Compositional analysis and control of dynamical systems

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Simulation relations and compositional analysis for nonlinear systems

Compositional reasoning for nonlinear systems generalizes the main results obtained for linear systems in the previous chapter. To start with, the idea to formulate the existence of a simulation relation between two systems as a geometric control problem can be adopted to the nonlinear case, as outlined in [74]. However, the differential geometric characteristics of submanifolds impose additional technical conditions compared to the linear case. For example, nonlinear simulation is in general (without extra technical conditions) not transitive. Constant rank assumptions on the relations between the defining submanifolds have to be made in order to ensure that nonlinear simulation is a preorder, and that compositional and assume-guarantee reasoning is sound.

4.1. Nonlinear simulation relations

Consider the class of input affine nonlinear systems Σ

$$\Sigma_i: \quad \begin{array}{rcl} \dot{x}_i & = & f_i(x_i) + g_i(x_i)u_i \\ y_i & = & h_i(x_i), \end{array} \qquad x_i \in \mathcal{X}_i, \quad u_i \in \mathcal{U}_i, \quad y_i \in \mathcal{Y}_i, \tag{4.1}$$

with $\mathcal{X}_i$ an n_i- dimensional manifold, and $\mathcal{U}_i$ and $\mathcal{Y}_i$ the linear input and output spaces of dimension m_i and p_i, respectively. We assume smoothness of all the vector fields f_i, g_i and mappings h_i.
Nonlinear simulation relations are treated in several publications, see [74, 69, 24] for an overview. Here we follow [74] where simulation relations are defined as regular submanifolds.

Definition 4.1. A subset S of a manifold N of dimension n is a regular submanifold of dimension k if for every $p \in S$ there is a coordinate neighborhood $(U, \varphi) = (U, x^1, \ldots, x^n)$ of N around p such that

$$U \cap S := \{q \in U \mid x^i(p) = x^i(q),\ i = k+1, \ldots, n\}$$

This implies that S is a manifold in its own right, with coordinate chart $\phi_{|S}$ on $U \cap S$ given by

$$\phi_{|S} = (x^1, \ldots, x^k, 0, \ldots, 0)$$

From now on, "submanifold" will always mean "regular submanifold". We define nonlinear simulation relations as the following invariance condition [74]:

Definition 4.2. A submanifold $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2, $\Sigma_i, i = 1, 2$, as defined in (4.1), if and only if for all $(x_1, x_2) \in S$ and all $u \in \mathcal{U}$ the following properties are satisfied:

$$\begin{aligned} &(1) \quad (f_1(x_1) + g_1(x_1)u, f_2(x_2) + g_2(x_2)u) \in T_{(x_1,x_2)}S \\ &(2) \quad h_1(x_1) = h_2(x_2) \end{aligned} \tag{4.2}$$

where $T_{(x_1,x_2)}S$ denotes the tangent space to S at the point $(x_1, x_2) \in S$. S is called a *full* simulation relation, denoted by $\Sigma_1 \preccurlyeq \Sigma_2$, if $\Pi_1 S = \mathcal{X}_1$. In this case, Σ_1 *is simulated by* Σ_2.

Definition 4.2 can be reformulated as

Proposition 4.3. *A submanifold $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if for all $(x_1, x_2) \in S$ the following properties hold:*

$$\begin{aligned} (i) & \quad \begin{bmatrix} f_1(x_1) \\ f_2(x_2) \end{bmatrix} \in T_{(x_1,x_2)}S \\ (ii) & \quad \mathrm{im} \begin{bmatrix} g_1(x_1) \\ g_2(x_2) \end{bmatrix} \subset T_{(x_1,x_2)}S \\ (iii) & \quad h_1(x_1) = h_2(x_2) \end{aligned} \tag{4.3}$$

Remark 4.4. Note that we defined nonlinear bisimulation relations directly as the invariance condition (4.2), and not as a trajectory-based notion as in the linear case. It is easily seen that a trajectory-based notion of bisimulation *implies* the invariance condition. For the converse, however, we need an extra constant rank assumption, see [24] for details.

4.2. Compositional analysis for nonlinear systems

Like in Section 3.2.2, open and closed feedback interconnections are defined in the following way: For any two nonlinear systems $\Sigma_i, i = 1, 2$, let

- $\Sigma_1 \|_{\mathrm{o}} \Sigma_2$ denote the *open* feedback interconnection where

$$u_1 = y_2 + e_1 \quad , \quad u_2 = y_1 + e_2$$

and

- $\Sigma_1 \|_{\mathrm{cl}} \Sigma_2$ denote the *closed* feedback interconnection where

$$u_1 = y_2 \quad , \quad u_2 = y_1$$

To develop a theory for nonlinear compositional reasoning, we collect some important facts from differential geometry that can be found in any textbook such as [65].

Proposition 4.5. *Let $S_i, i = 1, 2$, be regular submanifolds of the manifolds M_i with dimensions r_i. Then $S_1 \times S_2$ is a regular submanifold of $M_1 \times M_2$ of dimension $r_1 + r_2$.*

Given how simulation relations of nonlinear systems are characterized by Proposition 4.3, the following theorem assures that under a regularity assumption, zero level sets of smooth functions define smooth submanifolds.

