1^T \\ \vdots \\ B_1^T (A_1^n)^T \end{bmatrix} Q_{12} = 0 \qquad \begin{bmatrix} B_2^T \\ B_2^T A_2^T \\ \vdots \\ B_2^T (A_2^n)^T \end{bmatrix} Q_{21} = 0$$

Thus, if (A_1, B_1) and (A_2, B_2) are controllable, the off-diagonal terms Q_{12}, $Q_{21} = Q_{12}^T$ in (5.16) necessarily vanish which proves that the storage function of $\Sigma_1 \|_o \Sigma_2$ is indeed of the form (5.24). □

Remark 5.8. As stated in Proposition 5.3 the storage function of a lossless and controllable linear system Σ is always unique up to a constant. Hence, if $\Sigma_1 \|_o \Sigma_2$ is controllable and lossless, then it has a unique storage function $V(x_1, x_2)$ of the form (5.24). Observe that the assumption of Proposition 5.7, namely that the components $\Sigma_i, i = 1, 2$, are controllable, is equivalent to the assumption that their interconnection $\Sigma_1 \|_o \Sigma_2$ is controllable, which is needed to prove the same result using passivity theory.

In the previous chapters we observed differences between open and closed interconnections with respect to compositionality. This discrepancy also becomes apparent here as the following proposition shows.

Proposition 5.9. *Consider two passive linear systems $\Sigma_i, i = 1, 2$, and their closed negative feedback interconnection $\Sigma_1 \|_{cl} \Sigma_2$. Then it holds that*

$$\left. \begin{array}{lccc} S_1 : & \Sigma_1 & \preccurlyeq & \Xi_1 \\ S_2 : & \Sigma_2 & \preccurlyeq & \Xi_2 \end{array} \right\} \implies S : \Sigma_1 \|_{cl} \Sigma_2 \preccurlyeq \Xi_1 \|_{cl} \Xi_2 \tag{5.26}$$

The converse of (5.26) does not hold for closed negative feedback interconnection.

Proof. Analogous to Theorem 5.5, construct the relation S from S_1 and S_2 as in (5.12). Since $\Sigma_i, i = 1, 2$ are passive, it holds that $\dot{\xi}_i \leq u_i^T y_i$. For the interconnection $\Xi_1 \|_{\mathrm{cl}} \Xi_2$, we obtain

$$\Xi_1 \|_{\mathrm{cl}} \Xi_2 : \quad \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} \leq \begin{bmatrix} -y_2^T y_1 \\ y_1^T y_2 \end{bmatrix} \tag{5.27}$$

and thus,

$$\dot{\xi}_1 + \dot{\xi}_2 \leq 0 \tag{5.28}$$

Since the inputs of $\Sigma_1 \|_{\mathrm{cl}} \Sigma_2$ are closed by the feedback interconnection, there is no energy flow between the system and its environment, $u^T y = 0$.
For the converse, consider the example of two damped mass-spring systems $\Sigma_i, i = 1, 2$,

$$\Sigma_i : \quad \begin{array}{rcl} \begin{bmatrix} \dot{x}_i^1 \\ \dot{x}_i^2 \end{bmatrix} & = & \begin{bmatrix} 0 & 1 \\ \frac{k_i}{m_i} & \frac{d_i}{m_i} \end{bmatrix} \begin{bmatrix} x_i^1 \\ x_i^2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_i} \end{bmatrix} u_i \\ y_i & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_i^1 \\ x_i^2 \end{bmatrix} \end{array} \quad , i = 1, 2 \tag{5.29}$$

with damping coefficients $d_1 < 0$ in Σ_1 and $d_2 > 0$ in Σ_2 such that $d_1 + d_2 > 0$. Although Σ_1 is not passive, the interconnection $\Sigma_1 \|_{\mathrm{cl}} \Sigma_2$ is passive. Setting $d_1 = -1$, $d_2 = 2$ and $m_1 = m_2 = k_1 = k_2 = 1$, the quadratic storage function $V(x_1, x_2) = \frac{1}{2} \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is given by the numerically determined matrices

$$Q_{11} = \begin{bmatrix} 3.2856 & -5.3163 \\ -5.3163 & 8.6019 \end{bmatrix}, Q_{12} = \begin{bmatrix} -2.0306 & 0 \\ 3.2856 & 0 \end{bmatrix}, Q_{22} = \begin{bmatrix} 1.2250 & 0 \\ 0 & 0 \end{bmatrix}$$

