Compositional analysis and control of dynamical systems

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Compositional analysis for linear systems

3.1. Introduction

Compositional and assume-guarantee reasoning [49, 31, 21] was first introduced in the area of formal verification to provide strategies to decompose a verification task for a labeled transition system into several tasks involving individual components or components restricted to a specific environment. Thus, the complexity of the original verification task can be significantly reduced. The burden of complexity is also encountered in the area of systems and control. Many engineering systems such as chemical plants or cyber-physical systems are modeled as a large number of interacting subsystems which leads to high state space dimensions. Combining formal concepts and modeling techniques from both areas offers a huge potential for analysis and controller design problems. As a first step, we develop compositional analysis techniques for linear systems in this chapter. Our approach is based on the representation of linear systems by differential equations which allows us to make use of structural notions of (bi)simulation relations. Extending previous results [37, 39, 40], we treat the following range of topics: At first, feedback interconnections are considered. We prove that compositional and assume-guarantee reasoning rules are sound and present an illustrative example from circuit theory. The complementary issue of completeness is investigated distinguishing between open and closed feedback interconnections. We also show that the proof rules for compositional and assume-guarantee reasoning are sound when replacing simulation by bisimulation relations. A generalization to interconnections of more than two systems rounds off the section on feedback interconnections. In the second part of this chapter, parallel compositions of linear systems are studied. The resulting algebraic constraints on the system variables are characteristic for models of physical processes and can be modeled in the form of DAE systems. We give a linear-algebraic characterization of simulation relations for systems with state constraints and use this notion to prove soundness of compositional and assume-guarantee reasoning. Finally, we present a special proof rule for parallel compositions based on the decomposition of the global specification. This leads to the result that verifying the global specification is equivalent to verifying each subspecification of a parallel composition.

3.2. Feedback interconnections

In this section, we want to analyze feedback interconnections of linear systems Σ_i,

$$\Sigma_i : \quad \begin{array}{rcl} \dot{x}_i & = & A_i x_i + B_i u_i + G_i e_i + L_i d_i \\ y_i & = & C_i x_i \\ z_i & = & H_i x_i \end{array} \tag{3.1}$$

where all variables belong to finite dimensional vector spaces, $x_i \in \mathcal{X}_i = \mathbb{R}^{n_i}, u_i \in \mathcal{U}_i, e_i \in \mathcal{E}_i, d_i \in \mathcal{D}_i, y_i \in \mathcal{Y}_i, z_i \in \mathcal{Z}_i$. Compared to Section 2.2, we add a pair of external variables e and z to specify performance targets[1], see Figure 3.1. These external variables remain accessible even after interconnecting two systems, see Figure 3.2. By contrast, the variables u and y are used

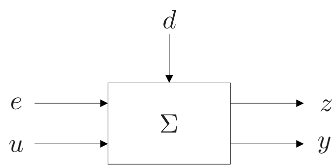

Figure 3.1.: Linear system Σ with internal and external variables.

for feedback interconnections between linear systems. As before, x represents the state variable and d an input that can be thought of as a disturbance as explained in Section 2.2. The temporal evolution of all system variables is characterized by functions of an appropriate function class, e.g. $\mathcal{C}^\infty$.

Definition 3.1. The feedback interconnection $\|$ of two linear continuous-time systems $\Sigma_i, i = 1, 2$, is defined as

$$u_2 = y_1 \ , \ u_1 = y_2 \ .$$

[1]This framework captures optimal control problems such as $H_\infty-$ optimal control where the performance target is specified as a norm bound on the transfer function between the external variables e and z.

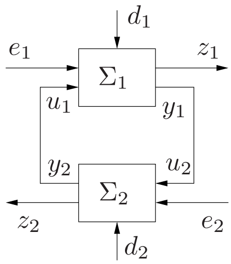

Figure 3.2.: Interconnection $\Sigma_1 \| \Sigma_2$

The dynamics of the interconnected system $\Sigma_1 \| \Sigma_2$ are then given by (see Figure 3.2)

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & B_1C_2 \\ B_2C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \\ &+ \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

(Bi)simulation relations will be used to formulate proof rules relating actual systems models to models representing their desired properties. To that end, we specialize the general concept of simulation relations as introduced in Chapter 2 as follows.

Definition 3.2. A simulation relation S of Σ_1 by Σ_2 is a linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ with the following property: For any $(x_{10}, x_{20}) \in S$, any joint input function $e_1(\cdot) = e_2(\cdot) = e(\cdot)$, any joint interconnection input $u_1(\cdot) = u_2(\cdot) = u(\cdot)$ and any disturbance function $d_1(\cdot)$ there should exist a disturbance $d_2(\cdot)$ such that the resulting state trajectories $x_i(.)$ with $x_i(0) = x_{i0}, i = 1, 2$, satisfy

$$\begin{aligned} (i) &\quad (x_1(t), x_2(t)) \in S, \ \forall t \geq 0 \\ (ii) &\quad H_1x_1(t) = H_2x_2(t), \ \forall t \geq 0 \\ (iii) &\quad C_1x_1(t) = C_2x_2(t), \ \forall t \geq 0 \end{aligned}$$

Σ_1 is simulated by Σ_2, denoted by $\Sigma_1 \preccurlyeq \Sigma_2$, if there exists a simulation relation S fulfilling $\Pi_1 S = \mathcal{X}_1$ with $\Pi_1 : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_1$ the canonical projection from $\mathcal{X}_1 \times \mathcal{X}_2$ to $\mathcal{X}_1$. In this case, S is called a *full simulation relation*.
A subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a *bisimulation relation* between Σ_1 and Σ_2 if it is a simulation relation of Σ_1 by Σ_2 and in addition $R^{-1} := \{(x_2, x_1) \mid (x_1, x_2) \in R\}$ defines a simulation relation of Σ_2 by Σ_1. Moreover, if $\Pi_i R = \mathcal{X}_i, i = 1, 2$, then R is called a *full bisimulation relation* and Σ_1 and Σ_2 are called *bisimilar*, denoted by $\Sigma_1 \approx \Sigma_2$.

Proposition 3.3. *A subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if for all $(x_1, x_2) \in S$, all $u \in \mathcal{U}$ and all $e \in \mathcal{E}$ the following holds:*

(i): $\forall\, d_1 \in \mathcal{D}_1 \;\exists\, d_2 \in \mathcal{D}_2 \;:\; \begin{bmatrix} A_1x_1 + B_1u + G_1e + L_1d_1 \\ A_2x_2 + B_2u + G_2e + L_2d_2 \end{bmatrix} \in S$

(ii): $H_1x_1 = H_2x_2$

(iii): $C_1x_1 = C_2x_2$

The subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ defines a bisimulation relation between Σ_1 and Σ_2 if and only if it fulfills conditions (i) - (iii) and additionally

(iv): $\forall\, d_2 \in \mathcal{D}_2 \;\exists\, d_1 \in \mathcal{D}_1 \;:\; \begin{bmatrix} A_1x_1 + B_1u + G_1e + L_1d_1 \\ A_2x_2 + B_2u + G_2e + L_2d_2 \end{bmatrix} \in R$

Theorem 3.4. *A linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if the following holds:*

1. $\operatorname{im}\begin{bmatrix} L_1 \\ 0 \end{bmatrix} \subset S + \operatorname{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} := S_e$

2. $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} S \subset S_e$

3. $\operatorname{im}\begin{bmatrix} G_1 & B_1 \\ G_2 & B_2 \end{bmatrix} \subset S_e$

4. $S \subset \ker\begin{bmatrix} H_1 & -H_2 \\ C_1 & -C_2 \end{bmatrix}$

The subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a bisimulation relation between Σ_1 and Σ_2 if and only if it satisfies conditions 2 - 4 and in addition

$1'$. $R + \operatorname{im}\begin{bmatrix} L_1 \\ 0 \end{bmatrix} = R + \operatorname{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} =: R_e$

As a first compositional result, observe that feedback interconnections of linear systems as in Definition 3.1 are commutative.

Proposition 3.5. *For any two linear systems Σ_P and Σ_Q,*

$$\Sigma_P \| \Sigma_Q \approx \Sigma_Q \| \Sigma_P \;,$$

Proof. Permuting the state vector of $\Sigma_Q \| \Sigma_P$, the relation

$$\begin{aligned} S \;=\; & \{((x_P, x_Q), (\bar{x}_P, \bar{x}_Q)) \mid (x_P, x_Q) \in \mathcal{X}_P \times \mathcal{X}_Q, (\bar{x}_Q, \bar{x}_P) \in \mathcal{X}_Q \times \mathcal{X}_P, \\ & x_P = \bar{x}_P, x_Q = \bar{x}_Q\} \end{aligned}$$

defines a bisimulation relation between $\Sigma_P \| \Sigma_Q$ and $\Sigma_Q \| \Sigma_P$. □

3.2.1. Soundness of compositional proof rules

Consider a complex linear plant system Σ_P which we assume to be given in the form of interconnected subsystems $\Sigma_{P_i}, i = 1, \ldots, k$, that is $\Sigma_P = \Sigma_{P_1} \| \ldots \| \Sigma_{P_k}$. We want to check whether Σ_P has the desired behavior specified by Σ_Q which again we assume to be given in the form of interconnected subspecifications Σ_{Q_i}, $\Sigma_Q = \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_k}$ where the interconnection variables are elements of the same vector spaces. First, we will restrict ourselves to interconnections of two subsystems only. However, the compositional techniques described in the following can be generalized to an arbitrary number of subsystems thanks to their modular structure, which will be treated in Section 3.2.4.
Using simulation relations, the verification task can be expressed as

$$\Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{3.2}$$

If there exists a full simulation relation of the given system $\Sigma_{P_1} \| \Sigma_{P_2}$ by the specification $\Sigma_{Q_1} \| \Sigma_{Q_2}$ the behavior of the given system is included in the behavior of the specification which means that the specification will always be satisfied. In order to ensure that the given system fulfills the specification exactly, simulation in (3.2) has to be replaced by bisimulation, see Section 3.2.3. However, both Σ_P and Σ_Q are complex systems given as interconnections of subsystems. The verification task (3.2) will therefore be decomposed into two subtasks for the component systems in order to reduce its complexity. We call such a proof rule *sound* if the original verification task (3.2) can be inferred from the two subtasks it was split into.

Compositional reasoning

We start with the first pillar for compositional analysis.

Theorem 3.6. *For any four linear systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$, the* compositionality *property*

$$\left.\begin{array}{lcl} \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_2} \end{array}\right\} \implies \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \tag{3.3}$$

holds.

Proof. Let S_i denote the full simulation relations of Σ_{P_i} by $\Sigma_{Q_i}, i = 1, 2$. Construct the relation

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_1}) \in S_1, (x_{P_2}, x_{Q_2}) \in S_2\} \tag{3.4}$$

Then for every $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$, every joint input $e_{P_i} = e_{Q_i} = e_i, i = 1, 2$, and every disturbances d_{P_1}, d_{P_2}, there exist disturbances d_{Q_1}, d_{Q_2} such that

$$\begin{bmatrix} A_{P_1} x_{P_1} + B_{P_1} C_{P_2} x_{P_2} + G_{P_1} e_1 + L_{P_1} d_{P_1} \\ A_{Q_1} x_{Q_1} + B_{Q_1} C_{Q_2} x_{Q_2} + G_{Q_1} e_1 + L_{Q_1} d_{Q_1} \end{bmatrix} \in S_1$$

and

$$\begin{bmatrix} A_{P_2}x_{P_2} + B_{P_2}C_{P_1}x_{P_1} + G_{P_2}e_2 + L_{P_2}d_{P_2} \\ A_{Q_2}x_{Q_2} + B_{Q_2}C_{Q_1}x_{Q_1} + G_{Q_2}e_2 + L_{Q_2}d_{Q_2} \end{bmatrix} \in S_2$$

Furthermore, since $C_{P_i}x_{P_i} = C_{Q_i}x_{Q_i}$ for all $(x_{P_i}, x_{Q_i}) \in S_i$, $H_{P_i}x_{P_i} = H_{Q_i}x_{Q_i}$. Moreover, S as defined in (3.4) is the product of the simulation relations S_1 and S_2 after reordering the vectors x_{Q_1} and x_{P_2}. Since $\Pi_{P_1}S_1 = \mathcal{X}_1$ and $\Pi_{P_2}S_2 = \mathcal{X}_2$ (because S_1 and S_2 are full) $\Pi_{P_1P_2}S = \mathcal{X}_1 \times \mathcal{X}_2$ and therefore S is full. □

Remark 3.7. In general, the converse implication in (3.3) does not hold. Take as a counterexample the following systems

$$\Sigma_{P_1}: \begin{array}{rcl} \dot{x}_{P_1} & = & 2u_{P_1} + e_{P_1} \\ y_{P_1} & = & x_{P_1} \\ z_{P_1} & = & x_{P_1} \end{array}, \qquad \Sigma_{P_2}: \begin{array}{rcl} \dot{x}_{P_2} & = & u_{P_2} + e_{P_2} \\ y_{P_2} & = & \frac{1}{2}x_{P_2} \\ z_{P_2} & = & x_{P_2} \end{array}$$

$$\Sigma_{Q_i}: \begin{array}{rcl} \dot{x}_{Q_i} & = & u_{Q_i} + e_{Q_i} \\ y_{Q_i} & = & x_{Q_i} \\ z_{Q_i} & = & x_{Q_i} \end{array}, \quad i = 1, 2 .$$

