Compositional analysis and control of dynamical systems

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2

Simulation theory: from labeled transition systems to linear systems

Many notions of equivalence, whether for discrete processes or continuous-time systems, are based on the external behavior. I.e., two processes or systems are equivalent if they cannot be distinguished with respect to their interaction with a common environment. In the theory of concurrent processes (bi)simulation theory is one of the most prominent concepts of external equivalence. In this chapter, we give a short introduction to (bi)simulation theory starting with its origin in labeled transition systems. In the second part, we describe how this concept can be adopted to linear input-state-output systems and how it is further developed using geometric control theory.

2.1. Simulation and bisimulation relations for labeled transition systems

Discrete-event systems in computer science are most commonly described as labeled transition systems.

Definition 2.1. A labeled transition system D is a triple $D = (Q, V, E)$ with

- a set of states Q,
- a set of transition labels V,
- a transition relation $E \subset Q \times V \times Q$ and

The semantics of a labeled transition system are described by executions.

Definition 2.2. An execution σ of a labeled transition system $D = (Q, V, E)$ is a finite or infinite sequence

$$\sigma = (q_0, v_0, q_1, v_1, \ldots)\ q_i, \in Q,\ v_i \in V,\ i \in \mathbb{N},$$

such that

$$(q_i, v_i, q_{i+1}) \in E$$

Executions define the external behavior of labeled transition systems. In particular, the sequence of transitions represented by their labels is used to define equivalences between labeled transition systems. Originally introduced by Park [57] and Milner [48] (bi)simulation relations formalize this idea.

Definition 2.3. For any two labeled transition systems $D_i = (Q_i, V_i, E_i), i = 1, 2$, with the same set of labels V, a relation $S \subset Q_1 \times Q_2$ is a *simulation relation* of D_1 by D_2 if and only if for all $(q_1, q_2) \in S$, $v \in V$ and $q_1' \in Q_1$ the following holds:

$$(q_1, v, q_1') \in E_1 \Longrightarrow \exists q_2' \in Q_2 : (q_2, v, q_2') \in E_2, (q_1', q_2') \in S$$

If there exists a simulation relation S of D_1 by D_2 we say that D_2 *simulates* D_1, denoted by $D_1 \preccurlyeq D_2$.
A subset $R \subset Q_1 \times Q_2$ is a *bisimulation relation* between D_1 and D_2 if and only if for all $(q_1, q_2) \in R$ it holds that

(i) for every $q_1' \in Q_1$ and every $v \in V$ such that $(q_1, v, q_1') \in E_1$ there exists a $q_2' \in Q_2$ such that $(q_2, v, q_2') \in E_2$ and $(q_1', q_2') \in R$ and conversely,

(ii) for every $q_2' \in Q_2$ and every $v \in V$ such that $(q_2, v, q_2') \in E_2$ there exists a $q_1' \in Q_1$ such that $(q_1, v, q_1') \in E_1$ and $(q_1', q_2') \in R$.

If there exists a bisimulation relation between D_1 and D_2 we say that D_1 and D_2 are *bisimilar*, denoted by $D_1 \approx D_2$.

Remark 2.4. Hence bisimilarity implies mutual similarity. However, it is a well-known fact that mutual similarity does *not* imply bisimilarity for labeled transition systems. I.e., if there exist simulation relations R_1 and R_2 of D_1 by D_2 and of D_2 by D_1, respectively, there need not exist any bisimulation relation between D_1 and D_2. This happens if one cannot construct $R_i, i = 1, 2$, in such a way that $R_1 = R_2^{-1}$, $R_2^{-1} = \{(q_1, q_2) \mid (q_2, q_1) \in R_2\}$, see e. g. [21] for a counterexample.

The notion of simulation relations is instrumental for compositional analysis since it provides a tool to relate labeled transition systems with the same transition structure. To finish this introductory section about (bi)simulation relations for labeled transition systems, we state without proof their most important properties.