□

Remark 5.10. Lyapunov stability is closely related to passivity. Indeed, a storage function $V(x)$ satisfying the dissipation inequality can serve as a Lyapunov function under certain conditions. In particular, if Σ is passive with storage function $V(x)$ and $V(x)$ has a strict (local) minimum at the equilibrium point $x^\star$, $x^\star$ is stable with Lyapunov function $L(x) = V(x) - V(x^\star)$, cf. [72]. Under this condition stability can then be specified in the same way as passivity, namely as the existence of a full nonlinear simulation relation between the system under consideration and the nonlinear system given by the Lyapunov condition for stability $\frac{d}{dt}V(x) \leq 0$. This holds in particular for closed interconnections where $u^T y = 0$. Thus, Proposition 5.9 can be translated into a result for *compositional stability analysis*. This reasoning is also valid for nonlinear systems as we will show in Corollary 5.13.

In the next step, we generalize the previous results to nonlinear input-affine systems Σ_i of the form (4.1). As an additional assumption, let $x_i = 0$ be an equilibrium of Σ_i, i.e.

$$f_i(0) = 0 \ . \tag{5.30}$$

Derived from the differential dissipation inequality, the Hill-Moylan conditions [33] will be used in the remainder as a characterization of passivity.

Proposition 5.11. *Let Σ be a nonlinear system of the form* (4.1) *and let $V(x)$ be a $\mathcal{C}^1$ storage function of Σ. Then Σ is passive (lossless) if and only if*

$$\begin{aligned} \frac{dV}{dx}(x)f(x) &\leq 0 (= 0) \\ \frac{dV}{dx}(x)g(x) &= h^T(x) \end{aligned} \tag{5.31}$$

We restate the classical result that negative feedback interconnections of passive systems are again passive using compositional reasoning.

Theorem 5.12. *For any two passive (lossless) nonlinear systems $\Sigma_i, i = 1, 2$, passivity (losslessness) is compositional under open negative feedback interconnection.*

Proof. We use again the differential dissipation inequality to prove compositionality of passivity. Assume the two systems $\Sigma_i, i = 1, 2$ are passive, then there exists a simulation relation $\mathcal{S}_1$ of Σ_i by Ξ_i,

$$\mathcal{S}_i := \{(x_i, \xi_i) \mid \xi_i = V(x_i)\} \tag{5.32}$$

where $V(x_i)$ is an unspecified storage function for system Σ_i such that

$$\dot{V}(x_1) \leq u_i^T y_i = u_i h_i^T(x_i) \tag{5.33}$$

The open feedback interconnection $\Sigma_1\|_o\Sigma_2$ is given by

$$\Sigma_1\|_o\Sigma_2 : \quad \begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} f_1(x_1) - g_1(x_1)h_2(x_2) \\ f_2(x_2) + g_2(x_2)h_1(x_1) \end{bmatrix} + \begin{bmatrix} g_1(x_1)e_1 \\ g_2(x_2)e_2 \end{bmatrix} \\ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} h_1(x_1) \\ h_2(x_2) \end{bmatrix} \end{aligned} \tag{5.34}$$

The simulation relation $\mathcal{S}$ of $\Sigma_1\|_o\Sigma_2$ by $\Xi_1\|_o\Xi_2$ is again constructed as

$$\mathcal{S} = \{((x_1, x_2), (\xi_1, x_2)) \mid \xi_1 + \xi_2 = V(x_1) + V(x_2)\} \tag{5.35}$$

since

$$\dot{\xi}_1 + \dot{\xi}_2 = \dot{V}_1(x_1) + \dot{V}_2(x_2) \leq (e_1 - y_2)^T y_1 + (e_2 + y_1)^T y_2 = e_1^T z_1 + e_2^T z_2 \tag{5.36}$$

which proves that $\Sigma_1\|_o\Sigma_2$ is passive. □

We can also formulate compositional reasoning for closed feedback interconnections as.