Then there exists a simulation relation S of $\Sigma_{P_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, namely

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid x_{P_1} = x_{Q_1}, x_{P_2} = x_{Q_2}\}$$

since the state space descriptions of $\Sigma_{P_1}\|\Sigma_{P_2}$ and $\Sigma_{Q_1}\|\Sigma_{Q_2}$ are identical. However, there do not exist any simulation relations of Σ_{P_1} by Σ_{Q_1} nor of Σ_{P_2} by Σ_{Q_2} since for the former

$$\text{im} \begin{bmatrix} B_{P_1} \\ B_{Q_1} \end{bmatrix} = \text{im} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \not\subseteq \ker \begin{bmatrix} C_{P_1} & -C_{Q_1} \\ H_{P_1} & -H_{Q_1} \end{bmatrix} = \text{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and for the latter

$$\text{im} \begin{bmatrix} B_{P_2} \\ B_{Q_2} \end{bmatrix} = \text{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \not\subseteq \ker \begin{bmatrix} C_{P_2} & -C_{Q_2} \\ H_{P_2} & -H_{Q_2} \end{bmatrix} = \text{im} \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We note that as a special case of compositionality, *invariance under composition* also holds:

Corollary 3.8. *Consider two linear systems $\Sigma_{P_1}, \Sigma_{Q_1}$ of the form* (3.1)*. Then for any linear system Σ_{Q_2} it holds that*

$$\Sigma_{P_1} \preccurlyeq \Sigma_{Q_1} \quad \Longrightarrow \quad \Sigma_{P_1}\|\Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2} \tag{3.5}$$

Assume-guarantee reasoning

In the case that one or both of the components $\Sigma_{P_i}, i = 1, 2$, do not fulfill their subspecifications Σ_{Q_i} directly, compositional reasoning cannot be applied to simplify the verification task (3.2). However, rather than considering the component systems Σ_{P_i} in isolation one can restrict them to a particular environment. More concretely, one can replace the assumption $\Sigma_{P_2} \preccurlyeq \Sigma_{Q_2}$ by $\Sigma_{Q_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}$, that is, Σ_{P_2} is compared with Σ_{Q_2} while it is already assumed that Σ_{P_1} may be replaced by Σ_{Q_1}, and similarly, $\Sigma_{P_1} \preccurlyeq \Sigma_{Q_1}$ can be replaced by $\Sigma_{P_1} \| \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}$ thus assuming that Σ_{P_2} satisfies its specification Σ_{Q_2}. This describes the general principle of *assume-guarantee reasoning*, a divide and conquer scheme based on mutual assumptions and guarantees to reduce the complexity of verification tasks. We first present two types of non-circular assume-guarantee reasoning rules, each replacing one of the assumptions $\Sigma_{P_i} \preccurlyeq \Sigma_{Q_i}$ as described. Hence, we avoid examining the more complex interconnection $\Sigma_{P_1} \| \Sigma_{P_2}$ yet are still able to guarantee that (3.2) holds.

Theorem 3.9. *For any given linear systems Σ_i,$i \in \{P_1, P_2, Q_1, Q_2\}$, non-circular assume-guarantee reasoning is sound, i.e. the following deduction scheme*

$$\left.\begin{array}{rrcl} S_1: & \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_{II}: & \Sigma_{Q_1} \parallel \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \parallel \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S : \Sigma_{P_1} \parallel \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \parallel \Sigma_{Q_2} \quad (3.6)$$

and its symmetric counterpart

$$\left.\begin{array}{rrcl} S_2: & \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_2} \\ S_{I}: & \Sigma_{P_1} \parallel \Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1} \parallel \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S : \Sigma_{P_1} \parallel \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \parallel \Sigma_{Q_2} \quad (3.7)$$

hold.

Proof. The proof only requires the relation $\preccurlyeq$ to be a preorder and the interconnection $\|$ to be invariant under composition. To prove (3.6), reflexivity of simulation and invariance under composition yield

$$\left.\begin{array}{rcl} \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ \Sigma_{P_2} & \preccurlyeq & \Sigma_{P_2} \end{array}\right\} \quad \Longrightarrow \quad \Sigma_{P_1} \parallel \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \parallel \Sigma_{P_2}$$

which due to S_{II} and transitivity of simulation yields the desired result

$$\Sigma_{P_1} \parallel \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \parallel \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \parallel \Sigma_{Q_2}$$

Exploiting commutativity of the interconnection, the same arguments hold for the non circular rule (3.7),

$$\Sigma_{P_1} \parallel \Sigma_{P_2} \preccurlyeq \Sigma_{P_2} \parallel \Sigma_{P_1} \preccurlyeq \Sigma_{Q_2} \parallel \Sigma_{P_1} \preccurlyeq \Sigma_{P_1} \parallel \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1} \parallel \Sigma_{Q_2}$$

□

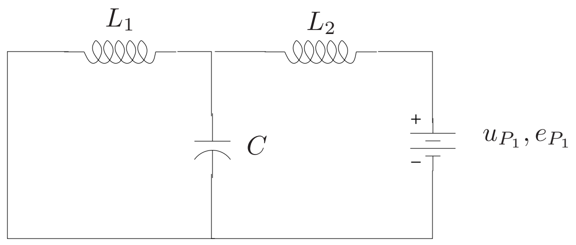

Figure 3.3.: Σ_{P_1}: $LC-$ circuit.

Example 3.10. Consider as Σ_{P_1} the $LC-$ circuit in Figure 3.3 with two inductors L_1 and L_2, one capacitor C, a voltage source as input u_{P_1} and the current over the capacitor as output y_{P_1}. The external in- and outputs are chosen to be the same as the interconnection variables, $u_{P_1} = e_{P_1}$ and $y_{P_1} = z_{P_1}$, while there are no external disturbances, i.e., d_{P_1} and d_{P_2} are absent.

Then, Σ_{P_1} is given by

$$\Sigma_{P_1}: \begin{array}{rcl} \begin{bmatrix} \dot{q}_C \\ \dot{\phi}_{L_1} \\ \dot{\phi}_{L_2} \end{bmatrix} & = & \begin{bmatrix} 0 & \frac{1}{L_1} & -\frac{1}{L_2} \\ -\frac{1}{C} & 0 & 0 \\ \frac{1}{C} & 0 & 0 \end{bmatrix} \begin{bmatrix} q_C \\ \phi_{L_1} \\ \phi_{L_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_{P_1} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e_{P_1} \\ y_{P_1} & = & \begin{bmatrix} \frac{1}{C} & 0 & 0 \end{bmatrix} x_{P_1} = z_{P_1} \end{array}$$

where $x_{P_1} = \begin{bmatrix} q_C & \phi_{L_1} & \phi_{L_2} \end{bmatrix}^T$ denotes the state vector. In the remainder, all the parameter values are set to 1 for simplicity. To stabilize the electrical circuit Σ_{P_1} we apply a simple feedback controller Σ_{P_2}, given as

$$\Sigma_{P_2}: \begin{array}{rcl} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} & = & \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_{P_2} \\ y_{P_2} & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{array}$$

Observe that $e_{P_2}, z_{P_2}, d_{P_2}$ are all void.
The verification goal is to simulate the 5-dimensional interconnection $\Sigma_{P_1} \| \Sigma_{P_2}$ by a less complex specification $\Sigma_{Q_1} \| \Sigma_{Q_2}$. The components of this specification are given by the $LC-$ circuit Σ_{Q_1} as in Figure 3.10 and an abstracted controller Σ_{Q_2}. In particular,

$$\Sigma_{Q_1}: \begin{array}{rcl} \begin{bmatrix} \dot{\phi}_{Q_1} \\ \dot{q}_{Q_1} \end{bmatrix} & = & \begin{bmatrix} 0 & -\frac{1}{C_{Q_1}} \\ \frac{1}{L_{Q_1}} & 0 \end{bmatrix} \begin{bmatrix} \phi_{Q_1} \\ q_{Q_1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{Q_1} + \\ & & + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_{Q_1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d_{Q_1} \\ y_{Q_1} & = & \begin{bmatrix} 0 & \frac{1}{C_{Q_1}} \end{bmatrix} x_{Q_1} = z_{Q_1} \end{array}$$

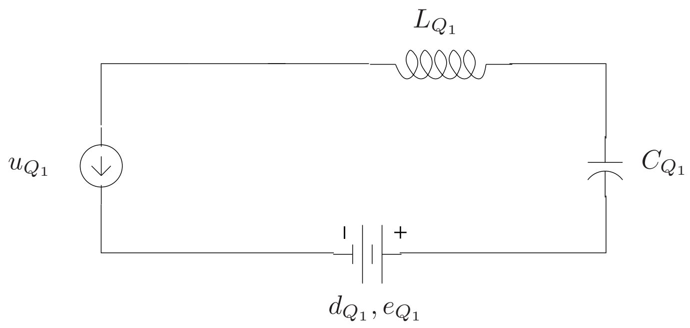

Figure 3.4.: Σ_{Q_1}

where $x_{Q_1} = \begin{bmatrix} \phi_{Q_1} & q_{Q_1} \end{bmatrix}^T$ and all parameter values are again set to 1. The controller Σ_{Q_2} is described by

$$\Sigma_{Q_2}: \quad \begin{array}{rcl} \dot{x}_{Q_2} & = & -5x_{Q_2} + u_{Q_2} + d_{Q_2} \\ y_{Q_2} & = & x_{Q_2} \end{array}$$

The first observation is that compositionality is not applicable since there does not exist any simulation relation of Σ_{P_1} by Σ_{Q_1}. The physical explanation is that the disturbance input d_{Q_1} represents a voltage source which cannot mimic the behavior of the inductor L_2. However, the controller systems Σ_{P_2} and Σ_{Q_2} can be related by means of a full simulation relation S_2 given as

$$S_2 = \{(z_1, z_2), x_{Q_2}) \mid z_1 = x_{Q_2}\} \tag{3.8}$$

Moreover, the interconnected system $\Sigma_{P_1} \| \Sigma_{Q_2}$ can be simulated by $\Sigma_{Q_1} \| \Sigma_{Q_2}$ using the simulation relation

$$\begin{array}{rcl} S_I & = & \{((q_C, \phi_{L_1}, \phi_{L_2}, x_{Q_2}), (x_1, x_2, x'_{Q_2})) \mid x_{Q_2} = x'_{Q_2}, \\ & & q_C = x_2, \phi_{L_1} = \phi_{L_2} + x_{Q_2} + x_1\} \end{array} \tag{3.9}$$

By Theorem 3.9, we can therefore conclude that there exists a full simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, given by

$$\begin{array}{rcl} S & = & \{((q_C, \phi_{L_1}, \phi_{L_2}, z_1, z_2), (x_1, x_2, x_{Q_2})) \mid z_1 = x_{Q_2}, \\ & & q_C = x_2, \phi_{L_1} = \phi_{L_2} + z_1 + x_1\} \end{array} \tag{3.10}$$

This shows that it is possible to abstract the behavior of the 5 dimensional controlled electrical circuit by a 3-dimensional electrical circuit with disturbances.

In *circular* assume-guarantee reasoning neither of the relations $S_i : \Sigma_{P_1} \preccurlyeq \Sigma_{Q_1}, i = 1, 2$, is assumed to hold. Instead, we consider interconnections of subsystems Σ_{P_i} of the plant and subspecifications Σ_{Q_i}. Thus, the behavior of the subsystems Σ_{P_i} is restricted since their interconnection variables are determined by the respective in- and outputs of Σ_{Q_j}. Replacing S_1 with the relation $S_I : \Sigma_{P_1} \| \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}$ and S_2 with $S_{II} : \Sigma_{Q_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2}$ results in a circular rule that is used to prove (3.2). That is, in S_I we *assume* that Σ_{P_2} fulfills Σ_{Q_2} to prove that under this assumption, Σ_{P_1} restricted by Σ_{Q_2} satisfies the specification Σ_{Q_1} (which is equally restricted by Σ_{Q_2}) while in S_{II}, we take for granted that Σ_{P_1} fulfills Σ_{Q_1} (the *guarantee*) to prove that Σ_{P_2} restricted by Σ_{Q_1} satisfies its specification Σ_{Q_2} and vice versa. Since the subspecifications are usually of much lower complexity than the corresponding subsystems of the plant the resulting reasoning scheme (3.15) is more efficient than direct verification. Due to the circular dependencies of assumptions and guarantees, circular assume-guarantee reasoning in general is only sound under additional conditions [21]. For linear systems, however, it turns out to *always* hold true. The main idea in this proof is to enlarge the simulation relations S_1 and S_2 in a suitable way and then to construct, based on these extended simulation relations, a simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$. Since the proofs of the following lemmas and the main theorem are quite technical, we defer them to the appendix.

Lemma 3.11. *Given full simulation relations $S_i, i = I, II$, of $\Sigma_{P_1} \| \Sigma_{Q_2}$ and $\Sigma_{Q_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, respectively, and define the following linear subspaces*

$$\begin{aligned} \bar{S}_I \quad &:= \quad \{(x_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, -x_{Q_2}) \mid x_{Q_2}, \bar{x}_{Q_2} \in \ker C_{Q_2} \cap \ker H_{Q_2}, \quad (3.11) \\ &\qquad x_{P_1} \in \ker C_{P_1} \cap \ker H_{P_1}, x_{Q_1} \in \ker C_{Q_1} \cap \ker H_{Q_1}, \\ &\qquad (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I\} \\ \bar{S}_{II} \quad &:= \quad \{(\bar{x}_{Q_1}, x_{Q_2}, -x_{Q_1}, x_{Q_2}) \mid x_{Q_1}, \bar{x}_{Q_1} \in \ker C_{Q_1} \cap \ker H_{Q_1}, \quad (3.12) \\ &\qquad x_{P_2} \in \ker C_{P_2} \cap \ker H_{P_2}, x_{Q_2} \in \ker C_{Q_2} \cap \ker H_{Q_2}, \\ &\qquad (x_{Q_1}, x_{Q_1}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}\} \end{aligned}$$

Then $S_I + \bar{S}_I$ and $S_{II} + \bar{S}_{II}$ also define full simulation relations of $\Sigma_{P_1} \| \Sigma_{Q_2}$ and $\Sigma_{Q_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, respectively.