- Simulation relations of labeled transition systems are preorders, i.e. they are

reflexive: $D \preccurlyeq D$ for any D

transitive: $$\left.\begin{array}{ccc} D_1 & \preccurlyeq & D_2 \\ D_2 & \preccurlyeq & D_3 \end{array}\right\} \Longrightarrow D_1 \preccurlyeq D_3 \quad \text{for any } D_i, i = 1, 2, 3\ .$$

Bisimulation relations are equivalence relations ([14]), i.e. they are reflexive, transitive and

$$\text{symmetric:} \quad D_1 \approx D_2 \implies D_2 \approx D_1 \quad \text{for any } D_i, i = 1, 2\ .$$

- Let D_1 and D_2 be two labeled transition systems and $(R_i)_{i \in I}$ a family of bisimulation relations with I an index set. Then the relation $R = \cup_{i \in I} R_i$ is also a bisimulation relation between D_1 and D_2, see [3]. Moreover, if $D_1 \approx D_2$ then there exists a unique *maximal* bisimulation relation $R^\star$ between D_1 and D_2 such that

$$R \quad \subset \quad R^\star$$

for any bisimulation relation R between D_1 and D_2.

2.2. Simulation and bisimulation theory for linear systems

Consider the following linear input-state-output system

$$\Sigma_i : \quad \begin{array}{rcl} \dot{x}_i & = & A_i x_i + B_i u_i + L_i d_i \\ y_i & = & C_i x_i \end{array} \tag{2.1}$$

where $x_i \in \mathcal{X}_i = \mathbb{R}^{n_i}$ represents the state variables, $u_i \in \mathcal{U}_i$ and $y_i \in \mathcal{Y}_i$ are a pair of inputs and outputs used for interconnection with other linear systems and $d_i \in \mathcal{D}_i$ is an additional input acting as a generator for non-determinism capturing e.g. unmodeled dynamics. We will therefore refer to d_i as “disturbances” influencing the modeled system behavior. Reference [55] adopted (bi)simulation theory to linear systems of the form (2.1) by interpreting them as general transition systems. Indeed, the set of states can be associated with the finite-dimensional vector space $\mathcal{X}_i$ (or in general a manifold), the transition relation is linked with the flow of the differential equation (2.1) and the set of variables corresponds to the in- and outputs u_i, y_i. Like in Definition 2.3, (bi)simulation relations for linear systems are then defined with respect to their executions, i.e. the trajectories of systems $\Sigma_i, i = 1, 2$ of the form (2.1).

Definition 2.5. Given two systems $\Sigma_i, i = 1, 2$ as in (2.1). A *simulation relation* S of Σ_1 by Σ_2 is a linear subspace of the product state space $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ with the following properties:
For any $(x_{10}, x_{20}) \in S$, any joint input function $u_1(\cdot) = u_2(\cdot)$ and any disturbance $d_1(\cdot)$ there should exist a disturbance $d_2(\cdot)$ such that the resulting state trajectories $x_i(\cdot), i = 1, 2$ with $x_i(0) = x_{i0}$ satisfy

$$\begin{array}{ll} (i): & (x_1(t), x_2(t)) \in S \quad \forall t \geq 0 \\ (ii): & C_1 x_1(t) = C_2 x_2(t) \quad \forall t \geq 0 \end{array} \tag{2.2}$$

Furthermore , Σ_1 is *simulated* by Σ_2, denoted $\Sigma_1 \preccurlyeq \Sigma_2$, if there exists a simulation relation $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ fulfilling $\Pi_{\mathcal{X}_1} S = \mathcal{X}_1$ where $\Pi_{\mathcal{X}_i} : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_i$ denotes the canonical projection on $\mathcal{X}_i$. In this case, S is called a *full simulation relation*.
A subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a *bisimulation relation* between Σ_1 and Σ_2 if it is a simulation relation of Σ_1 by Σ_2 and $R^{-1} := \{(x_2, x_1) \mid (x_1, x_2) \in R\}$ is a simulation relation of Σ_2 by Σ_1. If R satisfies $\Pi_{\mathcal{X}_i} R = \mathcal{X}_i, i = 1, 2$, then Σ_1 and Σ_2 are *bisimilar* and R is called a *full bisimulation relation*.

