Size effects in cellular solids

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6 Discussion

In this chapter, we discuss our overall results and connect them to the goal of this thesis. We distinguish stiffening and weakening size effects in cellular solids and comment on the predictive power of the generalized continuum theories that we analyzed. Finally, we conclude the thesis by addressing the limitations of the generalized continuum theories and give recommendations for improvement.

In the introduction of this thesis, we stated our goal as follows:
1) To explore the physical mechanisms that are responsible for the size-dependent elastic behaviour of cellular solids by using a discrete microstructural model.
2) To assess the capability of generalized continuum theories to capture size effects through a careful comparison with the discrete simulations.

To establish these goals, we first reviewed the deformation mechanisms of regular and irregular two-dimensional cellular solids in chapter 2. We performed simple shear, uniaxial compression and pure bending tests on discrete samples with the regular and irregular microstructures. For all of the microstructures tested, we detected two kinds of size effects: i) The macroscopic shear stiffness increases with decreasing sample size. ii) The macroscopic (uniaxial) compressive and bending stiffness decrease with decreasing sample size. The first, stiffening under simple shear, is associated with the strong boundary layers that form adjacent to the top and bottom boundaries, to which the cell walls are perfectly bonded. The smaller the sample size, the larger the area fraction of the strong boundary layers and thus the macroscopic shear stiffness. The second kind of size effect, weakening in the compressive and bending stiffness, on the other hand, is a result of weak boundary layers that form adjacent to the traction free edges, where the cells are much more compliant compared to the bulk.

Stiffness can be defined as the resistance of an elastic body to deformation by an applied traction. In the classical continuum theory, three displacement degrees of freedom are used to quantify the change in the position of a material point and deformation is expressed in terms of the symmetric part of the displacement gradients. For most (i.e. dense) materials, classical continuum theory suffices to accurately describe the elastic deformation, since most sample sizes and loading wavelengths are much larger than the characteristic material length scales (e.g. atoms, grains). However, in cellular solids, where the cell size sets the material length, this no longer holds. Consequently, higher-order deformation modes (whose effect was negligible in dense materials) become important at the macroscopic scale and should be accounted for. The generalized continuum theories studied here use additional degrees of freedom and/or additional deformation modes to add higher-order terms to the internal strain energy density. If in a boundary value problem these higher order terms are triggered, the overall response will be stiffer compared to the classical theory. The simple shear problem is such a case, where gradients in shear strain (or rotation) develop near the sample edges resulting in strong boundary layers and an overall stiffening compared to classical theory. The size effects of the second kind, or free edge effects leading to weakening, are different in character from the first kind. They are not associated with an additional deformation mode that is absent in the classical continuum theory, but arise due to the fact that near the traction free edges the cells cannot transfer the forces as efficient as in the core, resulting in a layer of reduced material stiffness. The generalized continuum theories mentioned in this thesis (see Table 1.1) cannot capture these free edge effects.

In chapter 3 we analyzed the micropolar theory, featuring extra degrees of freedom, the microrotations, and the associated higher-order deformations, the microrotation gradients. We derived the analytical solutions of the simple shear and pure bending problems. By comparing the discrete and analytical results for the simple shear problem in terms of the macroscopic shear stiffness, we fitted the coupling factor m and the characteristic length l_c. It turned out that for the best fit, the microrotations should be constrained to be equal to the macrorotations ($m \to \infty$), so that the micropolar theory reduces to the couple stress theory, featuring one additional material parameter, l_c, that was found to be on the order of the cell size for the microstructures analyzed. The corresponding local response, i.e. the macrorotation and shear strain fields through the thickness of the samples, was found to be in excellent agreement with the discrete fields. For the pure bending problem, however, the analytical solution predicted an increasing bending stiffness with decreasing sample thickness, which is opposite to the weakening observed for the discrete analyses. The reason for this is that rotation gradients develop in both the simple shear problem and the pure bending problem. In the simple shear problem, due to the symmetries, the rotation gradients through the thickness account for the gradient in shear strain, which is responsible for the extra energy consumed. Thus, only the rotation gradients in shear trigger a higher-order response in cellular solids, while the rotation gradients in bending do not. In the well-known Euler-Bernoulli beam theory, the normal stress gradients through the thickness are replaced by a bending moment and its constitutive relation to the rotation gradient (curvature). In micropolar theory, however, both the (classical) stress gradients and (higher order) bending moments are present, resulting in double counting in case of bending of small samples.