Proof. The proof can be found in A.1. □

Lemma 3.12. *Given full simulation relations S_I and S_{II} of $\Sigma_{P_1} \| \Sigma_{Q_2}$ and $\Sigma_{Q_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, respectively, then also their symmetrical closures*

$$S_I^{sym} := S_I + \hat{S}_I, \quad S_{II}^{sym} := S_{II} + \hat{S}_{II} \tag{3.13}$$

where

$$\begin{aligned} \hat{S}_I &:= \{(x_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I\} \qquad (3.14) \\ \hat{S}_{II} &:= \{(\bar{x}_{Q_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}\} \end{aligned}$$

define full simulation relations of $\Sigma_{P_1} \| \Sigma_{Q_2}$ and $\Sigma_{Q_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$.

Proof. The proof is very similar to the previous one and can be found in A.2. □

Adding the subspaces $\bar{S}_i, i = I, II$, and then taking the symmetrical closure ${S_i}^{\mathrm{sym}}$ ensures that the extended simulation relations $(S_i + \bar{S}_i)^{\mathrm{sym}}$ include elements of a particular form.

Lemma 3.13. *Consider full simulation relation $(S_I + \bar{S}_I)^{sym}$ and $(S_{II} + \bar{S}_{II})^{sym}$ of $\Sigma_{P_1} \| \Sigma_{Q_2}$ and $\Sigma_{Q_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$ as defined in the previous lemmas. Then for every $x \in \ker C_{Q_2} \cap \ker H_{Q_2}$, $(0, x, 0, x) \in (S_I + \bar{S}_I)^{sym}$ and analogously, for every $y \in \ker C_{Q_1} \cap \ker H_{Q_1}$, $(y, 0, y, 0) \in (S_{II} + \bar{S}_{II})^{sym}$.*

Proof. The proof can be found in A.3. □

Using the extended full simulation relations $(S_I + \bar{S}_I)^{\mathrm{sym}}$ and $(S_{II} + \bar{S}_{II})^{\mathrm{sym}}$ it is possible to construct a full simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, as formulated in the following main theorem.

Theorem 3.14. *For any given linear systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$, circular assume-guarantee reasoning is sound, i.e. the deduction*

$$\left.\begin{array}{llcl} S_I: & \Sigma_{P_1} \parallel \Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1} \parallel \Sigma_{Q_2} \\ S_{II}: & \Sigma_{Q_1} \parallel \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \parallel \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S: \ \Sigma_{P_1} \| \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \| \Sigma_{Q_2} \quad (3.15)$$

holds. The full simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$ is given by

$$\begin{aligned} S \ := \ & \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists \bar{x}_{Q_1}, \bar{x}_{Q_2} : (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in \\ & (S_I + \bar{S}_I)^{sym}, (x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in (S_{II} + \bar{S}_{II})^{sym}\} \end{aligned} \quad (3.16)$$

Proof. The proof is in A.4. □

Symmetrization of simulation relations as in Lemma 3.12 has been used for labeled transition systems as well. In [21], the following lemma can be found:

Lemma 3.15. *Consider a full simulation relation S_I of $\Sigma_{P_1} \| \Sigma_{Q_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$. Then also*

$$\tilde{S}_I := \{(x_{P_1}, x_{Q_2}, x_{Q_1}, x_{Q_2}) \mid \exists \bar{x}_{Q_2} \ : \ (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I\} \quad (3.17)$$

is a full simulation relation of $\Sigma_{P_1} \| \Sigma_{Q_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$. The same holds for the symmetric counterpart of a simulation relation S_{II} of $\Sigma_{Q_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$.

Proof. This can be concluded simply from Definition 3.2 by substituting $\bar{x}_{Q_2}$ with x_{Q_2}. □

Lemma 3.15 gives rise to an alternative construction of a full simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2}$ in the case of circular assume-guarantee reasoning.

Proposition 3.16. *Consider any four linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ of the form (3.1). Let simulation relations S_I and S_{II} of $\Sigma_{P_1} \| \Sigma_{Q_2}$ and $\Sigma_{Q_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, respectively, be given such that the relation S, given by*

$$\begin{aligned} S \quad := \quad & \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists \bar{x}_{Q_1}, \bar{x}_{Q_2} : (x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in \\ & S_I, (x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}\} \end{aligned} \tag{3.18}$$

is a full simulation relation of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$. Then

$$\begin{aligned} \tilde{S} \quad := \quad & \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_2}, x_{Q_1}, x_{Q_2}) \in \tilde{S}_I, \\ & (x_{Q_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in \tilde{S}_{II}\} \end{aligned} \tag{3.19}$$

with $\tilde{S}_i, i = I, II$ as in (3.17)*, defines the same simulation relation of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$, i.e. $S = \tilde{S}$.*

Proof. Observe first that due to Theorem 3.14, there always exist relations S_I and S_{II} large enough such that S as in (3.16) is a full simulation relation of $\Sigma_{P_1} \| \Sigma_{P_2}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2}$. Clearly, $\tilde{S} \subset S$ since the components $\bar{x}_{Q_2}$, $\bar{x}_{Q_1}$ do not play any role in constructing S. For the converse, recall that $\tilde{S}_i, i = I, II$, as defined in (3.17) are full simulation relations. Take any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. Then there exists $\bar{x}_{Q_1}, \bar{x}_{Q_2}$ such that $(x_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in S_I$ and $(x_{Q_1}, x_{P_2}, \bar{x}_{Q_1}, x_{Q_2}) \in S_{II}$. But then $(x_{P_1}, x_{Q_2}, x_{Q_1}, x_{Q_2}) \in \tilde{S}_I$ and similarly, $(x_{Q_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in \tilde{S}_{II}$. This in turn implies that $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in \tilde{S}$. □

3.2.2. Completeness considerations

So far we used compositional analysis techniques to split a global verification task into several less complex tasks for the components involved. In other words, verifying properties of component systems allowed us to infer properties of their interconnection. This gives rise to the question whether the converse strategy also works, i.e., if we know that the overall system satisfies a certain property can we then conclude that all the components have this property? In the terminology of formal verification, *completeness* means that the converse implication of a compositional proof rule holds true as well. In this section, we will investigate completeness of both compositional and assume-guarantee reasoning.

Completeness of compositional reasoning with respect to open and closed interconnections

Recall the earlier Example 3.10 of two controlled $LC-$circuits. We showed that the interconnection $\Sigma_{P_1} \| \Sigma_{P_2}$ did satisfy the lower-dimensional specification $\Sigma_{Q_1} \| \Sigma_{Q_2}$ even though $\Sigma_{P_1} \not\preceq \Sigma_{Q_1}$. Thus, compositionality cannot be complete in general. As it turns out, it depends on the presence of additional inputs in the feedback interconnection whether or not compositional reasoning is complete. To illustrate this, we modify Definition 3.1 by adding inputs v_i to the feedback interconnection.

Definition 3.17. Given two linear systems $\Sigma_i, i = 1, 2$, of the form

$$\Sigma_i : \begin{array}{rcl} \dot{x}_i &=& A_i x_i + B_i u_i + L_i d_i \\ y_i &=& C_i x_i \end{array} \tag{3.20}$$

The *open* feedback interconnection, denoted by $\Sigma_1 \parallel_o \Sigma_2$, is given by

$$u_1 = y_2 + v_1 \quad , \quad u_2 = y_1 + v_2$$

yielding the closed loop dynamics

$$\Sigma_1 \parallel_o \Sigma_2 : \begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &=& \begin{bmatrix} A_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ && + \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &=& \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array} \tag{3.21}$$

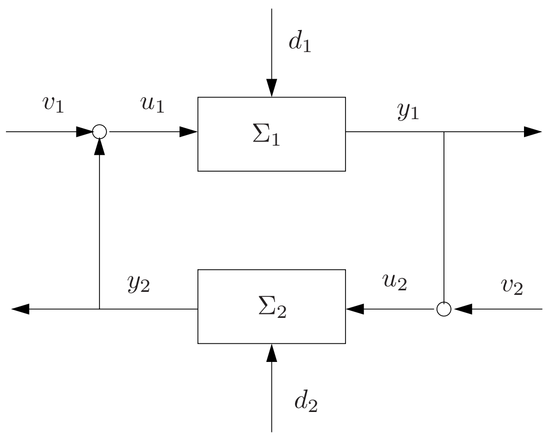

Figure 3.5.: Open feedback interconnection $\Sigma_1 \parallel_o \Sigma_2$.

Proposition 3.18. *For linear systems* (3.1) *interconnected by open feedback* (3.19), *the following equivalence holds:*

$$\left. \begin{array}{rcl} \Sigma_{P_1} &\preccurlyeq& \Sigma_{Q_1} \\ \Sigma_{P_2} &\preccurlyeq& \Sigma_{Q_2} \end{array} \right\} \Longleftrightarrow \Sigma_{P_1} \parallel_o \Sigma_{P_2} \preccurlyeq \Sigma_{Q_1} \parallel_o \Sigma_{Q_2} \tag{3.22}$$

Proof. "$\Longrightarrow$": The proof is the same as for Theorem 3.6.
"$\Longleftarrow$": Let a simulation relation S of $\Sigma_{P_1}\|\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|\Sigma_{Q_2}$ be given, i.e. for any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$, any joint v_1, v_2 and any d_{P_1}, d_{P_2} there exists d_{Q_1}, d_{Q_2} such that

$$(i): \begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}(v_1 + C_{P_2}x_{P_2}) + G_{P_1}d_{Q_1} \\ A_{P_2}x_{P_2} + B_{P_2}(v_2 + C_{P_1}x_{P_1}) + G_{P_2}d_{P_2} \\ A_{Q_1}x_{Q_1} + B_{Q_1}(v_1 + C_{Q_2}x_{Q_2}) + G_{Q_1}d_{Q_1} \\ A_{Q_2}x_{Q_2} + B_{Q_2}(v_2 + C_{Q_1}x_{Q_1}) + G_{Q_2}d_{Q_2} \end{bmatrix} \in S \tag{3.23}$$
$$(ii): C_{P_1}x_{P_1} = C_{Q_1}x_{Q_1}\ ,\ C_{P_2}x_{P_2} = C_{Q_2}x_{Q_2}$$

Define the relation $S_1 := \{(x_{P_1}, x_{Q_1}) \mid \exists\, x_{P_2}, x_{Q_2} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S\}$. It has to be shown that for any $(x_{P_1}, x_{Q_1}) \in S_1$, any u and any d_1 there exists a d_2 such that

$$(i): \begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u + G_{P_1}d_1 \\ A_{Q_1}x_{Q_1} + B_{Q_1}u + G_{Q_1}d_2 \end{bmatrix} \in S_1$$
$$(ii): C_{P_1}x_{P_1} = C_{Q_1}x_{Q_1}$$

Take any $(x_{P_1}, x_{Q_1}) \in S_1$ and fix u and d_1. Since (3.23) holds for any v_1, v_2 and d_{P_1}, d_{P_2}, in particular it holds for $v_1 = u - C_{P_2}x_{P_2}$ and $d_{P_1} = d_1$. Thus, S_1 defines a simulation relation of Σ_{P_1} by Σ_{Q_1}. The same arguments also hold for a relation $S_2 := \{(x_{P_2}, x_{Q_2}) \mid \exists\, x_{P_1}, x_{Q_1} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S\}$.
□

The consequence of Proposition 3.18 is immediate. For open feedback interconnections, the problem of checking $\Sigma_{P_1}\|_o\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_o\Sigma_{Q_2}$ can be reduced to (and, in fact, is equivalent to)

$$\Sigma_{P_1} \preccurlyeq \Sigma_{Q_1}\ ,\ \Sigma_{Q_1} \preccurlyeq \Sigma_{Q_2}$$

Hence, it is enough to verify properties of the individual components to ensure that these properties hold for the interconnection. Stated differently, assume-guarantee reasoning is not needed for *open* feedback interconnections since one can always resort to compositional reasoning which is less complex. This result also simplifies the computation of the maximal simulation relation for linear systems without disturbance inputs.

Proposition 3.19. *Consider four linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$, of the form* (3.1) *with d_i void and without external variables, $e_i = z_i \equiv 0$. Assume that the transfer matrices of the open feedback interconnections$\Sigma_{P_1}\|_o\Sigma_{P_2}$and $\Sigma_{Q_1}\|_o\Sigma_{Q_2}$ are* *the same. Then the maximal simulation relation $R^\star_{P_1P_2Q_1Q_2}$ between $\Sigma_{P_1}\|_o\Sigma_{P_2}$ and $\Sigma_{Q_1}\|_o\Sigma_{Q_2}$ is given by*

$$R^\star_{P_1P_2Q_1Q_2} = \ker \begin{bmatrix} C_{P_1} & 0 & -C_{Q_1} & 0 \\ 0 & C_{P_2} & 0 & C_{Q_2} \\ C_{P_1}A_{P_1} & 0 & C_{Q_1}A_{Q_1} & 0 \\ 0 & C_{P_2}A_{P_2} & 0 & C_{Q_2}A_{Q_2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{P_1}A^n_{P_1} & 0 & C_{Q_1}A^n_{Q_1} & 0 \\ 0 & C_{P_2}A^n_{P_2} & 0 & C_{Q_2}A^n_{Q_2} \end{bmatrix} \tag{3.24}$$

where $n = \max\{n_{P_1}, n_{P_2}, n_{Q_1}, n_{Q_2}\} - 1$.