Following the same principle, (bi)simulation relations can be defined for other classes of systems such as nonlinear ([56], [24]) or hybrid systems ([2]). From an application point of view, however, Definition 2.5 has the drawback to rely on solutions of differential equations which in general are not explicitly computable. Using geometric control theory, [74] characterized simulation relations of linear systems by the following invariance condition. [1]

Proposition 2.6. *A subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if for all $(x_1, x_2) \in S$ and all $u \in \mathcal{U}$ the following holds:*

(i): For all $d_1 \in \mathcal{D}_1$ there should exist a $d_2 \in \mathcal{D}_2$ such that

$$\begin{bmatrix} A_1 x_1 + B_1 u + L_1 d_1 \\ A_2 x_2 + B_2 u + L_2 d_2 \end{bmatrix} \in S \tag{2.3}$$

(ii):

$$C_1 x_1 = C_2 x_2 \tag{2.4}$$

A subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ defines a bisimulation relation between Σ_1 and Σ_2 if and only if for every $d_1 \in \mathcal{D}_1$ there exists a $d_2 \in \mathcal{D}_2$ such that R satisfies (2.3) *and* (2.4) *and vice versa, for every d_2 there exists a d_1 such that* (2.3) *and* (2.4) *hold.*

Conditions (2.3) and (2.4) give rise to a linear-algebraic characterization of (bi)simulation subspaces.

Theorem 2.7. *A linear subspace $S \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 if and only if the following holds:*

1. $\operatorname{im} \begin{bmatrix} L_1 \\ 0 \end{bmatrix} \subset S + \operatorname{im} \begin{bmatrix} 0 \\ L_2 \end{bmatrix} := S_e$

2. $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} S \subset S_e$

[1] Note that in the remainder we will use the relational notation $(x_1, x_2) \in S \subset \mathcal{X}_1 \times \mathcal{X}_2$ and the subspace notation $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S$ interchangeably.

3. $\operatorname{im}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset S_e$

4. $S \subset \ker\begin{bmatrix} C_1 & -C_2 \end{bmatrix}$

A subspace $R \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a bisimulation relation between Σ_1 and Σ_2 if and only if R satisfies conditions 2 – 4 and additionally

1′. $R + \operatorname{im}\begin{bmatrix} L_1 \\ 0 \end{bmatrix} = R + \operatorname{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} =: R_e$

The above subspace characterizations of (bi)simulation relations for linear systems are easy to check and to apply. Moreover, an algorithmic procedure to compute maximal (bi)simulation relations can be derived from Theorem 2.7.

Algorithm 2.8. Given two linear systems $\Sigma_i, i = 1, 2$ of the form (2.1). Define the following sequence of subspaces $S^j \subset \mathcal{X}_1 \times \mathcal{X}_2$:

$$\begin{aligned}
S^0 &= \mathcal{X}_1 \times \mathcal{X}_2 \\
S^1 &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S^0 \,\middle|\, (x_1, x_2) \in \ker\begin{bmatrix} C_1 & -C_2 \end{bmatrix} \right\} \\
S^2 &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S^1 \,\middle|\, \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \operatorname{im}\begin{bmatrix} L_1 \\ 0 \end{bmatrix} \subset \right. \\
&\qquad \left. S^1 + \operatorname{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} \right\} \qquad (2.5) \\
&\vdots \\
S^{j+1} &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S^j \,\middle|\, \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \operatorname{im}\begin{bmatrix} L_1 \\ 0 \end{bmatrix} \subset \right. \\
&\qquad \left. S^j + \operatorname{im}\begin{bmatrix} 0 \\ L_2 \end{bmatrix} \right\}
\end{aligned}$$

Theorem 2.9. *The sequence of subspaces $S^j, j \in \mathbb{N}$, has the following properties:*

- $S^0 \supset S^1 \supset S^2 \supset \ldots \supset S^j \supset S^{j+1} \supset \ldots$
- *there exists a finite k such that $S^k = S^{k+1} := S^\star = S^i$, $i \geq k + 1$.*
- *If $S^\star$ is non-empty and fulfills condition 3 in Theorem 2.7 then $S^\star$ is the maximal simulation relation of Σ_1 by Σ_2. Conversely, if $S^\star$ is empty or does not fulfill condition 3 in Theorem 2.6 then there does not exist any simulation relation of Σ_1 by Σ_2.*

