Aspects of algorithmic algebra

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Chapter 6

Geometrically continuous surface complexes

6.1 Introduction

A fundamental problem in CAGD (Computer Aided Geometric Design) is the construction of smooth surfaces in a Euclidean space by patching together several pieces of parametric surfaces. One usually considers parametric surface patches given by a polynomial map from a closed polygonal domain in $\mathbb{R}^2$ to $\mathbb{R}^m$. If several patches join along their common boundary in such a way that they form a C^1 surface in the sense of differential geometry, then the obtained surface is called *geometrically* (or *visually*, CG^1) *continuous* in CAGD. A special case of a geometrically continuous surface is a *parametrically continuous* surface. It is the graph of a C^1 spline on a triangulated region Δ in $\mathbb{R}^2$ as in the previous chapter 5. More generally, a tuple of such splines defines a C^1 map from Δ to a Euclidean space, and in general the image of this map is a C^1 smooth surface.

A general theory of geometric continuity in terms of polygonal patches is developed in [26, 29], including higher order CG^r geometric continuity. Explicit conditions for two or several polygonal patches to join with CG^1 or CG^2 continuity were worked out in [18, 15, 65]. In this chapter CG^1 continuity is presented in a slightly different setting, using more terminology from differential geometry. One purpose is to define "abstract" CG^1 surface complexes and CG^1 functions (and splines) on them without a reference to their realizations by polygonal patches in a Euclidean space. Similar to the definitions of a CG^1 surface patch complex in [29], or of an abstract CG^1 surface in [17], a CG^1 *surface complex* is given by a collection of polygons, an equivalence relation on their edges, and some CG^1 *glueing data* for each pair of equivalent edges. The CG^1 glueing data is given by a diffeomorphism μ between the two equivalent edges, and a continuous isomorphism Θ between the "tangent" bundles on them (more precisely, the restrictions of the tangent bundle of $\mathbb{R}^2$ to them). We use the differential geometry definition of the tangent space at a point $P \in \mathbb{R}^2$ as the vector space of all derivations (or derivatives) at P. Then Θ can be seen as an identification of the derivations at the points identified by

μ. (For higher order CG^r continuity we would take Θ to be a continuous isomorphism between the *r-jet bundles* on the pairs of identified edges.) Additionally, one requires a special structure for CG^1 glueing of several polygons at a *common vertex*, i.e. so that a set of their vertices has to be identified as one point on the resulting topological space, see [29, 26]. We reformulate the definitions of J.M. Hahn as the requirement for the CG^1 glueing data coming from the edges "incident" to the common vertex to be *consistent*.

A CG^1 *function* on a CG^1 surface complex is an assignment of a C^1 function to every polygon of the complex in such a way that for every pair of identified edges the two C^1 functions assigned to their polygons "satisfy" the CG^1 glueing data associated to this pair of edges. A CG^1 *spline* is a CG^1 function such that the C^1 functions assigned to the polygons are polynomials. The main result of this chapter is theorem 6.4.6, which gives the dimension formula for some vector spaces of CG^1 splines on certain CG^1 surface complexes. Specifically, we consider splines given by polynomials of bounded degree, and the CG^1 surface complexes (named *Bézier complexes*) are formed exclusively by triangles and have *rational* CG^1 glueing data for the equivalent edges. In particular, we specialize our formula to the case of a triangulation of a region in $\mathbb{R}^2$ and the usual bivariate C^1 splines on it, and reproduce the classical results [43, 50] of J. Morgan, R. Scott and L.L. Schumaker. In fact our method for computing the dimensions is very similar to the one in [43].

Some of our results suggest efficient ways to construct CG^1 splines and surfaces, see the algorithm 6.2.1 and the discussion at the end of subsections 6.4.1 and 6.4.3. It looks more practical to work with "one-dimensional" CG^1 continuity conditions for splines than with the similar ones for polygonal patches in $\mathbb{R}^m$. This is illustrated by a number of relevant examples of closed CG^1 surfaces.

To motivate our approach we consider an m-tuple $(f_1, \ldots, f_m)$ of CG^1 functions on a CG^1 surface complex $\mathcal{M}$. In general such a tuple gives us a collection of polygonal patches in $\mathbb{R}^m$ joining with CG^1 continuity, so that they form a geometrically continuous patch complex in the sense of [29]. If these patches do not intersect themselves and each other, then they form a C^1 surface $\mathcal{S}$ in $\mathbb{R}^m$, as it is proved in [29]. For us this C^1 surface (or the tuple $(f_1, \ldots, f_m)$) is a *realization* of $\mathcal{M}$ in $\mathbb{R}^m$. The motivating observation is that the space of C^1 functions on $\mathcal{S}$ is determined by our CG^1 glueing data (or less specifically, by the *connecting diffeomorphisms* from the usual definitions of CG^1 joining of two patches, see [26, 29]), and does not further depend on the realization. This allows us to define CG^1 functions and splines without such a realization.

Our approach is reminiscent of the idea of universal splines in [52]. There H.-P. Seidel defines the space of geometrically continuous CG^r curves with the same geometric continuity constraints (i.e. fixed CG^r *shape parameters* in CAGD terminology). In particular, he shows that this space is determined by a single curve of this kind, called a *universal spline*, in the sense that any other such a curve is the image of a C^r map from the universal spline. Actually every differentiable CG^r curve is a universal spline. With our approach we would characterize such a space of curves by the ring of CG^r functions on the abstract "CG^r curve complex", where the "CG^r glueing data" is given by the same shape parameters as in [52]. This space of CG^r functions looks like a simpler and more intrinsic object to represent all the curves with the same collection of shape parameters. The spaces of functions satisfying a fixed set of geometric continuity constraints for curves is also considered in [25, 24].

This chapter is organized as follows. In the next section we define CG^1 *glueing data* for two polygons and two identified edges of them, *consistent* CG^1 *glueing data* for several polygons at a common vertex, and CG^1 *surface complexes*. At the same time we compare these definitions with the similar definitions used in CAGD. An algorithm is given to produce consistent CG^1 glueing data at a complex vertex with desired properties. In the second half of subsection 6.2.4 we reformulate a few results of J.M. Hahn in [29] in our setting. Specifically, we define a CG^1 *surface* as the topological space obtained by glueing the polygons of a CG^1 surface complex according to the given CG^1 glueing data. We do not give it a structure of a C^1 surface (like in [17]), but we define CG^1 functions on it, and prove (theorem 6.2.5) that a CG^1 surface has a unique structure of a C^1 surface such that the notions of CG^1 functions and C^1 functions on it coincide. In the third section we define the *(triangular) Bézier complexes* for which we are going to compute the dimension of spline spaces. For this purpose we recall the representation of polynomial functions on triangles as homogeneous polynomials in their *barycentric coordinates*, and rewrite the CG^1 glueing conditions in them. The last section contains our main results (theorems 6.4.6 and 6.4.7) — the dimension formula for CG^1 spline spaces of sufficiently large degree on the mentioned Bézier complexes, or a lower bound for the dimension. The definitions and results of this chapter are supplemented by several examples in subsections 6.2.5 and 6.4.4.

6.2 CG^1 surface complexes

In this section CG^1 surface complexes are defined in two steps, like in [29]. First CG^1 glueing data for two polygons and their two edges is defined, and then — consistent CG^1 glueing data of several polygonal patches at a common vertex (in subsections 6.2.2 and 6.2.3). The third subsection also contains the algorithm 6.2.1 for constructing consistent CG^1 glueing data at the common vertex. In the fourth subsection we define CG^1 surface complexes, CG^1 surfaces, and CG^1 functions and splines on them. There we also prove theorem 6.2.5 which refines a few results in [29] in our setting. The last subsection contains a number of examples.

6.2.1 Some terminology and notation

Definition 6.1 In differential geometry (see [63]) a *differential surface* of class C^1 is a Hausdorff space $\mathcal{X}$ together with a collection $\{(U_i, \varphi_i)\}_i$ such that
(a) $\{U_i\}_i$ is an open covering of $\mathcal{X}$.
(b) Each φ_i is a homeomorphism $\varphi : \mathcal{U} \to U_i$, where $\mathcal{U} \subset \mathbb{R}^2$ is the open unit disk.
(c) For each pair $i \neq j$ with $U_i \cap U_j \neq \emptyset$, let $U_{i,j} := \varphi_i^{-1}(U_i \cap U_j)$ and $U_{j,i} := \varphi_j^{-1}(U_i \cap U_j)$. Then the map $\varphi_j^{-1}\varphi_i : U_{i,j} \to U_{j,i}$ is required to be a C^1-diffeomorphism.

We also use the following definitions. Consider a point $P \in \mathbb{R}^2$. The ring of germs of the C^1 functions at P is denoted by C^1_P. Let x, y be the standard coordinates of $\mathbb{R}^2$. The *tangent space* $T_{\mathbb{R}^2,P}$ is defined to be the vector space of all point derivations at P. Thus the elements of $T_{\mathbb{R}^2,0}$ are the $\mathbb{R}$-linear maps $D : C^1_P \to \mathbb{R}$ satisfying the property $D(fg) = f(P)D(g) + g(P)D(f)$. Each point derivation D can be written in the form

$D = a\frac{\partial}{\partial x}|_P + b\frac{\partial}{\partial y}|_P$ for unique real numbers a, b. Similarly one defines the tangent space $T_{\mathcal{X},\mathcal{P}}$ at a point P of any differentiable manifold $\mathcal{X}$.

A (germ of a) C^1 morphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, which maps $A \in \mathbb{R}^2$ to $B \in \mathbb{R}^2$, induces an $\mathbb{R}$-algebra homomorphism $C^1_B \to C^1_A$ by $f \mapsto f \circ \phi$. This homomorphism induces an $\mathbb{R}$-linear map $T_A(\phi) : T_{\mathbb{R}^2,A} \to T_{\mathbb{R}^2,B}$. For a C^1 morphism $\Psi : \mathbb{R}^2 \to \mathbb{R}^m$ which maps $A \in \mathbb{R}^2$ to $B \in \mathbb{R}^m$ we obtain a similar $\mathbb{R}$-linear map $T_A(\Psi) : T_{\mathbb{R}^2,A} \to T_{\mathbb{R}^m,B}$. In particular, for a C^1-function $f : \mathbb{R}^2 \to \mathbb{R}$ we can identify the tangent space at $f(A) \in \mathbb{R}$ with $\mathbb{R}$ itself. Then the obtained linear map $T_{\mathbb{R}^2,A} \to \mathbb{R}$ is the *gradient* (or the *1-form*) of f at the point A. This linear map is denoted by df or $df(A)$.

Consider a closed polygon $\Omega \subset \mathbb{R}^2$. For a point $P \in \Omega$ the tangent space $T_{\Omega,P}$ is defined as $T_{\mathbb{R}^2,P}$. For a boundary point $P \in \Omega$, which lies on only one edge, the tangent space $T_{\Omega,P}$ is given as an extra structure a closed half plane $H_{\Omega,P}$ through the origin $(0,0) \in T_{\Omega,P}$ of the derivatives with respect to the infinitesimal directions lying in Ω. For a vertex P of the polygon Ω the space $T_{\Omega,P}$ is given as an extra structure the closed cone $C_{\Omega,P} \subset T_{\Omega,P}$ with its vertex in the origin $(0,0)$, of the derivatives with respect to the infinitesimal directions which lie in Ω. The tangent space $T_{\Omega,P}$ of an interior point $P \in \Omega$ is not given an extra structure.

Let E be an edge of Ω. The restriction $\{T_{\Omega,P}|\ P \in E\}$ of the tangent bundle of $\mathbb{R}^2$ onto E is considered as a continuous *plane bundle* (or family of planes). It is a trivial two-dimensional vector bundle on E. This plane bundle has as a subbundle the tangent bundle $\{T_{E,P}|\ P \in E\}$ of E as a 1-dimensional C^1-manifold. For a diffeomorphism $\mu : E_1 \to E_2$ we have the isomorphism $T(\mu) : \{T_{E_1,P_1}|\ P_1 \in E_1\} \to \{T_{E_2,P_2}|\ P_2 \in E_2\}$ of their tangent bundles. We denote $T(\mu)(P_1)$ by $T_{P_1}(\mu)$, with the same meaning as above.

In this chapter a C^1 *regular patch* in $\mathbb{R}^m$ (with $m \geq 3$) is an injective map $\Psi : \Omega \to \mathbb{R}^m$ of a closed polygon Ω into $\mathbb{R}^m$, such that Ψ extends to a C^1 map on some open neighbourhood of Ω and such that the rank of the induced tangent map $T_P(\Psi)$ is two at every point P of Ω.

6.2.2 Geometrically continuous glueing along two edges

Definition 6.2 Let Ω_1 and Ω_2 be two polygons in $\mathbb{R}^2$, and for $i = 1, 2$ let E_i be an edge of Ω_i. We define the CG^1 *glueing data* of Ω_1 and Ω_2 along their edges E_1, E_2 as a pair (μ, Θ), where:

(1) $\mu : E_1 \to E_2$ is a C^1-diffeomorphism, i.e. edges E_1 and E_2 are seen as closed segments of the lines L_1 and L_2 in the plane, and μ extends to a C^1-isomorphism between open neighbourhoods $\breve{E}_1$ of E_1 on L_1 and $\breve{E}_2$ of E_2 on L_2. Such an extension is also denoted by μ.

(2) Θ is a continuous isomorphism of the plane bundles $\{T_{\Omega_1,P_1}|\ P_1 \in E_1\}$ and $\{T_{\Omega_2,P_2}|\ P_2 \in E_2\}$, i.e. for every $P_1 \in E_1$ we have an $\mathbb{R}$-linear isomorphism $\Theta(P_1) : T_{\Omega_1,P_1} \to T_{\Omega_2,\mu(P_1)}$, and $\Theta(P_1)$ depends continuously on P_1.

Besides, we impose the following conditions on Θ.

(i) Recall that the tangent bundles of E_1 and E_2 (or more precisely, the restrictions of the tangent bundles of L_1 and L_2 to these edges) are subbundles of $\{T_{\Omega_1,P_1}|\ P_1 \in E_1\}$ and $\{T_{\Omega_2,P_2}|\ P_2 \in E_2\}$ respectively. We require that Θ maps the tangent bundle of E_1 to the tangent bundle of E_2, and that the restriction of Θ to these

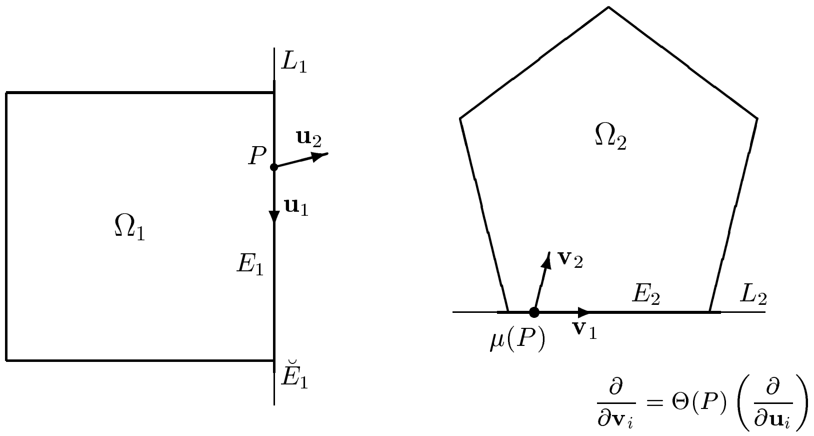

Figure 6.1: CG^1 glueing of two polygons

tangent bundles coincides with $T(\mu)$.

(ii) If P_1 is an interior point of E_1, then the closed half planes $\Theta(P_1)\, H_{\Omega_1,P_1}$ and $H_{\Omega_2,\mu(P_1)}$ have as intersection a line through the origin $(0,0)$, and have as union all of $T_{\Omega_2,\mu(P_1)}$.

(iii) If P_1 is a vertex of Ω_1 then the intersection of the cones $C_{\Omega_2,\mu(P_1)}$ and $\Theta(P_1)\, C_{\Omega_1,P_1}$ (see the previous subsection) is a half line through $(0,0)$ in $T_{\Omega_2,\mu(P_1)}$.

Note that condition *(i)* leaves two possibilities for the closed half plane $\Theta(P_1)\, H_{\Omega_1,P_1}$, namely $H_{\Omega_2,\mu(P_1)}$ or the one required in *(ii)* . In the first situation the glueing does not lead to a "good" CG^1-surface. The first condition also implies that the two cones in the third condition have at least a half line through $(0,0)$ in common. Condition *(iii)* excludes the case in which the two cones fill up the whole tangent plane or, even worse, have a cone as intersection. The latter condition is unnecessary if all angles of the polygons are less than π, because then the cones C_{Ω_1,P_1} and $C_{\Omega_2,\mu(P_1)}$ are convex.

The CG^1 glueing data above gives us a topological space $\mathcal{X}$, which is $\Omega_1 \cup \Omega_2$ with the edges E_1 and E_2 being identified according to μ. Occasionally we will use the same notation Ω_i for the images of the two polygons in $\mathcal{X}$. We allow the possibility that $\Omega_1 = \Omega_2$, but the edges E_1 and E_2 must be distinct. (Think of the classical construction of the Möbius strip from a rectangle, for example.) We also allow the polygons Ω_1 and Ω_2 to overlap, and even coincide. In the last case the topological space $\mathcal{X}$ is formed by two copies of the same polygon, and these copies have distinct sets of edges and vertices.

Definition 6.3 A *continuous function* $f : \mathcal{X} \to \mathbb{R}$ is the same thing as a pair of continuous functions (f_1, f_2) on Ω_1 and Ω_2 such that $f_2 \circ \mu = f_1$ holds on E_1. A continuous function f is called a *CG^1-function* if:

(i) For $i = 1,2$ the function f_i is a C^1 function, i.e. it extends to a C^1-function on some open neighborhood of Ω_i.

(ii) For every $P_1 \in E_1$ one has $df_2\,(\mu(P_1)) \circ \Theta(P_1) = df_1\,(P_1)$.

The second condition means that for all $P_1 \in E_1$ and all derivations $D \in T_{\Omega_1,P_1}$ we have $\Theta(P_1)(D)\, f_2 = D\, f_1$. This condition does not depend on the choice of the extensions of each f_i to a neighbourhood of Ω_i, since for $i = 1, 2$ the gradients of an extension of f_i at points on E_i are limits of the gradients of f_i at interior points of Ω_i.

We compare our definition of the CG^1 glueing data of two polygons with the usual definition of CG^1 join of two polygonal patches. Let $\Psi_1 : \Omega_1 \to \mathbb{R}^m$ and $\Psi_2 : \Omega_2 \to \mathbb{R}^m$ be two regular C^1 patches, and let $\lambda_1 : [0,1] \to \mathbb{R}^2$ be a regular parametrization of the edge E_1. Like in [29, 26], the patches Ψ_1 and Ψ_2 are said to *join with geometric continuity* CG^1 along the edge E_1, if there is a connecting diffeomorphism φ from an open neighbourhood of E_1 to $\mathbb{R}^2$ such that:

(Domain continuation:) $\lambda_2 := \varphi \circ \lambda_1$ is a parametrization of $E_2 \subset \Omega_2$, and φ maps interior points of Ω_1 into the exterior of Ω_2.

(Patch continuation:) We have $\Psi_1 \circ \lambda_1 = \Psi_2 \circ \lambda_2$, and for any (first order) derivation D on the whole $\mathbb{R}^2$ we have that $D\,(\Psi_2 \circ \varphi)$ coincides with $D\Psi_1$ on the edge E_1.

Having two patches $\Psi_1 = (g_1, \ldots, g_m)$ and $\Psi_2 = (h_1, \ldots, h_m)$ joining with CG^1 continuity along the edges E_1 and E_2, let $\mathcal{S}$ be the surface in $\mathbb{R}^m$ formed by them. We take the CG^1 glueing data $\mathcal{D} = (\mu, \Theta)$ for the polygons Ω_1 and Ω_2, defined by $\mu = \varphi|_{E_1}$ and $\Theta(P) = T_P(\varphi)$ for all $P \in E_1$. Let $\mathcal{X} = \Omega_1 \cup \Omega_2 / \sim$ be the surface obtained by glueing Ω_1 and Ω_2 according to this CG^1 glueing data. Then for $i = 1, \ldots, m$ the pair (g_i, h_i) of functions on the polygons is a CG^1 function on $\mathcal{X}$, because the patch continuation condition translates to the condition (ii) for CG^1 smoothness of each (g_i, h_i).

If the two patches Ψ_1 and Ψ_2 do not intersect, except $\Psi_1(E_1) = \Psi_2(E_2)$, then the surface $\mathcal{S}$ formed by them is a C^1 surface in $\mathbb{R}^m$, as it was proved by J.M. Hahn. Then the two patches Ψ_1, Ψ_2 give us an m-tuple of CG^1 functions on $\mathcal{X}$, which *realizes* the CG^1 glueing data $\mathcal{D}$ (or the surface $\mathcal{X}$), i.e. give us an injective map $\mathcal{S} \to \mathbb{R}^m$. Assuming the same situation, let f be a C^1 function on $\mathcal{S}$ as a C^1 surface. Then the two C^1 functions $f \circ \Psi_1$ and $f \circ \Psi_2$ on the polygons also satisfy the definition of a CG^1 function for $\mathcal{D}$. It follows that the C^1 functions on $\mathcal{S}$, viewed as pairs of their pull-backs to the polygons, depend only on the CG^1 glueing data $\mathcal{D}$ (or less specifically, on the connecting diffeomorphism φ).

On the other hand, lemma 6.2.3 in subsection 6.2.4 below implies that starting from a CG^1 glueing data of two polygons Ω_1 and Ω_2 along their edges E_1, E_2, one can construct a diffeomorphism φ such that a tuple of CG^1 functions on them in general gives us two patches joining with CG^1 continuity, and φ is a connecting diffeomorphism for these patches. Also note that our CG^1 glueing data (μ, Θ) is unique for two patches joining with CG^1 continuity (up to replacing μ by μ^{-1}, etc.), whereas the connecting diffeomorphism φ is not defined uniquely.