Theorem 4.6. *Let $f_1, \ldots, f_m, m \leq n$, be smooth functions on $\mathbb{R}^n$. Define*

$$N := \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_m(x) = 0\} \tag{4.4}$$

and assume $N \neq \varnothing$. Suppose that

$$\operatorname{rank}\left[\frac{\partial f_i}{\partial x_j}(x)\right] = m, \quad \forall\, x \in N \tag{4.5}$$

Then N is a smooth submanifold of $\mathbb{R}^n$ of dimension $n - m$. Moreover, all submanifolds of $\mathbb{R}^n$ can be represented in this way.

In the remainder, we assume that all simulations relations will be defined by submanifolds of the form (4.4). The following lemma, taken from [71], shows how by means of a coordinate transform, variables can be partially eliminated from a set of nonlinear equations without changing the solution set.

Lemma 4.7 (compare with Lemma 2.1 in [71])**.** *Let $f_1(x, z), \ldots, f_m(x, z)$ be smooth functions of $x \in \mathbb{R}^n$, $z \in \mathbb{R}^k$ and let $(a, b) \in \mathbb{R}^n \times \mathbb{R}^m$ be such that $f_1(a, b) = \ldots = f_m(a, b) = 0$. Suppose that*

$$\operatorname{rank}\left[\frac{\partial f_i}{\partial x_j}(x, z)\right]_{\substack{i=1,\ldots,m,\\ j=1,\ldots,n}} = m - p,\ \forall\ (x, z) \in U(a, b),$$

with $U(a,b)$ are neighborhood of (a,b).
Then there exist functions $\psi_1(y,z),\ \ldots,\ \psi_p(y,z)$, $y \in \mathbb{R}^m,\ z \in \mathbb{R}^k$, defined around a neighborhood $V(0,b)$ and independent as functions of y, such that

$$f_1(x,z) = \cdots = f_m(x,z) = 0 \qquad (4.6)$$
$$\Longleftrightarrow$$
$$\begin{cases} \psi_i(f_1(x,z),\ldots,f_m(x,z),z) = 0, & i = 1,\ldots,p \\ f_j(x,z) = 0, & j = p+1,\ldots,m \end{cases}$$

for all (x,z) around (a,b) where

$$\frac{\partial \psi_i}{\partial x_j}(f_1(x,z),\ldots,f_m(x,z),z) = 0\ ,\ i = 1,\ldots,p,\ j = 1,\ldots,n$$

and, after permuting the functions $f_1,\ldots,f_m$,

$$\operatorname{rank}\left[\ \frac{\partial f_{p+i}}{\partial x_j}\ \right]_{\substack{i=1,\ldots,m-p \\ j=1,\ldots,m}} = m - p\ .$$

Remark 4.8. We want to illustrate Lemma 4.7 and its use for nonlinear simulation theory with the following example. Consider three linear systems $\Sigma_i, i = 1,2,3$, and suppose that S_{12} and S_{23} are linear simulation relations of Σ_1 by Σ_2 and of Σ_2 by Σ_3, respectively, each given as

$$\begin{aligned} S_{12} &= \{(x_1,x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid F_1x_1 + F_2x_2 = 0,\ F_i \in \mathbb{R}^{f \times n_i},\ i = 1,2\} \\ S_{23} &= \{(x_2,x_3) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \mid G_2x_2 + G_3x_3 = 0,\ G_j \in \mathbb{R}^{g \times n_j},\ j = 2,3\}\ . \end{aligned}$$

Combining S_{12} and S_{23} as a system of $f + g$ linearly independent equations we obtain

$$S(x_1,x_2,x_3) = \left\{ (x_1,x_2,x_2) \mid \begin{bmatrix} F_1 & F_2 & 0 \\ 0 & G_2 & G_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}. \qquad (4.7)$$

Assume now that $\operatorname{rank}\begin{bmatrix} F_2 \\ G_2 \end{bmatrix} = f + g - p$. Then there exists an annihilating matrix $\psi = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \in \mathbb{R}^{p \times (f+g)}$ of rank p such that

$$\psi(S(x_1,x_2,x_3)) = \{(x_1,x_3) \mid M_1F_1x_1 + M_2G_3x_3 = 0\} \qquad (4.8)$$

since

$$\begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} F_2 \\ G_2 \end{bmatrix} = 0\ .$$

Hence, (4.8) is independent of x_2. In fact, (4.8) defines the subspace $S_{13} := \{(x_1,x_3) \mid \exists x_2 :\ (x_1,x_2) \in S_{12}, (x_2,x_3) \in S_{23}\}$ and is therefore a simulation relation of Σ_1 by Σ_3. Moreover, the solution subspace of (4.7) is equivalent to the solution of (4.8) combined with $f+g-p$ equations of (4.7). If the systems $\Sigma_i, i = 1,2,3$, and the corresponding simulations are nonlinear, Lemma 4.7 can be applied accordingly to eliminate the variables x_2. However, the resulting system $\psi_i(f(x_1,x_2), g(x_2,x_3), x_1, x_3) = 0,\ i = 1,\dots,p$, need not have submanifold properties as Example 4.14 will show.

The first main result for nonlinear compositional analysis is that nonlinear simulation relations – as their linear counterparts – are compositional.