Remark 2.10. Similar to (2.5), the sequence of subspaces $R^j \subset \mathcal{X}_1 \times \mathcal{X}_2, j = 0, 1, \ldots,$ with

$$\begin{aligned} R^0 &= \mathcal{X}_1 \times \mathcal{X}_2, \\ R^1 &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R^0 \,\middle|\, (x_1, x_2) \in \ker \begin{bmatrix} C_1 & -C_2 \end{bmatrix} \right\}, \\ R^j &= \Bigg\{ (x_1, x_2) \in R^{j-1} \mid \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \operatorname{im} \begin{bmatrix} L_1 \\ 0 \end{bmatrix} \subset \\ & \quad R^j + \operatorname{im} \begin{bmatrix} 0 \\ L_2 \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \operatorname{im} \begin{bmatrix} 0 \\ L_2 \end{bmatrix} \subset \\ & \quad R^j + \operatorname{im} \begin{bmatrix} L_1 \\ 0 \end{bmatrix} \Bigg\}, \ j = 2, \ldots \end{aligned} \tag{2.6}$$

yields a candidate for the maximal bisimulation relation $R^\star$ between Σ_1 and Σ_2. The results of Theorem 2.9 also hold for the subspaces defined by (2.6), in particular the condition to determine whether or not $R^\star$ is indeed the maximal bisimulation relation between Σ_1 and Σ_2.

Some useful relational properties of (bi)simulations of linear systems as known from their discrete counterparts are retained.

Proposition 2.11. *Simulation relations $\preccurlyeq$ for linear systems are preorders whereas bisimulation relations $\approx$ are equivalence relations.*

Proof. Consider linear systems $\Sigma_i, i \in \{1, 2, 3\}$, of the form (2.1).
Reflexivity: The relation $S = \{(x_1, x_1) \mid x_1 \in \mathcal{X}_1\}$ fulfills conditions (i) and (ii) of Definition 2.5 and therefore defines a full simulation relation of Σ_1 by Σ_1.
Transitivity: Assume S_1 defines a full simulation relation of Σ_1 by Σ_2 and S_2 of Σ_2 by Σ_3. Then $S_{12} = \{(x_1, x_3) \mid \exists x_2 : (x_1, x_2) \in S_1, (x_2, x_3) \in S_2\}$ defines a full simulation relation of Σ_1 by Σ_3.
Both properties are also valid with respect to bisimulation.
Symmetry: Suppose that Σ_1 and Σ_2 are bisimilar and R defines a full bisimulation relation between them. Then R^{-1} is a full bisimulation relation between Σ_1 and Σ_2. □

Consider now the case of linear systems Σ_i without disturbances d_i. The above results specialize as follows.

Proposition 2.12. *For any two linear systems $\Sigma_i, i = 1, 2$, of the form (2.1) with d_i void, there exists a simulation relation of Σ_1 by Σ_2 if and only if there exists a bisimulation relation between Σ_1 and Σ_2. Moreover,*

$$\Sigma_1 \preccurlyeq \Sigma_2 \iff \Sigma_1 \approx \Sigma_2 \tag{2.7}$$

Proposition 2.13. *Consider* $\Sigma_i, i = 1, 2$, *as in* (2.1) *with* d_i *void. There exists a bisimulation relation* R *between* Σ_1 *and* Σ_2 *as in* (2.1) *with* d_i *void, if and only if the Markov parameters of* Σ_1 *and* Σ_2 *are equal, that is*

$$C_1 A_1^k B_1 \quad = \quad C_2 A_2^k B_2 \; \forall k = 0, 1, 2, \ldots \tag{2.8}$$

or equivalently, if their transfer matrices $G_i(s) := C_i(Is - A_i)^{-1} B_i, i = 1, 2$, *are the same. Moreover, if* $\Sigma_i, i = 1, 2$, *are controllable,*

$$\Sigma_1 \approx \Sigma_2 \quad \Longleftrightarrow \quad G_1(s) = G_2(s) \tag{2.9}$$

Proposition 2.13 also indicates that the maximal bisimulation relation between Σ_1 and Σ_2 is related to the unobservability space $\mathcal{O}_{12}$ of the augmented system Σ_{12}, given by

$$\Sigma_{12}: \begin{array}{rcl} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = & \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y_{12} & = & \begin{bmatrix} C_1 & -C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array}$$