Instead of the conditions for CG^1 joining of two polygonal patches we have conditions on CG^1 functions. For writing them down explicitly it is convenient to include a vector tangent to E_i into a basis of T_{Ω_i,P_i} for $i = 1, 2$ and all $P_i \in E_i$. Specifically, let $\mathbf{E}_1$ be a (non-vanishing) vector field[1] tangent to E_1, i.e. a section of the tangent bundle of

[1] Adapting terminology of CAGD, by a *vector field* along an edge $E \subset \Omega$ we mean a continuous family on E of vectors in $\mathbb{R}^2$, i.e. a section of $\{T_{\Omega,P} |\ P \in E\}$. In differential geometry a vector field

E_1. Let $\mathbf{E}_2$ be the tangent vector field along E_2 defined by $\mathbf{E}_2(\mu(P_1)) = T_{\mathbf{E}_1(P_1)}(\mu)$. For $i = 1, 2$ let $\mathbf{U}_i$ be a "transversal" vector field along E_i, i.e. $\mathbf{U}_i(P_i) \notin T_{E_i,P_i}$ for all $P_i \in E_i$. If we take the directional derivatives with respect to $\mathbf{E}_i(P_i)$ and $\mathbf{U}_i(P_i)$ as a basis of T_{E_i,P_i} for $i = 1, 2$, where $P_1 \in E_1$ and $P_2 = \mu(P_1)$, then the matrix of $\Theta(P_1)$ has the form $\begin{pmatrix} 1 & b(P_1) \\ 0 & c(P_1) \end{pmatrix}$, where $b(P_1), c(P_1)$ are continuous functions on E_1. From this one derives that a pair of C^1 functions (f_1, f_2) is a CG^1 function on $\mathcal{X}$ if and only if $f_2 \circ \mu = f_1$ and

$$\frac{\partial f_2}{\partial\, \mathbf{U}_2(P_2)}(P_2) = b(P_1)\, \frac{\partial f_1}{\partial\, \mathbf{E}_1(P_1)}(P_1) + c(P_1)\, \frac{\partial f_1}{\partial\, \mathbf{U}_1(P_1)}(P_1), \tag{6.1}$$

for all $P_1 \in E_1$ and $P_2 = \mu(P_1)$. Here $\partial f_2/\partial \mathbf{U_2}(P_2)$ denoted the directional derivative of f_2 with respect to the vector $\mathbf{U_2}(P_2)$, etc. Note that if for $i = 1, 2$ and all $P_i \in E_i$ the directional derivative with respect to $\mathbf{U}_i(P_i)$ is in H_{Ω_i,E_i}, then $c(P_1)$ is a negative function. These conditions are completely analogous to the explicit tangent plane and geometric continuity conditions for patches used in [18], [15] and [29].

Definition 6.4 A CG^1 glueing data (μ, Θ) of two polygons Ω_1 and Ω_2 along their edges E_1, E_2 is called *rational* if:

(i) μ is a linear diffeomorphism.

(ii) For $i = 1, 2$ choose "constant" vector fields $\mathbf{E}_i$, $\mathbf{U}_i$ as above, i.e. these vector fields do not depend on $P_i \in E_i$. In other words, for $i = 1, 2$ we identify all the tangent spaces T_{Ω_i,P_i} with $T_{\mathbb{R}^2,O}$ and choose the same basis for all of them. Then we require that the entries $b(P_1)$, $c(P_1)$ in the matrix $\begin{pmatrix} 1 & b(P_1) \\ 0 & c(P_1) \end{pmatrix}$ of $\Theta(P_1)$ in these bases are continuous rational functions in a linear parameter of E_1.

This definition does not depend on the choice of the constant vector fields and the direction of μ (because μ must be linear). The *degree* of a rational glueing data (μ, Θ) is defined as follows. After the choice of the vector fields $\mathbf{E}_i$, $\mathbf{U}_i$, and a linear parametrization $\lambda_1 : [0, 1] \to E_1$ we write $P_1 = \lambda_1(t)$ for $t \in [0, 1]$. The rational functions $b \circ \lambda_1$ and $c \circ \lambda_1$ can be expressed as $b(P_1) = -\beta(t)/\gamma(t)$ and $c(P_1) = -\alpha(t)/\gamma(t)$, where α, β and γ are polynomials in $\mathbb{R}[t]$ without a common divisor. Then the degree of (μ, Θ) is the maximum of the degrees of α, β and γ.

In particular, for a rational CG^1 glueing data of degree 0 the isomorphism Θ of the plane bundles is *constant*, i.e. after the identification of the tangent spaces with $T_{\mathbb{R}^2,O}$ the map $\Theta(P_1)$ is the same for all $P_1 \in E_1$. For a realization of such a CG^1 glueing data by two patches Ψ_1 and Ψ_2 one can always choose a linear connecting diffeomorphism φ defined on an open neighbourhood of Ω_1. Then $\varphi(\Omega_1)$ is also a polygon, and it has the edge E_2 in common with Ω_2. If $\varphi(\Omega_1)$ and Ω_2 have no other common points, then a CG^1 function on $\mathcal{X}$ is the same thing as a C^1 function on $\varphi(\Omega_1) \cup \Omega_2$. Then the two patches Ψ_1 and Ψ_2 join with C^1 *parametric continuity*, i.e. they are defined by a single C^1 map $\varphi(\Omega_1) \cup \Omega_2 \to \mathbb{R}^m$.

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along an edge is usually a family of vectors tangent to E.

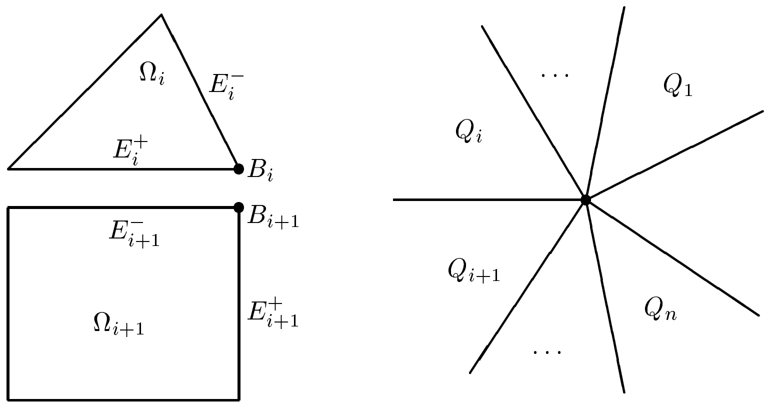

Figure 6.2: CG^1 glueing at the common vertex.

6.2.3 CG^1 glueing at a vertex

Definition 6.5 The next thing to define is the consistent CG^1 glueing data of several polygons at a common vertex. Given are polygons $\Omega_1, \ldots, \Omega_n$ in $\mathbb{R}^2$, and for $i = 1, \ldots, n$ we have a vertex B_i of Ω_i and two distinct edges E_i^-, E_i^+ of Ω_i such that $E_i^- \cap E_i^+ = \{B_i\}$, see the left-hand part of figure 6.2. The polygons are to be glued in such a way that their vertices B_i become identified on the resulting topological space as one point, their *common vertex*. We talk about *consistent CG^1 glueing data* around a common vertex in the following two situations (1–1a) and (2–2b).

(1) Suppose that for $i = 1, \ldots, n-1$ we have a CG^1 glueing data (μ_i, Θ_i) of the polygons Ω_i and Ω_{i+1} along their edges E_i^+ and E_{i+1}^-, as in the previous subsection. We require that $\mu_i(B_i) = B_{i+1}$. Then the collection of the CG^1 glueing data (μ_i, Θ_i) is *consistent* (or forms *consistent CG^1 glueing data* at the common vertex) if the following condition holds.

(1a) Let $Q_n = C_{\Omega_n, B_n}$, and for $i = 1, \ldots, n-1$ let Q_i be the image of the cone C_{Ω_i, B_i} under the composition $\Theta_{n-1}(B_{n-1}) \circ \ldots \circ \Theta_i(B_i)$. The requirement is that the cones Q_i form a subdivision of a proper closed cone Q in the tangent space T_{Ω_n, B_n}, i.e. their union is the cone Q, and the intersection of any two of them does not have a subset homeomorphic to an open disk, see the right-hand part of figure 6.2.

(2) Suppose that, additionally to the CG^1 glueing data (μ_i, Θ_i) for $i = 1, \ldots, n-1$ as in *(1)* , we are given a CG^1-gluing data (μ_n, Θ_n) of polygons Ω_n and Ω_1 along their edges E_n^+ and E_1^-, of course with $\mu_n(B_n) = B_1$. This collection of the CG^1 glueing data (μ_i, Θ_i) is (also) *consistent*, if the following two "closure conditions" are satisfied:

(2a) Let $Q_1, \ldots, Q_n$ be the cones defined in (1a). We require that these cones form a subdivision of the whole tangent space T_{Ω_n, B_n}.

(2b) The composition $\Theta_n(B_n) \circ \Theta_{n-1}(B_{n-1}) \circ \cdots \circ \Theta_2(B_2) \circ \Theta_1(B_1)$ is the identity.

In the second situation we also say that the pairs (μ_i, Θ_i) form CG^1 *surround glueing data* around the "common vertex". Note that the conditions (1a) and (2a-2b) are restrictions on the linear isomorphisms $\Theta_i(B_i)$ and the angles of the polygons at B_i's only.

Let $\mathcal{X}$ be the topological space obtained from $\Omega_1 \cup \ldots \cup \Omega_n$ by identifying their edges according to the homeomorphisms $\mu_1, \ldots, \mu_{n-1}$ and perhaps μ_n. The vertices $B_1, \ldots, B_n$ are identified into the *common vertex* on $\mathcal{X}$. We do admit that some of the polygons Ω_i are the same. However, the vertices B_i must be distinct. We also allow some of the Ω_i's overlap or to coincide set-theoretically, like in the previous subsection.

Definition 6.6 A *continuous function* f on $\mathcal{X}$ is an n-tuple of continuous functions $(f_1, \ldots, f_n)$ on the polygons Ω_i such that $f_{i+1} \circ \mu_i$ is equal to f_i on E_i^+ for $i = 1, \ldots, n-1$, and, in the case of a CG^1 surround glueing data, $f_1 \circ \mu_n = f_n$ on E_n^+. A continuous function f is called a CG^1*-function* if moreover:

(i) Each f_i extends to a C^1-function on some open neighbourhood of Ω_i.

(ii) For $i = 1, \ldots, n-1$ one has $df_{i+1}(\mu_i(P)) \circ \Theta_i(P) = df_i(P)$ for all $P \in E_i^+$.

(iii) In the case of a CG^1 surround glueing data, also $df_1(\mu_n(P)) \circ \Theta_n(P) = df_n(P)$ for all $P \in E_n^+$.

Note that conditions *(ii)* and *(iii)* do not depend on the choice of the extensions of each f_i to a C^1-functions on some neighbourhood of Ω_i.

For comparison we mention that in [29] J.H.Hahn considers n polygonal smooth patches $\Psi_1 : \Omega_1 \to \mathbb{R}^m, \ldots, \Psi_n : \Omega_n \to \mathbb{R}^m$ having a common vertex $B \in \mathbb{R}^m$. For $i = 1, \ldots, n$ let B_i be the point on Ω_i identified with B, and let E_i^-, E_i^+ be the edges as above. J.H.Hahn defines the *tangent sector* of Ψ_i at B to be the set of tangent vectors[2] $d\Psi_i(\sigma(s))/ds|_{s=0}$ to regular curves $\sigma(s) : [0,1] \to \Omega_i$ with $\sigma(0) = B_i$. Then the patches $\Psi_1, \ldots, \Psi_n$ are said to *meet with geometric continuity* CG^1 at B, if:

(a) For $i = 1, \ldots, n-1$ the patches Ψ_i and Ψ_{i+1} join with CG^1 continuity along the edges E_i^+ and E_{i+1}^-.

(b) The tangent sectors of distinct patches do not overlap at B.

(c) Condition (a) implies that all the patches Ψ_i have a common tangent plane at B. The union of their tangent sectors at B must be a proper subset of the common tangent plane.

The same patches *surround* B *with geometric continuity* CG^1 when also Ψ_n and Ψ_1 join with CG^1 continuity along E_n^+ and E_1^-, and (c) is replaced by the requirement that the union of the tangent sectors of Ψ_i's is the whole (common) tangent plane.

It is quite clear that if n patches $\Psi_1, \ldots, \Psi_n$ as above are given, and they meet at the common vertex B, or surround it, then we have consistent CG^1 glueing data for the polygons $\Omega_1, \ldots, \Omega_n$. Indeed, every pair of identified polygonal edges give us a CG^1 glueing data (μ_i, Θ_i), and we have the first type (1–1a) or second type (2–2b) of the consistent CG^1 glueing data depending on whether the patches meet at B or surround it. In both cases the tangent sectors of Ψ_i's form a subdivision of a proper cone in $\mathbb{R}^2$ or the whole $\mathbb{R}^2$, and this subdivision is a linear transformation of the subdivision formed

[2] Here we use the "set-theoretical" notion of the *tangent plane* of a smooth surface at a point in $\mathbb{R}^m$, which is equivalent to the notion of the tangent space we use, of course.

by the cones Q_i in *(1a)* or (2a). Condition (2b) follows from (2a) and the existence[3] of the common tangent plane of Ψ_i's at B.

Let $\mathcal{X}$ be the topological surface defined by the induced CG^1 glueing data. Like in the previous section, the components of the Ψ_i's give us an m-tuple of CG^1 functions on $\mathcal{X}$. If the patches do not have common points except the specified common edges, then such a tuple of functions *realizes* the consistent CG^1 glueing data (or the surface $\mathcal{X}$), i.e., define an injective differentiable map $\mathcal{X} \to \mathbb{R}^m$. On the other hand, any m-tuple of CG^1 functions, realizing $\mathcal{X}$ gives us a collection of polygonal patches joining with CG^1 continuity at a common vertex, according to the given definition of J.M. Hahn (compare this also with lemma 6.2.4 below).

Definition 6.7 For consistent CG^1 glueing data $\mathcal{D}$ at a common vertex we call any linear transformation of the collection of the cones Q_i in (1a) or (2a) in definition 6.5 a *tangent subdivision* for the glueing data $\mathcal{D}$.

Note that, according to the above definitions of J.M. Hahn, for any realization of $\mathcal{D}$ in $\mathbb{R}^m$ the tangent sectors of the corresponding patches at the common point form a tangent subdivision for $\mathcal{D}$. In particular, if $\mathcal{D}$ satisfies (1–1a) and the angle of the cone $Q \subset T_{\Omega_n,B_n}$ in (1a) is less, equal or greater than π, then for any realization of $\mathcal{D}$ the boundary angle (of the obtained surface in $\mathbb{R}^m$) at the common vertex is respectively less, equal or greater than π as well.

Here we present an algorithm for choosing consistent CG^1 glueing data at a vertex. The difficulty is to choose the isomorphisms $\Theta_i(B_i)$ of the plane bundles so that the conditions (1a) or (2a–2b) are satisfied, so we suppose that the other data is given. For simplicity we assume that all angles of the given polygons are less than π. The algorithm also receives a subdivision of $\mathbb{R}^2$ (or subdivision of a proper closed cone in $\mathbb{R}^2$, if the glueing data of type (1-1a) is required) into cones with angles $< \pi$ as the input, and produces consistent CG^1 glueing data $\mathcal{D}$ such that the chosen subdivision is a tangent subdivision for $\mathcal{D}$. In other words, we predetermine (up to linear transformations) the tangent sectors of possible realizations of $\mathcal{D}$. The algorithm is described completely only for the construction of CG^1 surround glueing data.

Algorithm 6.2.1 *Given are:*

- *polygons* $\Omega_1, \ldots, \Omega_n$ *with vertices* $B_i \in \Omega_i$ *and edges* E_i^-, E_i^+ *as in definition 6.5.*
- *A subdivision of* $\mathbb{R}^2$ *into cones* $K_1, K_2, \ldots, K_n$ *with the common vertex* $(0,0) \in \mathbb{R}^2$*, such that the angle of each* Q_i *is less than* π*, see the left-hand side of figure 6.3.*

The algorithm produces a collection of linear isomorphisms $\Theta_i(B_i)$*, satisfying the conditions* (2a-2b) *of definition 6.5, such that supplemented by the diffeomorphisms* μ_i *and the* Θ_i*'s defined on the corresponding edges (but with the produces values at* B_i*'s) they give* CG^1 *surround glueing data* $\mathcal{D}$ *around a vertex, such that the given subdivision of* $\mathbb{R}^2$ *is a tangent subdivision for* $\mathcal{D}$*.*

[3]The conditions (2a) and (2b) are independent, because (2a) allows the composition $\Theta_n(B_n) \circ \cdots \circ \Theta_2(B_2) \circ \Theta_1(B_1)$ to be a scalar multiple of the identity, and (2b) allows the cones Q_i to wrap around B_n more than once. One may consider a glueing around the common vertex without one of these conditions, but then the obtained "CG^1 functions" *do not separate directions* at the common vertex, in a sense that the gradients of their components f_i are proportional to each other.

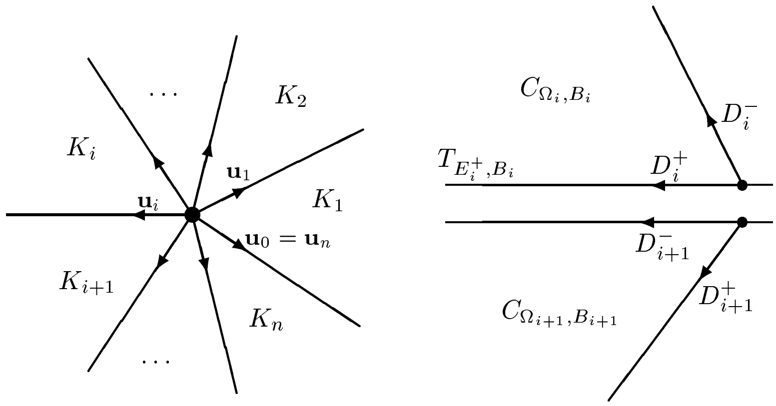

Figure 6.3: Choosing consistent CG^1 glueing data

1. Choose a vector $\mathbf{u}_0 = (x_0, y_0)$ on the common half-line of K_n and K_1, and for $i = 1, \dots, n-1$ choose a vector $\mathbf{u}_i = (x_i, y_i)$ on the common half-line of K_i and K_{i+1}. Also set $\mathbf{u}_n = \mathbf{u}_0$.
2. For $i = 1, \dots, n$ choose a derivation D_i^+ in $C_{\Omega_i,B_i} \cap T_{E_i^+,B_i}$. Also define the derivations $D_1^- := T_{B_n}(\mu_n)(D_n^+)$ and $D_{i+1}^- := T_{B_i}(\mu_i)(D_i^+)$ for $i = 1, \dots, n-1$, see the right-hand side of figure 6.3. Then we identify T_{Ω_i,B_i} with $\mathbb{R}^2$ by the linear isomorphism σ_i which maps $D_i^- \mapsto \mathbf{u}_{i-1}$ and $D_i^+ \mapsto \mathbf{u}_i$.
3. For $i = 1, \dots, n-1$ let θ_i be the linear map which maps $\mathbf{u}_{i-1} \mapsto \mathbf{u}_i$ and $\mathbf{u}_i \mapsto \mathbf{u}_{i+1}$. Then we choose $\Theta_i(B_i)$ to be $\sigma_{i+1}^{-1} \circ \theta_i \circ \sigma_i$. Explicitly, the matrix of $\Theta_i(B_i)$ in the bases (D_j^-, D_j^+) of the tangent spaces T_{Ω_j,B_j} for $j = i, i+1$, is equal to the matrix of θ_i in the bases $(\mathbf{u}_{i-1}, \mathbf{u}_i)$ and $(\mathbf{u}_i, \mathbf{u}_{i+1})$ for the range and the image of θ_i respectively, that is $\begin{pmatrix} 0 & a_i \\ 1 & b_i \end{pmatrix}$ with

$$a_i = \frac{x_i y_{i+1} - x_{i+1} y_i}{x_i y_{i-1} - x_{i-1} y_i}, \qquad b_i = \frac{x_{i+1} y_{i-1} - x_{i-1} y_{i+1}}{x_i y_{i-1} - x_{i-1} y_i}. \tag{6.2}$$

   Similarly, choose $\Theta_n(B_n)$ to be $\sigma_1^{-1} \circ \theta_n \circ \sigma_n$, where θ_n is the linear map which sends $\mathbf{u}_{n-1} \mapsto \mathbf{u}_0$ and $\mathbf{u}_0 \mapsto \mathbf{u}_1$.

One can check that the produced $\Theta_i(B_i)$'s give us a subdivision of T_{Ω_n,B_n} into cones Q_i as in (2a) of definition 6.5, and this subdivision is the linear transformation (under σ_n^{-1}) of the given subdivision. Also condition (2b) is easily satisfied because $\theta_n \circ \cdots \circ \theta_2 \circ \theta_1$ is the identity map. As we see, the linear maps $\Theta_i(B_i)$ are uniquely[4] defined by the vectors $\mathbf{u}_i$. The algorithm can produce any possible consistent CG^1 glueing data at a vertex,

[4] If the angle of some Ω_i at P_i is equal to π, then we should require that the angle of K_i must be also π, and that σ_i would map vectors directed into (close) interior of Ω_i to the cone K_i. In this case the maps $\Theta_{i-1}(B_{i-1})$ and $\Theta_i(B_i)$ are not defined uniquely, but we must have $\theta_{i-1}(\mathbf{u}_{i-2}) = \mathbf{u}_{i-1}$, $\theta_{i-1}(\mathbf{u}_{i-1}) \in B_i$ and $\theta_i(\mathbf{u}_i) = \mathbf{u}_{i+1}$, $\theta_i \circ \theta_{i-1}(\mathbf{u}_{i-1}) = \mathbf{u}_{i+1}$. If the angle of Ω_i at P_i is greater than π, then the angle of K_i must be $> \pi$ as well.

and improves the recipe in [29] (subsection 8.2) which gives only the most symmetric collection of $\Theta_i(B_i)$'s.

Example 6.8 Consider CG^1 surround glueing of three polygons Ω_1, Ω_2, Ω_3 at the common vertex, which corresponds to three vertices $B_i \in \Omega_i$ of the polygons. We keep the same notation for the edges which have to be identified (and the assumption on the angles of the polygons). The first step of algorithm 6.2.1 produces three vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ in $\mathbb{R}^2$. There must be a linear relation

$$\zeta_1\mathbf{u}_1 + \zeta_2\mathbf{u}_2 + \zeta_3\mathbf{u}_3 = 0 \tag{6.3}$$

between these vectors. Moreover, the three real numbers ζ_1, ζ_2 and ζ_3 must be positive, because the angles of the cones in the given subdivision are less than π. The choice of the derivations D_i^+, D_i^- in step 2 gives us bases for the tangent spaces T_{Ω_i,B_i}. The relation 6.3 determines the matrices of the θ_i's in the mentioned bases involving $\mathbf{u}_i$'s. For example, the matrix of θ_1 is $\begin{pmatrix} 0 & -\zeta_2/\zeta_3 \\ 1 & -\zeta_1/\zeta_3 \end{pmatrix}$, which is also the matrix of $\Theta_1(B_1)$ in the chosen bases of T_{Ω_i,B_i}'s. Similarly are the maps $\Theta_2(B_2)$ and $\Theta_3(B_3)$ are defined.