Corollary 5.13. *For any two passive nonlinear systems $\Sigma_i, i = 1, 2$, it holds that*

$$\left.\begin{array}{lll} S_1: & \Sigma_1 & \preccurlyeq \Xi_1 \\ S_2: & \Sigma_2 & \preccurlyeq \Xi_2 \end{array}\right\} \implies S: \quad \Sigma_1\|_{cl}\Sigma_2 \preccurlyeq \Xi_1\|_{cl}\Xi_2 \tag{5.37}$$

If we assume that the storage functions $V_i(x_i)$ of the nonlinear systems $\Sigma_i, i = 1, 2$, have a strict (local) minimum at $x_i^\star$, we can associate with the systems Ξ_i one-dimensional systems

$$\tilde{\Xi}_i : \quad \dot{\tilde{\xi}}_i \leq 0$$

Using the full nonlinear simulation relations

$$\tilde{S}_i := \{(x_i, \tilde{\xi}_i) \mid \tilde{\xi}_i = V(x_1) - V(x_1^\star)\}$$

Corollary 5.13 can then be interpreted as a result for compositional stability, namely that if the subsystems Σ_i are stable with Lyapunov functions $L(x_i) = V(x_1) - V(x_1^\star)$ then also their closed negative feedback interconnection is stable with Lyapnuov function $L(x_1, x_2) = L(x_1) + L(x_2)$. The converse, however, does not hold.
Theorem 5.12 states that feedback interconnections of passive systems are themselves passive. The converse implication has not yet been investigated in the literature but is equally valid. As shown for the linear case in Proposition 5.6, passivity as a compositional reasoning technique is complete. For the nonlinear case, we obtain the same result.

Theorem 5.14. *Let two nonlinear systems $\Sigma_i, i = 1, 2$, be given such that $\Sigma_1\|_o\Sigma_2$ is passive (lossless). Then also the systems Σ_i are passive (lossless).*

Proof. We will only prove the passive case, the same arguments hold for the lossless case. By Proposition 5.2, the interconnection $\Sigma_1 \|_o \Sigma_2$ being passive is equivalent to the existence of a full simulation relation $\mathcal{S}$ of $\Sigma_1 \|_o \Sigma_2$ by Ξ. Here, Ξ is given by

$$\Xi : \quad \dot{\xi} \leq e_1^T z_1 + e_2^T z_2 \tag{5.38}$$

and the simulation relation S is defined as

$$S = \{((x_1, x_2), \xi) \mid \xi = V(x_1, x_2)\} \tag{5.39}$$

with $V(x_1, x_2)$ an unspecified storage function of $\Sigma_1 \|_o \Sigma_2$. Define

$$V_1(x_1) = V(x_1, 0) \quad , \quad V_2(x_2) = V(0, x_2) \tag{5.40}$$

as candidate storage functions for the component systems $\Sigma_i, i = 1, 2$. From (5.39) we obtain

$$\begin{aligned} \dot{\xi} \quad = \quad & \frac{\partial V}{\partial x_1}(x_1, x_2)\,(f_1(x_1) - g_1(x_1)h_2(x_2) + g_1(x_1)e_1) + \\ & \frac{\partial V}{\partial x_2}(x_1, x_2)\,(f_2(x_2) + g_2(x_2)h_1(x_1) + g_2(x_2)e_2) \leq \\ \leq \quad & e_1^T z_1 + e_2^T z_2 = e_1^T h_1(x_1) + e_2^T h_2(x_2) \end{aligned} \tag{5.41}$$