Proof. Observe first that the maximal simulation relation $R^\star_{P_1P_2Q_1Q_2}$ can be written as

$$R^\star_{P_1P_2Q_1Q_2} = \bigcap_{i=0}^{n} \ker \Big[C_{P_1P_2}\left(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2}\right)^i \quad - C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^i\Big]$$

where

$$A_{j_1j_2} = \begin{bmatrix} A_{j_1} & 0 \\ 0 & A_{j_2} \end{bmatrix}, B_{j_1j_2} = \begin{bmatrix} 0 & -B_{j_1} \\ B_{j_2} & 0 \end{bmatrix}, C_{j_1j_2} = \begin{bmatrix} C_{j_1} & 0 \\ 0 & C_{j_2} \end{bmatrix},$$

$j \in \{P, Q\}$. Secondly, for all $(x_{P_1P_2}, x_{Q_1Q_2})$, $x_{j_1j_2} = (x_{j_1}, x_{j_2})$, such that

$$C_{P_1P_2}x_{P_1P_2} = C_{Q_1Q_2}x_{Q_1Q_2}$$

the following equivalence holds for all $k = 1, 2, \ldots$:

$$\begin{aligned} C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^k x_{P_1P_2} &= C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^k x_{Q_1Q_2} \\ &\Longleftrightarrow \\ C_{P_1P_2}A^k_{P_1P_2}x_{P_1P_2} &= C_{Q_1Q_2}A^k_{Q_1Q_2}x_{Q_1Q_2} \end{aligned} \tag{3.25}$$

Indeed, for $k = 1$, we obtain due to equality of the Markov parameters $C_{P_1P_2}B_{P_1P_2} = C_{Q_1Q_2}B_{Q_1Q_2}$ that

$$\begin{aligned} C_{P_1P_2}A_{P_1P_2}x_{P_1P_2} - C_{P_1P_2}B_{P_1P_2}C_{P_1P_2}x_{P_1P_2} &= C_{Q_1Q_2}A_{Q_1Q_2}x_{Q_1Q_2} - C_{Q_1Q_2}B_{Q_1Q_2}C_{Q_1Q_2}x_{Q_1Q_2} \\ &\Longleftrightarrow \\ C_{P_1P_2}A_{P_1P_2}x_{P_1P_2} &= C_{Q_1Q_2}A_{Q_1Q_2}x_{Q_1Q_2} \end{aligned}$$

Assuming that (3.25) holds for a certain k, we conclude

$$C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^{k+1}x_{P_1P_2} = C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^{k+1} x_{Q_1Q_2}$$

$$\Longleftrightarrow$$

$$C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^{k}A_{P_1P_2}x_{P_1P_2} - C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^{k} B_{P_1P_2}C_{P_1P_2}x_{P_1P_2} = C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^{k}A_{Q_1Q_2}x_{Q_1Q_2} - C_{Q_1Q_2} (A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^{k}B_{Q_1Q_2}C_{Q_1Q_2}x_{Q_1Q_2}$$

$$\Longleftrightarrow$$

$$C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^{k}A_{P_1P_2}x_{P_1P_2} = C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^{k} A_{Q_1Q_2}x_{Q_1Q_2}$$

$$\Longleftrightarrow$$

$$C_{P_1P_2}A^{k+1}_{P_1P_2}x_{P_1P_2} = C_{Q_1Q_2}A^{k+1}_{Q_1Q_2}x_{Q_1Q_2}$$

where we made use of

$$C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^{k}B_{P_1P_2}C_{P_1P_2}x_{P_1P_2} = C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^{k}B_{Q_1Q_2}C_{Q_1Q_2}x_{Q_1Q_2}.$$

As before,

$$C_{P_1P_2}\tilde{x}_{P_1P_2} := C_{P_1P_2}B_{P_1P_2}C_{P_1P_2}x_{P_1P_2} = C_{Q_1Q_2}B_{Q_1Q_2}C_{Q_1Q_2}x_{Q_1Q_2} =: C_{Q_1Q_2}\tilde{x}_{Q_1Q_2}$$

so that we can apply the hypothesis (3.25) on

$$C_{P_1P_2}(A_{P_1P_2} - B_{P_1P_2}C_{P_1P_2})^{k}\tilde{x}_{P_1P_2} = C_{Q_1Q_2}(A_{Q_1Q_2} - B_{Q_1Q_2}C_{Q_1Q_2})^{k}\tilde{x}_{Q_1Q_2} \ .$$

Equation (3.26) then reduces to the equality of the $k+1-$th Markov parameters. The maximal simulation relation $R^{\star}_{P_1P_2Q_1Q_2}$ can thus be constructed as the product of the maximal simulation relations $R^{\star}_{P_1Q_1}$ and $R^{\star}_{P_2Q_2}$ between Σ_{P_1} and Σ_{Q_1} on the one hand and Σ_{P_2} and Σ_{Q_2} on the other hand since the influence of the feedback terms vanishes. By Proposition 2.12 $R^{\star}_{P_1P_2Q_1Q_2}$ is also the maximal bisimulation relation between $\Sigma_{P_1}\|_o\Sigma_{P_2}$ and $\Sigma_{Q_1}\|_o\Sigma_{Q_2}$. □

Another interesting fact about open feedback interconnections becomes apparent if we consider a version of simulation relations that does not require the inputs $v_i, i = 1, 2$, to be the same. In this case Definition 3.2 is modified such that *for any* input $u_1(\cdot)$ *there should exist* an input $u_2(\cdot)$ such that (i) and (iii) hold. Then, for any two linear systems $\Sigma_i, i = 1, 2$, of the form (3.20) the open feedback interconnection $\Sigma_1\|_o\Sigma_2$ is in fact bisimilar to the two systems running in parallel [2], denoted by $\Sigma_1 \vDash \Sigma_2$ and given as

$$\Sigma_1 \vDash \Sigma_2 :$$
$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} + \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$
$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} .$$

In fact, the inputs v_i of the open feedback interconnection $\Sigma_1\|_o\Sigma_2$ and $\tilde{v}_i$ of the interconnection $\Sigma_1 \vDash \Sigma_2$ are related,

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} C_2 x_2 \\ C_1 x_1 \end{bmatrix} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} \tag{3.26}$$

Construct the bisimulation relation S between $\Sigma_1\|_o\Sigma_2$ and $\Sigma_1 \vDash \Sigma_2$ as

$$S = \{((x_1, x_2), (\tilde{x}_1, \tilde{x}_2)) \mid x_i = \tilde{x}_i,\ i = 1, 2\} .$$

It follows immediately that for any input $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ there always exists an input $\begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}$ given by (3.26) such that conditions (i) and (iii) in Definition 3.2 are fulfilled and vice versa. Hence,

$$\Sigma_1\|_o\Sigma_2 \approx \Sigma_1 \vDash \Sigma_2$$

Thus, the open feedback interconnection does not restrict the behavior of the individual components since the influence of the feedback terms can always be compensated by the additional inputs v_i.

Completeness of circular assume-guarantee reasoning

Circular assume-guarantee reasoning is known not to be complete for labeled transition systems, see e.g. the counterexample in [21]. We will show, however, that circular assume-guarantee reasoning is indeed complete for linear systems.

Theorem 3.20. *Circular assume-guarantee reasoning is complete, i.e. for any given linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ such that there exists a full simulation relation S of $\Sigma_{P_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, there also exist full simulation relations S_I and S_{II} of $\Sigma_{P_1}\|\Sigma_{Q_2}$ and $\Sigma_{Q_1}\|\Sigma_{P_2}$ by $\Sigma_{Q_1}\|\Sigma_{Q_2}$, respectively.*
The relations S_I and S_{II} are given by

$$\begin{aligned} S_I &= \{(x_{P_1}, x_{Q_2}, x_{Q_1}, x_{Q_2}) \mid \exists x_{P_2} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S\} + \\ &\quad +\{(0, x, 0, x) \mid x \in \mathcal{X}_{Q_2}\} \\ S_{II} &= \{(x_{Q_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists x_{P_1} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S\} + \\ &\quad +\{(x, 0, x, 0) \mid x \in \mathcal{X}_{Q_1}\} \end{aligned} \tag{3.27}$$

[2]This result was contributed by Paulo Tabuada, whose authorship is gratefully acknowledged

Proof. The proof is deferred to Appendix B. □

3.2.3. Proof rules involving bisimulation relations

In the previous sections, we presented compositional analysis techniques for linear systems using simulation relations to reduce the complexity of the verification task. The target system in these deduction schemes represents an abstraction of the actual system model as illustrated by Example 3.10. By contrast, bisimulation relations constitute *equivalence relations*. Hence, they can be used to exactly reduce system models by quotienting out parts of the state space, see Example 2.14. In this section, we want to investigate whether the deduction schemes of Section 3.2 also work when replacing simulation with bisimulation.

Proposition 3.21. *Consider four linear systems* $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ *of the form* (3.1)*. Then the following deduction schemes are sound:*

1. *(compositional reasoning)*

$$\left.\begin{array}{lccc} S_1: & \Sigma_{P_1} & \approx & \Sigma_{Q_1} \\ S_2: & \Sigma_{P_2} & \approx & \Sigma_{Q_2} \end{array}\right\} \implies S: \Sigma_{P_1}\|\Sigma_{P_2} \approx \Sigma_{Q_1}\|\Sigma_{Q_2}$$

2. *(non-circular assume-guarantee reasoning)*

$$\left.\begin{array}{lccc} S_1: & \Sigma_{P_1} & \approx & \Sigma_{Q_1} \\ S_{II}: & \Sigma_{Q_1}\|\Sigma_{P_2} & \approx & \Sigma_{Q_1}\|\Sigma_{Q_2} \end{array}\right\} \implies S: \Sigma_{P_1}\|\Sigma_{P_2} \approx \Sigma_{Q_1}\|\Sigma_{Q_2}$$

3. *(circular assume-guarantee reasoning)*

$$\left.\begin{array}{lccc} S_I: & \Sigma_{P_1}\|\Sigma_{Q_2} & \approx & \Sigma_{Q_1}\|\Sigma_{Q_2} \\ S_{II}: & \Sigma_{Q_1}\|\Sigma_{P_2} & \approx & \Sigma_{Q_1}\|\Sigma_{Q_2} \end{array}\right\} \iff S: \Sigma_{P_1}\|\Sigma_{P_2} \approx \Sigma_{Q_1}\|\Sigma_{Q_2}$$

Proof. Notice that in all three cases we only need to check condition (iv) in Proposition 3.3 (respectively (4) in Theorem 3.4) and fullness of the candidate relation. For rule 1, observe that the construction of S, namely taking the product $S = S_1 \times S_2$ after reordering the state variables, ensures that $\Pi_{Q_1Q_2}S = \mathcal{X}_{Q_1} \times \mathcal{X}_{Q_2}$ since both S_1 and S_2 are full bisimulation relations. Moreover, condition (iv) in Proposition 3.3 is fulfilled since for every d_{P_2} there exists a d_{P_1} such that (3.5) holds and similarly, for every d_{Q_2} there exists a d_{Q_1} such that (3.5) holds.
Since bisimulation relations are equivalence relations, they are also transitive which together with rule 1 guarantees soundness of rule 2.
Rule 3 will be proved more explicitly. Use the same construction of S as in (3.16). To show condition (4) of Theorem 3.4, observe that since $(S_I + \bar{S}_I)^{\text{sym}}$ is

a full bisimulation relation, for every $d_{Q_2} \in \text{im} L_{Q_1}$ there exists $x_{P_1} \in \text{im} L_{P_1}$, $x_{Q_2} \in \text{im} L_{Q_2} \cap \ker C_{Q_2} \cap \ker H_{Q_2}$ such that

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ d_{Q_2} \end{bmatrix} = \begin{bmatrix} x_{P_1} \\ x_{Q_2} \\ 0 \\ d_{Q_2} \end{bmatrix} + \text{im} \begin{bmatrix} L_{P_1} & 0 \\ 0 & L_{Q_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Due to symmetrization, also

$$\begin{bmatrix} x_{P_1} \\ d_{Q_2} \\ 0 \\ x_{Q_2} \end{bmatrix} \in (S_I + \bar{S}_I)^{\text{sym}} \; \forall d_{Q_2} \in \text{im} L_{Q_2}$$

Since $(S_{II} + \bar{S}_{II})^{\text{sym}}$ is also a full bisimulation relation, for every $d_{Q_2} \in \mathrm{L}_{Q_2}$ there exists $x_{Q_1} \in \text{im} L_{Q_1} \cap \ker C_{Q_1} \cap \ker H_{Q_1}$ such that

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ d_{Q_2} \end{bmatrix} = \begin{bmatrix} x_{Q_1} \\ x_{P_2} \\ 0 \\ d_{Q_2} \end{bmatrix} + \text{im} \begin{bmatrix} L_{Q_1} & 0 \\ 0 & L_{P_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_{Q_1} \\ x_{P_2} \\ 0 \\ d_{Q_2} \end{bmatrix} \in (S_{II} + \bar{S}_{II})^{\text{sym}}$$