Theorem 4.9. *Consider four nonlinear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ of the form* (4.1). *Assume there exists full nonlinear simulation relations S_i of Σ_{P_i} by $\Sigma_{Q_i}, i = 1,2$. Then nonlinear simulation is compositional under both open and closed feedback, i.e.*

$$\left.\begin{array}{lclcl} S_1: & \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_2: & \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_2} \end{array}\right\} \implies S:\ \Sigma_{P_1}\|_{o,cl}\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_{o,cl}\Sigma_{Q_2} \tag{4.9}$$

Proof. Construct the nonlinear simulation relation S by taking the product $S_1 \times S_2$ with the state variables reordered afterwards,

$$S = \{x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_1}) \in S_1, (x_{P_2}, x_{Q_2}) \in S_2\}$$

Since S_1 and S_2 are both submanifolds, their product also defines a submanifold according to Proposition 4.5. Condition 2 in Definition 4.2 is fulfilled by construction while

$$\begin{bmatrix} f_{P_1}(x_{P_1}) + g_{P_1}(x_{P_1})u_1 \\ f_{Q_1}(x_{Q_1}) + g_{Q_1}(x_{Q_1})u_2 \\ f_{P_2}(x_{P_2}) + g_{P_2}(x_{P_2})u_1 \\ f_{Q_2}(x_{Q_2}) + g_{Q_2}(x_{Q_2})u_2 \end{bmatrix} \in T_{x_{P_1}x_{Q_1}}S_1 \times T_{x_{P_2}x_{Q_2}}S_2$$

where $u_1 = e_1 + h_{P_2}(x_{P_2}) = e_1 + h_{Q_2}(x_{Q_2})$, $u_2 = e_2 + h_{P_1}(x_{P_1}) = e_2 + h_{Q_1}(x_{Q_1})$ for open and $u_1 = h_{Q_2}(x_{Q_2}) = h_{Q_2}(x_{Q_2})$, $u_2 = h_{Q_1}(x_{Q_1}) = h_{P_1}(x_{P_1})$ for closed feedback interconnections, respectively. Fullness of S follows from taking the construction as the product of S_1 and S_2, both full relations themselves. □

The converse of Theorem 4.9 does not hold in general as we know already from the linear case, see Remark 3.7. For open feedback interconnections of nonlinear systems, however, compositional reasoning is again, as in the linear case (cf. Proposition 3.18), complete. To obtain this result, the maximal simulation relation of two open feedback interconnections has to be expressed explicitly. Recall that as a consequence of Proposition 2.13, the maximal simulation relation of two linear systems without disturbance inputs d_i equals the unobservability subspace of their augmented system. The nonlinear version of this result has been proved in [74].

Proposition 4.10. *Let $\Sigma_i, i = 1, 2$, be two nonlinear systems of the form* (4.1). *A submanifold S is a simulation relation of Σ_1 by Σ_2 if and only if all $(x_1, x_2) \in S$ satisfy*

$$L_{X_1^1} L_{X_1^2} \dots L_{X_1^k} h_1(x_1) = L_{X_2^1} L_{X_2^2} \dots L_{X_2^k} h_2(x_2) \; , k = 1, 2 \dots , \tag{4.10}$$

where $X_i^j \in \{f_i, g_i\}, i = 1, 2, \; j = 1, 2, \dots$.
Moreover, the maximal bisimulation relation $S^\star$ is given by

$$S^\star \quad = \quad \{(x_1, x_2) \mid (x_1, x_2) \text{ satisfy } (4.10)\} \tag{4.11}$$

Condition (4.10) can be interpreted as the nonlinear analogon of the equality of the Markov parameters (2.8). Proposition 4.10 now allows us to explicitly compute the maximal simulation relation for open feedback interconnections of nonlinear systems.

Lemma 4.11. *Let $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$, be four nonlinear input affine systems of the form* (4.1) *such that*

$$\Sigma_{P_1} \|_o \Sigma_{P_2} \quad \preccurlyeq \quad \Sigma_{Q_1} \|_o \Sigma_{Q_2} \tag{4.12}$$

Then the maximal simulation relation $S^\star$ of $\Sigma_{P_1} \|_o \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_o \Sigma_{Q_2}$ is given by

$$\begin{aligned} S^\star \quad = \quad & \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \\ & L_{X^1_{P_1}} \dots L_{X^k_{P_1}} h_{P_1}(x_{P_1}) = L_{X^1_{Q_1}} \dots L_{X^k_{Q_1}} h_{Q_1}(x_{Q_1}), \\ & L_{X^1_{P_2}} \dots L_{X^k_{P_2}} h_{P_2}(x_{P_2}) = L_{X^1_{Q_2}} \dots L_{X^k_{Q_2}} h_{Q_2}(x_{Q_2}), \; k = 1, 2, \dots\} \end{aligned} \tag{4.13}$$

Proof. By Proposition 4.10, the maximal simulation relation $S^\star$ of $\Sigma_{P_1} \|_o \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_o \Sigma_{Q_2}$ is the set of points

$$\begin{aligned} S^\star = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) | & L_{X^1_{P_1 P_2}} \dots L_{X^k_{P_1 P_2}} h_{P_1 P_2}(x_{P_1}, x_{P_2}) \\ & = L_{X^1_{Q_1 Q_2}} \dots L_{X^k_{Q_1 Q_2}} h_{Q_1 Q_2}(x_{Q_1}, x_{Q_2})\} \end{aligned}$$

where $X_{j_1 j_2} \in \left\{ \begin{bmatrix} f_{j_1} + g_{j_1} h_{j_2} \\ f_{j_2} + g_{j_2} h_{j_1} \end{bmatrix}, \begin{bmatrix} g_{j_1} \\ g_{j_2} \end{bmatrix} \right\}, h_{j_1 j_2} = \begin{bmatrix} h_{j_1} \\ h_{j_2} \end{bmatrix}, j \in \{P, Q\}$. We claim that the repeated Lie derivatives in (4.14) can be split into smooth functions depending on either x_{P_1}, x_{Q_1} or x_{P_2}, x_{Q_2}, respectively. Start with condition (iii) in Theorem 4.3, i.e.