which is equivalent to

$$\begin{aligned} \frac{\partial V}{\partial x_1}(x_1, x_2)\,(f_1(x_1) - g_1(x_1)h_2(x_2)) + & \\ \frac{\partial V}{\partial x_2}(x_1, x_2)\,(f_2(x_2) + g_2(x_2)h_1(x_1)) \quad & \leq \quad 0 \end{aligned} \tag{5.42}$$

$$\frac{\partial V}{\partial x_1}(x_1, x_2)g_1(x_1) \quad = \quad h_1^T(x_1) \tag{5.43}$$

$$\frac{\partial V}{\partial x_2}(x_1, x_2)g_2(x_2) \quad = \quad h_2^T(x_2) \tag{5.44}$$

Plugging (5.43) and (5.44) in (5.42) results in

$$\begin{aligned} & \frac{\partial V}{\partial x_1}(x_1, x_2)f_1(x_1) - \underbrace{\frac{\partial V}{\partial x_1}(x_1, x_2)g_1(x_1)}_{=h_1^T(x_1)}\,h_2(x_2) + \frac{\partial V}{\partial x_2}(x_1, x_2)f_2(x_2) + \\ & \underbrace{\frac{\partial V}{\partial x_2}(x_1, x_2)g_2(x_2)}_{=h_2^T(x_2)}\,h_1(x_1) = \frac{\partial V}{\partial x_1}(x_1, x_2)f_1(x_1) + \frac{\partial V}{\partial x_2}(x_1, x_2)f_2(x_2) \leq 0 \end{aligned} \tag{5.45}$$

For $x_2 = 0$, (5.45) then becomes

$$\frac{\partial V}{\partial x_1}(x_1, 0)f_1(x_1) + \frac{\partial V}{\partial x_2}(x_1, 0)f_2(0) = \frac{\partial V}{\partial x_1}(x_1, 0)f_1(x_1) = \frac{dV_1}{dx_1}(x_1)f_1(x_1) \leq 0$$

since $f_2(0) = 0$ because of (5.30) while (5.43) becomes

$$\frac{\partial V}{\partial x_1}(x_1, 0)g_1(x_1) = \frac{dV_1}{dx_1}(x_1)g_1(x_1) = h_1^T(x_1) \tag{5.46}$$

Hence, $V_1(x_1) = V(x_1, 0)$ is a storage function for Σ_1 fulfilling the Hill-Moylan conditions (5.31). Similar arguments lead to Σ_2 being passive with storage function $V_2(x_2) = V(0, x_2)$. Consider now the relations $S_i, i = 1, 2$, of Σ_i by Ξ_i,

$$\begin{aligned} S_1 &:= \{(x_1, \xi_1) \mid \xi_1 = V_1(x_1)\} \\ S_2 &:= \{(x_2, \xi_2) \mid \xi_2 = V_2(x_2)\} \end{aligned} \tag{5.47}$$

with Ξ_i as in (5.4). By Proposition 4.6, $S_i, i = 1, 2$, are submanifolds since rank $\begin{bmatrix} \frac{\partial S_i}{\partial x_i} & \frac{\partial S_i}{\partial \xi_i} \end{bmatrix} = 1$ for all $(x_i, \xi_i) \in S_i$. Since $V_i(x_i)$ is a storage function of Σ_i, invariance of the submanifolds S_i is guaranteed. Fullness follows directly since there exists for every x_i a $\xi_i = V_i(x_i)$ such that $(x_i, \xi_i) \in S_i$. Hence, S_i defines a full simulation relation of Σ_i by Ξ_i which is equivalent to Σ_i being passive. □

Like in the linear case, an important implication of Theorem 5.12 is that whenever the open interconnection of two nonlinear systems is passive there exists a decoupled storage function

$$V(x_1, x_2) = V_1(x_1) + V_2(x_2) + c \tag{5.48}$$

where $V_i(x_i)$ are the storage functions of the components $\Sigma_i, i = 1, 2$ and $c \in \mathbb{R}$ a constant. For interconnections involving lossless components, we obtain even stronger results under accessibility assumptions.