Due to Lemma 3.13, also $(x_{Q_1}, 0, x_{Q_1}, 0) \in (S_{II} + \bar{S}_{II})^{\text{sym}}$ and therefore

$$\begin{bmatrix} x_{P_1} \\ d_{Q_2} \\ 0 \\ x_{Q_2} \end{bmatrix} \in (S_I + \bar{S}_I)^{\text{sym}}, \quad \begin{bmatrix} x_{Q_1} \\ x_{P_2} \\ 0 \\ d_{Q_2} \end{bmatrix} - \begin{bmatrix} x_{Q_1} \\ 0 \\ x_{Q_1} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ x_{P_2} \\ -x_{Q_1} \\ d_{Q_2} \end{bmatrix} \in (S_{II} + \bar{S}_{II})^{\text{sym}}$$

Hence, for any $d_{Q_2} \in \text{im} L_{Q_2}$,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ d_{Q_2} \end{bmatrix} = \underbrace{\begin{bmatrix} x_{P_1} \\ x_{P_2} \\ 0 \\ d_{Q_2} \end{bmatrix}}_{\in S} + \text{im} \begin{bmatrix} L_{P_1} & 0 \\ 0 & L_{P_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Analogously, one can proof that $\text{im} \begin{bmatrix} 0 \\ 0 \\ L_{Q_1} \\ 0 \end{bmatrix} \in S + \text{im} \begin{bmatrix} L_{P_1} & 0 \\ 0 & L_{P_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

To show that $\Pi_{Q_1 Q_2} S = \mathcal{X}_{Q_1} \times \mathcal{X}_{Q_2}$, note that for every every x_{Q_1}, x_{Q_2} there exists $\bar{x}_{P_1}, \bar{x}_{Q_2}$ such that $(\bar{x}_{P_1}, \bar{x}_{Q_2}, x_{Q_1}, x_{Q_2}) \in (S_I + \bar{S}_I)^{\text{sym}}$. Due to Lemma 3.12, then also $(\bar{x}_{P_1}, x_{Q_2}, x_{Q_1}, \bar{x}_{Q_2}) \in (S_I + \bar{S}_I)^{\text{sym}}$. Similarly, since $(S_{II} + \bar{S}_{II})^{\text{sym}}$ is full, for every x_{Q_1}, x_{Q_2} there exists $\hat{x}_{Q_1}, \hat{x}_{Q_1}$ such that $(\hat{x}_{Q_1}, \hat{x}_{P_2}, x_{Q_1}, x_{Q_2}) \in (S_{II} + \bar{S}_{II})^{\text{sym}}$, and by symmetry also $(x_{Q_1}, \hat{x}_{P_2}, \hat{x}_{Q_1}, x_{Q_2}) \in (S_{II} + \bar{S}_{II})^{\text{sym}}$. Hence, $(\bar{x}_{P_1}, \hat{x}_{P_2}, x_{Q_1}, x_{Q_2}) \in S$ for every x_{Q_1}, x_{Q_2}. □

Remark 3.22. Since bisimulation implies mutual simulation, Proposition 3.21 also proves soundness of the following type of reasoning:

$$\left.\begin{array}{ccccc}\underline{\Sigma_{Q_1}} & \preccurlyeq & \Sigma_{P_1} & \preccurlyeq & \overline{\Sigma_{Q_1}} \\ \underline{\Sigma_{Q_2}} & \preccurlyeq & \Sigma_{P_2} & \preccurlyeq & \overline{\Sigma_{Q_2}}\end{array}\right\} \Longrightarrow \left(\underline{\Sigma_{Q_1}} \| \underline{\Sigma_{Q_2}}\right) \preccurlyeq (\Sigma_{P_1} \| \Sigma_{P_2}) \preccurlyeq \left(\overline{\Sigma_{Q_1}} \| \overline{\Sigma_{Q_2}}\right)$$

Here, $\underline{\Sigma_{Q_i}}, i = 1, 2$, could represent an under-approximation of a specification or a *refinement* of Σ_{P_i} and $\overline{\Sigma_{Q_i}}$ an over-approximation or *abstraction*. A complementary notion of abstraction, refinements express distinct features of a model in more detail thus adding additional information about the system, which in the case of control systems usually involves more state variables and differential equations. Some verification schemes e.g. in [42] use both abstraction and refinement.

3.2.4. Generalization to k systems

Having completed our analysis of compositional techniques for feedback interconnections of two systems, a natural generalization is to consider interconnections of *more than two* systems. As before, the interconnection variables u and y will be used to interconnect linear systems with each other. Furthermore, we want to consider series interconnections. Hence, we partition the in- and output matrices related to the variables u_i, y_i as follows:

$$B_i = \begin{bmatrix} B_i^- & B_i^+ \end{bmatrix} \quad , \quad C_i = \begin{bmatrix} C_i^- \\ C_i^+ \end{bmatrix}, i = 1, \ldots, k$$

Every system $\Sigma_i, i = 2, \ldots, k-1$, is thus connected to a predecessor Σ_{i-1} on the left and a successor system Σ_{i+1} on the right:

$$\begin{array}{ccccccc} y_i^- & = & u_{i-1}^+ & , & y_i^+ & = & u_{i+1}^- \\ u_i^- & = & y_{i-1}^+ & , & u_i^+ & = & y_{i+1}^- \end{array}$$

The interconnection variables $u_1 = u_1^-, y_1 = y_1^-$ of the first and the variables $u_k^+ = u_k, y_k^+ = y_k$ of the last system Σ_k remain accessible, see Figure 3.6.

Remark 3.23. In the remainder of this chapter we restrict ourselves to series of feedback interconnections of linear systems. However, all the results presented here also hold for other network topologies, e.g. for the full interconnection case where each system is interconnected to every other systems by feedback.

Compositional reasoning for k systems can then be derived inductively based on the result of Section 3.2.1.

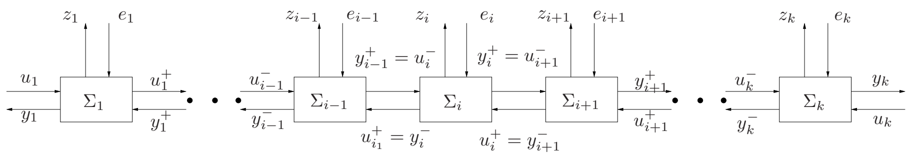

Figure 3.6.: Series interconnection $\Sigma_1\|\ldots\|\Sigma_k$.

Theorem 3.24. *Consider k linear systems Σ_{P_i} and $\Sigma_{Q_i}, i = 1\ldots,k$, of the form* (3.1)*. Then compositional reasoning is sound for series interconnections of k systems, i.e.*

$$\begin{aligned} \Sigma_{P_i} \quad &\preccurlyeq \quad \Sigma_{Q_i} \qquad\qquad \forall\, i = 1,\ldots,k \\ &\Longrightarrow \\ \Sigma_{P_1}\|\ldots\|\Sigma_{P_k} \quad &\preccurlyeq \quad \Sigma_{Q_1}\|\ldots\|\Sigma_{Q_k} \end{aligned} \tag{3.28}$$

Proof. The proof uses induction over k. Theorem 3.6 contains the proof for the case $k = 2$. Assume now that the series interconnection Σ_P of k plant-controller systems,

$$\Sigma_P := \Sigma_{P_1}\|\ldots\|\Sigma_{P_k}$$

fulfills a composed specification Σ_Q of the form

$$\Sigma_S := \Sigma_{S_1}\|\ldots\|\Sigma_{Q_k} \ ,$$

i.e.,

$$\Sigma_P \quad \preccurlyeq \quad \Sigma_Q \tag{3.29}$$

Moreover, let there exist a full simulation relation of $\Sigma_{P_{k+1}}$ by $\Sigma_{Q_{k+1}}$,

$$\Sigma_{P_{k+1}} \quad \preccurlyeq \quad \Sigma_{Q_{k+1}} \tag{3.30}$$

Taking the product of the full simulation relations for (3.29) and (3.30) yields, after reordering the state components, a full simulation relation of

$$\Sigma_P\|\Sigma_{P_{k+1}} \preccurlyeq \Sigma_Q\|\Sigma_{Q_{k+1}} \tag{3.31}$$

This proves the induction step. □

In contrast to this straightforward generalization, proving soundness of circular assume-guarantee reasoning for k linear systems is more involved. We need an auxiliary result that extends the proof rule of Theorem 3.14 by interconnecting arbitrary systems from the left and right.

Lemma 3.25. *Consider six linear control systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2, L, R\}$ of the form* (3.1). *Then the following reasoning is sound:*

$$\begin{array}{rlcl} S_I: & \Sigma_L\|\Sigma_{P_1}\|\Sigma_{Q_2}\|\Sigma_R & \preccurlyeq & \Sigma_L\|\Sigma_{Q_1}\|\Sigma_{Q_2}\|\Sigma_R \\ S_{II}: & \Sigma_L\|\Sigma_{Q_1}\|\Sigma_{P_2}\|\Sigma_R & \preccurlyeq & \Sigma_L\|\Sigma_{Q_1}\|\Sigma_{Q_2}\|\Sigma_R \\ & & \Downarrow & \\ S: & \Sigma_L\|\Sigma_{P_1}\|\Sigma_{P_2}\|\Sigma_R & \preccurlyeq & \Sigma_L\|\Sigma_{Q_1}\|\Sigma_{Q_2}\|\Sigma_R \end{array} \tag{3.32}$$

Proof. In order to prove this lemma, we make use of Theorem 3.14 and Proposition 3.16. That is, we extend $S_i, i = I, II$, in two steps. First, consider

$$\begin{aligned} S'_I &:= \{(x_L, x_{P_1}, x_{Q_2}, x_R, x_L, x_{Q_1}, \bar{x}_{Q_2}, x_R) \mid \exists x'_L, x'_R : \\ &\quad (x_L, x_{P_1}, x_{Q_2}, x_R, x'_L, x_{Q_1}, \bar{x}_{Q_2}, x'_R) \in S_I\} \\ S'_{II} &:= \{(x_L, x_{Q_1}, x_{P_2}, x_R, x_L, \bar{x}_{Q_1}, x_{Q_2}, x_R) \mid \exists x'_L, x'_R : \\ &\quad (x_L, x_{Q_1}, x_{P_2}, x_R, x'_L, \bar{x}_{Q_1}, x_{Q_2}, x'_R) \in S_{II}\} \end{aligned}$$

The fact that $S'_i, i = I, II$, are also full simulation relations is a consequence of Proposition 3.16 and will not be proved explicitly. Second, add suitable subspaces to obtain the relations $\left(S'_i + \bar{S}'_i\right)^{\text{sym}}$ with

$$\begin{aligned} \bar{S}'_I &= \{(x_L, x_{P_1}, \bar{x}_{Q_2}, x_R, x_L, x_{Q_1}, -x_{Q_2}, x_L) \mid \bar{x}_{Q_2} \in \ker H_{Q_2} \cap \ker C_{Q_2}, \\ &\quad x_i \in \ker H_i \cap \ker C_i, i \in \{L, P_1, Q_2, Q_1, R\}, \\ &\quad (x_L, x_{P_1}, x_{Q_2}, x_R, x_L, x_{Q_1}, \bar{x}_{Q_2}, x_L) \in S'_I\} \\ \bar{S}'_{II} &= \{(x_L, \bar{x}_{Q_1}, x_{P_2}, x_R, x_L, -x_{Q_1}, x_{Q_2}, x_L) \mid \bar{x}_{Q_1} \in \ker H_{Q_1} \cap \ker C_{Q_1}, \\ &\quad x_i \in \ker H_i \cap \ker C_i, i \in \{L, P_2, Q_1, Q_2, R\}, \\ &\quad (x_L, x_{Q_1}, x_{P_2}, x_R, x_L, \bar{x}_{Q_1}, x_{Q_2}, x_L) \in S'_{II}\} \end{aligned}$$

and

$$\begin{aligned} \left(S'_O + \bar{S}'_I\right)^{\text{sym}} &= \{(x_L, x_{P_1}, \bar{x}_{Q_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_L) \mid \\ &\quad (x_L, x_{P_1}, x_{Q_2}, x_R, x_L, x_{Q_1}, \bar{x}_{Q_2}, x_L) \in \left(S'_I + \bar{S}'_I\right)\} \\ \left(S'_{II} + \bar{S}'_{II}\right)^{\text{sym}} &= \{(x_L, \bar{x}_{Q_1}, x_{Q_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_L) \mid \\ &\quad (x_L, x_{Q_1}, x_{P_2}, x_R, x_L, \bar{x}_{Q_1}, x_{Q_2}, x_L) \in \left(S'_{II} + \bar{S}'_{II}\right)\} \end{aligned}$$

Similarly as before, construct S as

$$\begin{aligned} S &= \{(x_L, x_{P_1}, x_{P_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_R) \mid \exists\, \bar{x}_{Q_1}, \bar{x}_{Q_2} : \\ &\quad (x_L, x_{P_1}, \bar{x}_{Q_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_R) \in \left(S'_I + \bar{S}'_I\right)^{\text{sym}}, \\ &\quad (x_L, \bar{x}_{Q_1}, x_{P_2}, x_R, x_L, x_{Q_1}, x_{Q_2}, x_R) \in \left(S'_{II} + \bar{S}'_{II}\right)^{\text{sym}}\} \end{aligned} \tag{3.33}$$

- The proofs that $S'_i + \bar{S}'_i, i = I, II$, and $(S'_i)^{\text{sym}}$ are full simulation relations of $\Sigma_L\|\Sigma_{P_1}\|\Sigma_{Q_2}\|\Sigma_R$ and of $\Sigma_L\|\Sigma_{Q_1}\|\Sigma_{P_2}\|\Sigma_R$ by $\Sigma_L\|\Sigma_{Q_1}\|\Sigma_{Q_2}\|\Sigma_R$, respectively, are analogous to the proofs of Lemma 3.11 and Lemma 3.12,

- The proof that for every $x \in \ker H_{Q_2} \cap C_{Q_2}, y \in \ker H_{Q_1} \cap \ker C_{Q_1}$, there exist elements $(0,0,x,0,0,0,x,0) \in \left(S'_I + \bar{S}'_I\right)^{\mathrm{sym}}$ and $(0,y,0,0,0,y,0,0) \in \left(S'_{II} + \bar{S}'_{II}\right)^{\mathrm{sym}}$ is similar to the proof of Lemma 3.13.