$$h_{P_1}(x_{P_1}) = h_{Q_1} x_{Q_1} \quad , \quad h_{P_2}(x_{P_2}) = h_{Q_2} x_{Q_2} \tag{4.14}$$

The first time derivative becomes

$$L_{f_{P_1} + g_{P_1} h_{P_2}} h_{P_1} + L_{g_{P_1}} h_{P_1} \cdot e_1 = L_{f_{Q_1} + g_{Q_1} h_{Q_2}} h_{Q_1} + L_{g_{Q_1}} h_{Q_1} \cdot e_1$$

which is equivalent to

$$L_{f_{P_1}} h_{P_1} - L_{f_{Q_1}} h_{Q_1} = (L_{g_{Q_1}} h_{Q_1} - L_{g_{P_1}} h_{P_1})(h_{P_2} + e_1) \tag{4.15}$$

using the fact that $h_{P_2} = h_{Q_2}$. Since (4.15) has to hold for all e_1, we obtain

$$L_{f_{P_1}} h_{P_1} = L_{f_{Q_1}} h_{Q_1} \tag{4.16}$$

from setting $e_1 = h_{P_2}$ and consequently also

$$L_{g_{P_1}} h_{P_1} = L_{g_{Q_1}} h_{Q_1} \tag{4.17}$$

Similarly, the second equation in (4.14) yields

$$L_{f_{P_2}} h_{P_2} = L_{f_{Q_2}} h_{Q_2} \quad , \quad L_{g_{P_2}} h_{P_2} = L_{g_{Q_2}} h_{Q_2} \tag{4.18}$$

Further time derivatives of (4.16), (4.17) and (4.17) result in (4.13). □

Remark 4.12. For linear systems without disturbance inputs d_i, Example 3.19 contains a similar result as (4.13).

Lemma 4.11 means that the submanifold $S^\star$ defining the maximal simulation relation of $\Sigma_{P_1} \|_o \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_o \Sigma_{Q_2}$ can be written as the zero-level set of smooth functions

$$\begin{aligned} v &: \mathcal{X}_{P_1} \times \mathcal{X}_{Q_1} \to \mathbb{R}^v \\ w &: \mathcal{X}_{P_2} \times \mathcal{X}_{Q_2} \to \mathbb{R}^w \end{aligned} \tag{4.19}$$

as

$$\begin{aligned} S^\star \quad = \quad & \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) | v_i(x_{P_1}, x_{Q_1}) = 0,\ i = 1, \ldots, v, \\ & \qquad w_j(x_{P_2}, x_{Q_2}) = 0,\ j = 1, \ldots, w\}\,. \end{aligned} \tag{4.20}$$

Since the maximal simulation relation of the interconnections $\Sigma_{P_1} \|_o \Sigma_{P_2}$ and $\Sigma_{Q_1} \|_o \Sigma_{Q_2}$ can be constructed simply as the product of the maximal simulation relations of Σ_{P_1} by Σ_{Q_1} and of Σ_{P_2} by Σ_{Q_2}, the converse also holds true, i.e. that the simulation relations of the components can be reconstructed from $S^\star$.

Proposition 4.13. *Consider four nonlinear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ such that*

$$\Sigma_{P_1} \|_o \Sigma_{P_2} \quad \preccurlyeq \quad \Sigma_{Q_1} \|_o \Sigma_{Q_2} \tag{4.21}$$

with the maximal simulation relation $S^\star$ of $\Sigma_{P_1} \|_o \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_o \Sigma_{Q_2}$ given by (4.20)*. Then there exist full simulation relations $S_i, i = 1, 2,$ of Σ_{P_1} by Σ_{Q_1} and of Σ_{P_2} by Σ_{Q_2}, respectively.*

Proof. Construct the relations $S_i, i = 1, 2$, as follows:

$$\begin{aligned} S_1 &= \{(x_{P_1}, x_{Q_1}) \mid v_i(x_{P_1}, x_{Q_1}) = 0 \ , \ i = 1, \dots, v\} \\ S_2 &= \{(x_{P_2}, x_{Q_2}) \mid w_j(x_{P_2}, x_{Q_2}) = 0 \ , \ j = 1, \dots, w\} \end{aligned}$$

where v_i and w_j are the smooth functions defining $S^\star$. By (4.20), $S^\star$ is defined as the zero-level set of smooth functions v_i, w_j, which according to Theorem 4.6 means that $S^\star$ is a submanifold. Hence, the rank of the Jacobian Jac$S^\star$ is constant:

$$\text{rank} \begin{bmatrix} \frac{\partial f_i}{\partial x_{P_1}} & 0 & \frac{\partial f_i}{\partial x_{Q_1}} & 0 \\ 0 & \frac{\partial g_j}{\partial x_{P_2}} & 0 & \frac{\partial g_j}{\partial x_{Q_2}} \end{bmatrix}_{\substack{i=1,\dots,l, \\ j=1,\dots,m}} (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) = \text{const} \tag{4.22}$$

for all $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S^\star$. Furthermore, (4.22) is equivalent to