Definition 5.15. Consider a nonlinear system Σ of the form (4.1). Then the *accessibility algebra* $\mathcal{C}$ is the smallest subalgebra of the Lie algebra of vector fields on $\mathcal{X}$ that contains f and all input vector fields $[g_1, \ldots, g_p] =: g$. Define $\mathcal{C}_0$ as the smallest subalgebra containing g and satisfying $[f, X] \in \mathcal{C}_0$ for all $X \in \mathcal{C}_0$.
Σ is *locally strongly accessible* if the reachable set $R_T^V(x_0) = \cup_{\tau \leq T} R^V(x_0, \tau)$,

$$\begin{aligned} R^V(x_0, T) &= \{(x \in \mathcal{X} \mid \exists u : [0, T] \to \mathcal{U} \text{ s. t. } x(t) \in V, 0 \leq t \leq T, \\ &\qquad x(0) = x_0, x(T) = x\} \end{aligned}$$

for all $x_0 \in \mathcal{X}$ contains a non-empty open set of $\mathcal{X}$ for all neighborhoods V of x_0 and any sufficiently small $T > 0$.
Σ is *reachable* from x_0 if $R_T^V(x_0) = \mathcal{X}$ for some $T \geq 0$.

As shown in [54], every element of the subalgebra $\mathcal{C}_0$ is a linear combination of repeated Lie brackets $[X_k, [X_{k-1}, [\ldots, [X_1, g] \ldots]]], k = 0, 1, \ldots$.

Proposition 5.16. *Let Σ be a nonlinear system of the form* (4.1)*. If Σ is locally strongly accessible then* $\dim(\mathrm{span}\{X(x_0) \mid X \in \mathcal{C}_0\}) = n = \dim\mathcal{X}$ *for x_0 in an open and dense subset of $\mathcal{X}$.*

We are now able to state the first result concerning the negative feedback interconnection of a passive and a lossless component.

Proposition 5.17. *Consider two nonlinear systems $\Sigma_i, i = 1,2$ of the form* (4.1) *and let Σ_1 be passive and Σ_2 lossless. Assume that Σ_1 is locally strongly accessible. Then all storage functions $V(x_1, x_2)$ of the interconnection $\Sigma_1\|_o\Sigma_2$ are of the form $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ where $V_1(x_1)$ is a storage function of Σ_1 and $V_2(x_2)$ the unique storage function of Σ_2. The same also holds when interchanging Σ_1 and Σ_2.*

Proof. Since Σ_1 is passive and Σ_2 lossless, the interconnection $\Sigma_1\|_o\Sigma_2$ by Theorem 5.12 is also passive with storage function $V(x_1, x_2)$. Rewriting the dissipation inequality, this implies

$$L_{f_1}V + L_{g_1}V(e_1 - y_2) + L_{f_2}V + L_{g_2}V(e_2 + y_1) = e_1^T y_1 + e_2^T y_2 - W$$

with $W = W(x_1)$ a nonnegative function of x_1 or, equivalently,

$$\begin{aligned} L_{f_1}V + L_{f_2}V + W(x_1) &= 0 \\ L_{g_1}V &= h_1^T \\ L_{g_2}V &= h_2^T \end{aligned}$$

We claim that $L_X V$ is a function of x_1 only for all $X \in \mathcal{C}_0^1$. Clearly, $L_{g_1}V = h_1^T$ is a function of x_1. Moreover, $L_{[f_1,g_1]}V = L_{f_1}L_{g_1}V - L_{g_1}L_{f_1}V = L_{f_1}h_1^T + L_{g_1}L_{f_2}V + L_{g_1}W$ is a function of x_1 only since $L_{g_1}L_{f_2}V = L_{f_2}L_{g_1}V = L_{f_2}h_1^T = 0$. In fact, $L_{X_i}L_{X_j}V = L_{X_j}L_{X_i}V, (i,j) \in \{(1,2),(2,1)\}$ due to $[f_i, g_j] = 0$. Assume now that $L_{[X_k,[X_{k-1},[\ldots,[X_1,g]\ldots]]]}V$ is a function of x_1 only denoted by $R(x_1)$. To complete the induction step, consider first the case $X_{K+1} = g_1$. Then