Finally, we have to prove that S as constructed by (3.33) is indeed a full simulation relation of $\Sigma_L \| \Sigma_{P_1} \| \Sigma_{P_2} \| \Sigma_R$ by $\Sigma_L \| \Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_R$. To do so, the proof of Theorem 3.14 can be repeated in all its steps. □

We are now in the position to prove soundness of circular assume-guarantee reasoning for feedback interconnections of arbitrarily yet finitely many linear systems.

Theorem 3.26. *Consider k linear systems Σ_{P_i} and $\Sigma_{Q_i} i = 1, \ldots, k$ of the form* (3.1). *Let k circularly dependent conditions*

$$\begin{array}{llcl} S_I : & \Sigma_{P_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{Q_k} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{Q_k} \\ S_{II} : & \Sigma_{Q_1} \| \Sigma_{P_2} \| \ldots \| \Sigma_{Q_k} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{Q_k} \\ \vdots & \vdots & \vdots & \vdots \\ S_k : & \Sigma_{Q_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{P_k} & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{Q_k} \end{array} \tag{3.34}$$

be fulfilled. Then the global interconnected plant

$$\Sigma_P := \Sigma_{P_1} \| \ldots \| \Sigma_{P_k}$$

fulfills its global specification

$$\Sigma_Q := \Sigma_{Q_1} \| \Sigma_{Q_2} \| \ldots \| \Sigma_{Q_k} \ ,$$

i.e.,

$$S : \ \Sigma_P \ \preccurlyeq \ \Sigma_Q$$

Proof. The idea of this proof is to successively combine the k conditions (3.34) and apply Lemma 3.25 at every step. Let k simulation relations $S_i, i = I, \ldots, k$ as in (3.34) be given, i.e.

$$S_i : \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_{i-1}} \| \Sigma_{P_i} \| \Sigma_{Q_{i+1}} \| \ldots \| \Sigma_{Q_k} \preccurlyeq \Sigma_Q \tag{3.35}$$

Starting with $i = 1$, consider $\lfloor \frac{k}{2} \rfloor$ pairs of two relations S_i and $S_{i+1}, i = 1, \ldots, k-1$ where $\lfloor x \rfloor$ is the greatest natural number less or equal to x. Apply Lemma 3.25 to each of the $\lfloor \frac{k}{2} \rfloor$ pairs to obtain simulation relations $S_{i,i+1}$ of the form

$$S_{i,i+1} : \Sigma_{Q_1} \| \ldots \| \Sigma_{Q_{i-1}} \| \Sigma_{P_i} \| \Sigma_{P_{i+1}} \| \ldots \| \Sigma_{Q_k} \preccurlyeq \Sigma_Q \tag{3.36}$$

After this first step, $\lceil \frac{k}{2} \rceil$ simulation relations are left where $\lceil x \rceil$ is the smallest natural number greater or equal to x.

$$\begin{cases} \{S_{i,i+1}, i = 1, \ldots, k-1\}, & \text{if } \lfloor \frac{k}{2} \rfloor = \frac{k}{2} \\ \{\{S_{i,i+1}, i = 1, \ldots, k-2\}, S_k\}, & \text{otherwise} \end{cases} \tag{3.37}$$

Continue by forming $\lfloor \lceil \frac{k}{2} \rceil \rfloor$ pairs of two simulation relations $S_j, S_j \in$ (3.37) to apply Lemma 3.25 on them. Repeating this procedure $\lceil \frac{k}{2} \rceil$- times in total, the desired result follows in the last step. We formalize this approach in the following

Algorithm 3.27. Compute S from k simulation relations $S_i, i = 1, \ldots, k$ of the form (3.34)

$k = \lfloor \frac{N}{2} \rfloor$
$R = \{S_i, i = I, \ldots, k\}$
for $i = 1$ to $\lceil \frac{k}{2} \rceil$ **do**
  $k = |R|$
  for $j = 1$ to $\lfloor \frac{k}{2} \rfloor$ **do**
    apply Lemma 3.25 to $S_{2j-1}, S_{2j}, S_j \in R$ to obtain the relations $S_{2j-1,2j}$ as given by (3.36)
  end for
  if $\frac{k}{2} == \lfloor \frac{k}{2} \rfloor$ **then**
    $R = \{S_{2j-1,2j}, j = 1, \ldots, \lfloor \frac{k}{2} \rfloor\}$
  else
    $R = \left\{\{S_{2j-1,2j}, j = 1, \ldots, \lfloor \frac{k}{2} \rfloor\}, S_k\right\}$
  end if
end for
$S = R$

□

We illustrate Theorem 3.26 and Algorithm 3.27 with the following example.

Example 3.28. Let the linear systems $\Sigma_{P_i}, \Sigma_{Q_i}, i = 1, \ldots, 5$, of the form (3.1) be given such that there exist full simulation relations S_i as follows:

$$\begin{array}{rlcl}
S_I: & \Sigma_{P_1} \| \Sigma_{Q_2} \| (\Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5}) & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \| (\Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5}) \\
S_{II}: & \Sigma_{Q_1} \| \Sigma_{P_2} \| (\Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5}) & \preccurlyeq & \Sigma_{Q_1} \| \Sigma_{Q_2} \| (\Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5}) \\
S_{III}: & (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| \Sigma_{P_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5} & \preccurlyeq & (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| \Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5} \\
S_{IV}: & (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| \Sigma_{Q_3} \| \Sigma_{P_4} \| \Sigma_{Q_5} & \preccurlyeq & (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| \Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5} \\
S_V: & (\Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \| \Sigma_{Q_4}) \| \Sigma_{P_5} & \preccurlyeq & (\Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \| \Sigma_{Q_4}) \| \Sigma_{Q_5}
\end{array}$$

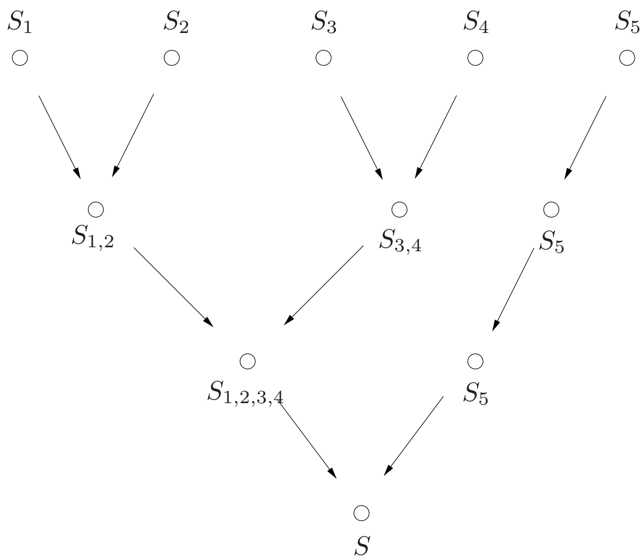

Figure 3.7.: Algorithm 3.27 applied to Example 3.28.

In the first step, we pair the relations S_I, S_{II} and S_{III}, S_{IV}. Applying Lemma 3.25 to each pair, we conclude that there exist full simulation relations $S_{I,II}$ and $S_{III,IV}$ such that

$$S_{I,II} : (\Sigma_{P_1} \| \Sigma_{P_2}) \| (\Sigma_{Q_3} \| \Sigma_{Q_4}) \| \Sigma_{Q_5} \preccurlyeq (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| (\Sigma_{Q_3} \| \Sigma_{Q_4}) \| \Sigma_{Q_5}$$
$$S_{III,IV} : (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| (\Sigma_{P_3} \| \Sigma_{P_4}) \| \Sigma_{Q_5} \preccurlyeq (\Sigma_{Q_1} \| \Sigma_{Q_2}) \| (\Sigma_{Q_3} \| \Sigma_{Q_4}) \| \Sigma_{Q_5}$$

In the second step, Lemma 3.25 can now be applied to $S_{I,II}$ and $S_{III,IV}$ to obtain

$$S_{I,II,III,IV} : (\Sigma_{P_1} \| \Sigma_{P_2} \| \Sigma_{P_3} \| \Sigma_{P_4}) \| \Sigma_{Q_5} \quad \preccurlyeq \quad (\Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \| \Sigma_{Q_4}) \| \Sigma_{Q_5}$$

In the third step, consider the relations $S_{I,II,III,IV}$ and S_V. As a special case of Lemma 3.25, they fulfill the circular assume-guarantee rule of Theorem 3.14. Hence, there indeed exists a full simulation relation S of $\Sigma_{P_1} \| \Sigma_{P_2} \| \Sigma_{P_3} \| \Sigma_{P_4} \| \Sigma_{P_5}$ by $\Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5}$,

$$S : \Sigma_{P_1} \| \Sigma_{P_2} \| \Sigma_{P_3} \| \Sigma_{P_4} \| \Sigma_{P_5} \quad \preccurlyeq \quad \Sigma_{Q_1} \| \Sigma_{Q_2} \| \Sigma_{Q_3} \| \Sigma_{Q_4} \| \Sigma_{Q_5}$$

Remark 3.29. Without formalizing it in form of a theorem it is worth pointing out that apart from compositional and circular assume-guarantee reasoning rules also triangular proof rules based on non-circular assume-guarantee reasoning can be developed for more than two systems. Like in Theorem 3.9, soundness of non-circular assume-guarantee reasoning is ensured by compositionality of series interconnections (Theorem 3.24) and transitivity of simulation. Example 6.8 in Chapter 6 describes a decentralized control scheme based on a triangular rule for 3 systems.

Finally, we investigate whether circular assume-guarantee reasoning for k linear systems is also complete.

Proposition 3.30. *Let S be a full simulation relation of $\Sigma_{P_1}\|\ldots\|\Sigma_{P_k}$ by $\Sigma_{Q_1}\|\ldots\|\Sigma_{Q_k}$. Then it holds for every $i = 1,\ldots,k$ that*

$$\Sigma_{Q_1}\|\ldots\Sigma_{Q_{i-1}}\|\Sigma_{P_i}\|\Sigma_{Q_{i+1}}\|\ldots\|\Sigma_{P_k} \preccurlyeq \Sigma_{Q_1}\|\ldots\|\Sigma_{Q_k}\ , \tag{3.38}$$

where the full simulation relation S_i of $\Sigma_{Q_1}\|\ldots\Sigma_{Q_{i-1}}\|\Sigma_{P_i}\|\Sigma_{Q_{i+1}}\|\ldots\|\Sigma_{P_k}$ by $\Sigma_{Q_1}\|\ldots\|\Sigma_{Q_k}$ can be constructed as

$$\begin{aligned} S_i \quad = \quad & \{(x_{Q_1},\ldots,x_{Q_{i-1}},x_{P_i},x_{Q_{i+1}},\ldots,x_{Q_k},x_{Q_1},\ldots,x_{Q_k}) \mid \exists x_{P_j}, \\ & j=1,\ldots,k, j\neq i : (x_{P_1},\ldots,x_{P_k},x_{Q_1},\ldots,x_{Q_k}) \in S\} \\ & +\{(a_1,\ldots,a_{i-1},0,a_{i+1},\ldots,a_k,a_1,\ldots,a_{i-1},0,a_{i+1},\ldots,a_k), a_j \in \mathcal{X}_{Q_j}\} \end{aligned} \tag{3.39}$$

Proof. This statement can be proved in the same way as Theorem 3.20. □

It follows immediately that circular assume-guarantee reasoning for k systems is indeed complete.

Theorem 3.31. *Given k linear systems Σ_{P_i} and Σ_{Q_i}, $i = 1,\ldots,k$, of the form* (3.1). *Suppose there exists a full simulation relation S of $\Sigma_{P_1}\|\ldots\|\Sigma_{P_k}$ by $\Sigma_{Q_1}\|\ldots\|\Sigma_{Q_k}$. Then the following holds:*

$$\begin{array}{lrcl} S: & \Sigma_{P_1}\|\ldots\|\Sigma_{P_k} & \preccurlyeq & \Sigma_{Q_1}\|\ldots\|\Sigma_{Q_k} \\ & & \Downarrow & \\ S_I: & \Sigma_{P_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k} & \preccurlyeq & \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k} \\ S_{II}: & \Sigma_{Q_1}\|\Sigma_{P_2}\|\ldots\|\Sigma_{Q_k} & \preccurlyeq & \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k} \\ \vdots & \vdots & \vdots & \vdots \\ S_k: & \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{P_k} & \preccurlyeq & \Sigma_{Q_1}\|\Sigma_{Q_2}\|\ldots\|\Sigma_{Q_k} \end{array} \tag{3.40}$$

3.3. Interconnections with algebraic constraints

In many applications, the interconnection of subsystems introduces *algebraic constraints* on the state variables. In this section, we consider *parallel compositions* of linear systems as a specific type of interconnection inducing algebraic constraints on the states. One way to incorporate these constraints is to use the representation of a DAE system. We first show that the notions of simulation relations for parallel compositions derived from the original definition for linear systems (Definition 2.5) on the one hand and from the definition for DAE systems [75] on the other hand are equivalent. Once this relation is established, we investigate compositional and assume-guarantee reasoning rules for parallel compositions. Specifically for parallel compositions a reasoning scheme based on the decomposition of the global specification into subspecifications is shown to hold true to round off this section.