$$\begin{aligned} \text{rank} \begin{bmatrix} \frac{\partial f_i}{\partial x_{P_1}} & \frac{\partial f_i}{\partial x_{Q_1}} \end{bmatrix}_{i=1,\dots,l} (x_{P_1}, x_{Q_1}) = \text{const.} \\ \text{rank} \begin{bmatrix} \frac{\partial g_j}{\partial x_{P_2}} & \frac{\partial g_j}{\partial x_{Q_2}} \end{bmatrix}_{j=1,\dots,m} (x_{P_2}, x_{Q_2}) = \text{const.} \end{aligned}$$

for all $\{(x_{P_i}, x_{Q_i}) \mid \exists \, x_{P_j}, x_{Q_j} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S^\star\}$, $(i, j) \in \{(1, 2), (2, 1)\}$. By Theorem 4.6, both S_1 and S_2 are therefore submanifolds. Since by Lemma 4.11, the functions f_i, g_j are given by all the independent repeated Lie derivatives,

$$\begin{aligned} f_i(x_{P_1}, x_{Q_1}) &= L_{X^1_{P_1}} \dots L_{X^k_{P_1}} h_{P_1}(x_{P_1}) - L_{X^1_{Q_1}} \dots L_{X^k_{Q_1}} h_{Q_1}(x_{Q_1}) \ , \\ g_j(x_{P_2}, x_{Q_2}) &= L_{X^1_{P_2}} \dots L_{X^k_{P_2}} h_{P_2}(x_{P_2}) - L_{X^1_{Q_2}} \dots L_{X^k_{Q_2}} h_{Q_2}(x_{Q_2}) \end{aligned}$$

the relations S_1 and S_2 in fact define the maximal simulation relations $S_1^\star$ and $S_2^\star$ of Σ_{P_1} by Σ_{Q_1} and of Σ_{P_2} by Σ_{Q_2}, respectively, see Proposition 4.10. Finally, fullness of $S^\star$ ensures fullness of $S_i, i = 1, 2$. □

Thus, the results for soundness and completeness of compositional reasoning of linear systems also hold for nonlinear input-affine systems. However, more effort is needed as far as other properties are concerned. E.g., simulation for nonlinear systems cannot be expected to be a preorder in general as the following example illustrates.

Example 4.14. Consider the nonlinear systems

$$\Sigma_P : \begin{aligned} \dot{x}_P &= 0 \\ y_P &= x_P^3 \end{aligned} \ , \qquad \Sigma_Q : \begin{aligned} \dot{x}_Q &= 0 \\ y_Q &= x_Q \end{aligned} \ , \qquad \Sigma_R : \begin{aligned} \dot{x}_R &= 0 \\ y_R &= x_R^2 \end{aligned}$$

The simulation relations S_{PQ} and S_{QR} are given by

$$\begin{aligned} S_{PQ} &= \{(x_P, x_Q) \mid x_P^3 - x_Q = 0\} \\ S_{QR} &= \{(x_Q, x_R) \mid x_Q - x_R^2 = 0\} \end{aligned}$$

where both S_{PQ} and S_{QR} are smooth submanifolds of $\mathbb{R}^2$. We construct a relation S_{PR} by setting

$$S_{PR} = \{(x_P, x_R) \mid x_P^3 - x_R^2 = 0\} \tag{4.23}$$

S_{PR} as in (4.23) fulfills the conditions (i) –(iv) of Theorem 4.3 but is not smooth in $(0, 0)$. Hence, S_{PR} is not a submanifold and therefore does not define a simulation relation of Σ_P by Σ_R according to Definition 4.2.

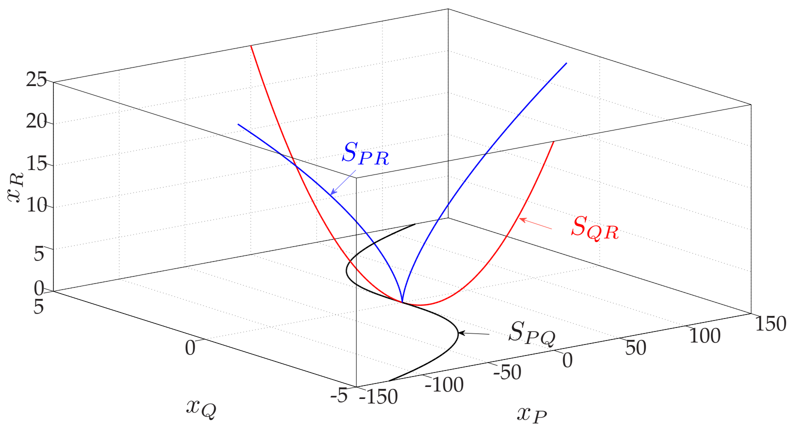

Figure 4.1.: Simulation relations S_{PQ}, S_{QR} and S_{PR}.

In order to obtain results that are valid at least locally we need assumptions on the regularity of the composition of the submanifolds that define simulation relations.