$$\begin{aligned} L_{[g_1,[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]]}V = L_{g_1}L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}V - \\ L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}L_{g_1}V = L_{g_1}R(x_1) - L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}L_{g_1}V \\ = L_{g_1}R(x_1) - L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}h_1^T \end{aligned}$$

is a function of x_1 only since all $X_i, i = 1, 2, \ldots,$ depend on x_1 only. If $X_{k+1} = f_1$, then

$$\begin{aligned} L_{[f_1,[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]]}V = L_{f_1}L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}V - \\ L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}L_{f_1}V = L_{f_1}R(x_1) - L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}L_{f_1}V \\ = L_{f_1}R(x_1) + L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}(L_{f_2}V + W(x_1)) = \\ L_{f_1}R(x_1) + L_{f_2}L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}V + L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}W(x_1) \\ = L_{f_1}(x_1) + L_{f_2}R_{x_1} + L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}W(x_1) \end{aligned}$$

is also a function of x_1 only. Thus, $L_{\mathcal{C}_0^1}V$ is indeed a function of x_1 only, i.e.

$$\frac{\partial}{\partial x_2}\left\{L_{g_1}V, L_{[f_1,g_1]}V, \ldots, L_{[X_k,[X_{k-1},[\ldots,[X_1,g_1]\ldots]]]}V\right\} = 0 \tag{5.49}$$

Since Σ_1 is locally strongly accessible, $\dim\left(\operatorname{span}\{X_1(x_1) \mid X_1 \in \mathcal{C}_0^1\}\right) = n_1$ for x_1 in an open and dense subset of $\mathcal{M}_1$. By continuity of $V(x_1, x_2)$, (5.49) thus implies that $\frac{\partial^2}{\partial x_1 \partial x_2}V(x_1, x_2) = 0$ and thus the storage function $V(x_1, x_2)$ of $\Sigma_1\|_{\text{o}}\Sigma_2$ is of the form $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ up to a constant. As a consequence of Theorem 5.14, $V_i(x_i)$ are storage functions of $\Sigma_i, i = 1, 2$. Since Σ_2 is lossless, the storage function $V_2(x_2)$ is unique up to a constant. □

Next, we consider interconnections of lossless systems to obtain an even stronger result.

Proposition 5.18. *Consider two nonlinear systems $\Sigma_i, i = 1, 2$ of the form* (4.1)*. Let $\Sigma_i, i = 1, 2$, be lossless and locally strongly accessible. Then every storage function $V(x_1, x_2)$ of the interconnection $\Sigma_1\|_{\text{o}}\Sigma_2$ is of the form* (5.48)*.*

Proof. Observe first that by Theorems 5.12 and 5.14, $\Sigma_1\|_{\text{o}}\Sigma_2$ being lossless with storage function $V(x_1, x_2)$ is equivalent to both $\Sigma_i, i = 1, 2$ being lossless with storage functions $V_i(x_i)$. Furthermore, $\Sigma_1\|_{\text{o}}\Sigma_2$ being lossless implies by (5.42) – (5.44) that

$$\begin{aligned} L_{f_1}V(x_1, x_2) + L_{f_2}V(x_1, x_2) = 0, \\ L_{g_1}V(x_1, x_2) = h_1^T(x_1), \quad L_{g_2}V(x_1, x_2) = h_2^T(x_2) \end{aligned} \tag{5.50}$$