Definition 3.32. Given two linear dynamical systems Σ_i, $i = 1, 2$, of the form

$$\Sigma_i : \begin{array}{rcl} \dot{x}_i & = & A_i x_i + B_i u_i + L_i d_i \\ y_i & = & C_i x_i \end{array} \tag{3.41}$$

where $x_i \in \mathcal{X}_i = \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^p, d_i \in \mathcal{D}_i$ and $y_i \in \mathbb{R}^q$.
Then the parallel composition $\Sigma_1 \|_{\mathrm{pc}} \Sigma_2$ is given by

$$\Sigma_1 \|_{\mathrm{pc}} \Sigma_2 : \begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \\ & & + \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ y & = & C_1 x_1 = C_2 x_2 \end{array} \tag{3.42}$$

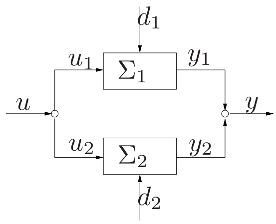

Figure 3.8.: $\Sigma_1 \|_{\mathrm{pc}} \Sigma_2$

Since parallel composition entails the algebraic constraint $C_1 x_1 = C_2 x_2$, depicted in Figure 3.8, the equations (3.42) can be rewritten in differential-algebraic form as

$$\Sigma_{12} : \begin{array}{rcl} E_{12}\dot{z}_{12} & = & A_{12} z_{12}, z_{12} \in \mathcal{Z}_{12} \\ w_{12} & = & C_{12} z_{12} \end{array} \tag{3.43}$$

where the matrices E_{12}, A_{12}, C_{12} and the state and output vectors z_{12} and w_{12} are given by

$$\begin{aligned} z_{12} &= \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}, A_{12} = \begin{bmatrix} L_1^{\perp} A_1 & 0 & L_1^{\perp} B_1 \\ 0 & L_2^{\perp} A_2 & L_2^{\perp} B_2 \\ C_1 & -C_2 & 0 \end{bmatrix}, \\ w_{12} &= \begin{bmatrix} y \\ u \end{bmatrix}, E_{12} = \begin{bmatrix} L_1^{\perp} & 0 & 0 \\ 0 & L_2^{\perp} & 0 \\ 0 & 0 & 0 \end{bmatrix}, C_{12} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \end{aligned} \tag{3.44}$$

respectively, where $L_1^{\perp}$ and $L_2^{\perp}$ are left annihilating matrices of L_1 and L_2 of maximal rank. The set of states and inputs consistent with these constraints is defined by the consistent subspace.

Definition 3.33. Consider a system Σ_{12} of the form (3.43). Then the consistent subspace $\mathcal{V}_{12}^{\star}$ for Σ_{12} is the largest subspace $\mathcal{V}_{12} \subset \mathcal{Z}_{12}$ such that

$$A_{12}\mathcal{V}_{12} \subset E_{12}\mathcal{V}_{12} \tag{3.45}$$

Furthermore, denote by $\mathcal{W}_{12}^{\star}$ and $\mathcal{U}_{12}^{\star}$ the projections

$$\mathcal{W}_{12}^{\star} = \Pi_{\mathcal{X}_1\mathcal{X}_2}\mathcal{V}_{12}^{\star} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; \exists u : \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} \in \mathcal{V}_{12}^{\star} \right\} \tag{3.46}$$

$$\mathcal{U}_{12}^{\star} = \Pi_{\mathcal{U}}\mathcal{V}_{12}^{\star} = \left\{ u \;\middle|\; \exists x_1, x_2 : \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} \in \mathcal{V}_{12}^{\star} \right\} \tag{3.47}$$

This allows us to specialize the general definition of simulation relations (Definition 3.2) to linear systems of the form (3.42) respectively (3.43), compare also with [75].

Definition 3.34. Given two linear systems Σ_i, $i = \{P_1P_2, Q_1Q_2\}$ of the form (3.43) with consistent subspaces $\mathcal{V}_i^{\star}$. Then a subspace $\tilde{S} \subset \mathcal{Z}_{P_1P_2} \times \mathcal{Z}_{Q_1Q_2}$ with $\Pi_{P_1P_2}\tilde{S} \subset \mathcal{V}_{P_1P_2}^{\star}$ is a simulation relation of $\tilde{\Sigma}_{P_1P_2}$ by $\tilde{\Sigma}_{Q_1Q_2}$ if and only if for all $(z_{P_1P_2}, z_{Q_1Q_2}) \in S$,

1. for all $v_{P_1P_2} \in \mathcal{V}_{P_1P_2}^{\star}$ such that $E_{P_1P_2}v_{P_1P_2} = A_{P_1P_2}z_{P_1P_2}$ there should exist a $v_{Q_1Q_2} \in \mathcal{V}_{Q_1Q_2}^{\star}$ such that $E_{Q_1Q_2}v_{Q_1Q_2} = A_{Q_1Q_2}z_{Q_1Q_2}$ and $(v_{P_1P_2}, v_{Q_1Q_2}) \in \tilde{S}$
2. $C_{P_1P_2}z_{P_1P_2} = C_{Q_1Q_2}z_{Q_1Q_2}$

The simulation relation $\tilde{S}$ is *full*, denoted by $\Sigma_{P_1P_2} \preccurlyeq \Sigma_{Q_1Q_2}$, if the projection on $\mathcal{Z}_{P_1P_2}$ equals the consistent subspace, that is, $\Pi_{P_1P_2}\tilde{S} = \mathcal{V}_{12}^{\star}$.

The linear algebraic characterization is derived similarly to Proposition 3.3 and Theorem 3.4.

Proposition 3.35. *There exists a simulation relation* $S \subset \mathcal{X}_{P_1} \times \mathcal{X}_{P_2} \times \mathcal{X}_{Q_1} \times \mathcal{X}_{Q_2}$ *of* $\Sigma_{P_1} \|_{pc} \Sigma_{P_2}$ *by* $\Sigma_{Q_1} \|_{pc} \Sigma_{Q_2}$ *if and only if for all* $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$ *and all* $u \in \mathcal{U}^{\star}_{P_1 P_2}$ *the following holds:*

1. $\forall \begin{bmatrix} d_{P_1} \\ d_{P_2} \end{bmatrix} \in \mathcal{D}_{P_1} \times \mathcal{D}_{P_2} \quad \exists \quad \begin{bmatrix} d_{Q_1} \\ d_{Q_2} \end{bmatrix} \in \mathcal{D}_{Q_1} \times \mathcal{D}_{Q_2}$: $\begin{bmatrix} A_{P_1} x_{P_1} + B_{P_1} u + L_{P_1} d_{P_1} \\ A_{P_2} x_{P_2} + B_{P_2} u + L_{P_2} d_{P_2} \\ A_{Q_1} x_{Q_1} + B_{Q_1} u + L_{Q_1} d_{Q_1} \\ A_{Q_2} x_{Q_2} + B_{Q_2} u + L_{Q_2} d_{Q_2} \end{bmatrix} \in S$
2. $C_{P_1} x_{P_1} = C_{P_2} x_{P_2} = C_{Q_1} x_{Q_1} = C_{Q_2} x_{Q_2}$

Proof. With the system matrices (3.44), condition 2. in Definition 3.34 yields

$$u_{P_1} = u_{Q_1} \tag{3.48}$$

and

$$C_{P_1} x_{P_1} = C_{Q_1} x_{Q_1} \tag{3.49}$$

Writing out condition 1. from Definition 3.34 results in

$$\begin{bmatrix} A_{P_1} x_{P_1} + B_{P_1} u_{P_1} + L_{P_1} d_{P_1} \\ A_{P_2} x_{P_2} + B_{P_2} u_{P_1} + L_{P_2} d_{P_2} \\ A_{Q_1} x_{Q_1} + B_{Q_1} u_{Q_1} + L_{Q_1} d_{Q_1} \\ A_{Q_2} x_{Q_2} + B_{Q_2} u_{Q_1} + L_{Q_2} d_{Q_2} \end{bmatrix} \in S \tag{3.50}$$

and

$$C_{P_1} x_{P_1} = C_{P_2} x_{P_2}, C_{Q_1} x_{Q_1} = C_{Q_2} x_{Q_2} \tag{3.51}$$

for all $(x_{P_1}, x_{P_2}, u_{P_1}, x_{Q_1}, x_{Q_2}, u_{Q_1}) \in \tilde{S}$ and

$$u_{P_1} \in \mathcal{U}^{\star}_{P_1 P_2} \tag{3.52}$$

Thus, equations (3.48) – (3.52) are equivalent to the conditions 1. and 2. in Definition 3.34. □

Proposition 3.36. *There exists a simulation relation* $S \subset \mathcal{X}_{P_1} \times \mathcal{X}_{P_2} \times \mathcal{X}_{Q_1} \times \mathcal{X}_{Q_2}$ *of* $\Sigma_{P_1} \|_{pc} \Sigma_{P_2}$ *by* $\Sigma_{Q_1} \|_{pc} \Sigma_{Q_2}$ *if and only if the following conditions hold:*

1. *diag* $\{A_{P_1}, A_{P_2}, A_{Q_1}, A_{Q_2}\} S \subset S + \operatorname{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}$

2. $\operatorname{im}\begin{bmatrix} L_{P_1} & 0 \\ 0 & L_{P_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \subset S + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}$

3. $\operatorname{im}\begin{bmatrix} B_{P_1} \\ B_{P_2} \\ B_{Q_1} \\ B_{Q_2} \end{bmatrix} \cap \left(\mathcal{W}^\star_{P_1P_2} \times \mathcal{W}^\star_{Q_1Q_2}\right) \subset S + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}$

4. $S \subset \ker\begin{bmatrix} C_{P_1} & -C_{P_2} & 0 & 0 \\ 0 & 0 & C_{Q_1} & -C_{Q_2} \\ C_{P_1} & 0 & -C_{Q_1} & 0 \end{bmatrix}$

Proof. Condition 2 in Proposition 3.35 is equivalent to condition 4 in Proposition 3.36. Condition 1 in Proposition 3.35 results in

$$\operatorname{diag}\{A_{P_1}, A_{P_2}, A_{Q_1}, A_{Q_2}\}S + \operatorname{im}\begin{bmatrix} B_{P_1} \\ B_{P_2} \\ B_{Q_1} \\ B_{Q_2} \end{bmatrix} \subset S + \operatorname{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{Q_1} & 0 \\ 0 & L_{Q_2} \end{bmatrix}.$$

Since u is restricted to $u \in \mathcal{U}^\star_{P_1P_2}$, the image of the input map has to be restricted to the subspace of all admissible inputs, which is given by

$$\{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid \exists u : (x_{P_1}, x_{P_2}, u) \in \mathcal{V}^\star_{P_1P_2}, (x_{Q_1}, x_{Q_2}, u) \in \mathcal{V}^\star_{Q_1Q_2}\} = \mathcal{W}^\star_{P_1P_2} \times \mathcal{W}^\star_{Q_1Q_2}$$

Therefore, conditions 1 – 3 in Proposition 3.36 are equivalent to condition 1 in Proposition 3.35. ☐

Proposition 3.37. *Simulation of systems interconnected by parallel composition defines a preorder.*

Proof. (*reflexivity*): Clearly, the identity relation

$$S_{\text{id}} = \{((x_P, x_Q), (x_p, x_Q)) \mid (x_P, x_Q) \in \mathcal{W}_{PQ}\}$$

defines a full simulation relation of $\Sigma_P \|_{\text{pc}} \Sigma_Q$ by itself.
(*transitivity*): Let S_1 and S_2 be full simulation relations of $\Sigma_{P_1}\|_{\text{pc}}\Sigma_{P_2}$ by $\Sigma_{Q_1}\|_{\text{pc}}\Sigma_{Q_2}$ and of $\Sigma_{Q_1}\|_{\text{pc}}\Sigma_{Q_2}$ by $\Sigma_{R_1}\|_{\text{pc}}\Sigma_{R_2}$, respectively. Then

$$\begin{aligned} S \;:=\; & \{(x_{P_1}, x_{P_2}, x_{R_1}, x_{R_2}) \mid \exists x_{Q_1}, x_{Q_2} : (x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S_1, \\ & (x_{Q_1}, x_{Q_2}, x_{R_1}, x_{R_2}) \in S_2\} \end{aligned}$$

defines a full simulation relation of $\Sigma_{P_1}\|_{\text{pc}}\Sigma_{P_2}$ by $\Sigma_{R_1}\|_{\text{pc}}\Sigma_{R_2}$. Take any $(x_{P_1}, x_{P_2}, u) \in \mathcal{V}_{P_1P_2}$. Then there exists a (x_{Q_1}, x_{Q_2}) such that $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in$