6.2.4 Geometrically continuous surfaces

To define the "abstract" CG^1 surface complexes we first define "continuous" CG^0 surface complexes and introduce the necessary combinatorial terminology.
Definition 6.9 A CG^0 *surface complex* $\mathcal{M}$ is given by the following data:

(i) A collection of polygons $\Omega_1, \ldots, \Omega_n$.
(ii) An equivalence relation between edges of Ω_i's, such that each edge is equivalent to at most one other polygonal edge.
(iii) For any two equivalent edges E_1, E_2 there is a homeomorphism $\mu : E_1 \to E_2$.

This data defines a topological surface $\mathcal{X}$ (possibly with a boundary) obtained as a union of all polygons Ω_i with their equivalent edges identified according to the identifying homeomorphisms μ in *(iii)* . For simplicity we always *assume* that the considered surface complexes give us *connected* topological spaces (see [41]).

Here we introduce the combinatorial notions by extending the terminology for simplicial (or polyhedral) complexes in [66]. A *complex edge* of $\mathcal{M}$ is an equivalence class of polygonal edges. Such an equivalence class is called an *interior edge* if there are exactly two polygonal edges in it, and it is called a *boundary edge* otherwise. The equivalence relation on the polygonal edges and the homeomorphisms in *(iii)* induce an equivalence relation between vertices of the polygons Ω_i, which is generated by the relations "polygonal vertices E_1, E_2 are equivalent if there is a homeomorphism μ with $\mu(E_1) = E_2$". A *complex vertex* of $\mathcal{M}$ is an equivalence class of polygonal vertices. To avoid confusion, from *here on* we specify whether we mean complex (or interior, boundary) edges or vertices, or polygonal (or equivalent) edges and vertices.

We say that a polygon Ω_i is *incident* to a complex edge (or a complex vertex) and vice versa, if Ω_i has an edge (or a vertex, respectively) in the corresponding equivalence class. Two polygons are *adjacent* if there is a complex edge incident to both of them.

For an interior edge e there are two complex vertices p_1 and p_2 (which may coincide) such that the end-points of polygonal edges in the equivalence class e lie in p_1 and p_2. We say that e is *incident* to these complex vertices and vice versa, and that e *connects* them. Finally, we call a complex vertex v *interior* if all complex edges incident to it are interior, and v is a *boundary* vertex otherwise.

Note that the complex vertices and edges are represented by points and continuous curves on the topological surface $\mathcal{X}$. The introduced notions have an obvious natural interpretation in terms of the "surface edges" and "vertices" on $\mathcal{X}$. Our dimension formulas for splines on the surface complexes formed by triangles naturally involve the cardinality of the sets of polygons, complex edges and complex vertices[5].

Now we are ready to give the definition of a CG^1 surface complex.

Definition 6.10 A CG^1 *surface complex* $\mathcal{M}$ is given by:

(i) A CG^0 surface complex, such that all the homeomorphisms μ in definition 6.9 are C^1-diffeomorphisms.

(ii) For each interior edge e we are given an isomorphism Θ between the plane bundles on the corresponding polygonal edges, so that we have a complete CG^1 glueing data (μ, Θ) along the two polygonal edges in e.

Besides, we *require* that:

(iii) For each complex vertex p the CG^1 glueing data chosen for the interior edges incident to p form consistent CG^1 glueing data at the common vertex as defined in subsection 6.2.3. (Note that we have here a CG^1 surround data precisely if p is an interior vertex.)

A *continuous* function f on a CG^0 surface complex is defined as an assignment of a continuous function $f|_\Omega$ to each polygon Ω, such that for each interior edge e we have $f|_{\Omega_1} = f|_{\Omega_2} \circ \mu_e$ on E_1, where μ_e is the homeomorphism between the edges E_1 and E_2 identified in e, and Ω_i's are the polygons on which E_i's lie. Here the continuous functions $f|_\Omega$ are the *restrictions* of f onto polygons. The continuous function f is a *continuous (polynomial) spline* if on each polygon Ω the restriction $f|_\Omega$ is a polynomial function. We do not consider other kinds (rational, trigonometric, etc.) of splines.

Definition 6.11 A continuous function f on a CG^1 surface complex is a CG^1 *function* if:

(i) On each polygon Ω the restriction $f|_\Omega$ is a C^1 function.

(ii) For each interior edge e we have $d\,f|_{\Omega_2}(\mu_e(P)) \circ \Theta_e(P) = d\,f|_{\Omega_1}(P)$ for all $P \in E_1$, where (μ_e, Θ_e) is the CG^1 glueing data along the edges $E_1 \subset \Omega_1$ and $E_2 \subset \Omega_2$ identified in e.

The CG^1 function f is a CG^1 *spline* if, moreover, all the restrictions $f|_\Omega$ are polynomial functions.

[5]One can say that the polygons, complex edges and vertices with the introduced relations form a *multicomplex*, i.e. a generalization of a polyhedral complex, where one allows an edge to connect a vertex with itself, or two sides of the same polygon to be identified into a (multicomplex) edge. We remark that our definition gives a multicomplex which is automatically *pure two-dimensional* and *hereditary*, because of the restriction in *(ii)* and of the chosen equivalence relation on polygonal vertices.

The space of splines on a CG^1 surface complex $\mathcal{M}$ is denoted by $S^1(\mathcal{M})$. It is a subring of the space of all CG^1 functions on $\mathcal{M}$ (as we will see in this subsection, the space of CG^1 functions is the same as the space of C^1 functions on the underlying topological space with the proper C^1 structure, hence it is a ring.) The space of splines, defined by polynomials of bounded degree k, is denoted by $S^1_k(\mathcal{M})$. It is a finite dimensional vector space over $\mathbb{R}$. Our main interest is the dimension of these spline spaces on certain surface complexes[6]. Especially we are interested in the spline spaces which separate complex vertices and directions at all these vertices, as defined below.

Definition 6.12 Let $\mathcal{M}$ be a CG^1 surface complex, and let p be an interior vertex of $\mathcal{M}$. Suppose that p identifies n polygonal vertices $P_i \subset \Omega_i$ (for $i = 1, \ldots, n$). Let $\Theta_j(P_j)$ be the isomorphisms of the tangent planes T_{Ω_j,P_j} induced by the consistent CG^1 glueing data $\mathcal{D}$ at p (for $j = 1, \ldots, n-1$ or n). These isomorphisms define an equivalence relation between the derivations on different tangent planes T_{Ω_j,P_j}, so that an equivalence class contains a derivation from each tangent plane. Such an equivalence class $\mathbf{D} = (D_1, \ldots, D_n)$ is called a *complex derivation at the vertex* p. It is indeed a derivation on the CG^1 functions on $\mathcal{M}$, because for a CG^1 function f we have well a well-defined derivative $\mathbf{D}f := D_1 f|_{\Omega_1} = \ldots = D_n f|_{\Omega_n}$ with the required properties. If a complex derivation is represented by a derivation in T_{E,P_i} for some polygonal edge E, then we call it a *complex derivation at* p *along the complex edge* corresponding to E. We may say that the complex derivations form the "tangent space" of $\mathcal{M}$ at p.

If p is an interior vertex, then one can represent a complex derivation at p by a derivation $D_i \in C_{\Omega_i,P_i}$ on some polygon Ω_i. The complex derivations at p along the complex edges are represented by derivations on two polygons. In order to make linear combinations of the complex derivations represented in this way, one can use a tangent subdivision (see definition 6.7) for $\mathcal{D}$ and linear identifications of its cones with the cones C_{Ω_i,P_i}. This construction is independent of the choice of a tangent subdivision.

The following lemma "reminds" us that the dimension of the space of the complex derivations at a complex vertex is two. We use it in the proof of the main theorem 6.4.6.

Lemma 6.2.2 *Let p be a complex vertex of a surface complex $\mathcal{M}$. Let $\tilde{S}$ be a vector space of CG^1 functions, and let $\mathbf{D}_1, \ldots, \mathbf{D}_m$ be complex derivations at p. Then the image of the linear map $\psi : \tilde{S} \to \mathbb{R}^m$ defined by $f \mapsto (\mathbf{D}_1 f, \ldots, \mathbf{D}_m f)$ is at most two-dimensional.*

Proof. All the complex derivations at p form a vector space of dimension two. Two generators for it can be represented by two independent derivations in C_{Ω_1,P_1}. □

Example 6.13 To support a few examples later, consider a complex vertex p which identifies three polygonal vertices $B_1 \in \Omega_1$, $B_2 \in \Omega_2$ and $B_3 \in \Omega_3$. Let $\mathcal{D}$ be the CG^1

[6] It makes most sense to speak about splines on a CG^1 surface if the μ's are linear homeomorphisms and the maps Θ of the plane bundles are rational. However, one can obtain interesting spline spaces if each homeomorphism $\mu : E_1 \to E_2$ has form $\lambda_2 \circ \lambda_1^{-1}$, where $\lambda_i : [0,1] \to E_i$ for $i = 1, 2$ are polynomial parametrizations of the equivalent polygonal edges. To get an example, consider two polygonal patches $\Psi_1 : [1,4] \times [0,1] \to \mathbb{R}^3$ and $\Psi_2 : [1,8] \times [-1,0] \to \mathbb{R}^3$ defined by $\Psi_1(x,y) = (x^3, y, 0)$ and $\Psi_2(x,y) = (x^2, y, 0)$.

surround glueing data around p. We keep the same notations (and the assumption on the polygonal angles at the B_i's) as in algorithm 6.2.1 and example 6.8. For $i = 1, 2, 3$ choose the derivations D_i^+, D_i^- on the boundary of C_{Ω_i,B_i} as in step 2 of algorithm 6.2.1. The derivations D_1^+, D_2^- are equivalent and represent a complex derivation $\mathbf{D}_1$. Similarly we have the complex derivations $\mathbf{D}_2$, $\mathbf{D}_3$. We identify the space of the complex derivations at p with $\mathbb{R}^2$ by taking a tangent subdivision for $\mathcal{D}$ and identifying $\mathbf{D}_1$ and $\mathbf{D}_2$ with vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ on the corresponding half-lines of the subdivision. Then the derivation $\mathbf{D}_3$ is identified with a vector $\mathbf{u}_3$ on the third half-line. Like in example 6.8, the vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ satisfy the linear relation (6.3) with some positive constants ζ_1, ζ_2, ζ_3. It follows that the three complex derivations satisfy

$$\zeta_1\mathbf{D}_1 + \zeta_2\mathbf{D}_2 + \zeta_3\mathbf{D}_3 = 0. \tag{6.4}$$

The triple $(\zeta_1, \zeta_2, \zeta_3)$ is determined up to a constant multiple. One can normalize the three numbers so that $\zeta_1 + \zeta_2 + \zeta_3 = 1$.

Definition 6.14 Let $\tilde{S}$ be a subset of CG^1 functions on a CG^1 surface complex $\mathcal{M}$. We say that $\tilde{S}$ *separates vertices* of $\mathcal{M}$ if for any two complex vertices there is a CG^1 function which vanishes at one vertex and does not vanish at the other one. The space $\tilde{S}$ *separates directions* at a complex vertex p if for any two independent complex derivations $\mathbf{D}_1$, $\mathbf{D}_2$ at p the linear map $\tilde{S} \to \mathbb{R}^2$ defined by $f \mapsto (\mathbf{D}_1 f,\, \mathbf{D}_2 f)$ is surjective.

Definition 6.15 The topological space $\mathcal{X}$ obtained from a CG^1 surface complex $\mathcal{M}$ is called a CG^1 *surface*. We identify the polygons of $\mathcal{M}$ as subsets of $\mathcal{X}$. If all the complex edges of $\mathcal{M}$ are interior, then $\mathcal{X}$ is a *closed* CG^1 *surface*. A CG^1 *function* on the CG^1 surface $\mathcal{M}$ is just a CG^1 function on the surface complex $\mathcal{M}$ seen as a function on $\mathcal{X}$. The *tangent space* $T_{\mathcal{X},P}$ of $\mathcal{X}$ at its interior point P is defined as follows:
(a) If P lies in an interior of a polygon Ω, then the tangent space is $T_{\Omega,P}$.
(b) If P corresponds to an interior vertex p of $\mathcal{M}$, then $T_{\mathcal{X},P}$ is the linear space of the complex derivations at p.
(c) If P corresponds to a point on an interior edge e of $\mathcal{M}$, let (μ, Θ) be the CG^1 glueing data assigned to e, and let $P_1 \in \Omega_1, P_2 = \mu(P_1) \in \Omega_2$ be the two points on the corresponding polygonal edges which are identified as P on $\mathcal{X}$. Then $T_{\mathcal{X},P}$ is the space of derivations in T_{Ω_1,P_1} and T_{Ω_2,P_2} identified by the linear transformation $\Theta(P_1)$.
Let $\Psi : \mathcal{X} \to \mathbb{R}^m$, $\Psi = (f_1, \ldots, f_m)$ be a mapping given by CG^1 functions f_i on $\mathcal{X}$. For a point $P \in \mathcal{X}$ corresponding to "polygonal" points $P_i \in \Omega_i$ one can correctly define the linear map $T_P(\Psi) : T_{\mathcal{X},P} \to T_{\mathbb{R}^m,\Psi(P)}$ such that restricted on the tangent spaces T_{Ω_i,P_i} it coincides with $T_{P_i}(\Psi|_{\Omega_i})$. The map $\Psi : \mathcal{X} \to \mathbb{R}^m$ as above is a *realization* of the CG^1 surface $\mathcal{X}$ (or of the CG^1 surface complex $\mathcal{M}$) if it is injective, and at all points $P \in \mathcal{X}$ the linear map $T_P(\Psi)$ is injective as well. (In particular, the set of coordinate functions f_i separates directions at the complex vertices.)

In the rest of this subsection we show that a closed CG^1 surface has a C^1 surface structure such that the notions of C^1 functions and CG^1 functions coincide. Essentially we rewrite several results of J.M. Hahn in [29] in our slightly different setting. In particular, a realization of a CG^1 surface complex in $\mathbb{R}^m$ gives us a number of C^1

polygonal patches joining with CG^1 continuity (see subsections 6.2.2 and 6.5). From the results in [29] it follows that such a CG^1 patch complex forms a C^1 surface in $\mathbb{R}^m$. Besides, the restriction of a C^1 function on $\mathbb{R}^m$ is a CG^1 function on the realized CG^1 surface. For simplicity we do not consider CG^1 surfaces with boundary. Besides, definition 6.1 does not include C^1 surfaces with boundary. "Good" boundary of CG^1 surfaces is ensured by condition (1a) of definition 6.5.

Lemma 6.2.3 *Suppose that CG^1 glueing data (μ, Θ) for two polygons Ω_1 and Ω_2 and their edges E_1, E_2 is given. Then there exist open neighbourhoods U_1 of E_1, U_2 of E_2 in $\mathbb{R}^2$ and a C^1-isomorphism $\varphi : U_1 \to U_2$ such that:*
(a) *$\varphi(E_1) = E_2$ and the restriction of φ to E_1 coincides with μ.*
(b) *For all $P \in E_1$ the map $T_P(\varphi) : T_{\Omega_1,P} \to T_{\Omega_2,\mu(P)}$ coincides with $\Theta(P)$.*

Proof. For notational convenience we may suppose that $E_1 = E_2 = [0,1] \times \{0\} \subset \mathbb{R}^2$ and that $\Omega_1 \subset \{(x,y) \in \mathbb{R}^2 | y \geq 0\}$, $\Omega_2 \subset \{(x,y) \in \mathbb{R}^2 | y \leq 0\}$. The diffeomorphism μ has the form $(x,0) \mapsto (a(x), 0)$, where a is a C^1-function, so it extends to a C^1-function on $(-\epsilon, 1+\epsilon)$ for some $\epsilon > 0$. This extension is also given the name a. Moreover, $a : [0,1] \to [0,1]$ is a bijection, and if we choose $a(0) = 0$, $a(1) = 1$, then the derivative of a is strictly positive.

On all the tangent planes we will use the basis $\partial/\partial x$, $\partial/\partial y$. The map Θ has the matrix $\begin{pmatrix} a'(x) & b(x) \\ 0 & c(x) \end{pmatrix}$ with respect to these bases, where $b(x)$ and $c(x)$ are continuous functions, and $c(x)$ is strictly positive. We have to produce a C^1-map $\varphi(x,y) = (\varphi_1(x,y), \varphi_2(x,y))$ such that:
(1) $\varphi(x,0) = (a(x), 0)$.
(2) $T_{(x,0)}(\varphi) = \Theta(x,0)$.
(3) φ is a C^1-isomorphism between neighbourhoods of E_1 and E_2.
Recall that the matrix of $T(\phi)$ is given by $\begin{pmatrix} \partial\varphi_1/\partial x & \partial\varphi_1/\partial y \\ \partial\varphi_2/\partial x & \partial\varphi_2/\partial y \end{pmatrix}$. If b and c are differentiable, then one can take $\varphi(x,y) = (a(x) + b(x)y, c(x)y)$. For general b and c we extend these functions to continuous functions on $(-\infty, +\infty)$. Let B and C denote primitive functions for b and c, also defined on $(-\infty, +\infty)$. For $x \in (-\epsilon, 1+\epsilon)$ and y with $|y| < \epsilon$, we define

$$\varphi(x,y) = (a(x) + B(x+y) - B(x), C(x+y) - C(x)).$$

Conditions (1) and (2) are easily verified. Further φ is "locally" a C^1-isomorphism since for small $\epsilon > 0$ the determinant of $T(\varphi)$ does not vanish. In order to show that φ is injective for small enough $\epsilon > 0$, we consider the inverse glueing data (μ^{-1}, Θ^{-1}). There is a corresponding C^1-map $\tilde{\varphi}$ defined on a neighborhood of E_2. The composition $J = \tilde{\varphi} \circ \varphi$ corresponds to the "identity" glueing data $(id,\ id)$ from E_1 to E_1. A simple calculation[7] shows that J is injective on some neighbourhood of E_1. Hence φ is also injective for small enough $\epsilon > 0$, so condition (3) is also satisfied. □

[7] Indeed, the composition J has the form $(x + r_1(x,y), y + r_2(x,y))$, where r_1, r_2 are C^1 functions such that, in particular, $\partial r_i/\partial x = \partial r_i/\partial y = 0$ on $[0,1]$ for $i = 1, 2$, so for any $\delta > 0$ these derivatives are in $(-\delta, \delta)$ for small enough neighbourhood of $[0,1]$ in $\mathbb{R}^2$. From here one derives $\| J(P_1) - J(P_2) \| \geq (1 - 2\delta) \| P_1 - P_2 \|$. By taking $\delta < 1/2$ one gets the injectivity of J.

The promised C^1 surface structure in this situation is given as follows. As in subsection 6.2.2, let $\mathcal{X}$ denote the glueing of two polygons Ω_1 and Ω_2 along edges E_1 and E_2 with glueing data (μ, Θ). Let $f : \mathcal{X} \to \mathbb{R}$ be a continuous function. The condition that f is a CG^1-function is a local one. For a point $x \in \mathcal{X}$ which lies in the interior of Ω_1 or Ω_2 we have by definition the condition of a C^1 function. Consider a point $P \in \mathcal{X}$ which corresponds to an interior point P_1 on E_1 and $\mu(P_1) \in E_2$. Using the C^1-isomorphism φ of lemma 6.2.3, one can identify a neighbourhood U of P in $\mathcal{X}$ with a neighbourhood V of the point $\mu(E_1)$ in $\mathbb{R}^2$. The homeomorphism $\psi : U \to V$, which produces this identification, is given by $\psi(P) = \varphi(P)$ for $P \in U \cap \Omega_1$ and $\psi(P) = P$ for $P \in U \cap \Omega_2$. One can easily check that if f is a CG^1 function then $g = f|_U \circ \psi^{-1}$ is a C^1 function on V, and vice versa, if g is a C^1 function on V, then $f = g \circ \psi$ is a CG^1 function on U. In particular, let $\mathcal{X}^0$ be obtained from $\mathcal{X}$ by deleting the other edges of Ω_1 and Ω_2. Then $\mathcal{X}^0$ has a unique structure of a C^1-surface such that the notions of CG^1-function and C^1-function coincide.

The next lemma proves the similar statement for the situation of subsection 6.2.3.

Lemma 6.2.4 *Let $\mathcal{D}$ be a CG^1 surround glueing data of n polygons at a common vertex, and let $\mathcal{X}$ be the topological surface obtained from $\mathcal{D}$. Let $B \in \mathcal{X}$ be the common vertex. Then there exists a homeomorphism $\xi : U \to V$ of an open neighbourhood $U \subset \mathcal{X}$ of B with an open subset of $\mathbb{R}^2$, such that: $f : U \to \mathbb{R}$ is a CG^1-function if and only if there is a C^1-function $g : V \to \mathbb{R}$ with $f = g \circ \xi$.*

Let $\mathcal{X}^0$ be obtained from $\mathcal{X}$ by deleting all other edges. Then $\mathcal{X}^0$ has a unique structure of C^1-surface such that the notions of a CG^1-function and of a C^1-function coincide.