Indeed, if the transfer matrices of $\Sigma_i, i = 1, 2$, are the same (which by Proposition 2.13 is equivalent to the existence of a bisimulation relation), the unobservability space

$$\mathcal{O}_{12} = \ker \begin{bmatrix} C_1 & -C_2 \\ C_1 A_1 & -C_2 A_2 \\ \vdots & \vdots \\ C_1 A_1^n & -C_2 A_2^n \end{bmatrix}, \; n = \max\{n_1, n_2\} - 1$$

equals the *maximal bisimulation relation* $R^\star$, since by equality of the Markov parameters

$$\operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subset \mathcal{O}_{12}$$

One of the main applications for simulation relations is to *abstract* a system of higher state space dimension by a lower dimensional one. Adopted from the discrete case, abstractions have been proposed in [55] and [74] for linear system of the form (2.1). More concretely, a system

$$\begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ y & = & \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array} \tag{2.10}$$

can be abstracted by the lower order model

$$\begin{aligned} \dot{z} &= A_{11}z + A_{12}d \\ y &= C_1 z \end{aligned} \tag{2.11}$$

in the sense that (2.10) is simulated by (2.11). Here, the internal disturbance d captures the influence of the x_2-dynamics on the the x_1-dynamics thus hiding details of the original system.
Similarly, bisimulation relations can be used to *reduce* linear systems Σ of the form (2.1). This idea was first proposed in [74] and later extended to other classes of systems, see e.g. [60] for reduction of switching linear systems. At first, the maximal bisimulation relation $R^\star_{\mathrm{id}}$ between Σ and itself is constructed. Associated with $R^\star_{\mathrm{id}}$ is the linear subspace

$$\bar{R} := \{x - y \mid (x, y) \in R^\star_{\mathrm{id}}\} \tag{2.12}$$

$\bar{R}$ can easily be seen to satisfy

$$\begin{aligned} A\bar{R} &\subset \bar{R} + \mathrm{im} G \\ \bar{R} &\subset \ker C \end{aligned} \tag{2.13}$$

The system Σ can then be reduced by factoring out $\bar{R}$ from the state space $\mathcal{X}$ using the canonical projection $\Pi_{\bar{R}} : \mathcal{X} \to \mathcal{X}/\bar{R}$. Thus, one obtains the reduced system

$$\Sigma_{\bar{R}} : \begin{aligned} \dot{x}_{\bar{R}} &= A_{\bar{R}} x_{\bar{R}} + B_{\bar{R}} u + L_{\bar{R}} d \\ y_{\bar{R}} &= C_{\bar{R}} x_{\bar{R}} \end{aligned}$$

with $A_{\bar{\mathcal{R}}}\Pi_{\bar{\mathcal{R}}} = \Pi_{\bar{\mathcal{R}}}(A + LK)$ for some matrix K computable from the first line of (2.13), $B_{\bar{\mathcal{R}}} = \Pi_{\bar{\mathcal{R}}} B$, $C = C_{\bar{\mathcal{R}}}\Pi_{\bar{\mathcal{R}}}$ and $L_{\bar{\mathcal{R}}} = \Pi_{\bar{\mathcal{R}}} L$. The following example illustrates this reduction procedure.

Example 2.14. Consider the linear system

$$\Sigma : \begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d_1 \\ y_1 &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Compute the maximal bisimulation relation $R^\star_{\mathrm{id}}$ between Σ and itself,

$$R^\star_{\mathrm{id}} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

The associated subspace $\bar{R}$ is given by

$$\bar{R} = \operatorname{im} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Factoring out $\bar{R}$ using the canonical projection $\Pi_{\bar{R}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} = C$ yields the reduced system $\Sigma_{\bar{R}}$ of dimension 1,

$$\Sigma_{\bar{R}} : \begin{array}{rcl} \dot{x}_{\bar{R}} & = & -x_{\bar{R}} + u + \frac{1}{2}d \\ y_{\bar{R}} & = & x_{\bar{R}} \, . \end{array}$$