S_1 and $(x_{Q_1}, x_{Q_2}, x_{R_1}, x_{R_2}) \in S_2$. Thus, for every d_{P_1}, d_{P_2} there exist d_{Q_1}, d_{Q_2} such that

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u + L_{P_1}d_{P_1} \\ A_{P_2}x_{P_2} + B_{P_2}u + L_{P_2}d_{P_2} \\ A_{Q_1}x_{Q_1} + B_{Q_1}u + L_{Q_1}d_{Q_1} \\ A_{Q_2}x_{Q_2} + B_{Q_2}u + L_{Q_2}d_{Q_2} \end{bmatrix} \in S_1$$
$$C_{P_1}x_{P_1} = C_{P_2}x_{P_2} = C_{Q_1}x_{Q_1} = C_{Q_2}x_{Q_2}, (x_{Q_1}, x_{Q_2}, u) \in \mathcal{U}^{\star}_{Q_1Q_2}$$

since S_1 is full. Moreover, since S_2 is also full, for the same u and d_{Q_1}, d_{Q_2} there exist d_{R_1}, d_{R_2} such that

$$\begin{bmatrix} A_{Q_1}x_{Q_1} + B_{Q_1}u + L_{Q_1}d_{Q_1} \\ A_{Q_2}x_{Q_2} + B_{Q_2}u + L_{Q_2}d_{Q_2} \\ A_{R_1}x_{R_1} + B_{R_1}u + L_{R_1}d_{R_1} \\ A_{R_2}x_{R_2} + B_{R_2}u + L_{R_2}d_{R_2} \end{bmatrix} \in S_2$$
$$C_{R_1}x_{R_1} = C_{R_2}x_{R_2} = C_{Q_1}x_{Q_1} = C_{Q_2}x_{Q_2}, (x_{R_1}, x_{R_2}, u) \in \mathcal{U}^{\star}_{Q_1Q_2}$$

and hence $(x_{P_1}, x_{P_2}, x_{R_1}, x_{R_2}) \in S$ as well as

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u + L_{P_1}d_{P_1} \\ A_{P_2}x_{P_2} + B_{P_2}u + L_{P_2}d_{P_2} \\ A_{R_1}x_{R_1} + B_{R_1}u + L_{R_1}d_{R_1} \\ A_{R_2}x_{R_2} + B_{R_2}u + L_{R_2}d_{R_2} \end{bmatrix} \in S$$
$$C_{R_1}x_{R_1} = C_{R_2}x_{R_2} = C_{P_1}x_{P_1} = C_{P_2}x_{P_2}$$

Moreover, since this holds for any $(x_{P_1}, x_{P_2}, u) \in \mathcal{V}^{\star}_{P_1P_2}$, S is also full. □

3.3.1. Compositional reasoning

We begin our analysis of parallel compositions by examining the compositionality property.

Theorem 3.38. *Given any four systems Σ_i, $i \in \{P_1, P_2, Q_1, Q_2\}$ of the form (3.41) respectively* (3.43). *Then parallel composition is compositional, i.e.*

$$\left.\begin{matrix} S_1: & \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_2 & :\Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_2} \end{matrix}\right\} \implies \Sigma_{P_1}\|_{pc}\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_{pc}\Sigma_{Q_2} \qquad (3.53)$$

Proof. Construct the relation S from given full simulation relations S_1 and S_2 of Σ_{P_1} and Σ_{P_2} by Σ_{Q_1}, respectively Σ_{Q_2}, as the product relation

$$S = \{(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \mid (x_{P_1}, x_{Q_1}) \in S_1, (x_{P_2}, x_{Q_2}) \in S_2\} \qquad (3.54)$$

Then for any $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$, any joint input $u \in \mathcal{U}^{\star}_{P_1P_2}$ and any d_{P_1}, d_{P_2} there exist d_{Q_1}, d_{Q_2} such that

$$\begin{bmatrix} A_{P_1}x_{P_1} + B_{P_1}u + G_{P_1}d_{P_1} \\ A_{P_2}x_{P_2} + B_{P_2}u + G_{P_2}d_{P_2} \\ A_{Q_1}x_{Q_1} + B_{Q_1}u + G_{Q_1}d_{Q_1} \\ A_{Q_2}x_{Q_2} + B_{Q_2}u + G_{Q_2}d_{Q_2} \end{bmatrix} \in S ,$$

since for any d_{P_1} there exists a d_{Q_1} such that

$$\begin{bmatrix} A_{P_i}x_{P_i} + B_{P_i}u + G_{P_1}d_{P_1} \\ A_{Q_i}x_{Q_i} + B_{P_i}u + G_{Q_1}d_{Q_1} \end{bmatrix} \in S_i, i = 1, 2$$

for all $u \in \mathcal{U}$. Moreover, since $y_{P_1} = y_{Q_1}$ due to S_1 and $y_{P_2} = y_{Q_2}$ due to S_2 and $y_{P_1} = y_{P_2}$ as well as $y_{Q_1} = y_{Q_2}$ enforced by parallel composition, condition (ii) in Proposition 3.35 is also fulfilled which proves that S is indeed a simulation relation of $\Sigma_{P_1}\|_{\mathrm{pc}}\Sigma_{P_2}$ by $\Sigma_{Q_1}\|_{\mathrm{pc}}\Sigma_{Q_2}$.
To show that S as defined in (3.54) is full, observe that (3.55) holds for all u. Since both S_1 and S_2 are full, we can find for every $u \in \mathcal{U}^{\star}_{P_1P_2}$ and every $(x_{P_1}, x_{P_2}) \in \mathcal{W}^{\star}_{P_1P_2}$ elements x_{Q_1}, x_{Q_2} such that $(x_{P_i}, x_{Q_i}) \in S_i, i = 1, 2$ and thus $(x_{P_1}, x_{P_2}, x_{Q_1}, x_{Q_2}) \in S$. ☐

The converse is in general not true since the consistent subspace $\mathcal{V}^{\star}_{P_1P_2}$ restricts the choice of inputs u depending on the states x_{P_1}, x_{P_2}. Thus, contrary to the open feedback interconnection, compositional reasoning is not complete for parallel compositions.

3.3.2. Assume-guarantee reasoning

Since compositional reasoning is not complete for parallel compositions, assume-guarantee reasoning schemes are to be investigated.

Theorem 3.39. *Consider four linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ of the form* (3.41). *For parallel compositions, the non-circular assume-guarantee reasoning schemes*

$$\left.\begin{array}{lrcl} S_1: & \Sigma_{P_1} & \preccurlyeq & \Sigma_{Q_1} \\ S_2: & \Sigma_{Q_1} \|_{pc} \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1} \|_{pc} \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S : \Sigma_{P_1}\|_{pc}\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_{pc}\Sigma_{Q_2}$$

and

$$\left.\begin{array}{lrcl} S_1': & \Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_2} \\ S_2': & \Sigma_{P_1} \|_{pc} \Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1} \|_{pc} \Sigma_{Q_2} \end{array}\right\} \Longrightarrow S : \Sigma_{P_1}\|_{pc}\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_{pc}\Sigma_{Q_2}$$

are sound.

Proof. Compositionality of parallel composition, see Theorem 3.38, and transitivity of simulation, c. f. Proposition 3.37, prove this result. ☐

Finally, circular assume-guarantee reasoning also works for parallel compositions.

Theorem 3.40. *Consider four linear systems $\Sigma_i, i \in \{P_1, P_2, Q_1, Q_2\}$ of the form* (3.41). *Circular assume-guarantee reasoning is sound for parallel compositions,*

$$\left.\begin{array}{llll} S_1: & \Sigma_{P_1}\|_{pc}\Sigma_{Q_2} & \preccurlyeq & \Sigma_{Q_1}\|_{pc}\Sigma_{Q_2} \\ S_2: & \Sigma_{Q_1}\|_{pc}\Sigma_{P_2} & \preccurlyeq & \Sigma_{Q_1}\|_{pc}\Sigma_{Q_2} \end{array}\right\} \Longrightarrow S: \Sigma_{P_1}\|_{pc}\Sigma_{P_2} \preccurlyeq \Sigma_{Q_1}\|_{pc}\Sigma_{Q_2} \quad (3.55)$$

Proof. Performing the same steps as in the proof for soundness of circular assume-guarantee reasoning for linear systems (Theorem 3.14 and Lemmas 3.11 – 3.13) yields the desired result. □

3.3.3. Verification of the global specification by decomposition

Throughout this chapter we have assumed that the given overall specification Σ_Q can be decomposed into subspecifications $\Sigma_{Q_i}, i = 1, \ldots k,$, in the same way as the modeled system Σ_P consists of interconnected components $\Sigma_{P_i}, i = 1, \ldots, k$. For parallel compositions, this decomposition of the specification makes it possible to verify the global specification by verifying all the subspecifications. In other words, it is enough to prove that Σ_P is simulated by each of the subspecifications Σ_{Q_i} to guarantee that it also fulfills the overall specification $\Sigma_Q = \Sigma_{Q_1}\|_{\mathrm{pc}}\Sigma_{Q_2}\|_{\mathrm{pc}} \ldots \|_{\mathrm{pc}}\Sigma_{Q_k}$. Since each subspecification is expected to be of lower complexity than the global specification, this reasoning scheme reduces the complexity of the overall verification task (3.2). Before proving the main result we collect some basic facts about parallel compositions.

Proposition 3.41. *For any system Σ_P it holds that*

$$\Sigma_P \preccurlyeq \Sigma_P\|_{pc}\Sigma_P \quad (3.56)$$

Proof. Construct a simulation relation S by setting all state variables equal to each other,

$$S = \{(x_1, (x_2, x_3)) \mid x_1 = x_2 = x_3 \in \Sigma_P\}$$

Then, S defines a full simulation relation of Σ_P by $\Sigma_P\|\Sigma_P$ since the evolution remains within the constrained subspace $Cx_1 = Cx_2 = Cx_3$ for all times. □

Proposition 3.42. *For any two systems Σ_P, Σ_Q, it holds that*

$$\Sigma_P\|_{pc}\Sigma_Q \preccurlyeq \Sigma_P \quad (3.57)$$

Proof. The relation

$$S = \{((x_P, x_Q), \bar{x}_P) \mid x_P = \bar{x}_P, (x_P, x_Q) \in \mathcal{W}^{\star}_{PQ}\}$$

defines a full simulation relation of $\Sigma_P\|_{\mathrm{pc}}\Sigma_Q$ by Σ_P. □

The main result to decompose a given global specification Σ_Q into an interconnection of local specifications Σ_{Q_1} and Σ_{Q_2} can now be stated as follows:

Theorem 3.43. *Given a system Σ_P and specifications $\Sigma_{Q_i}, i = 1, 2$, of the form (3.41). Then*

$$\Sigma_P \preccurlyeq \Sigma_{Q_1} \|_{pc} \Sigma_{Q_2} \tag{3.58}$$

if and only if

$$\Sigma_P \preccurlyeq \Sigma_{Q_1} \text{ and } \Sigma_P \preccurlyeq \Sigma_{Q_2} \tag{3.59}$$

Proof. $\Longrightarrow$: Given a full simulation relation of Σ_P by $\Sigma_{Q_1} \|_{\mathrm{pc}} \Sigma_{Q_2}$, Proposition 3.42 allows us to conclude that

$$\Sigma_P \preccurlyeq \Sigma_{Q_1} \|_{\mathrm{pc}} \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_1} \Longrightarrow \Sigma_P \preccurlyeq \Sigma_{Q_1}$$

and by symmetry,

$$\Sigma_P \preccurlyeq \Sigma_{Q_1} \|_{\mathrm{pc}} \Sigma_{Q_2} \preccurlyeq \Sigma_{Q_2} \|_{\mathrm{pc}} \Sigma_{Q_1} \preccurlyeq \Sigma_{Q_2} \Longrightarrow \Sigma_P \preccurlyeq \Sigma_{Q_2}$$

$\Longleftarrow$: Compositionality and Proposition 3.41 yield

$$\Sigma_P \|_{\mathrm{pc}} \Sigma_P \preccurlyeq \Sigma_{Q_1} \|_{\mathrm{pc}} \Sigma_{Q_2}, \Sigma_P \preccurlyeq \Sigma_P \|_{\mathrm{pc}} \Sigma_P \quad \Longrightarrow \quad \Sigma_P \preccurlyeq \Sigma_{Q_1} \|_{\mathrm{pc}} \Sigma_{Q_2}$$

□

3.4. Conclusions

The aim of this chapter is to demonstrate how compositional techniques that were developed in computer science for verification of labeled transition systems can be extended to linear systems. We analyzed two types of interconnections, namely feedback interconnections and parallel compositions of linear systems. For each one, we studied compositional and assume-guarantee reasoning rules and proved their soundness. In the feedback case, we also showed that these proof rules are complete. Furthermore, they could be extended firstly by considering bisimulation instead of simulation relations and secondly by allowing for interconnections of more than two systems. For parallel compositions we derived a proof rule based on the decomposition of the specification.
The next step will be to generalize this methodology to other classes of systems; in Chapter 7 results for switching linear systems will be presented. Further generalizations could also be obtained using the more abstract framework presented in [67]. Compositional analysis and decentralized control are strongly related. We will therefore apply the techniques developed in this

chapter to a decentralized control setting in Chapter 6. Another important direction of research is to investigate how to formulate system properties such as stability by means of simulations. In Chapter 5 we will establish a link between simulation and passivity theory that promotes the idea of model checking for linear and nonlinear systems.