Proof. The second statement follows directly form the first one. The main issue is the construction of the homeomorphism ξ. For notational convenience we consider the case $n = 4$ and the polygons $\Omega_1, \ldots, \Omega_4$ given as follows. We assume that Ω_1 lies in the first quadrant, and two of its edges are $\{0\} \times [0,1]$, $[0,1] \times \{0\}$. Similarly $\Omega_2, \Omega_3, \Omega_4$ lie in the second, third and fourth quadrant and have similar edges of length 1 on the coordinate axes. The glueing data are given for the pairs of edges $\{0\} \times [0,1] \subset \Omega_2$ and $\{0\} \times [0,1] \subset \Omega_1$, $[-1,0] \times \{0\} \subset \Omega_2$ and $[-1,0] \times \{0\} \subset \Omega_3$ and so on. By lemma 6.2.3, one can define a C^1-isomorphism φ_2 from an open neighbourhood (always in $\mathbb{R}^2$) of the edge $\{0\} \times [0,1]$ of Ω_2 to an open neighbourhood of the edge $\{0\} \times [0,1]$ of Ω_1, which fits the glueing data. The maps $\varphi_3, \varphi_4, \varphi_1$ are obtained in the same way. One would like to define the homeomorphism $\xi : U \to V$ by:

(1) ξ is the identity on $U \cap \Omega_1$.
(2) ξ is equal to φ_2 on $U \cap \Omega_2$.
(3) ξ is equal to $\varphi_2 \circ \varphi_3$ on $U \cap \Omega_3$.
(4) ξ is equal to $\varphi_2 \circ \varphi_3 \circ \varphi_4$ on $U \cap \Omega_4$.

This works well for all the gluing of the edges, except for the edge $[0,1] \times \{0\}$ of Ω_4 and the edge $[0,1] \times \{0\}$ of Ω_1. The map φ_1^{-1} performs the last glueing, but in general φ_1^{-1} is not equal to $\psi := \varphi_2 \circ \varphi_3 \circ \varphi_4$. The CG^1 surround glueing conditions (2a–2b) of definition 6.5 only prescribe that the tangent maps of φ_1^{-1} and ψ at $(0,0)$ are the same and the four cones form a tangent subdivision (see definition 6.7). The remedy is to change ξ on $U \cap \Omega_4$ into a suitable C^1-isomorphism ξ_4 (always locally at $(0,0)$)

which is equal to ψ near the edge $\{0\} \times [-1, 0]$ of Ω_4, and is equal to φ_1^{-1} near the edge $[0, 1] \times \{0\}$ of Ω_4.

The construction of ξ_4 can be given as follows. We note that $\psi = \varphi_1^{-1} \circ (id+\omega)$, where ω is a C^1-map at a neighbourhood of $(0,0)$, satisfying $\omega(0,0) = (0,0)$ and $T_{(0,0)}(\omega) = 0$. Let $\mathbf{S}^1$ be the circle with radius 1 and center $(0,0)$. On this circle we consider a C^∞ function h with values in $[0,1]$ such that h is equal to 1 in a neighbourhood of $(0,-1) \in \mathbf{S}^1$, and it is equal to 0 in a neighbourhood of $(1,0) \in \mathbf{S}^1$. We define the map $\bar{\omega} : U \to U$ by $\bar{\omega}(0,0) = (0,0)$ and $\bar{\omega}(x,y) := h\left((x,y)/\sqrt{x^2+y^2}\right)\, \omega(x,y)$ for $(x,y) \neq (0,0)$. Then one can check $\bar{\omega}$ is a C^1 map and that $T_{(0,0)}(\bar{\omega}) = 0$. We define now ξ_4 by $\xi_4 := \varphi_1^{-1} \circ (id + \bar{\omega})$. By considering the cases when (x,y) lies in the interior of Ω_1 in U, or (x,y) lies on the edge $[0,1] \times \{0\}$, or $(x,y) = (0,0)$, one verifies that the map ξ, defined by replacing the original ψ on P_4 by ξ_4, has the required properties. □

Theorem 6.2.5 *Let $\mathcal{X}$ be a closed CG^1-surface. Then $\mathcal{X}$ has a unique structure of C^1-surface such that the notions of a CG^1 function and of a C^1 function coincide.*

Let $\Phi : \mathcal{X} \to \mathbb{R}^m$ be a realization. Then the image $\Phi(\mathcal{X}) \subset \mathbb{R}^m$ is, with respect to the differential structure induced by $\mathbb{R}^m$, a C^1 surface. Moreover, Φ is a C^1 diffeomorphism $\mathcal{X}$ (as a C^1 surface) with $\Psi(\mathcal{X})$.

Proof. We have to show that every point P of $\mathcal{X}$ has a neighbourhood U and a homeomorphism ξ of U with an open disk $\mathcal{U}$ in $\mathbb{R}^2$, such that ξ identifies the CG^1-functions on U with the C^1-functions on $\mathcal{U}$. In this situation the covering of $\mathcal{X}$ by these open sets U, and the homeomorphisms ξ satisfy the axioms for a C^1 surface, and that every ξ is unique up to a C^1-diffeomorphisms of $\mathcal{U}$. Moreover, each ξ is unique up to a C^1 diffeomorphism of $\mathcal{U}$. Then the first statement follows.

If $P \in \mathcal{X}$ is an interior point of some polygon then the statement is obvious. The next situation is where P comes from an interior point of some edge of a polygon. This case is dealt with in lemma 6.2.3 and the remarks after it. Finally one has to consider a $P \in \mathcal{X}$ which is the image of a vertex of some polygon. This situation is treated in lemma 6.2.4. The second part of the theorem is rather obvious. Indeed, after the identification of $\mathcal{X}$ as a C^1-surface, the map Ψ is a C^1-diffeomorphism and thus provides a C^1-isomorphism of $\mathcal{X}$ with its image $\Psi(\mathcal{X})$. □

6.2.5 Examples

In this subsection we consider a few examples of CG^1 continuous surface complexes. Two surface complexes are formed by just one rectangle, and the opposite edges are chosen to be equivalent. These two examples represent two closed topological surfaces, a torus and a Klein bottle. In both examples we find all possible choices of rational CG^1 glueing data (μ, Θ) of degree zero (see definition 6.4) for the identified edges so that we obtain a CG^1 surface complex. For the torus we give the generators for the ring of CG^0 and CG^1 splines, and examples of realizations. For the Klein bottle we do this only for the easiest choice of the differential data. We also briefly consider the construction of a projective plane from a rectangle, and argue why one can not put a CG^1 surface structure on this CG^0 surface complex.

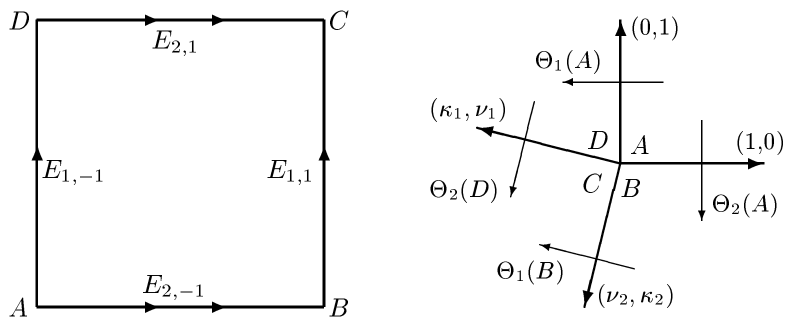

Figure 6.4: Defining a CG^1 torus

Our last example is a surface complex formed by four triangles, which glue together in the same way as the four triangular faces of a tetrahedron in $\mathbb{R}^3$ do, see figure 6.8. Topologically the obtained CG^1 surface is a sphere. We show that one cannot choose a rational CG^1 glueing data of degree 0 on any of the six interior edges, and give the most general way to choose the rational data of degree 1 on each interior edge. This example will be continued in subsection 6.4.4.

Example 6.16 (The torus) Consider a CG^0 surface complex $\mathcal{T}$ formed by one rectangle $\Omega = [-1,1] \times [-1,1]$ whose pairs of opposite edges are linearly identified as in figure 6.4. This is a classical construction of a torus (see [41]). For $i \in \{-1,1\}$ let $E_{1,j} = \{j\} \times [-1,1]$ and $E_{2,j} = [-1,1] \times \{j\}$ denote the edges of Ω. We identify all the tangent spaces $T_{\Omega,P}$ for $P \in \Omega$ with $T_{\mathbb{R}^2,0}$ by matching partial derivatives $\partial/\partial x$ and $\partial/\partial y$ on each of them. The four vertices of Ω are "identified" into one complex vertex $a = \{A_1, A_2, A_3, A_4\}$. There are two interior edges $e_i = \{E_{i,-1}, E_{i,1}\}$ for $i = 1, 2$, and $\mu_i : E_{i,-1} \to E_{i,1}$ sends $(-1, s)$ to $(1, s)$, or $(s, -1)$ to $(s, 1)$ respectively.

We want to assign all possible combinations of "constant" isomorphisms Θ_1, Θ_2 of the plane bundles on the identified edges, so that with the obtained rational CG^1 glueing data (μ_i, Θ_i) of degree 0 the torus $\mathcal{T}$ is a CG^1 surface. We have to choose consistent CG^1 surround data around the common vertex a. Applying algorithm 6.2.1, in the first two steps we choose four vectors $(1,0)$, $(0,1)$, (κ_1, ν_1) and (ν_2, κ_2) in $\mathbb{R}^2$ so that they define a subdivision of $\mathbb{R}^2$ into four convex cones. Thus $\kappa_i < 0$ for $i = 1, 2$, and $\kappa_1\kappa_2 - \nu_1\nu_2 > 0$. For each vertex $P \in \{A, B, C, D\}$ we take the basis $(\pm\partial/\partial x, \pm\partial/\partial y)$ for $T_{\Omega,P}$, where signs are chosen in such a way that the derivatives are in $C_{\Omega,P}$. Then equations (6.2) give us the matrices of $\Theta_1(A)$, $\Theta_1(B)$, $\Theta_2(A)$ and $\Theta_2(D)$ in these bases. For instance, $\Theta_1(A)$ and $\Theta_1(B)$ are given respectively by

$$\begin{pmatrix} -\partial/\partial x \\ \partial/\partial y \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 \\ 1 & \nu_1 \end{pmatrix} \begin{pmatrix} \partial/\partial y \\ \partial/\partial x \end{pmatrix}, \quad \begin{pmatrix} -\partial/\partial x \\ -\partial/\partial y \end{pmatrix} = \begin{pmatrix} 0 & \frac{\kappa_1\kappa_2 - \nu_1\nu_2}{\kappa_2} \\ 1 & \frac{\nu_1}{\kappa_2} \end{pmatrix} \begin{pmatrix} -\partial/\partial y \\ \partial/\partial x \end{pmatrix}.$$

Since we require that Θ_1 is constant, the matrices of $\Theta_1(A)$ and $\Theta_1(B)$ must be equal in the same basis $(\partial/\partial x, \partial/\partial y)$ of the tangent spaces, hence $-\kappa_1 = -\kappa_1 + \nu_1\nu_2/\kappa_2$ and

$X = (4 - 2\,x^2)(1 - 2\,y^2)$
$Y = (4 - 2\,x^2)(2\,y - 2\,y^3)$
$Z = 2\,(x - x^3)$
$(x, y) \in [-1,1] \times [-1,1]$

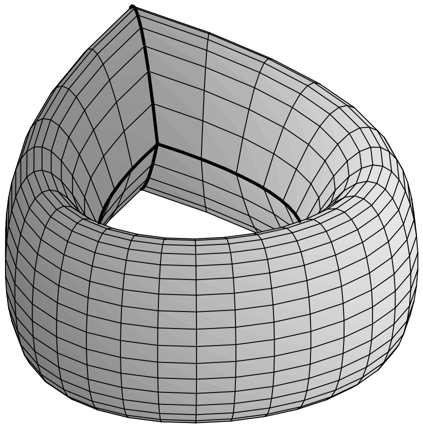

Figure 6.5: Continuous torus realized by one rectangular patch.

$\nu_1 = -\nu_1/\kappa_2$. We also have two similar equations coming from $\Theta_2(A)$ and $\Theta_2(D)$. From these equations we obtain the relations $\nu_1\nu_2 = 0$, $\nu_1(\kappa_2+1) = 0$ and $\nu_2(\kappa_1+1) = 0$. This gives us three families of CG^1 surface complexes with constant Θ_i's:
(a) $\nu_1 = 0$, $\nu_2 = 0$, and arbitrary κ_1, κ_2.
(b) $\kappa_1 = -1$, $\nu_1 = 0$, and arbitrary κ_2, ν_2.
(c) symmetrically to (2), $\kappa_2 = -1$, $\nu_2 = 0$.

Now we proceed to compute CG^1 splines on the torus for each combination of the differential data. First we note that a continuous spline on the torus is given by a polynomial on Ω of form

$$F(x,y) = c_0 + (x^2-1)f(x) + (y^2-1)g(y) + (x^2-1)(y^2-1)h(x,y), \tag{6.5}$$

for some real constant c_0 and polynomials $f(x)$, $g(y)$ and $h(x,y)$. Here c_0 is the value of the spline at the common vertex, and $c_0 + (x^2-1)f(x)$, $c_0 + (y^2-1)g(y)$ are the restrictions to the edges of Ω. One can easily check that continuous splines on this torus form the ring $\mathbb{R}[x^2, x^3-x, y^2, y^3-y]$, because $(x^2-1)f(x) \in \mathbb{R}[x^2, x^3-x]$ for any $f(x) \in \mathbb{R}[x]$, etc. To realize a continuous torus one can mimic the well known parametrization of a torus

$$(x,\, y) \mapsto ((a+\cos\pi x)\cos\pi y,\, (a+\cos\pi x)\sin\pi y,\, \sin\pi x) \tag{6.6}$$

(with real $a > 1$) by the rectangle $[-1,1]\times[-1,1]$. To do this, we approximate the functions $\cos\pi t$ and $\sin\pi t$ on the interval $[-1,1]$ by the polynomials $1-2t^2$ and $2(t-t^3)$ respectively. Then by replacing the trigonometric functions by these approximations one obtains a realization of a continuous torus like in figure 6.5.

From the CG^1 continuity relation (6.1) we derive that a polynomial F in (6.5) defines a CG^1 spline on the torus if and only if

$$\frac{\partial F}{\partial x}(-1,t) = -\kappa_1\frac{\partial F}{\partial x}(1,t) - \nu_1\frac{\partial F}{\partial y}(1,t) \tag{6.7}$$

$$\frac{\partial F}{\partial y}(t,-1) = -\kappa_2\frac{\partial F}{\partial y}(t,1) - \nu_2\frac{\partial F}{\partial x}(t,1) \tag{6.8}$$

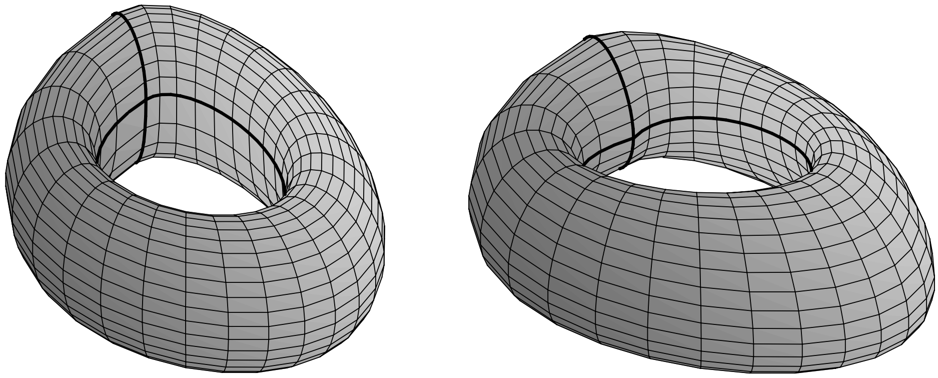

Figure 6.6: CG^1 tori realized by one rectangular patch.

We can refine the expression (6.5) by

$$F = c_0 + (c_1 + c_2 x)\,X_2 + f_0(x)\,X_2^2 + (c_3 + c_4 y)\,Y_2 + g_0(y)\,Y_2^2 + \\ +(c_5 + c_6 x + c_7 y + c_8 xy)\,X_2 Y_2 + (f_1(x) + y f_2(x))\,X_2^2 Y_2 + \\ +(g_1(y) + x g_2(y))\,X_2 Y_2^2 + h_0(x,y)\,X_2^2 Y_2^2,$$

where $X_2 = x^2 - 1$, $Y_2 = y^2 - 1$, and c_i are constants, f_i, g_i, h — polynomials. From equations (6.7–6.8) we derive the relations between undetermined constants and polynomials for all three families of possible CG^1 tori.

In the first case $\nu_1 = \nu_2 = 0$ we get the equations

$$\frac{c_1}{c_2} = \frac{c_5}{c_6} = \frac{c_7}{c_8} = \frac{g_1(y)}{g_2(y)} = \frac{1+\kappa_1}{1-\kappa_1}, \qquad \frac{c_3}{c_4} = \frac{c_5}{c_7} = \frac{c_6}{c_8} = \frac{f_1(x)}{f_2(x)} = \frac{1+\kappa_2}{1-\kappa_2}.$$

It follows that $c_0, c_2, c_4, c_8, f_0, f_2, g_0, g_2, h_0$ can be chosen freely. For $i = 1,2$ we introduce $\tilde\kappa_i = (1+\kappa_i)/(1-\kappa_i) \in (-1,1)$, and set $X_3 = (x+\tilde\kappa_1)(x^2-1)$ and $Y_3 = (y+\tilde\kappa_2)(y^2-1)$. One can check that polynomials F, which give us a CG^1 spline on the CG^1 torus in this case, form the ring $\mathbb{R}[X_3, X_2^2, X_2X_3, Y_3, Y_2^2, Y_2Y_3]$. One can again approximate $\sin\pi x \sim -2\,X_3$ and $\cos\pi x \sim 2\,X_2^2 - 1$ in the parametrization (6.6) to get a realization of a torus like in figure 6.6 on the left-hand side. Such a realization of the torus with the most symmetric choice $\kappa_1 = \kappa_2 = -1$ visually almost does not differ from the image of the trigonometric map (6.6).

In the second case $\kappa_1 = -1, \nu_1 = 0$, the first equation in (6.7) gives $c_1 = c_5 = c_7 = 0$ and $g_1(y) = 0$. We keep notations μ_2, X_2, Y_2, Y_3 and $X_3 = x^3 - x$ and introduce $\xi_2 = \nu_2/(1-\kappa_2)$. We assume $\nu_2 \neq 0$, because $\nu_2 = 0$ gives us the previous case. We also introduce $\tilde c_0$, $\tilde c_1$ and $\tilde f_0(x)$ from $f_0(x) = \tilde c_0 + \tilde c_1 + (x^2-1)\tilde f_0(x)$. Then one can derive

$$c_3 = \tilde\kappa_2 c_4 + \xi_2 c_2, \quad c_6 = \tilde\kappa_2 c_8 + 2\xi_2 \tilde c_0, \quad 4\tilde c_1 + 3c_2 = 0, \\ f_1(x) = \tilde\kappa_2 f_2(x) + \nu_2\left(3x\tilde f_0(x) + \tfrac{5}{2}\tilde c_1 + \tfrac{x^2-1}{2}\tilde f_0'(x)\right).$$

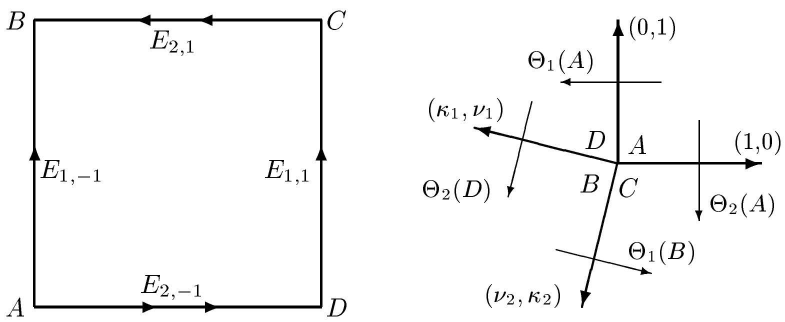

Figure 6.7: Defining a CG^1 Klein bottle

From here one can compute that the ring of CG^1 continuous splines on this torus is generated by Y_3, Y_2^2, Y_2Y_3, X_3Y_3, $X_3Y_2^2$, $X_2^2Y_3$ and

$$X_2^2 + 2\xi_2X_3Y_2, \quad 6X_2X_3 - 8X_3 + 15\xi_2X_2^2Y_2 - 8\xi_2Y_2,$$
$$X_2^3 + 3\xi_2X_2X_3Y_2, \quad 2X_2^2X_3 + 7\xi_2X_2^3Y_2 + 6\xi_2X_2^2Y_2.$$

A realization of a torus in this case is shown on the right-hand side of figure 6.4. It may seem that the surface is not smooth at the "common vertex", but we must keep in mind that the tangent sectors (see the definition of J.M. Hahn) of the patch need not be $\pi/2$, and the corresponding tangent subdivision is not formed by two intersecting lines (compare with definition 6.25 later.

Example 6.17 (The Klein bottle) Similarly we consider the Klein bottle $\mathcal{K}$ formed by the rectangle $\Omega = [-1,1] \times [-1,1]$ with identifications of opposite sides as in figure 6.7. Here $\mu_2 : E_{2,-1} \to E_{2,1}$ sends $(s,-1) \mapsto (-s,1)$. Like in the previous example, we are interested in CG^1 Klein bottles with rational CG^1 glueing data (μ_i, Θ_i) of degree 0 for the identified edges. We keep the same notations and identifications of the tangent spaces $T_{\Omega,P}$. Applying the algorithm 6.2.1 in the same way, we obtain the same matrices of $\Theta_1(A)$, $\Theta_2(A)$ and $\Theta_2(D)$ in the same bases, and the map $\Theta_1(B)$ is given by:

$$\begin{pmatrix} -\partial/\partial x \\ \partial/\partial y \end{pmatrix} = \begin{pmatrix} 0 & \frac{\kappa_2}{\kappa_1\kappa_2-\nu_1\nu_2} \\ 1 & -\frac{\nu_1}{\kappa_1\kappa_2-\nu_1\nu_2} \end{pmatrix} \begin{pmatrix} -\partial/\partial y \\ -\partial/\partial x \end{pmatrix}.$$

Comparing the matrices of $\Theta_1(A)$ and $\Theta_1(B)$ in the same bases gives the equation $\kappa_2 = (\kappa_1\kappa_2 - \nu_1\nu_2)\kappa_1$ and $\nu_1(\kappa_1\kappa_2 - \nu_1\nu_2) = \nu_1$. From the map Θ_2 we derive the same $\nu_1\nu_2 = 0$ and $\nu_2(\kappa_1 + 1) = 0$ as for the torus. From here we get two families of solutions:
(a) $\kappa_1 = -1$, $\nu_1 = 0$, and arbitrary κ_2, ν_2.
(b) $\kappa_1 = \kappa_2 = -1$, $\nu_2 = 0$, and arbitrary ν_1.
The most symmetric choice $\kappa_1 = \kappa_2 = -1$ and $\nu_1 = \nu_2 = 0$ gives the ring of splines generated by Y_3, Y_2^2, Y_2Y_3, X_3Y_2, $X_3(y^3-3y)$, $X_3Y_2^2$, X_2^2, $X_2X_3Y_2$, $X_2X_3(y^3-3y)$, $X_3Y_2^2$ and X_2^3, where $X_2 = 1-x^2$, $X_3 = x - x^3$, etc.