Another way of extending the classical continuum theory is to associate energy not only to strain, but also to its gradients (see Table 1.1). Among these higher-grade theories, Toupin-Mindlin's strain gradient theory is most often encountered in the literature. It has inspired many scientists to develop higher-grade theories to capture size effects in the mechanical behaviour, both for elasticity (e.g. Lam *et al*. [2003]) and for plasticity (e.g. Aifantis [1987], Fleck and Hutchinson [1993, 1997, 2001]). Toupin-Mindlin's strain gradient theory, however, is based on third order tensors (the strain gradients and the conjugate double stresses) and requires five additional material constants to be defined for linear elastic, centro-symmetric isotropic materials. Therefore, to give a complete and unique experimental delineation of strain gradient behaviour is a formidable task. A better strategy is to take into account only that part of the strain gradient that actually coincides with the extra deformation measure(s) related to the observed size effects. For this purpose, we developed a continuum theory that assigns energy to the divergence of strain, which coincides with the shear strain gradient in the case of the simple shear problem. This theory is based on vectors (the strain divergence and conjugate higher-order stress) and consequently requires only one additional constant for isotropic materials. We showed that this strain divergence theory is able to capture the stiffening under simple shear with decreasing sample size as accurate as the couple stress theory (in fact, both solutions coincide, see chapter 4). For the case of pure bending, however, as a result of satisfying both the classical and additional higher-order boundary conditions, the transverse strains (ε_{22} in chapter 4) turn out to be non-linear, in contrast to the classical solution, and this results in an increasing bending stiffness with decreasing specimen thickness.

We also briefly explored the role of the five constants of the full strain gradient theory in the shear and bending problems. The analytical solution for the simple shear problem for the full strain gradient theory (see e.g. Kouznetsova [2002]) can be shown to depend only on the second characteristic length, $l_2 = (a_3+2a_4+a_5)/2\mu$. At least one of the constants a_3, a_4, a_5 should be non-zero to exactly capture the shear stiffening. When a_5 is the only non-zero constant, however, the non-negativeness of the strain energy density is not satisfied. When a_3 is the only non-zero constant, the strain gradient theory falls back to the strain divergence theory. When a_4 is the only non-zero constant, we can choose it to be equal to l_c^{SD}/μ to ensure an excellent fit to the discrete results in shear (see chapter 4). However, by substituting this in the pure bending solution for the strain gradient theory (see the Appendix of chapter 4) it follows that the stiffening is larger compared to the strain divergence solution, both for isotropic and transverse isotropic solids.

Finally, in chapter 5 we analyzed the effects of a circular cylindrical hole in a field of uniaxial tension on the strain distribution. It was observed that the discrete strain fields remain largely unaffected for hole radii larger than approximately three cell sizes. For smaller hole radii the strain gradients around the hole considerably reduce. By comparing the discrete results with the couple stress and strain divergence solutions, it followed that they are both able to capture the effect of the hole size, except for a region very close to the hole boundary, where discrete effects prefail. Outside this regime the discrete strain distribution is successfully captured by these two theories, whereas micropolar continuum theory with a small coupling factor *m* is shown to be less accurate.

Table 6.1 summarizes the performance of the generalized continuum theories analyzed in this thesis. The micropolar theory with a small coupling factor *m* is not able to accurately capture the size-dependent response in simple shear and for the hole problem, whereas the couple stress and strain divergence theories perform much better and equally well. It should be emphasized that in the (planar) problems analyzed here, both theories feature only one additional higher-order constant (the characteristic length l_c). By fitting this constant to the overall discrete response in shear, it is shown that the discrete strain fields in shear and around the hole can be well captured by these theories. For uniaxial loading the discrete calculations predict a size-independent response when free-edge effects are disregarded, while also in a state of pure bending no higher-order modes are triggered in the cellular structures. The continuum theories (obviously) show a size-independent response under uniaxial loading, due to the absence of gradients. For pure bending, however, stiffening is predicted with decreasing sample size for all theories, in contrast to the discrete calculations. Note that the discrete structures analyzed here have an in-plane Poisson's ratio $\nu_{\mathrm{p}} = 0.94$, close to the limit of incompressibility, for which the couple stress and strain divergence theories are almost identical for all the boundary value problems analyzed. Real metal foams, however, are isotropic with a much lower Poisson's ratio, around $\nu \approx 0.3$ (see Ashby *et al*. [2000]). For such low Poisson's ratios, the two solutions for the hole problem are very different, while stiffening under pure bending is negligible for the strain divergence theory, whereas it increases even more for the couple stress theory.

Table 6.1

The performance of the generalized continuum theories analyzed in this thesis.

	Micropolar theory (with a small m)	Couple stress theory	Strain divergence theory
Simple shear	–	+	+
Hole problem	–	+	+
Uniaxial loading	+	+	+
Pure bending	–	–	–

The best one can expect from the generalized continuum theories studied in this thesis is to give a size-independent response for cases in which weakening size effects originate due to traction free edges. To capture weakening in a continuum setting, several routes can be followed. Brezny and Green [1990] accounted for weakening in bending by using a composite model with boundary layers having a lover stiffness than the bulk (see also Andrews *et al*. [2001]). Lakes [1995] used a non-local continuum theory which takes into account long range interactions of material points. Finally, one could use a generalized continuum theory that can account for the presence of surface stresses (see e.g. Gurtin and Murdoch [1975]). This theory has been successfully applied to predict weakening size effects in tension and bending of single crystals at the nanoscale (see e.g. Miller and Shenoy [2000]).