We want to show that

$$L_{g_i}V, L_{[f_i,g_i]}V, \ldots, L_{[X_1^i,[X_2^i,\ldots,[X_k^i,X_{K+1}^i],\ldots]]}V, \quad i = 1, 2, \tag{5.51}$$

are functions of x_i only for $i = 1, 2$ and all $X_j^i, j \in k$ from the set $\{f_i, g_i\}$, $k \geq 1$. Clearly, $L_{g_i}V = h_i^T$ is a function of x_i only. The proof that also $L_{[X_1^i,[X_2^i,\ldots,[X_k^i,X_{K+1}^i],\ldots]]}V$ is a function of x_i only relies on the same arguments as used in the proof of Proposition 5.17. Hence, differentiation of (5.51) with respect to x_j yields

$$\frac{\partial}{\partial x_j}\left\{L_{g_i}V, L_{[f_i,g_i]}V, \ldots, L_{[X_k,[X_{k-1},[\ldots,[X_1,g_i]\ldots]]]}V\right\} = 0, \qquad (i, j) \in \{(1, 2), (2, 1)\}\,. \tag{5.52}$$

Since Σ_i are locally strongly accessible, $\mathcal{C}_0^i(x)$ has full rank for x in an open and dense subset of $\mathcal{M}_i$. Hence, 5.52 implies by continuity of $V(x_1, x_2)$ that $\frac{\partial^2 V}{\partial x_i \partial x_j} = 0, (i, j) \in \{(1, 2), (2, 1)\}$. Hence, any storage function $V(x_1, x_2)$ of $\Sigma_1\|_{\text{o}}\Sigma_2$ is of the form $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ where $V_i(x_i)$ by Theorem 5.14 is a storage function of Σ_i. □

A result similar to Proposition 5.18 can be obtained using arguments from passivity theory. In fact, it is a well-known fact that if $\Sigma_i, i = 1, 2$, is reachable from at least one point $x_i^\star$ it has a unique storage function $V_i(x_i)$. But then the open feedback interconnection $\Sigma_1\|_o\Sigma_2$ is reachable from $(x_1^\star, x_2^\star)$ using the inputs $e_1 = u_1 + h_2(x_2), e_2 = u_2 - h_1(x_1)$ and thus $\Sigma_1\|_o\Sigma_2$ has a unique storage function $V(x_1, x_2)$ as well. Theorem 5.12 tells us that $V(x_1, x_2)$ is given as the sum of the unique storage functions $V_i(x_i)$.

Finally, we present a passivity result for interconnections of an arbitrary system with a passive one.

Corollary 5.19. *Consider a nonlinear system* Σ *of the form* (4.1) *and two systems* Ξ_2, Ξ *of the form* (5.4). *Then the following statements hold:*

1. *If* Ξ_2 *is lossless then*

$$\Sigma\|_o\Xi_2 \preccurlyeq \Xi \quad \Longrightarrow \quad \exists\, \Xi_1 : \Sigma \preccurlyeq \Xi_1 \tag{5.53}$$

with Ξ_1 *of the form* (5.4).

2. *If* Ξ_2 *is passive and the simulation relation of* $\Sigma\|_o\Xi_2$ *by* Ξ *is such that*

$$\frac{\partial \xi}{\partial \xi_2} \geq 0, \xi = \xi(x, \xi_2), \tag{5.54}$$

then (5.53) *holds.*

Proof. Let Ξ be a system with state variable ξ such that $\dot{\xi} \leq e_1^T y_1 + e_2^T y_2$. According to (5.53) there exists a full simulation relation $\mathcal{S}$ of $\Sigma\|_o\Xi_2$ by Ξ. By Theorem 4.6, $\mathcal{S}$ can be written as

$$\mathcal{S} = \{(x, \xi_2, \xi) \mid \xi = \xi(x, \xi_2)\} \,. \tag{5.55}$$

Since (5.55) defines a nonlinear simulation relation, it has to hold for all (x, ξ_2, ξ) $\in \mathcal{S}$ and all e_1, e_2 that

$$\dot{\xi} = L_f\xi + L_g\xi(e_1 - y_2) + \frac{\partial \xi}{\partial \xi_2}\dot{\xi}_2 y_2 \leq e_1^T y_1 + e_2^T y_2 \tag{5.56}$$

Consider first the case that Ξ_2 is lossless. Then the system $\Sigma\|_o\Xi_2$ is given by

$$\Sigma\|_o\Xi_2 : \begin{array}{rcl} \dot{x} & = & f(x) - g(x)y_2 + g(x)e_1 \\ \dot{\xi}_2 & = & y_1^T y_2 + e_2^T y_2 \end{array} \tag{5.57}$$