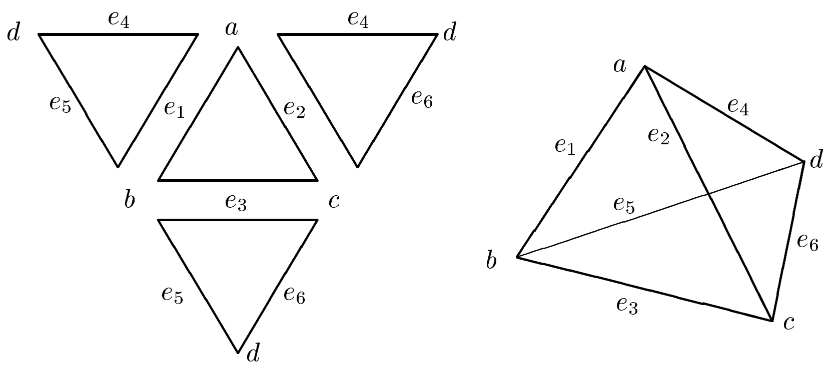

Figure 6.8: Defining a topological sphere by four triangles

Example 6.18 (The projective plane) There is one more classical closed surface obtained by identifying opposite edge of a rectangle. It is the projective plane, see [41]. With the same rectangle Ω with vertices A, B, C, D as in on the left hand side of figures 6.4 and 6.7, the linear homeomorphisms $\mu_1 : E_{1,-1} \to E_{1,1}$ and $\mu_2 : E_{2,-1} \to E_{2,1}$ are given by $(-1, s) \mapsto (1, -s)$ and $(s, -1) \mapsto (-s, 1)$ respectively. However, by glueing the rectangle Ω in this way we obtain two distinct complex vertices , namely $\{A, C\}$ and $\{B, D\}$. Both complex vertices are surrounded by just two polygonal angles $< \pi$, but two convex cones in a tangent subdivision cannot cover all $\mathbb{R}^2$. For this reason we cannot put consistent CG^1 glueing data "around" the complex vertices. To construct a CG^1 projective plane one has to subdivide the rectangle Ω.

Example 6.19 (A "tetrahedral" sphere) Consider a CG^1 surface complex $\mathcal{S}_4$ constructed from four triangles as follows. We mark the vertices of all triangles by four letters a, b, c and d in such a way that each triangle has a distinct set of three letters assigned to its vertices. For any two letters out of $\{a, b, c, d\}$ there are precisely two triangular edges whose end-points are marked by the two letters. We identify these pairs of polygonal edges so that their vertices marked by the same letter become equivalent. The obtained surface complex has four vertices (we name them a, b, c, d) and six edges $e_1, \ldots, e_6$, see figure 6.8. In fact we mimic the complex of triangular faces, edges and vertices of a tetrahedron in $\mathbb{R}^3$. Topologically the corresponding CG^0 surface is a sphere. We denote the triangles Ω_{abc}, Ω_{abd}, etc., according to the labels of their vertices. The polygonal edge of Ω_{abc} with end-vertices marked by a, b is denoted by $E_{ab,c}$, and similarly for the other polygonal edges.

To put a CG^1 structure on the surface we first choose CG^1 surround glueing data for the complex vertices. For the vertex a and an interior edge e_i with $i \in \{1, 2, 4\}$ we choose a complex derivation $\mathbf{D}_{e_i,a}$ at a along e_i. The vertex a is surrounded by three different triangles, hence a CG^1 surround structure is defined by one relation like (6.4) between the chosen complex derivations. Specifically, we rewrite it as

$$\zeta_{ab}\, \mathbf{D}_{e_1,a} + \zeta_{ac}\, \mathbf{D}_{e_2,a} + \zeta_{ad}\, \mathbf{D}_{e_4,a} = 0, \tag{6.9}$$

for some positive constants ζ_{ab}, ζ_{ac} and ζ_{ad}, normalized so that $\zeta_{ab}+\zeta_{ac}+\zeta_{ad}=1$. The other complex vertices are also surrounded by three polygons. We choose the complex derivations $\mathbf{D}_{e_1,b}$, $\mathbf{D}_{e_3,b}$, etc. Let ζ_{ba}, ζ_{bc}, ζ_{bd}, $\zeta_{ca},\ldots$ denote positive constants defined by relations like (6.9) at vertices b, c and d.

It remains to choose an isomorphism Θ_i of the plane bundles for each edge e_i on the surface. We note that it is impossible to choose a rational CG^1 glueing data of degree 0 for any complex edge. Indeed, for e_1 we have $\mathbf{D}_{e_1,b}=-\kappa_1\mathbf{D}_{e_1,a}$ for some $\kappa_1>0$, and

$$\mathbf{D}_{e_3,b}=\tilde{c}_1\mathbf{D}_{e_2,a}-\tilde{c}_2\mathbf{D}_{e_1,a},\quad \mathbf{D}_{e_5,b}=\tilde{c}_3\mathbf{D}_{e_4,a}-\tilde{c}_4\mathbf{D}_{e_1,a} \tag{6.10}$$

on the triangles Ω_{abc} and Ω_{abd} respectively, with positive constants $\tilde{c}_i$. Then the relation like (6.9) for the vertex b can be rewritten as

$$(-\kappa_1\zeta_{ba}-\tilde{c}_2\zeta_{bc}-\tilde{c}_4\zeta_{bd})\,\mathbf{D}_{e_1,a}+\tilde{c}_1\zeta_{bc}\,\mathbf{D}_{e_2,a}+\tilde{c}_3\zeta_{bd}\,\mathbf{D}_{e_4,a}=0. \tag{6.11}$$

For "constant" Θ_1 this expression should be a scalar multiple of (6.9). But this is impossible because all constants in these two expressions are positive.

One can choose a rational CG^1 glueing data (μ_1,Θ_1) of degree 1 as follows. Let $\lambda_1:[0,1]\to\mathbb{R}^2$ be the linear parametrization of the polygonal edge $E_{ab,c}$ such that $\lambda_1(0)\in a$. We choose two positive constants ν_{ab} and ν_{ba}, and use the constant κ_1 defined above. We consider the complex derivation $\mathbf{D}_{e_1,a}$ as a derivation at all points of the triangles Ω_{abc} and Ω_{abd} (by canonically identifying the tangent of all points on the triangles with $T_{\mathbb{R}^2,0}$), and we consider other chosen complex derivations in the similar way. For $t\in[0,1]$ let $P=\lambda_1(t)$. We take the basis $(\mathbf{D}_{e_1,a},t\nu_{ab}\zeta_{ac}\mathbf{D}_{e_2,a}+(1-t)\nu_{ba}\zeta_{bc}\mathbf{D}_{e_3,b})$ for the tangent space $T_{\Omega_{abc},P}$, and we take the basis $(\mathbf{D}_{e_1,a},t\nu_{ab}\zeta_{ad}\mathbf{D}_{e_4,a}+(1-t)\nu_{ba}\zeta_{bd}\mathbf{D}_{e_5,b})$ for $T_{\Omega_{abd},\mu_1(P)}$. Then we define $\Theta_1(P)$ by the matrix $\begin{pmatrix}1 & -t\nu_{ab}\zeta_{ab}+(1-t)\kappa_1\nu_{ba}\zeta_{ba}\\ 0 & -1\end{pmatrix}$ in the chosen bases for both tangent spaces. By evaluating these bases and the matrix at $t=0$ ant $t=1$ one can check that this Θ_1 is compatible with the chosen CG^1 surround structures at the vertices a and b. (Note that multiplication of ν_{ab} and ν_{ba} by the same constant gives us the same Θ_1.)

Analogously one can choose CG^1 structures on other edges of the surface. In section 6.4.4 we develop this example further by giving the dimension formula for the spline spaces (example 6.27). Using the most convenient of the derivations $\mathbf{D}_{e_1,a}$, etc., we will easily see that the above construction gives the most general way to choose rational CG^1 glueing data of degree 1 for all interior edges. A realization of a "tetrahedral" sphere is shown in figure 6.15 further.

6.3 Geometrically continuous Bézier complexes

From here on we restrict ourselves to the following class of CG^1 surface complexes.

Definition 6.20 A CG^1 surface complex is a CG^1 *(triangular) Bézier complex,* if

(i) All polygons of the complex are triangles.

(ii) The CG^1 glueing data assigned to each interior edge is rational.

In particular, all homeomorphisms μ in definition 6.10 (or definition 6.9) are linear, recall the definition 6.4 of the rational CG^1 glueing data.

Our main objective is counting dimensions of the spline spaces $S^1_k(\mathcal{M})$ on a Bézier complex. Especially we are interested in the spline spaces which separate complex vertices and directions at each complex vertex, because only they can be used to realize CG^1 surface complexes.

The purpose of this section is to recall the way in which the polynomial functions on triangles are represented in CAGD as homogeneous polynomials in three variables (*barycentric coordinates*). We use these techniques to represent a spline on a CG^1 Bézier complex $\mathcal{M}$ as a collection of homogeneous polynomials in three variables of the same degree, like in the theory of usual bivariate splines (see the previous chapter 5). In this way we get more algebraic structure on the space of splines. The definitions and notations in subsection 6.3.2 will help us to extract useful information characterizing a spline (i.e. its values at the vertices, derivatives, etc.) when needed.

6.3.1 Polynomial functions on triangles

Consider a non-degenerate triangle Ω in $\mathbb{R}^2$ with vertices A, B and C. There is a unique linear function $u : \mathbb{R}^2 \to \mathbb{R}$ such that $u(A) = 1$ and $u(B) = u(C) = 0$. Similarly, there are unique linear functions v and w such that $v(B) = w(C) = 1$ and they evaluate to zero at other vertices of Ω. These three functions are the *barycentric coordinates* on the real plane with respect to the triangle Ω, because at any point $P \in \mathbb{R}^2$ these functions evaluate to the classical barycentric coordinates of P with respect to Ω, see [20]. In particular, these functions satisfy $u+v+w = 1$, and the points on the triangle are given by $u \geq 0$, $v \geq 0$ and $w \geq 0$.

On the other hand, one can express the Cartesian coordinates x and y as homogeneous linear forms in the barycentric coordinates u, v and w. Explicitly, $x(u,v,w) = x_A u + x_B v + x_C w$, where x_A, x_B and x_C are the x-coordinates of the three points, and similarly for $y(u,v,w)$. Using this one can express any polynomial function $h(x,y)$ of degree $\leq n$ as a homogeneous polynomial in u, v and w of degree n:

$$h(x,y) = (u+v+w)^n \; h\left(\frac{x(u,v,w)}{u+v+w}, \frac{y(u,v,w)}{u+v+w}\right). \tag{6.12}$$

This is the *barycentric form* of $h(x,y)$, a special case of a homogenization of a polynomial with respect to a linear form, see [60]. To evaluate $h(x,y)$ at a point with barycentric coordinates (u,v,w) means to evaluate the barycentric form (6.12) at u,v,w.

Using barycentric coordinates one can consider the triangle Ω and functions on it without specifying its location[8] in $\mathbb{R}^2$. We introduce the *barycentric coordinate ring* $\mathbf{B}_2$ of Ω, which is just the graded polynomial ring $\mathbb{R}[u,v,w]$ (without the relation $u+v+w=1$). The nth graded part $\mathbf{B}_2(n)$ of this ring is isomorphic to the vector space of polynomial functions on Ω of degree $\leq n$ in the way given by (6.12).

It is convenient to express elements of $\mathbf{B}_2(n)$ in the *affine Bernstein-Bézier basis* of

[8]The barycentric coordinates also give an affine isomorphism of Ω with the standard two-dimensional simplex $\{(x,y) \in \mathbb{R}^2 \,|\, x,y, \geq 0,\ x+y \leq 1\}$ by $(x,y) \mapsto (u,v)$. More symmetrically, the map $\Omega \to \mathbb{R}^3$ given by $(x,y) \mapsto (u,v,w)$ identifies Ω with the standard two-dimensional simplex in $\mathbb{R}^3$, that is $\{(x,y,z) \in \mathbb{R}^3 \,|\, x,y,z \geq 0,\ x+y+z = 1\}$.

$\mathbf{B}_2(n)$. This basis consists of elements

$$B^n_{(i,j,k)}(u,v,w) = \frac{n!}{i!\,j!\,k!}\,u^i\,v^j\,w^k,$$

where i,j,k are non-negative integers such that $i+j+k=n$. Together these elements indeed form a basis of $\mathbf{B}_2(n)$ as a vector space. For an integer $n \geq 0$ let $\mathcal{J}_n$ denote the set of all triples (i,j,k) of non-negative functions such that $i+j+k=n$. Every polynomial in $\mathbf{B}_2(n)$ can then be written as

$$F(u,v,w) = \sum_{\mathbf{i}\in\mathcal{J}_n} c_{\mathbf{i}}\,B^n_{\mathbf{i}}(u,v,w). \tag{6.13}$$

The coefficients $c_{\mathbf{i}}$ are called the *control coefficients* of F. It is convenient to represent a polynomial in $\mathbf{B}_2$ by writing down its control coefficients in the triangular form, for example:

$$\begin{array}{ccccccc} c_{3,0,0} & & c_{2,1,0} & & c_{1,2,0} & & c_{0,3,0} \\ & c_{2,0,1} & & c_{1,1,1} & & c_{0,2,1} & \\ & & c_{1,0,2} & & c_{0,1,2} & & \\ & & & c_{0,0,3} & & & \end{array} \tag{6.14}$$

A control coefficient $c_{\mathbf{i}}$ is called a *boundary* control coefficient if at least one of the components of $\mathbf{i}$ is equal to zero. All other control coefficients are called *interior* control coefficients. If two indices of $\mathbf{i}$ are zero, then $c_{\mathbf{i}}$ is a *corner* control coefficient.

More generally, a tuple $\Psi = (F_1,\ldots,F_m)$ of polynomials in $\mathbf{B}_2(n)$ is called a *(triangular) Bézier patch*. The image of the map defined by Ψ is usually non-singular. One can write

$$\Psi(u,v,w) = \sum_{|\mathbf{i}|=n} P_{\mathbf{i}}\,B^n_{\mathbf{i}}(u,v,w),$$

where P_i are points in $\mathbb{R}^m$. These are called *control points* of the Bézier patch. One can perform *degree elevation*, i.e. represent the same polynomial map as a Bézier patch of larger degree $N>n$ (then the components of Ψ are multiplied $(u+v+w)^{N-n}$). As N tends to infinity, the sequence of their *control nets*, which by definition interpolate their control points piecewise linearly in a proper way (see [20]), converges uniformly to the polynomial map itself. In this sense control points of Ψ approximate the shape of the modeled surface. In CAGD Bernstein-Bézier bases are important because they give numerically stable and fast evaluation, differentiation and degree elevation algorithms (see [20]). We will use them to express geometric continuity constraints as convenient linear relations between control coefficients of polynomials defining splines.

Analogously, we use the representation of polynomial functions on a closed interval $[a,b]$ in $\mathbb{R}$ as homogeneous polynomials in two variables — barycentric coordinates $s=(t-a)/(b-a)$, $z=(b-t)/(b-a)$ on $\mathbb{R}$ with respect to the interval $[a,b]$. We also introduce the graded ring $\mathbf{B}_1 = \mathbb{R}[s,z]$ (without the relation $s+z=1$) as the *barycentric coordinate ring* on $[a,b]$. The nth graded part $\mathbf{B}_1(n)$ of $\mathbf{B}_1$ is isomorphic to the vector space of all polynomials in $\mathbb{R}[t]$ of degree $\leq n$. The *(affine) Bernstein-Bézier basis* of $\mathbf{B}_1(n)$ is the set of all elements $B^n_{(i,j)}(s,z) := \binom{n}{i} s^i z^j$, where i,j are non-negative integers such that $i+j=n$. If a polynomial $F \in \mathbf{B}_1(n)$ is expressed in this basis as

$$F(z,t) = \sum_{i=0}^{n} c_{(i,n-i)} \; B_{(i,n-i)},$$

then $c_{(i,n-i)}$ are *control coefficients* of F. The coefficients $c_{(0,n)}$ and $c_{(0,n)}$ are the *boundary* control coefficients, and the other ones are the *interior* control coefficients of F. Note that both barycentric coordinates give a linear parametrization $[0,1] \to [a,b]$ of the interval.

6.3.2 Splines on Bézier complexes

Let $\mathcal{M}$ be a CG^1 Bézier complex. It is natural to represent a spline f in $G^1_k(\mathcal{M})$ as an assignment of an element of $\mathbf{B}_2(k)$ to each triangle. From here on these homogeneous polynomials are the *restrictions* of the spline f to the triangles of the complex. We go a step further and we define the *restrictions* of f to the complex edges and the complex vertices as elements of $\mathbf{B}_1(n)$ or $\mathbb{R}$, respectively.

To define these restrictions, to every polygonal edge E we associate a ring homomorphism $\rho_E : \mathbf{B}_2 \to \mathbf{B}_1$ as follows. Suppose that E is given by $u_E = 0$ on its own triangle, where $u_E \in \{u,v,w\}$. Choose a permutation (v_E, w_E) of other two variables in the set $\{u,v,w\}$. We require that if E is equivalent to another polygonal edge $\tilde{E}$, then the corresponding linear diffeomorphism μ identifies the end-point $u_E = v_E = 0$ of E with the end-point $u_{\tilde{E}} = v_{\tilde{E}} = 0$ of $\tilde{E}$. Then ρ_E is defined by $\rho_E(u_E) = 0$, $\rho_E(v_E) = s$ and $\rho_E(w_E) = z$. Similarly, for a polygonal vertex P given by $u_P = v_P = 0$ on its own triangle (here u_P, v_P are distinct variables in $\{u,v,w\}$) we define a ring homomorphism $\rho_P : \mathbf{B}_2 \to \mathbb{R}$ by $\rho_P(u_P) = \rho_P(v_P) = 0$ and $\rho_P(w_P) = 1$, where w_P is the third variable.

Definition 6.21 We define the *restriction* $f|_e$ of a continuous spline f onto a complex edge e by $f|_e := \rho_E(f|_\Omega)$, where E is a polygonal edge in the equivalence class e, and Ω is the polygon of E. Similarly, the *restriction* $f|_p$ onto a complex vertex p is defined by $f|_p = \rho_P(f|_\Omega)$, where P is a polygonal vertex associated with p, and Ω is the polygon of P.

One can check that $f|_e$ does not depend on the choice of the polygonal edge E in the equivalence class e, and $f|_p$ does not depend on the choice of $P \in p$. Note that if a triangle Ω is incident to a complex edge e, then certain control coefficients of $f|_\Omega$ coincide with the control coefficients of $f|_e$, namely those which define the restrictions of $f|_\Omega$ to the polygonal edge identified[9] with e. We refer to the control coefficients of $f|_e$ as *edge control coefficients* of the spline f. The real numbers $f|_p$ are also called the *vertex control coefficients*. They determine the corner control coefficients of the restrictions $f|_\Omega$ and boundary control coefficients of the restrictions $f|_e$.

[9]One can associate to a complex edge e a "material" interval $[a,b] \subset R$, and consider elements of $\mathbf{B}_1$ as functions on the interval expressed in the barycentric coordinates with respect to $[a,b]$. The ring homomorphisms ρ_E give linear parametrizations $[a,b] \to E$ of polygonal edges E, such that they are compatible with μ's and give the "right" pull-backs of $f|_e$'s to the polygonal edges.

To express CG^1 continuity conditions of splines, we recall that directional derivatives on a triangle Ω are represented in CAGD by the derivations in the ring $\mathbf{B}_2$ of the form $c_1\partial/\partial u + c_2\partial/\partial u + c_3\partial/\partial w$ with $c_1 + c_2 + c_3 = 0$ (see [20]). These derivations are characterized by two properties: they decrease the degree of homogeneous polynomials by one, and map the "constant" polynomial $u+v+w$ to zero. In fact, we will only use the derivations like $\partial/\partial u - \partial/\partial v$, $\partial/\partial v - \partial/\partial w$ etc. These derivations gives us derivatives with respect to directions parallel to the sides of a triangle.

The CG^1 continuity conditions along an interior edge e for a spline can be expressed as follows. Suppose that the rational CG^1 glueing data (μ, Θ) has degree n, see definition 6.4. Let $E_1 \subset \Omega_1$ and $E_2 \subset \Omega_2$ be the polygonal edges identified by e. For $i = 1, 2$ and a point $P_i \in E_i$ we choose the basis $(\partial/\partial v_{E_i} - \partial/\partial w_{E_i}, \partial/\partial u_{E_i} - \partial/\partial w_{E_i})$ for the tangent space T_{Ω_i,P_i}. Then for $P_1 \in E_1$ the matrix of $\Theta(P_1)$ in the chosen bases is $\begin{pmatrix} 1 & -\beta(P_1)/\gamma(P_1) \\ 0 & -\alpha(P_1)/\gamma(P_1) \end{pmatrix}$, where α, β, γ are polynomials of degree at most n in a linear parameter of E_1. One can express these polynomials as elements of $\mathbf{B}_1(n)$, because the barycentric coordinates s, z are linear parameters of E_1.

From *here on* we represent the (rational) CG^1 glueing data (μ, Θ) for the interior edge e of a Bézier complex by the triple (α, β, γ) of homogeneous elements in $\mathbf{B}_1(n)$. Then from the explicit CG^1 glueing condition (6.1) one can derive the following condition on a CG^1 spline f represented in the barycentric coordinates of the triangles and edges[10] (keeping the notations E_i and Ω_i as above):

$$\alpha \cdot \rho_{E_1} \left(\frac{\partial f|_{\Omega_1}}{\partial u_{E_1}} - \frac{\partial f|_{\Omega_1}}{\partial w_{E_1}} \right) + \beta \cdot \left(\frac{\partial f|_e}{\partial s} - \frac{\partial f|_e}{\partial z} \right) + \gamma \cdot \rho_{E_2} \left(\frac{\partial f|_{\Omega_2}}{\partial u_{E_2}} - \frac{\partial f|_{\Omega_2}}{\partial w_{E_2}} \right) = 0. \tag{6.15}$$

The condition that the function $c(P_1)$ in formula (6.1) is negative on E_1 translates to the condition that $\alpha\,\gamma$, interpreted as a function on edge E_1 written in its barycentric coordinates, is a positive function.