Proposition 4.15. *Consider the nonlinear systems $\Sigma_i, i \in \{P, Q, R\}$. Let there exist full simulation relations S_{PQ} and Σ_{QR} of Σ_P by Σ_Q and of Σ_Q by Σ_R each given as the zero-level sets of smooth functions $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^k, g : \mathbb{R}^q \times \mathbb{R}^r \to \mathbb{R}^l$ in the following way:*

$$\begin{aligned} S_{PQ} &= \{(x_P, x_Q) \in \mathcal{X}_P \times \mathcal{X}_Q \mid f_1(x_P, x_Q) = \cdots = f_k(x_P, x_Q) = 0\} \\ S_{QR} &= \{(x_Q, x_R) \in \mathcal{X}_P \times \mathcal{X}_Q \mid g_1(x_Q, x_R) = \cdots = g_l(x_Q, x_R) = 0\} \end{aligned} \tag{4.24}$$

Suppose that

$$\text{rank} \begin{bmatrix} \frac{\partial f}{\partial x_Q} \\ \frac{\partial g}{\partial x_Q} \end{bmatrix} (x_P, x_q, x_r) = k + l - c, \ \forall (x_P, x_Q) \in S_{PQ}, (x_Q, x_R) \in S_{QR} \tag{4.25}$$

Then there exist c functions $\psi_i(y, x_P, x_R), i = 1, \ldots, c, y \in \mathbb{R}^{k+l}$ with

$$\frac{\partial \psi_i}{\partial x_Q}((f(x_P, x_Q), g(x_Q, x_R)), x_Q, x_R) = 0, i = 1, \ldots, c, \tag{4.26}$$

Moreover, if the rank assumption

$$\operatorname{rank}\begin{bmatrix} \frac{\partial\psi}{\partial x_P} & \frac{\partial\psi}{\partial x_R} \end{bmatrix}(x_P, x_R) = \text{const.} \tag{4.27}$$

holds for all x_P, x_R *such that there exists a* x_Q *such that* $(x_p, x_Q) \in S_{PQ}, (x_Q, x_R) \in S_{PR}$*, then*

$$S_{PR} := \{(x_P, x_R) \mid \psi_i(f(x_P, x_Q), g(x_Q, x_R), x_P, x_R) = 0, i = 1, \dots, c\} \tag{4.28}$$

defines a full simulation relation of Σ_P *by* Σ_R*.*

Proof. Let S_{PQ} and S_{QR} be simulation relations of Σ_P by Σ_Q and of Σ_Q by Σ_R, respectively, as in (4.24). Consider the relation

$$S_{PQR} = \{(x_P, x_Q, x_R) \mid (x_P, x_Q) \in S_{PQ}, (x_Q, x_R) \in S_{QR}\}$$

If assumption (4.25) holds, then Lemma 4.7 ensures that there exist c functions $\psi_i : \mathbb{R}^{k+l} \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^c$ such that (4.26). Since $\psi_i((f(x_P, x_Q), g(x_Q, x_R)), x_p, x_R), i = 1, \dots, c$, do not depend on x_Q, take them as the candidate simulation S_{PR} (4.28) of Σ_P by Σ_R. First of all, due to assumption (4.27), S_{PR} indeed defines a submanifold of $\mathcal{X}_P \times \mathcal{X}_R$. Moreover, due to (4.6), Lemma 4.7 implies that

$$S_{PR} = \{(x_P, x_R) \mid \exists x_Q : (x_P, x_Q) \in S_{PQ}, (x_Q, x_R) \in S_{QR}\}$$

Hence conditions (1) and (2) of Definition 4.2 are fulfilled by construction. Furthermore, $\Pi_{\mathcal{X}_P} S_{PR} = \mathcal{X}_P$ due to fullness of S_{PQ} and S_{QR}. Thus, S_{PR} indeed defines a full nonlinear simulation relation of Σ_P by Σ_R. □

It is now clear that the non-circular assume guarantee reasoning rule holds true for nonlinear simulation relations under regularity conditions.

Theorem 4.16. *Consider four nonlinear systems* $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$*, of the form* (4.1)*. Let* S_1 *be a full nonlinear simulation relation of* Σ_{P_1} *by* Σ_{Q_1}*, given as the zero level set of k smooth functions* $f_i, i = 1, \dots, k$*,*

$$S_1 = \{(x_{P_1}, x_{Q_1}) \mid f_1(x_{P_1}, x_{Q_1}) = \dots = f_k(x_{P_1}, x_{Q_1}) = 0\} ,$$

and S_{II} *a full simulation relation of* $\Sigma_{Q_1} \|_{cl} \Sigma_{P_2}$ *by* $\Sigma_{Q_1} \|_{cl} \Sigma_{Q_2}$*, given as the zero level set of l smooth functions* $g_j, j = 1, \dots, l$*,*

$$S_{II} = \{(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \mid g_j(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) = 0, j = 1, \dots, l\}.$$

If the rank assumption

$$\operatorname{rank}\begin{bmatrix} \frac{\partial f}{\partial x_{Q_1}} & 0 \\ \frac{\partial g}{\partial x_{Q_1}} & \frac{\partial g}{\partial x_{P_2}} \end{bmatrix} = k + l - c = \text{const.} \tag{4.29}$$

holds for all (x_{Q_1}, x_{P_2}) *such that there exist* $x_{P_1}, \bar{x}_{Q_1}, x_{Q_2}$ *such that* $(x_{P_1}, x_{Q_1}) \in S_1$ *and* $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}$*, then there exist* c *smooth functions* $\psi(x_{P_1}, x_{P_2}, y, \bar{x}_{Q_1}, x_{Q_2})$ *such that for all* $i = 1, \ldots, c$

$$\begin{aligned} \frac{\partial \psi_i}{\partial x_{Q_1}}(x_{P_1}, x_{P_2}, (f(x_{P_1}, x_{Q_1}), g(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2})), \bar{x}_{Q_1}, x_{Q_2}) &= 0 \\ \frac{\partial \psi_i}{\partial x_{P_2}}(x_{P_1}, x_{P_2}, (f(x_{P_1}, x_{Q_1}), g(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2})), \bar{x}_{Q_1}, x_{Q_2}) &= 0 \end{aligned} \tag{4.30}$$