Hence, (5.56) can be rewritten as

$$\dot{\xi} = L_f\xi + L_g\xi(e_1 - y_2) + \frac{\partial \xi}{\partial \xi_2}(y_1^T + e_2^T)y_2 \leq e_1^T y_1 + e_2^T y_2 \tag{5.58}$$

Since (5.56) has to hold for all $e_i, i = 1, 2,$, it follows that

$$\begin{aligned} L_f\xi - L_g\xi y_2 + \frac{\partial \xi}{\partial \xi_2} y_1^T y_2 &\leq 0 \\ L_g\xi &= h_1^T \\ \frac{\partial \xi}{\partial \xi_2} y_2 &= y_2 \end{aligned} \tag{5.59}$$

Conditions (5.59) are also obtained if Ξ_2 is passive and (5.54) holds. Indeed, (5.58) in this aces can be rewritten as

$$\begin{aligned} \dot{\xi} = L_f\xi + L_g\xi(e_1 - y_2) + \frac{\partial \xi}{\partial \xi_2}\dot{\xi}_2 y_2 \leq L_f\xi + L_g\xi(e_1 - y_2) + \frac{\partial \xi}{\partial \xi_2}(y_1^T + e_2^T)y_2 \\ \leq e_1^T y_1 + e_2^T y_2 \end{aligned}$$

The last line of (5.59) yields $\frac{\partial \xi}{\partial \xi_2} = 1$. Note that this is consistent with assumption (5.54) in the second case. Hence, the storage function ξ can be written as $\xi(x, \xi_2) = V(x) + \xi_2$. Substituting the last line of (5.59) into the first one results in

$$L_f V \leq 0, L_g V = h_1^T$$

which by Proposition 5.11 means that Σ is passive with storage function $V(x)$. Thus, Ξ_1 is given by

$$\Xi_1 : \xi_1 = V(x),\ \dot{\xi}_1 \leq u_1^T y_1$$

□

Remark 5.20. Corollary 5.19 states that if the open feedback interconnection of an arbitrary passive system with another arbitrary system Σ is passive, then Σ has to be passive as well. A similar result was presented in [15], namely if the *closed* negative feedback interconnection of an arbitrary passive system and Σ is stable then Σ must be passive.

5.3. Outlook

We showed in this chapter that passivity can be expressed as a nonlinear simulation relation between the system under consideration and the one-dimensional system given by the dissipation inequality. Based on this, we applied compositional analysis techniques to verify passivity properties of linear and nonlinear systems.

The close link between passivity and Lyapunov stability theory hints at further investigations to obtain similar result especially for closed feedback interconnections. In particular, assume-guarantee schemes could be applied to check stability properties of nonlinear systems. Consider as an example two nonlinear systems $\Sigma_i, i = 1, 2$, and two systems Ξ_i of the form (5.4). Assume that both systems Σ_i can be stabilized by interconnection with $\Xi_j, (i, j) \in \{(1, 2), (2, 1)\}$, i.e.

$$\begin{aligned} \Sigma_1 \|_{\text{cl}} \Xi_2 &\preccurlyeq \Xi_1 \|_{\text{cl}} \Xi_2 \\ \Xi_1 \|_{\text{cl}} \Sigma_2 &\preccurlyeq \Xi_1 \|_{\text{cl}} \Xi_2 \end{aligned} \tag{5.60}$$

Under which conditions is the closed negative feedback interconnection of $\Sigma_1 \|_{\text{cl}} \Sigma_2$ stable,

$$\Sigma_1 \|_{\text{cl}} \Sigma_2 \overset{?}{\preccurlyeq} \Xi_1 \|_{\text{cl}} \Xi_2 \tag{5.61}$$

A similar example is mentioned in Remark 5.20, which shows that compositional reasoning for stability properties could have many practical applications, for example in robotics.