These condition is equivalent to the conditions used in [18],[15] to derive explicit relations between the control points of two patches joining with CG^1 continuity. Instead of "coplanarity" and other conditions on some control points of two adjacent Bézier patches the equation 6.15 gives linear relations between certain control coefficients of the restriction of f, namely some of the control coefficients $c_{(i,j,k)}$ with $\min(i,j,k) \leq 1$. With some intuition these control coefficients of $f|_{\Omega_1}$ and $f|_{\Omega_2}$ can be easily "visually" identified when these polynomials in $\mathbf{B}_2$ are represented in the triangular form (6.14), see also figure 6.11 below. We say that these control coefficients are *restricted* by the interior edge e.

Definition 6.22 Here we specify the values of some complex derivation (see definition 6.12) at the complex vertices we will use. Consider a polygonal edge $E \in \Omega$ and a spline

[10] Note that the definition of $f|_e$ automatically implies

$$\frac{\partial f|_e}{\partial s} - \frac{\partial f|_e}{\partial z} = \rho_{E_1} \left(\frac{\partial f|_{\Omega_1}}{\partial u_{E_1}} - \frac{\partial f|_{\Omega_1}}{\partial v_{E_1}} \right) = \rho_{E_2} \left(\frac{\partial f|_{\Omega_2}}{\partial u_{E_2}} - \frac{\partial f|_{\Omega_2}}{\partial v_{E_2}} \right).$$

f. Let e be the complex edge associated to E, and let P_1 be the end-point $u_E = v_E = 0$ of E. Then we define the *edge-vertex derivative* (at P_1 along e)

$$\mathbf{D}_{e,P_1} f := \rho_{P_1} \left(\frac{\partial f|_\Omega}{\partial v_E} - \frac{\partial f|_\Omega}{\partial w_E} \right) = \left(\frac{\partial f|_e}{\partial s} - \frac{\partial f|_e}{\partial z} \right) \Big|_{s=0,z=1}.$$

Similarly, for the other end-point P_2 of E we define

$$\mathbf{D}_{e,P_2} f = \rho_{P_2} \left(\frac{\partial f|_\Omega}{\partial w_E} - \frac{\partial f|_\Omega}{\partial v_E} \right) = \left(\frac{\partial f|_e}{\partial z} - \frac{\partial f|_e}{\partial s} \right) \Big|_{s=1,z=0}.$$

If it is know that P_1 and P_2 correspond to distinct complex vertices p_1 and p_2, we denote these derivatives by $\mathbf{D}_{e,p_1} f$ and $\mathbf{D}_{e,p_2} f$ respectively.

Note that if E is equivalent to an other polygonal edge $E' \subset \Omega'$, and P'_1 is the vertex $u_{E'} = v_{E'} = 0$ of Ω', then $\mathbf{D}_{e,P_1} f = \mathbf{D}_{e,P'_1} f$. In fact the defined derivatives are the values of the complex derivations at p_1 (or p_2) along the edge e, represented by $\partial/\partial v_E - \partial/\partial w_E$, etc.

Finally, for a polygonal vertex $P \in \Omega$ we will use the *triangle-vertex derivative*

$$\mathbf{D}^2_{\Omega,P} f := \rho_P \left(\left(\frac{\partial}{\partial u_P} - \frac{\partial}{\partial w_P} \right) \left(\frac{\partial f|_\Omega}{\partial v_P} - \frac{\partial f|_\Omega}{\partial w_P} \right) \right).$$

If there are no other vertices on Ω equivalent to P, we denote this derivative by $\mathbf{D}^2_{\Omega,p}$, where p is the complex vertex of P.

As an application of Bernstein-Bézier bases and some other notions introduced here we compute the dimension of the space $S^0_k(\mathcal{M})$ of continuous splines on a CG^0 "Bézier" surface complex.

Lemma 6.3.1 *Let $\mathcal{M}$ be a (connected) CG^0 surface complex such that its polygons are triangles, and the homeomorphisms μ are linear. Let $\mathcal{P}$, $\mathcal{E}$ and $\mathcal{V}$ denote the sets of triangles, complex edges and complex vertices of $\mathcal{M}$. Then*

$$\dim S^0_k(\mathcal{M}) = \begin{cases} \binom{k-1}{2} \#\mathcal{P} + (k-1)\#\mathcal{E} + \#\mathcal{V}, & \text{if } k > 0, \\ 1, & \text{if } k = 0. \end{cases} \tag{6.16}$$

Proof. Let f denote a spline in $S^0_k(\mathcal{P})$. If $k \geq 1$ then one can choose the values $f|_p$ at all the vertices $p \in \mathcal{V}$ independently. Such a choice specifies all the corner control coefficients of $f|_\Omega$ for all triangles $\Omega \in \mathcal{P}$, and boundary control coefficients of $f|_e$ for all edges $e \in \mathcal{E}$. On each complex edge one can choose the $k-1$ interior control coefficients of $f|_e$ freely, specifying together all the boundary control coefficients for all $f|_\Omega$'s. On each triangle there are $\binom{k-1}{2}$ interior control coefficients which can be chosen independently. The formula (6.16) follows. □

Example 6.23 In figure 6.9 we depict a continuous torus realized by two Bézier patches of degree 3. The torus is constructed by choosing nine control points in such a way that their control nets form a polyhedral complex topologically equivalent to a torus. As we see, Bézier patches can mimic even such a non-convex control net of small degree.

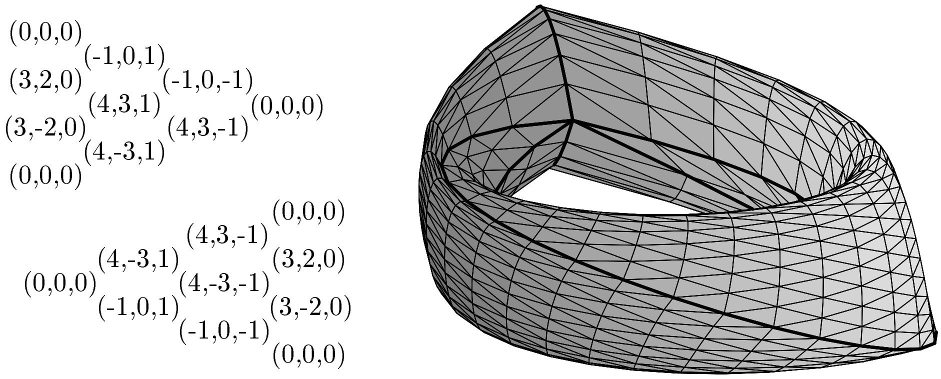

Figure 6.9: Continuous torus realized by 2 cubic patches.

6.4 Dimension of CG^1 spline spaces

In this section we compute a dimension formula for the spline spaces $S^1_k(\mathcal{M})$ (introduced in definition 6.11) on a CG^1 Bézier complex, which is the main result of this chapter. First an important special case is considered, namely a CG^1 Bézier surface complex made of two triangles and one interior edge. We call such a surface complex a *simple surface complex*, see figure 6.10.

6.4.1 CG^1 splines on two triangles

Consider a simple CG^1 surface complex $\mathcal{M}^0$ given by two triangles Ω_1 and Ω_2 with their two edges E_1 and E_2 being identified. Let $e_0 = \{E_1, E_2\}$ denotes the unique interior edge of $\mathcal{M}^0$. For $i = 1, 2$ let A_i be the end-point $u_{E_i} = v_{E_i} = 0$ of E_i, and let B_i be the other end-point of E_i. Notations for other polygonal vertices and complex edges are introduced in figure 6.10. If n is the degree of the rational CG^1 glueing data (μ, Θ) for e_0, then there are polynomials α, β and γ in $\mathbf{B}_1(n)$, which specify the glueing data as defined before equation (6.15).

The splines on the simple surface complex $\mathcal{M}^0$ are defined by two homogeneous polynomials $f|_{\Omega_1}$ and $f|_{\Omega_2}$ such that the restriction $f|_{e_0}$ is well defined and the equation (6.15) holds. Our first step in computing the dimension of $S^1_k(\mathcal{M}^0)$ for all k is to relate the splines on $\mathcal{M}^0$ to the polynomial solutions (F_1, F_2, F_3) of the equation $\alpha\, F_1 + \beta\, F_2 + \gamma\, F_3 = 0$, compare with equation (6.15). We use some commutative algebra, see [19]. In this subsection $\mathbf{B}_1\{-n\}^3$ denotes the graded free $\mathbf{B}_1$-module of rank three, whose all generators have degree n, so a homogeneous element of degree k is represented by a triple of polynomials in $\mathbf{B}_1(k-n)$.

Lemma 6.4.1 *Let (α, β, γ) be the triple of polynomials in $\mathbf{B}_1(n)$ which specify the CG^1 glueing data on e_0 as above. Let $\sigma : \mathbf{B}_1\{-n\}^3 \to \mathbf{B}_1$ be a homomorphism of free graded $\mathbf{B}_1$ modules, defined by $\sigma(F_1, F_2, F_3) = \alpha F_1 + \beta F_2 + \gamma F_3$. Then*

$$\dim S^1_k(\mathcal{M}^0) = k(k-1) + 1 + \dim \ker_{n+k-1} \sigma, \tag{6.17}$$

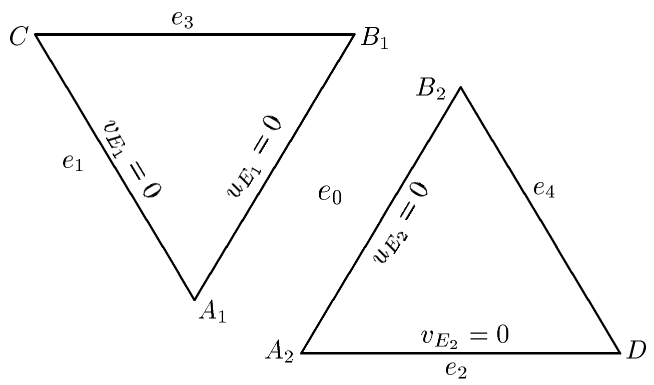

Figure 6.10: A simple CG^1 surface complex.

where $\ker_m \sigma$ *is the* mth *graded part of* $\ker\sigma$ *as a graded* $\mathbf{B}_1$ *module. (Note that an element of* $\ker\sigma_m$ *is represented by a triple of polynomials in* $\mathbf{B}_1(m-n)$*).*

Proof. Let $\phi : S^1_k(\mathcal{M}^0) \to \ker_{n+k-1}\sigma$ be an $\mathbb{R}$-linear map defined by

$$f \mapsto \left(\rho_{E_1}\left(\frac{\partial f|_{\Omega_1}}{u_{E_1}} - \frac{\partial f|_{\Omega_1}}{v_{E_1}}\right),\ \frac{\partial f|_e}{\partial s} - \frac{\partial f|_e}{\partial z},\ \rho_{E_2}\left(\frac{\partial f|_{\Omega_1}}{u_{E_2}} - \frac{\partial f|_{\Omega_2}}{v_{E_2}}\right)\right).$$

The image of ϕ is indeed in $\ker\sigma$ because of equation (6.15). We claim that ϕ is surjective. Take a triple $(F_1, F_2, F_3) \in \ker\sigma_{n+k-1}$, so that the degree of polynomials F_i is $k-1$. There is a polynomial $G \in \mathbf{B}_1(k)$ such that $\partial G/\partial s - \partial G/\partial z = F_2$. One can compute such G by considering F_2 as a polynomial function on $[0,1] \subset \mathbb{R}$ by putting $s=t$ and $z = 1-t$, taking its indefinite integral $\int F_2$, and writing the integrated function as the polynomial G in $\mathbf{B}_1(k)$. Then we define a spline f by

$$\begin{aligned} f|_{\Omega_1} &= G(v_{E_1}, w_{E_1}) + u_{E_1}\cdot\left(F_1(v_{E_1}, w_{E_1}) + \frac{\partial G}{\partial z}(v_{E_1}, w_{E_1})\right), \\ f|_{\Omega_2} &= G(v_{E_2}, w_{E_2}) + u_{E_2}\cdot\left(F_3(v_{E_2}, w_{E_2}) + \frac{\partial G}{\partial z}(v_{E_2}, w_{E_2})\right). \end{aligned}$$

Here $G(v_{E_1}, w_{E_1})$ denotes the polynomial in $\mathbf{B}_2(k)$, obtained from $G \in \mathbf{B}_1(k)$ by substituting $s = v_{E_1}$, $z = w_{E_1}$, and similarly for F_1, F_3 and $\partial G/\partial z$. This f is mapped to (F_1, F_2, F_3) by ϕ, hence ϕ is surjective.

The kernel of ϕ consists of the splines f such that all first order derivatives of $f|_{\Omega_1}$ and $f|_{\Omega_2}$ restricted to the interior edge e_0 are zero functions. The space of these splines is generated by a constant spline, the splines for which $f|_{\Omega_1} = 0$ and $f|_{\Omega_2} = B^k_{(i,j,m)}$ such that the component of (i,j,m) corresponding to u_{E_2} is greater than one, and the same kind of splines with Ω_1 and Ω_2 interchanged. Hence the dimension of $\ker\phi$ is $1 + 2\binom{k}{2} = k^2 - k + 1$, and formula (6.17) follows. □

Lemma 6.4.2 *Let $\alpha, \beta, \gamma \in \mathbf{B}_1$ be three homogeneous polynomials of the same degree n, without common divisors. Let $\sigma : \mathbf{B}_1\{-n\}^3 \to \mathbf{B}_1$ be a homomorphism between free graded $\mathbf{B}_1$ modules, defined like in the previous lemma. Then the kernel of σ is a free $\mathbf{B}_1$ module, generated by two elements of degree $n+d_1$ and $n+d_2$, respectively, for some non-negative integers d_1, d_2 satisfying $d_1 + d_2 = n$.*

Proof. Consider the exact sequence:

$$0 \longrightarrow \ker\sigma \longrightarrow \mathbf{B}_1\{-n\}^3 \stackrel{\sigma}{\longrightarrow} \mathbf{B}_1 \longrightarrow \mathbf{B}_1/(\alpha,\beta,\gamma) \longrightarrow 0.$$

The freeness of $\ker\sigma$ follows from the fact, that the homological dimension of $\mathbf{B}_1/(\alpha,\beta,\gamma)$ is at most 2 (according to the Hilbert basis theorem), so the image of σ has homological dimension ≤ 1 (see proposition 3.4 in [8]). The module $\mathbf{B}_1/(\alpha,\beta,\gamma)$ is zero-dimensional, because α, β, γ have no common divisors. Hence its Hilbert series is a polynomial. The Hilbert series of $\mathbf{B}_1\{-n\}^3$ is the series of $3\,t^n/(1-t)^2$. Hence the Hilbert series of $\ker\sigma$ is

$$H(\ker\sigma) = \frac{P(t)}{(1-t)^2} + (\text{polynomial}), \quad P(t) = 3\,t^n - 1.$$

It is known that $\ker\sigma$ should have $P(1) = 2$ free generators, and the sum of their degree should be $P'(1) = 3n$. Since the degree of the generators is at least n, the statement of the lemma follows. □

Theorem 6.4.3 *Let n be the degree of the rational CG^1 glueing data (μ, Θ) assigned to the interior edge e_0 of a simple CG^1 Bézier complex $\mathcal{M}^0$. Then for some non-negative $d_1 \leq n/2$*

$$\dim S^1_k(\mathcal{M}^0) = \begin{cases} k^2 - k + 1, & \text{if } k \leq d_1, \\ k^2 - d_1 + 1, & \text{if } d_1 \leq k \leq n - d_1, \\ k^2 + k - n + 1, & \text{if } k \geq n - d_1. \end{cases} \tag{6.18}$$

Proof. Let $\sigma : \mathbf{B}_1\{-n\}^3 \to \mathbf{B}_1$ be the linear map defined in lemma 6.4.1. From lemma 6.4.2 it follows that for some $d_1 \leq n/2$ and $d_2 = n - d_1$ we have

$$\dim\ker\sigma_m = \begin{cases} 0, & \text{if } m \leq n + d_1 - 1, \\ m - (n + d_1 - 1), & \text{if } n + d_1 - 1 \leq m \leq n + d_2 - 1, \\ 2m - 3n + 2, & \text{if } m \geq n + d_2 - 1 \end{cases} \tag{6.19}$$

It remains to use this expression in the formula (6.17). □

We note that a basis of $S^1_k(\mathcal{M}^0)$ is easy to compute from the two generators of the module $\ker\sigma$, introduced in lemma 6.4.1. Indeed, the module $\ker\sigma$ is free (see lemma 6.4.2), so a basis for its $(n+k-1)$th graded part is easy to compute from the two generators. Using the constructive arguments in the proof of lemma 6.4.1 one can obtain a basis of $S^1_k(\mathcal{M}^0)$ from a basis $\ker_{n+k-1}\sigma$. A more "locally supported" basis of $S^1_k(\mathcal{M}^0)$ (so that the basis elements have as much zero control coefficients as possible) can be generated using some recipes in the proof of lemma 6.4.5 of the next subsection.

Also note that the polynomials α, β, γ are uniquely (up to a constant) determined by two generators (G_1, G_2, G_3) and (H_1, H_2, H_3) of $\ker \sigma$, namely

$$(\alpha, \beta, \gamma) = (G_2 H_3 - G_3 H_2,\ G_3 H_1 - G_1 H_3,\ G_1 H_2 - G_2 H_1). \tag{6.20}$$

This fact can be useful in applications, because a couple of generators of $\ker \sigma$ describe the possible splines more explicitly than the "shape parameters" α, β, γ. The condition $\alpha\gamma > 0$ (introduced after formula 6.15) can be easily controlled in terms of the two generators of $\ker \sigma$. It might be practical to construct general CG^1 surface complexes when the CG^1 glueing data is given by pairs of generators of all the simple subcomplexes formed by two adjacent triangles. Compare this with algorithm 6.2.1, where consistent CG^1 glueing data at a vertex is constructed from a choice of more geometrical data.

In some situations (for example, checking the consistency of CG^1 glueing data at a complex vertex) it is useful to know how the triple (α, β, γ) and the generators of $\ker \sigma$ transform when we replace the bases $(\partial/\partial v_{E_i} - \partial/\partial w_{E_i}, \partial/\partial u_{E_i} - \partial/\partial w_{E_i})$ for the tangent spaces $T_{\Omega_i, P}$ on the edges E_i (for $i = 1, 2$) by the very similar bases $(\partial/\partial w_{E_i} - \partial/\partial v_{E_i}, \partial/\partial u_{E_i} - \partial/\partial v_{E_i})$, i.e. when we permute the end-vertices of E_i's and the barycentric coordinates s and z. From

$$\frac{\partial}{\partial u_{E_i}} - \frac{\partial}{\partial w_{E_i}} = \left(\frac{\partial}{\partial u_{E_i}} - \frac{\partial}{\partial v_{E_i}}\right) - \left(\frac{\partial}{\partial w_{E_i}} - \frac{\partial}{\partial v_{E_i}}\right), \quad \text{etc.,}$$

one easily derives that the triple $(\alpha, -\alpha - \beta - \gamma, \gamma)$ replaces (α, β, γ) in the similar representation of Θ in the alternative bases, compare with formulas (6.10) and (6.11) in example 6.19. Besides, an element (F_1, F_2, F_3) of $\ker \sigma$ gives an element $(F_1 - F_2, -F_2, F_3 - F_2)$ of the alternative $\ker \sigma$ defined by $\alpha\, G_1 + (-\alpha - \beta - \gamma)\, G_2 + \gamma\, G_3 = 0$. Without writing down this alternative representation of $S^1(\mathcal{M}^0)$ explicitly we use it (by "symmetry" argument) in the following subsection.

6.4.2 Separation of vertices

The idea of our proof of the dimension formula (theorem 6.4.6) on a general CG^1 Bézier complex $\mathcal{M}$ is to consider for each interior edge e a simple CG^1 complex $\mathcal{M}_e^0$ defined by the CG^1 glueing data assigned to e, and try to combine splines on the simple complexes $\mathcal{M}_e^0$ to "global" splines on the whole $\mathcal{M}$. An important question is the following. Given a simple surface complex with the interior edge e, and a space of splines on it, how independent are the values of these splines and their derivatives at the vertices of $\mathcal{M}_e^0$ incident to e. In particular, for which k the space $S_k^1(\mathcal{M}_e^0)$ separates the complex vertices and separates directions at them, recall definition 6.14.

Figure 6.11 gives a motivation in terms of control coefficients. Here a CG^1 surface $\mathcal{M}$ is formed by three triangles so that it has two interior edges e_1 and e_2, both incident to the complex vertex $a = \{A_1, A_2, A_3\}$. There are no further identifications of edges and vertices of triangles. For $i = 1, 2$ let $\mathcal{M}_{e_i}$ be the simple CG^1 surface complex formed by the interior edge e_i and two triangles incident to it. The control coefficients of a spline on $\mathcal{M}$ satisfy the relations coming from its both interior edges e_1 and e_2. The control coefficients which are restricted by each interior edge (due to formula (6.15)) are represented in figure 6.11 by circles of two different sizes. On the middle triangle Ω_2

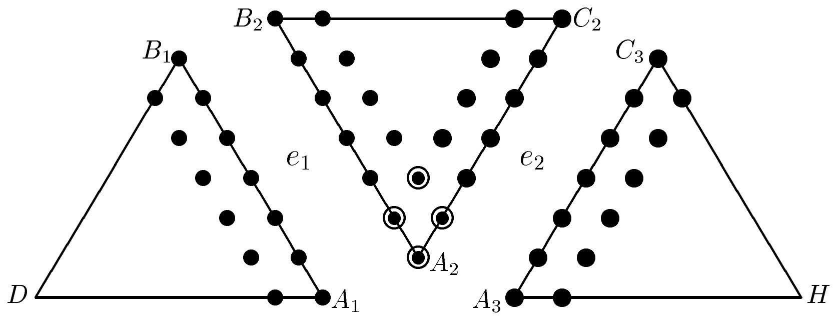

Figure 6.11: Control coefficients restricted by two interior edges.

there are four control coefficients restricted by both interior edges. Hence can combine an element in $S^1(\mathcal{M}_{e_1})$ and an element of $S^1(\mathcal{M}_{e_2})$ into a spline f on $\mathcal{M}$ only if they give the same four "common" control coefficients. In terms of definition 6.22 these four control coefficients give the values $f|_a$, $\mathbf{D}_{e_1,A_2}f$, $\mathbf{D}_{e_2,A_2}f$ and $\mathbf{D}^2_{\Omega_2,A_2}(f)$, and vice versa.