If moreover

$$\text{rank} \begin{bmatrix} \frac{\partial \psi}{\partial x_{P_1}} & \frac{\partial \psi}{\partial x_{P_2}} & \frac{\partial \psi}{\partial \bar{x}_{Q_1}} & \frac{\partial \psi}{\partial x_{Q_2}} \end{bmatrix} (x_{P_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) = \text{const.} \tag{4.31}$$

for all $x_{P_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}$ *such that there exists a* x_{Q_1} *such that* $(x_{P_1}, x_{Q_1}) \in S_1$ *and* $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}$*, then the non-circular assume-guarantee reasoning rule holds for nonlinear systems.*

Proof. First, we use the fact that nonlinear simulation is compositional with respect to feedback interconnection, see Theorem 4.9. Hence,

$$\Sigma_{P_1} \preccurlyeq \Sigma_{Q_1} \quad \Longrightarrow \quad \Sigma_{P_1} \|_{\text{cl}} \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \|_{\text{cl}} \Sigma_{P_2}$$

with the full simulation relation S_I of $\Sigma_{P_1} \|_{\text{cl}} \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_{\text{cl}} \Sigma_{P_2}$ given by

$$S_I = \{(x_{P_1}, x_{P_2}, x_{Q_1}, \bar{x}_{P_2}) \mid (x_{P_1}, x_{Q_1}) \in S_1, x_{P_2} = \bar{x}_{P_2}\}$$

Next, consider the relation

$$\begin{aligned} S' = \{ & (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, \bar{x}_{P_2}, \bar{x}_{Q_1}) \mid (x_{P_1}, x_{P_2}, x_{Q_1}, \bar{x}_{P_2}) \in S_I, \\ & (x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}\} \end{aligned}$$

with

$$\text{rank} S_I = \text{rank} \begin{bmatrix} \frac{\partial f}{\partial x_{P_1}} & 0 & \frac{\partial f}{\partial x_{Q_1}} & 0 & 0 & 0 \\ 0 & I & 0 & -I & 0 & 0 \\ 0 & 0 & \frac{\partial g}{\partial x_{Q_1}} & \frac{\partial g}{\partial x_{P_2}} & \frac{\partial g}{\partial \bar{x}_{Q_1}} & \frac{\partial g}{\partial x_{Q_2}} \end{bmatrix}$$

We now want to apply Proposition 4.15. If

$$\text{rank} \begin{bmatrix} \frac{\partial f}{\partial x_{Q_1}} & 0 \\ \frac{\partial g}{\partial x_{Q_1}} & \frac{\partial g}{\partial x_{P_2}} \end{bmatrix} (x_{Q_1}, x_{P_2}) = k + l - c$$

for all x_{Q_1}, x_{P_2} such that there exist $x_{P_1}, \bar{x}_{Q_1}, x_{Q_2}$ such that $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, \bar{x}_{P_2}, \bar{x}_{Q_1}) \in S'$, then there exist smooth functions $\psi_i, i = 1, \ldots, c$, fulfilling (4.30) due to Lemma 4.7. Taking

$$\begin{aligned} S := \{ & (x_{P_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \mid \exists \bar{x}_{P_2}, x_{Q_1} : \psi_i(x_{P_1}, x_{P_2}, (f(x_{P_1}, x_{Q_1}), \\ & g(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2})), \bar{x}_{Q_1}, x_{Q_2}) = 0, i = 1 \ldots, c, \} \end{aligned}$$

as the candidate relation, the rank assumption (4.30) ensures that S indeed defines a submanifold. Hence, Proposition 4.15 can be applied to conclude

$$\Sigma_{P_1}\|_{\mathrm{cl}}\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_{\mathrm{cl}}\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_{\mathrm{cl}}\Sigma_{Q_2}$$

due to transitivity of nonlinear simulation. □

For circular assume-guarantee reasoning , we only prove the existence of a nonlinear simulation relation but not its fullness.

Proposition 4.17. *Consider four nonlinear control systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ such that there exist simulation relation S_I, S_{II}, each given as the zero level sets of smooth functions $f_i, i = 1, \ldots, k$ and $g_j, j = 1, \ldots, l$, as follows:*

$$\begin{array}{llcl} S_I: & \Sigma_{P_1}\|_{cl}\Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1}\|_{cl}\Sigma_{Q_2} \\ S_{II}: & \Sigma_{Q_1}\|_{cl}\Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1}\|_{cl}\Sigma_{Q_2} \end{array} \tag{4.32}$$

with

$$\begin{aligned} S_I &:= \{x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \mid f_i(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) = 0, i = 1, \ldots, k\} \\ S_{II} &:= \{x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \mid g_j(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) = 0, j = 1, \ldots, l\} \end{aligned}$$

If

$$rank \begin{bmatrix} 0 & \frac{\partial f}{\partial \bar{x}_{Q_2}} \\ \frac{\partial g}{\partial \bar{x}_{Q_1}} & 0 \end{bmatrix} (\bar{x}_{Q_1}, \bar{x}_{Q_2}) = k + l - c \tag{4.33}$$

then there exist smooth functions $\psi_i(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, y), i = 1, \ldots, c$, such that

$$\frac{\partial \psi_i}{\partial \bar{x}_{Q_j}}(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, (f(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}), g(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}))) = 0$$
$$i = 1, \ldots, c, \; j = 1, 2$$

Moreover, if

$$\mathrm{rank} \begin{bmatrix} \frac{\partial \psi}{\partial x_{P_1}} & \frac{\partial \psi}{\partial x_{P_2}} & \frac{\partial \psi}{\partial x_{Q_1}} & \frac{\partial \psi}{\partial x_{Q_2}} \end{bmatrix} (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) = \text{const.} \tag{4.34}$$

for all $x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}$, then

$$\begin{aligned} S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \psi_i((x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, (f(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}), \\ g(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}))) = 0, i = 1, \ldots, c\} \end{aligned} \tag{4.35}$$

defines a simulation relation of $\Sigma_{P_1}\|_{cl}\Sigma_{P_2}$ by $\Sigma_{Q_1}\|_{cl}\Sigma_{Q_2}$.