In the rest of this subsection we consider a simple surface $\mathcal{M}^0$, where the CG^1 glueing data on its interior edge given by such a glueing data on an interior edge e of a "big" CG^1 surface complex $\mathcal{M}$. Note that even if in $\mathcal{M}$ there are more identifications of edges or vertices of the triangles incident to e, in the simple complex $\mathcal{M}^0$ only two triangular edges and their end-vertices are identified.

We keep the notations of figure 6.10 and the previous subsection. The control coefficients, corresponding to those control coefficients on $\mathcal{M}$ which may be also restricted by other interior edges of $\mathcal{M}$, are marked on figure 6.12. Here $a = \{A_1, A_2\}$ and $b = \{B_1, B_2\}$ denote two complex vertices. The important question is how independent on $\mathcal{M}^0$ are the twelve marked control coefficients. In terms of the derivatives in definition 6.22, for a spline f on $\mathcal{M}^0$ there is obviously a linear relation between the derivatives $\mathbf{D}_{e_1,a}f$, $\mathbf{D}_{e_0,a}f$ and $\mathbf{D}_{e_2,a}f$, given by evaluating (6.15) at the vertex a. If the component β from the triple (α, β, γ) evaluates to zero at a, then additionally there is a linear relation between these derivatives and $\mathbf{D}^2_{\Omega_1,a}$, $\mathbf{D}^2_{\Omega_2,a}$, which is obtained by taking the derivation $\partial/\partial s - \partial/\partial z$ of (6.15) and evaluating it at a. There are similar relations between the edge-vertex and triangle-vertex derivatives at the complex vertex b.

The main result (lemma 6.4.5, supplemented by lemma 6.4.4) of this subsection is that for big enough k there are no further relations between the values $f|_a$, $f|_b$ and these edge-vertex and triangle-vertex derivatives for splines in $S^1_k(\mathcal{M}^0)$, or the corresponding control coefficients marked in figure 6.12. In other words, one can choose these values freely, up to a couple of linear relations at both complex vertices incident to e_0, and find a spline in $S^1_k(\mathcal{M}^0)$ with the prescribed values, if k is big enough.

The following definitions specify the situations when the mentioned control coefficients are that much (or almost that much) independent. It might be useful to recall definition 6.14.

Definition 6.24 Suppose that a subspace $\tilde{S}$ of $S^1(\mathcal{M}^0)$ separates vertices of $\mathcal{M}_e$ incident to e_0. We say that $\tilde{S}$ ***strongly separates*** vertices, if for $p \in \{a, b\}$ the space of splines

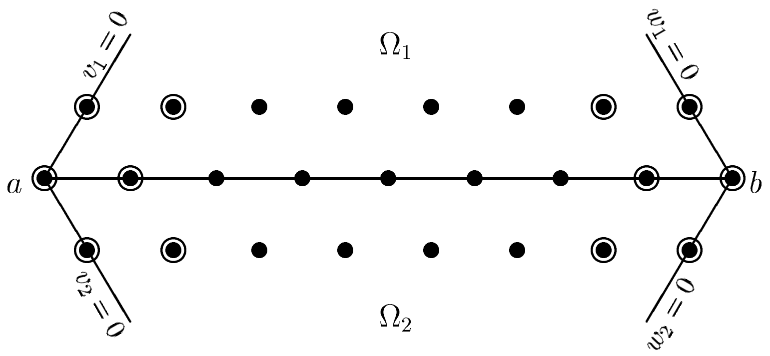

Figure 6.12: Control coefficients to separate

$f \in \tilde{S}$ which vanish at both end-vertices and such that all first order derivatives of $f|_{\Omega_1}$ vanish at p, separates directions at the other vertex. The same space *completely separates* vertices if, additionally, for $p \in \{a, b\}$ there are splines $f \in \tilde{S}$ such that:
(1) f and all first order derivatives of $f|_{\Omega_i}$'s vanish at both vertices.
(2) The triangle-vertex derivatives $\mathbf{D}^2_{\Omega_i,p} f$ are zero for $i = 1, 2$.
(3a) If β evaluates to zero at the other vertex $\tilde{p}$ incident to e_0, then the derivatives $\mathbf{D}^2_{\Omega_i,\tilde{p}} f$ are not zero.
(3b) If β does not evaluate to zero at the other vertex $\tilde{p}$, then the derivatives $\mathbf{D}^2_{\Omega_i,\tilde{p}} f$ attain any prescribed values.
With the first definition, the space $\tilde{S}$ strongly separates vertices, if (and only if) the values $f|_a$, $f|_b$, the derivatives $\mathbf{D}_{e_1,a} f$, $\mathbf{D}_{e_0,a} f$ $\mathbf{D}_{e_2,a} f$, and similar three derivatives at b of splines $f \in \tilde{S}$ are as independent as possible. The space $\tilde{S}$ completely separates vertices, if the twelve control coefficients marked in figure 6.12 are independent up to described necessary restrictions.

Lemma 6.4.4 *Let n be the degree of the rational CG^1 glueing data (μ, Θ) assigned to the interior edge of $\mathcal{M}^e$, and let d_1 be such that formula (6.18) in theorem 6.4.3 holds. Let $d_2 = n - d_1$. Then:*
(a) *The space $S^1_k(\mathcal{M}^0)$ separates directions at both vertices if and only if $k \geq d_2 + 1$.*
(b) *If $k \geq \max(2d_1 + 1, d_2 + 1)$ then $S^1_k(\mathcal{M}^0)$ separates vertices.*

Proof. Recall the map $\sigma : \mathbf{B}_1\{-n\}^3 \to \mathbf{B}_1$ of lemma 6.4.1, and the fact that the kernel $\ker \sigma$ is a free $\mathbf{B}_1$ module of rank 2 with generators of degree $n + d_1$ and $n + d_2$, see proof of lemma6.4.2. Let $G = (G_1, G_2, G_3)$ and $H = (H_1, H_2, H_3)$ be the generators of $\ker \sigma$ of degree $n + d_1$ and $n + d_2$ respectively. Also recall the map $\phi : S^1_k(\mathcal{M}^0) \to \ker \sigma_{n+k-1}$ in the proof of lemma 6.4.1.

To show the first statement consider the linear map $\varphi : S^1_k(\mathcal{M}^0) \to \mathbb{R}^2$ which sends $f \mapsto (\mathbf{D}_{e_1,a} f, \mathbf{D}_{e_0,a} f)$. By definition 6.14, the space $S^1_k(\mathcal{M}^0)$ separates directions at a precisely if this map is surjective. If $k \leq d_1$ then $S^1_k(\mathcal{M}^0)$ is a trivial linear space, see formula (6.18). If $d_1 < k \leq d_2$ then the image of φ is generated by one vector, namely (G_1, G_2) evaluated at a, because φ is also the map which evaluates the first two components of $\phi(f)$ at a. In these two cases $S^1_k(\mathcal{M}^0)$ does not separate directions at

a. If $k \geq d_2 + 1$ then also (H_1, H_2) evaluated at a is in Im φ. This new vector in $\mathbb{R}^2$ is independent of (G_1, G_2) evaluated at a, because according to (6.20) $G_1 H_2 - G_2 H_1$ is proportional to γ in the triple (α, β, γ), which is non-zero at a (due to the condition $\alpha\,\gamma > 0$, introduced together with formula 6.15). Hence $S^1_k(\mathcal{M}^0) \to \mathbb{R}^2$ separates directions at a precisely if $k \geq d_2 + 1$. By symmetry we get the same statement for b.

For the second statement we have to show that for $k \geq \max(2d_1 + 1, d_2 + 1)$ there are splines in $S^1_k(\mathcal{M}^0)$ which attain different values at a and b. If $F = (F_1, F_2, F_3) \in \ker \sigma_{n+k-1}$, and f is a spline in $S^1_k(\mathcal{M}^0)$ such that $\phi(f) = F$, then one can check (by considering the construction of "G" in the proof of lemma 6.4.1) that the values $f|_a$ and $f|_b$ differ by $\int_0^1 F_2(t, 1-t)\,dt$, where[11] $F_2(t, 1-t)$ denotes a polynomial function in t obtained by substituting $s = t$ and $z = 1 - t$. Now if $G_2 = 0$, then $H_2(t, 1-t)$ is a continuous function which is never zero for $t \in [0, 1]$ (because otherwise $\gamma(t, 1-t) = 0$ somewhere, contradicting the condition $\alpha\,\gamma > 0$), thus $\int_0^1 H_2(t, 1-t)\,dt \neq 0$. Otherwise $\int_0^1 G_2^2(t, 1-t)\,dt > 0$. Hence either H or $G_2 \cdot G$ give us a spline (through ϕ) of degree $d_2 + 1$ or $2d_1 + 1$, which[12] attains different values at the vertices a and b. □

Lemma 6.4.5 *As in the previous lemma, let n, d_1 be such that formula (6.18) in theorem 6.4.3 holds, and let $d_2 = n - d_1$ (like in the previous lemma). Suppose that $\tilde{S} = S^1_k(\mathcal{M}_e)$ separates vertices. Then:*
(a) *If $k \geq d_2 + 3$ then $\tilde{S}$ strongly separates vertices.*
(b) *If $k \geq d_2 + 5$ then $\tilde{S}$ completely separates vertices.*

Proof. Like in the previous proof, let G and H be the generators of the kernel of the map σ of lemma 6.4.1 of degree $n + d_1$ and $n + d_2$ respectively. However, in this proof (G_1, G_2, G_3) and (H_1, H_2, H_3) denote the $\mathbb{B}_1$-triples G and H evaluated at a.

For statement (a), let g_1 and g_2 be splines in $\tilde{S}$ such that they correspond to elements $z\,G$ and $z^2\,G$ of $\ker \sigma$ (due to the linear map $\phi : S^1_k(\mathcal{M}^0) \to \ker \sigma_{n+k-1}$ of lemma 6.4.1), and they vanish at b. In fact, all first order derivatives of these splines vanish at b. There is a linear combination g_3 of these two splines such that g_3 vanishes additionally at a. Similarly, starting from elements $t\,H$ and $t^2\,H$ of $\ker \sigma$ one produces a spline $h_3 \in \tilde{S}$ such that it vanishes at both vertices a and b, and all first order derivatives of it vanish at b. Like in the first part of the proof of lemma 6.4.4, values of two certain first order derivatives of g_3 and h_3 at a are (G_1, G_2) and (H_1, H_2) respectively, thus the spline space generated by g_3 and h_3 separates directions at a for $k = d_2 + 3$ or greater. Similarly, there is a subspace of $\tilde{S}$ of splines, vanishing at both a and b with their first order derivatives vanishing at a, which separates directions at b. The first statement follows.

Likewise, let $\tilde{g}_1$ and $\tilde{g}_2$ be splines in $\tilde{S}$ such that they correspond to elements $t\,z^2\,G$ and $t\,z^3\,G$ of $\ker \sigma$, and they vanish at b. Their all first order derivatives vanish at

[11] By a computation one can prove that this integral is equal to the arithmetic mean of the control coefficients of F_2.

[12] Note that the degree bound $2d_1 + 1$ can be achieved only when the component G_2 of G is (as a function on $[0, 1]$) orthogonal (with respect to $\mathcal{L}^2(0, 1)$ inner product $< f, g >= \int_0^1 f\,g\,dt$) to all polynomials in $\mathbb{R}[t]$ of degree $\leq d_1 - 1$. For generic choice of CG^1 glueing data μ, Θ) one has $\int_0^1 G_2(t, 1-t)\,dt \neq 0$ so already $S^1_{d_1+1}(\mathcal{M}^0)$ separates the vertices a and b generically.

both a and b, and their derivatives $\mathbf{D}^2_{\Omega_i,b}(f)$ (for $i = 1,2$) are zero. There is a linear combination $\tilde{g}_3$ of them such that $\tilde{g}_3$ vanishes also at a. In the same way, starting from $t\,z^2\,H$ and $t\,z^3\,H$, one obtains a spline $\tilde{h}_3 \in \tilde{S}$ which vanishes together with all first order derivatives at both a and b, and its derivatives $\mathbf{D}^2_{\Omega_i,b}\tilde{h}_3$ (for $i = 1,2$) is zero. Besides, the values $\mathbf{D}^2_{\Omega_i,a}\tilde{g}_3$ are $G_1/2$ and $G_3/2$ for $i = 1,2$ respectively, and the values $\mathbf{D}^2_{\Omega_i,a}\tilde{h}_3$,are $H_1/2$ and $H_3/2$. The vectors (G_1, G_3) and $(H_1, H_3) \in \mathbb{R}^2$ can not be both zero, and if they are linearly dependent, then $\beta|_a = 0$. Hence, for $k \geq d_2 + 5$ the conditions (1–2), (3a–3b) in the definition 6.24 of strong separation are satisfied for $p = b$, and similarly, for $p = a$. □

6.4.3 Splines on general CG^1 Bézier surfaces

Let $\mathcal{M}$ be a CG^1 Bézier surface complex. For an interior edge e let $\mathcal{M}^0_e$ denote the simple surface complex, such that the CG^1 glueing data (μ, Θ) on the interior edge of $\mathcal{M}^0_e$ is the same as on e. (Once more, in the simple complex $\mathcal{M}^0_e$ only two triangular edges and their end-vertices are identified, even if the are more identifications of point on the triangles incident to e.) Here we prove the main result of this section, that is, derive the dimension of the spaces $S^1_k(\mathcal{M})$ in the case when the corresponding spline spaces on each simple CG^1 surface $\mathcal{M}^0_e$ completely separate its vertices. From lemmas 6.4.5 and 6.4.4 it follows that this happens if k is big enough, certainly if for each edge $k \geq \max(d_2 + 5, 2d_1 + 1)$, or $k \geq n + 5$, where n, d_1 are defined in theorem 6.4.3, and $d_2 = n - d_1$. For arbitrary k we obtain a lower bound for the dimension of these spaces. In example 6.26 the obtained formula and inequality are specialized to the case of a triangulation of a region in $\mathbb{R}^2$ (without holes) by setting $n = 0$. In this special case the obtained results coincide with the known results of [43] and [50].

First we have to define a special local situation at a complex vertex which influences the dimension of these spline spaces.

Definition 6.25 An interior vertex a of $\mathcal{M}$ is called *particular* if (see the left part of figure 6.13):
(a) There are exactly four polygonal vertices $A_i \subset \Omega_i$ $(i = 1, \ldots, 4)$ identified in a.
(b) For each interior edge e_i incident to a, the "shape parameter" β in the triple (α, β, γ) vanishes at a.

Note that the four polygonal vertices in the first condition must belong to four distinct triangles. Indeed, if two of them would be vertices of the same triangle, then an interior edge of $\mathcal{M}$ would connect a with itself. Then by considering the topological space formed by the two triangles glued with such an interior edge, one concludes that the number of polygonal vertices incident to a must be at least six. It also follows that there are exactly four distinct interior edges incident to a. The second condition means that a tangent subdivision for the CG^1 surround glueing data at such a vertex is given by two lines intersecting at O, see the right-hand part of figure 6.13.

In the following theorem $\mathcal{P}$, $\mathcal{E}$ and $\mathcal{V}$ denote, respectively, the sets of polygons, complex edges and complex vertices of the CG^1 surface patch complex $\mathcal{M}$. Also $\mathcal{V}_{\text{prt}}$

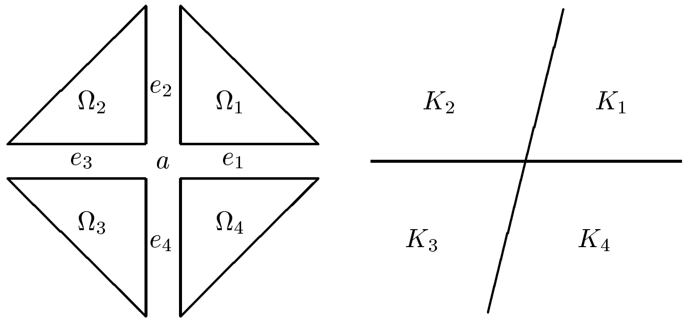

Figure 6.13: A particular vertex.

denotes the set of particular vertices in $\mathcal{M}$. For an interior edge e let n_e be the degree of the rational CG^1 glueing data assigned to e. For a boundary edge e we set $n_e = -2$.

Theorem 6.4.6 *Suppose that an integer k is such that for each interior edge e the spline space $S^1_k(\mathcal{M}^0_e)$ completely separates vertices incident to the interior edge of $\mathcal{M}^0_e$. Then*

$$\dim S^1_k(\mathcal{M}) = \left(\binom{k-4}{2} + 3\right) \#\mathcal{P} + \sum_{e\in\mathcal{E}} (2k - n_e - 9) + 3\#\mathcal{V} + \#\mathcal{V}_{\mathrm{prt}}. \tag{6.21}$$

Proof. Let $p \in \mathcal{V}$ be a complex vertex of $\mathcal{M}$, and $f \in S^1_k(\mathcal{M})$ is a spline. Here we summarize the facts about the relations between the edge-vertex and triangle-vertex derivatives at p.

(a) Consider the linear map $\psi_p : S^1_k(\mathcal{M}) \to \mathbb{R}^\star$, which given a spline f gives all the values $\mathbf{D}_{e,P} f$, for the polygonal vertex P identified in p and corresponding interior edges e. Lemma 6.2.2 and the condition that $S^1_k(\mathcal{M})$ separates directions at the vertices imply that the image of ψ_p is two-dimensional.

(b) If for an interior edge e incident to p the middle component β of the corresponding triple (α, β, γ) vanishes at p, then there is a linear condition between the corresponding values $\mathbf{D}^2_{\Omega,P} f$ (and $\mathbf{D}_{e,P} f$).

Let m_p be the number of those interior edges whose β vanishes at p. If p is not a particular vertex then $m_v \le 3$ and there is no relation between the m_v linear restrictions between $\mathbf{D}^2_{\Omega,P} f$'s. If p is a particular vertex, then $m_p = 4$, and one can check[13] that there is exactly one linear relation between the four linear restrictions on $\mathbf{D}^2_{\Omega,P} f$'s.

We claim that if one prescribes to each complex vertex $p \in \mathcal{V}$ a real value $f|_p$, an element in the image of the corresponding ψ_p, and the values $\mathbf{D}^2_{\Omega,P} f$ (with $P \in p$) satisfying the specified m_p linear restrictions, then there is a spline $f \in S^1_k(\mathcal{M})$ such that:

(a) It attains the prescribed values $f|_p$ at all the complex vertices p.

[13] Here a useful step is to consider only the splines whose first order derivatives vanish at p; then the restrictions on $\mathbf{D}^2_{\Omega,P} f$'s are very similar to CG^1 surround restrictions on $\mathbf{D}^2_{e,P} f$'s.

(b) All the derivatives $\mathbf{D}_{e,P}f$ match the prescribed elements in the images of the linear maps ψ_p.
(c) We get the prescribed values $\mathbf{D}^2_{P,\Omega}f$.
We construct such a spline as follows. At first, for each interior edge e we consider the "local" space $S^1_k(\mathcal{M}^0_e)$ of splines on the simple complex $\mathcal{M}^0_e$. For the two complex vertices a, b incident to the interior edge of $\mathcal{M}^0_e$ we have the induced prescriptions for the values $f_e|_p$, $\mathbf{D}_{\bar{e},p}f_e$ and $\mathbf{D}^2_{\bar{\Omega},p}f_e$, for the polygons $\bar{\Omega}$ and edges $\bar{e}$ of $\mathcal{M}^0_e$, and $p \in \{a, b\}$ (recall figure 6.12). Because $S^1_k(\mathcal{M}^0_e)$ completely separates the vertices a and b, there is a spline $\bar{f}$ in $S^1_k(\mathcal{M}^0_e)$ which satisfies the induced prescriptions. After choosing such a "local" spline on each interior edge, we construct a global spline f by taking the control coefficients restricted by an interior edge e from the local spline f_e, and taking arbitrary remaining control coefficients. The control coefficients of f restricted by several interior edges are well defined (because they are actually prescribed), and f satisfies the CG^1 continuity conditions on every interior edge. The claim follows.

Let us define a linear map $\Phi : S^1_k(\mathcal{M}) \to \mathbb{R}^{\#\mathcal{V}+2\#\mathcal{E}+3\#\mathcal{P}}$ by evaluating a spline f at all vertices of $\mathcal{M}$ and giving all the edge-vertex and triangle-vertex derivatives of f. The proved claim implies that

$$\dim \operatorname{Im}\, \Phi = \#\mathcal{V} + 2\#\mathcal{V} + 3\#\mathcal{P} - \sum_{p\in\mathcal{V}} m_p + \#\mathcal{V}_{\text{prt}}, \tag{6.22}$$

because there are in total $3\#\mathcal{P}$ triangle-vertex derivatives, and $\sum_{p\in\mathcal{V}} m_p - \#\mathcal{V}_{\text{prt}}$ independent linear restrictions on them.

It remains to compute the dimension of $\ker\Phi$. For a complex edge e let $\tilde{S}_e$ be the subspace of $S^1_k(\mathcal{M})$ of splines which vanish together with their first order derivatives at all edges of the complex, except e. (In particular, splines in $\tilde{S}_e$ and their first order derivatives vanish at all complex vertices.) Also, let $\tilde{S}_0$ be the subspace of $S^1_k(\mathcal{M})$ of splines which vanish together with their first order derivatives at each complex edge. It is clear that the kernel of Φ is generated by all $\tilde{S}_e$'s, and that the intersection of any two of these spaces is $\tilde{S}_0$. Hence

$$\dim \ker \Phi = \sum_{e\in\mathcal{E}} \dim \left(\tilde{S}_e/\tilde{S}_0\right) + \dim \tilde{S}_0. \tag{6.23}$$

The dimension of $\tilde{S}_0$ is $\binom{k-1}{2}\#\mathcal{P}$. If e is interior edge, let $m_e = 0, 1$ or 2, if e is particular at none, one or two complex vertices respectively. Then dim $\left(\tilde{S}_e/\tilde{S}_0\right)$ is equal to

$$\dim S^1_k(\mathcal{M}^0_e) - 2\binom{k-1}{2} - (10 - m_e) = (2k + 1 - n_e) - (10 - m_e), \tag{6.24}$$

because the linear map $S^1_k(\mathcal{M}^0_e) \to \mathbb{R}^{12}$ which gives the control coefficients marked in 6.12 (or the corresponding values and edge-vertex, triangle-vertex derivatives) has rank $10 - m_e$. If e is a boundary edge, then $\tilde{S}_e$ is generated by $\tilde{S}_0$ and $(2k+1) - 8$ independent splines, namely those whose restriction to the triangle incident to e has only one non-zero control point. Then $\dim(\tilde{S}_e/\tilde{S}_0)$ is equal to the right-hand side of (6.24) for the boundary edge e as well, if we set $n_e = -2$ and $m_e = 0$.