Proof. The proof uses the same arguments as in the previous theorem. Consider first the relation

$$S' = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, \bar{x}_{Q_1}, \bar{x}_{Q_2}) \mid f_i(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) = 0, \\ i = 1, \ldots, k, g_j(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) = 0, j = 1, \ldots, l\}$$

By Lemma 4.7, assumptions (4.33) ensures that there exist smooth functions $\psi_i(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}, y), i = 1, \ldots, c$ such that 4.34 holds. Defining the candidate relation S as in (4.35), the full rank assumption (4.34) guarantees that S is a smooth submanifold. By the equality of the solution sets of S' on the one hand and S plus the remaining $k + l - c$ equations of S' on the other hand, S can be rewritten as

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists\, \bar{x}_{Q_1}, \bar{x}_{Q_2} : (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I, \\ (x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}\}$$

Hence,

$$\begin{bmatrix} f_{P_1}(x_{P_1}) - g_{P_1}(x_{P_1}) h_{Q_2}(x_{Q_2}) \\ f_{P_2}(x_{P_2}) + g_{P_2}(x_{P_2}) h_{Q_1}(x_{Q_1}) \\ f_{Q_1}(x_{Q_1}) - g_{Q_1}(x_{Q_1}) h_{P_2}(x_{P_2}) \\ f_{Q_2}(x_{Q_2}) + g_{Q_2}(x_{Q_2}) h_{P_1}(x_{P_1}) \end{bmatrix} \in T_{x_{P_1} x_{P_2} x_{Q_1} x_{Q_2}} S$$

while $h_{P_1} x_{P_1} = h_{Q_1}(x_{Q_1}) = h_{Q_1}(\bar{x}_{Q_1})$ and $h_{P_2} x_{P_2} = h_{Q_2}(x_{Q_2}) = h_{Q_2}(\bar{x}_{Q_2})$ which proves the claim. □

Recall that in the linear case, we could prove fullness of S by ensuring that the relations S_I and S_{II} were large enough. To that end, suitable subspaces were added, see Theorem 3.14. Since we define nonlinear simulation relations as zero-level sets of smooth functions, this idea cannot be immediately generalized.

Example 4.18. Consider the systems

$$\Sigma_{P_1} : \begin{array}{rcl} \dot{x}_{P_1} & = & x_{P_1} \\ y_{P_1} & = & x_{P_1} \end{array}, \; \Sigma_{P_2} : \begin{array}{rcl} \dot{x}_{P_2} & = & x_{P_2} \\ y_{P_2} & = & x_{P_2}^2 \end{array}, \; \Sigma_i \; \begin{array}{rcl} \dot{x}_i & = & x_i \\ y_i & = & 0 \end{array}, i \in \{Q_1, Q_2\}\,.$$

Let the simulation relations S_1 and S_2 be given as the solution sets

$$S_I : \; x_{P_1} - x_{Q_1} + x_{Q_2} \;=\; 0 \quad , \quad S_{II} : \; x_{Q_1} + x_{P_2}^2 - x_{Q_2} \;=\; 0$$

Clearly, both $S_i, i \in \{I, II\}$, define submanifolds since $\mathrm{rank} S_I = 1 = \mathrm{rank} S_{II}$ for all $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I$ and all $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}$, respectively. Moreover, both S_I and S_{II} are full as for every (x_{P_1}, x_{Q_2}) there exists a $\bar{x}_{Q_2}$ and a $x_{Q_1} = x_{P_2} + x_{Q_2}$ such that $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I$ and similarly, for every $(x_{Q_1}, x_{P_2}$ there exists a $\bar{x}_{Q_1}$ and $x_{Q_2} = x_{Q_1} + x_{P_2}^2$ such that $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}$. The rank condition (4.33) is fulfilled,

$$\mathrm{rank}\begin{bmatrix} 0 & \frac{\partial S_I}{\partial \bar{x}_{Q_2}} \\ \frac{\partial S_{II}}{\partial \bar{x}_{Q_1}} & 0 \end{bmatrix} = 0 \; ,$$

and

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid x_{P_1} - x_{Q_1} + x_{Q_2} = 0 = x_{Q_1} + x_{P_2}^2 - x_{Q_2}\} \quad (4.36)$$

has full rank, $\mathrm{rank} S = 2$ for all $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. Hence, S defines a simulation relation of $\Sigma_{P_1} \|_{\mathrm{cl}} \Sigma_{P_2}$ by $\Sigma_{Q_1} \|_{\mathrm{cl}} \Sigma_{Q_2}$. However, S is not full since $x_{P_1} = -x_{P_2}^2$ is imposed by (4.36), and thus $x_{P_1} \leq 0$.

In the next chapter we will apply nonlinear simulation theory to passivity.