Besides, we have[14] $\sum_{v\in\mathcal{V}} m_v = \sum_{e\in\mathcal{E}} m_e$. Now the dimension formula (6.21) follows from $\dim S^1_k(\mathcal{M}) = \dim \mathrm{Im}\,\Phi + \dim\ker\Phi$. and the equations (6.22–6.24). □

The proof of theorem 6.4.6 is essentially constructive, i.e. it allows to construct a basis of $S^1_k(\mathcal{M})$. Moreover, the suggested basis is quite convenient for applications: for each standard basis element of Im Φ there corresponds a spline which vanish at all triangles and edges except those which are incident to one vertex. For a basis element of $\tilde{S}_e/\tilde{S}_0$ we have a spline which does not vanish only on edge e and adjacent triangles. Here some recipes can be found in the proof of lemmas 6.4.5 and 6.4.1 to find more "locally supported" splines. The space $\tilde{S}_0$ is generated by "very local" splines which vanish at all triangles except one.

For applications the dimension of spaces $S^1_k(\mathcal{M})$ with small k (but still separating the complex vertices and directions at them) are very interesting. Here we can prove that the right-hand side of the expression (6.21) is always a lower bound for the dimension of $S^1_k(\mathcal{M})$.

Theorem 6.4.7 *For any integer k we have*

$$\dim S^1_k(\mathcal{M}) \geq \left(\binom{k-4}{2} + 3\right)\#\mathcal{P} + \sum_{e\in\mathcal{E}}(2k - n_e - 9) + 3\#\mathcal{V} + \#\mathcal{V}_{\mathrm{prt}}. \tag{6.25}$$

Proof. We keep the same notations as in the previous proof. For an interior edge e let $\tilde{m}_e$ be the number of new independent relations between the edge-vertex and triangle-vertex derivatives relevant in $S^1_k(\mathcal{M}^0_e)$. By theorem 6.4.3 we have $\dim S^1_k(\mathcal{M}^0_e) \geq 2k + 1 - n_e$, thus

$$\dim\left(\tilde{S}_e/\tilde{S}_0\right) \geq (2k+1-n_e) - (10 - m_e) + \tilde{m}_e.$$

On the other hand, the total $\sum \tilde{m}_e$ of new relations defining ImΦ might be dependent from the old relations or between themselves. Hence (putting $\tilde{m}_e = 0$ for boundary edges)

$$\dim \mathrm{Im}\ \Phi \geq 3\#V + 3\#P - \sum_{p\in\mathcal{V}} m_v + \#\mathcal{V}_{\mathrm{prt}} - \sum_{e\in\mathcal{E}}\tilde{m}_e.$$

Summing up in the same way as in the previous theorem we get the statement. □

Example 6.26 Consider a triangulation Δ of a polygonal region without holes in $\mathbb{R}^2$ and the space of bivariate splines on it. The triangulation gives us a Bézier complex where we have the same triangles, interior (and boundary) edges and vertices, and rational CG^1 glueing data of degree 0 for each interior edge. Let $\mathcal{E}_0$ and $\mathcal{E}_\partial$ denote the sets of interior and boundary edges, and let $\mathcal{V}_0$, $\mathcal{V}_{\mathrm{prt}}$ and $\mathcal{V}_\partial$ denote the sets of interior, particular and boundary vertices respectively. From lemmas 6.4.4 and 6.4.5 we conclude that for the simple complexes $\mathcal{M}^0_e$ the spline spaces $S^1_k(\Delta_e)$ completely separate vertices if $k \geq 5$.

[14] Indeed, an interior edge e connects p with itself, then its β from the triple (α, β, γ) cannot vanish for $s = 0$ and for $z = 0$ at the same time, because otherwise two pairs of "tangent cones" K_i around p would already form two angles π, and this contradicts the CG^1 surround glueing structure either at p or at a neighbouring complex vertex incident to p.

Using the Euler relation $\#\mathcal{P} - \#\mathcal{E} + \#\mathcal{V} = 1$ and the relations $3\#\mathcal{P} = 2\#\mathcal{E}_0 + \#\mathcal{E}_\partial$, $\#\mathcal{E}_\partial = \#\mathcal{V}_\partial$ and $\#\mathcal{E} = \#\mathcal{E}_0 + \#\mathcal{E}_\partial$ etc., we can specialize the formula (6.21) to

$$\dim S_k^1(\Delta) = \binom{k-1}{2} \#\mathcal{P} + (k-1)\#\mathcal{E}_\partial + 3 + \#\mathcal{V}_{\mathrm{prt}}, \tag{6.26}$$

or, by manipulating further, to

$$\dim S_k^1(\Delta) = \binom{k+2}{2} \#\mathcal{P} - (2k+1)\,\#\mathcal{E}_0 + 3\,\mathcal{V}_0 + \#\mathcal{V}_{\mathrm{prt}}, \tag{6.27}$$

which both hold for $k \geq 5$. This precisely coincides with the result result of Morgan and Scott in [43], where they proved the last formula also for $k \geq 5$. Moreover, their method is essentially the same, because they similarly construct splines by prescribing their values, two derivatives at each vertex, their values at $k-5$ distinct points on each edge, etc.

Also the inequality 6.25 in the special case of planar triangulations is well known, it is the result of L.L. Schumaker in [50]. As it was shown in [1], the formula (6.27) still holds for $k = 4$, but the construction of "locally supported" splines here is more problematic, see also [42]. (In this context it is interesting to consider example 6.31 in this chapter below.) In [5] J.L.Billera has proved the conjecture of G.Strang [56] that this formula holds for all k and "generically" embedded triangulations in $\mathbb{R}^2$.

Summarising, we have seen a couple of times in this chapter that CG^1 glueing data (μ, Θ) can be given in terms of generators of certain spline spaces (see the remark after theorem 6.4.3), or in terms of the tangent subdivisions (algorithm 6.2.1). We can modify the general strategy of J.M. Hahn for constructing a CG^1 surface as follows. After the choice of the combinatorial data (triangles, identification of their edges) one can choose (any) consistent CG^1 glueing data at vertices of the surface using algorithm 6.2.1. As J.M. Hahn et al. noted, one can always use rational CG^1 glueing data of degree one on each interior edge to "connect" the chosen glueing data at the vertices. It might be easier to combine CG^1 glueing data on neighbouring edges if they are represented as pairs of generators of the corresponding $\ker\sigma$ of lemma 6.4.1. In this way one may try to choose several splines (or their "local" pieces) first, and then adjust the glueing data for them in order to get more global splines than may be expected in general.

6.4.4 More examples

Example 6.27 (A "tetrahedral" sphere again) Consider the CG^1 surface complex $\mathcal{S}_4$ of example 6.19 again. Recall that it is formed by four triangles in the same way as the faces of a tetrahedron do, see figure 6.8. In this example we assume the complex derivations $\mathbf{D}_{e_1,a}$, $\mathbf{D}_{e_2,a}$, etc., of example 6.19, applied to a spline f, give us the values $\mathbf{D}_{e_1,a}\,f$, $\mathbf{D}_{e_2,a}\,f$, etc., used in the last two sections. We keep the same notation for the triangles, (polygonal and complex) edges and complex vertices. We also denote the parameters $\zeta_{ab}, \zeta_{ac}, \zeta_{ad}, \zeta_{ba}, \ldots$ and $\nu_{ab}, \nu_{ba}, \ldots$ of the CG^1 glueing data (with respect to the specified derivations) in the same way as in the example of section 6.2.5.

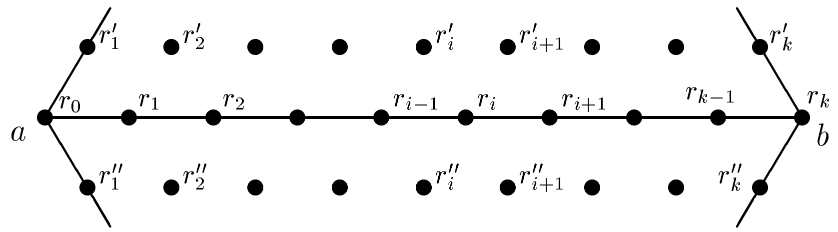

Figure 6.14: Deriving the relations between the control coefficients

In the notation for theorem 6.4.6 we have $n_e = 1$ for all complex edges. From lemmas 6.4.4 and 6.4.5 we conclude that the "local" simple complexes $\mathcal{M}_e^0$ strongly separate vertices for $k \geq 6$. For these k the theorem 6.4.6 tells us that

$$\dim S_k^1(\mathcal{S}_4) = \binom{k-4}{2} 4 + 12 + 6(2k-10) + 12 = 2(k-1)(k-2).$$

We show that this formula holds for $k \geq 3$, and that there are only constant splines on $\mathcal{S}_4$ of degree ≤ 2, independently of the chosen parameters.

To each interior edge e the are two possibilities for the triple (α, β, γ), because there are two possible choices for the homomorphisms ρ_{E_i} for the two polygonal edges E_i identified with e. If we assume that for $E = E_{ab,c}$ the polygonal edge $E_{ac,b}$ is given by $v_E = 0$ on the triangle Ω_{abc}, we have the following triple for the edge e_1:

$$\begin{aligned} \alpha &= \nu_{ab}\,\xi_{ac}\,s + \nu_{ba}\,\xi_{bc}\,z, \\ \beta &= \nu_{ab}\,\xi_{ab}\,s + \nu_{ba}\,(-\xi_{ba} - \xi_{bc} - \xi_{bd})\,z, \\ \gamma &= \nu_{ab}\,\xi_{ad}\,s + \nu_{ba}\,\xi_{bd}\,z. \end{aligned}$$

Note that this is the most general triple (α, β, γ) of polynomials in $\mathbf{B}_1(1)$, with the prescribes values (up to a common constant multiple) at $s = 0$ and $z = 0$.

Consider a spline f on $\mathcal{S}_4$ of degree $\leq k$, and let $r_0, r_1, \ldots, r_k$ be its control points on the edge e_1, so that $r_0 = f|_a$. Let $r'_1, \ldots, r'_k$ and $r''_1, \ldots, ''_k$ be the control points of $f|_{\Omega_{abc}}$ and $f|_{\Omega_{abd}}$ respectively, restricted by the edge e_1 and numbered as in figure 6.14. From (6.15) one can derive the following relations between these control coefficients:

$$\begin{aligned} r_0 &= \zeta_{ab}\,r_1 + \zeta_{ac}\,r'_1 + \zeta_{ad}\,r''_1, \\ r_k &= \zeta_{ba}\,r_{n-1} + \zeta_{bc}\,r'_n + \zeta_{bd}\,r''_n, \end{aligned} \tag{6.28}$$

and for $i = 1, \ldots, n-1$:

$$\begin{aligned} r_i = \; & \frac{(k-i)\,\nu_{ab}}{(k-i)\,\nu_{ab} + i\,\nu_{ba}} \left(\zeta_{ab}\,r_{i+1} + \zeta_{ac}\,r'_{i+1} + \zeta_{ad}\,r''_{i+1}\right) + \\ & \frac{i\,\nu_{ba}}{(k-i)\,\nu_{ab} + i\,\nu_{ba}} \left(\zeta_{ba}\,r_{i-1} + \zeta_{bc}\,r'_i + \zeta_{bd}\,r''_i\right). \end{aligned} \tag{6.29}$$

These equations express the control coefficient on the left hand side of the equality as a weighted average of several control coefficients "around" that control coefficient.

All together, for each edge control coefficient we get precisely one equation like in (6.28) or (6.29) where the control coefficient is on the left hand side of the equality. If $k \leq 2$ then all the control coefficients "lie" on the complex edges, and since each of them is a weighted average of some other ones, they all must be equal, i.e. they represent a constant spline.

If $k \geq 3$ then we claim that for any prescribed interior control coefficients for the four triangles there is a spline in $S_k^1(\mathcal{S}_4)$ with the prescribed control coefficients. Let $N = 6\,d - 2$, and let $V \in \mathbb{R}^N$ be the vector of all control coefficients on the surface edges of a spline F. Then the system of all equations of form (6.28-6.29) coming from all the complex edges can be written in the matrix form as $V = M\,V + W$, where $W \in \mathbb{R}^N$ is determined by the interior control coefficients, and M is a square matrix with non-negative entries. We have to show that the matrix $I - M$ is invertible (here I is the identity matrix). From the "averaging" properties of equations (6.28-6.29) it follows that the sum of entries on any row of M is ≤ 1, and this sum is equal to 1 precisely if the row corresponds to a vertex control coefficient. Let $|\cdot|_\infty$ be the norm on $\mathbb{R}^N$ defined by $|(b_1, \ldots, b_N)|_\infty = \max(|b_1|, \ldots, |b_N|)$. Let $\| M \| = \sup_{\mathbf{v} \in \mathbb{R}^N} |M\mathbf{v}|_\infty / |\mathbf{v}|_\infty$ be the corresponding norm of A as a linear operator. If $\| M \| < 1$ then by basic functional analysis (see [23]) the series $I + M + M^2 + \ldots$ converge and give the inverse matrix of $I - M$. We show that $\| M^2 \| < 1$, then this series give $(I - M)^{-1}$ as well. Let

$$C = \max_{\substack{p,q \in \{a,b,c,d\} \\ p \neq q}} \frac{(k-1)\,\nu_{pq}\,\zeta_{pq} + \nu_{qp}}{(k-1)\,\nu_{pq} + \nu_{qp}}.$$

If $\mathbf{v} \in \mathbb{R}^N$ then all entries of $M\mathbf{v}$, except possibly those which give the vertex control coefficients, are at most $C\,|\mathbf{v}|_\infty$. Hence all entries of $M^2\mathbf{v}$ are at most $C\,|\mathbf{v}|_\infty$, and $\| M^2 \| \leq C < 1$. It follows that $I - M$ is invertible, which implies the claim. Then the formula $\dim S_k^1(\mathcal{S}_4) = 4\binom{k-2}{2} = 2(k-1)(k-2)$ follows for $k \geq 3$.

In particular, if $k = 3$ then the dimension is 4. One can expect that for each complex vertex p there is a spline which has non-zero value at p and vanishes at all the other vertices. At least this is the case for the most symmetric choice of the parameters $\zeta_{pq} = 1/3$ and $\nu_{pq} = \nu_{qp}$ for all distinct p, q in $\{a, b, c, d\}$, so that all the triples (α, β, γ) are $(s + z, -3s + z, s + z)$. Such a spline f_a for the vertex a is represented on the left-hand side in figure 6.15. There the control coefficients of f_a on each triangle are given following schematically the triangular form the left-hand side of figure 6.8. The similar splines f_b, f_c and f_d are defined by the same control coefficients, but with the triangles accordingly permuted. On the right-hand side of 6.15 we see the image of the realization $\mathcal{S}_4 \to \mathbb{R}^3$ given by the vector $(f_a + f_b,\, f_a + f_c,\, f_b + f_c)$, so that the complex vertices are mapped to the points $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(0,0,0)$ which define a equal-sided tetrahedron in $\mathbb{R}^3$.

Example 6.28 (A sphere in general) If a CG^1 surface complex $\mathcal{M}$ gives us a topological surface homeomorphic to a sphere, then we have the Euler relation $\#\mathcal{P} - \#\mathcal{E} + \#\mathcal{V} = 2$, where $\mathcal{P}$, $\mathcal{E}$ and $\mathcal{V}$ are the sets of the polygons, the complex edges and the complex vertices respectively. Besides, we have $2\,\#\mathcal{E} = 3\,\#\mathcal{P}$, because every triangle

0 4 6 6   6   6 6 4 0
-2 7 6   6 6   6 7 -2
-2 4   4 7 4   4 -2
0   0 -2 -2 0   0

0 -2 -2 0
-2 -15 -2
-2 -2
0

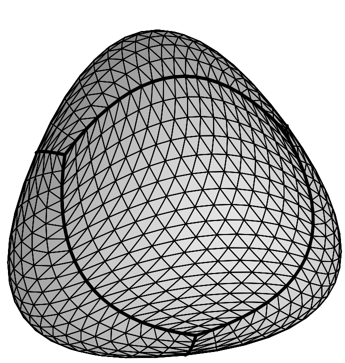

Figure 6.15: The sphere defined by four triangles

is incident to three complex edges, and every complex edge is incident to two triangles. From these two equations we derive $\#\mathcal{V} = 2 + \#\mathcal{P}/2$. Like in the previous example, we may choose rational CG^1 glueing data of degree 1 on each complex edge. In fact it can be possible to choose rational CG^1 data of degree 0 on a few complex edges. Let $\mathcal{E}_{\rm par}$ be the number of such complex edges. Then we can rewrite formula 6.21 as

$$\dim S_k^1(\mathcal{M}) = \frac{k^2 - 3\,k - 1}{2}\#\mathcal{P} + 6 + \#\mathcal{V}_{\rm prt} + \#\mathcal{E}_{\rm par}. \tag{6.30}$$

As in the previous example, this formula certainly holds for $k \geq 6$.

The construction of a geometrically continuous sphere by copying the polyhedral complex of the symmetric few Platonic "solid bodies", such as the tetrahedron, cube, octahedron or icosahedron, is considered in [46]. J. Peters and L. Kobbelt give a very similar representation (i.e. CG^1 surface patch complex) of a "tetrahedral" sphere, and the same representation of the "octaheral" sphere as in example 6.29 below. They also consider how the curvature varies on the constructed C^1 surfaces. They notice that along the common edges the curvature is quite small, and the most "curving" happens in the middle of the triangles. For this reason the shape of these spheroids approximate the *dual Platonic polyhedron* to the chosen one, e.g., the C^1 "octahedral" sphere look like the cube with smooth corners. This phenomenon can be noticed[15] in our pictures as well, see figures 6.15, 6.16 and 6.17. In the same "platonic" spirit we consider the following two examples.

Example 6.29 (An "octahedral" sphere) Consider a CG^0 surface complex $\mathcal{S}_8$ formed by 8 triangles glued in the same way as the triangular faces of a octahedron in $\mathbb{R}^3$ do. A glueing scheme is shown on the left-hand side of figure 6.16. There are 6 complex vertices which are denoted by a, b, c, d, p, q. The complex edges and polygons are determined by the set of complex vertices their are incident to. We will denote the

[15]Here we observe the following. Consider a simple Bézier complex $\mathcal{M}^0$ as in subsection 6.4.1 with rational CG^1 glueing data (μ, Θ) of degree n. For a fixed degree k one can derive explicit linear relations between the restricted control coefficients of the splines in $S_k^1(\mathcal{M}^0)$. It turns out that for $n > 0$ there are relations between the control coefficients of each single triangle of $\mathcal{M}^0$. This may explain why the realized CG^1 surfaces are usually "more flat" along the interior edges.

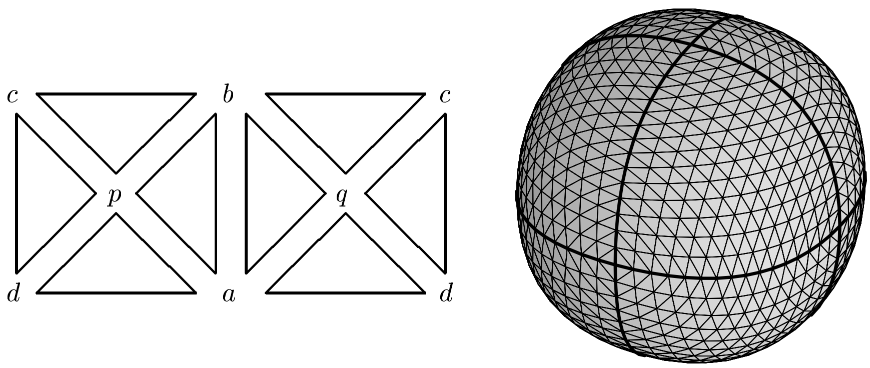

Figure 6.16: An "octahedral" sphere

complex edges and polygons by e_{ab}, Ω_{abp}, etc. Each complex vertex is surrounded by four triangles, so we may choose each complex vertex to be particular.

We make the most symmetric choice $(s+z, -2s, s+z)$ for the triple (α, β, γ) for each complex edge. By evaluating it at $s = 0$ (and evaluating the "reversed" triple $(s+z, -2z, s+z)$ at $z = 0$, see the remark at the end of subsection 6.4.1) one can check that this CG^1 glueing data is compatible with the "particularity" of the complex vertices. Then the formula (6.30) gives us $\dim S_k^1(\mathcal{S}_8) = 4(k-1)(k-2)$ for $k \geq 6$. Like in the previous example, we claim this dimension formula is true for $k \geq 3$.

We prove this claim in the same way as we did for the sphere $\mathcal{S}_4$. Namely, we show that for each edges control point there is a condition which expresses this control point as the weighted average of several control coefficients "around" it, and these conditions form a set of all independent relations on the control coefficients. Then using the same arguments about the obtained system of linear equations on the control coefficients, we conclude that a spline is determined by all interior control points of its restrictions to the polygons, so $\dim S_k^1(\mathcal{S}_8) = 8\binom{k-1}{2} = 4(k-1)(k-2)$ for $k \geq 3$ (and $\dim S_k^1(\mathcal{S}_8) = 1$ for $k \leq 2$ by the same arguments as in the previous example).

To derive the relations between the control coefficients, consider the interior edge E_{ab}, with the same notations for the control coefficients restricted by this edge as in figure 6.14. In our situation, the control coefficients r_i' and r_i'' are on the triangles Ω_{abp} and Ω_{abq}. Using equation (6.15) one can derive the following relations, similar to (6.29):

$$r_i = \frac{k-i}{2\,k}\left(r_{i+1}' + r_{i+1}''\right) + \frac{i}{2\,k}\left(r_i' + r_i''\right), \quad \text{for} \quad i = 0, \ldots, n. \tag{6.31}$$

One obtains the similar equations for other control coefficients. In this way each edge control coefficient is a weighted average of a few control coefficients "around" the edge control coefficient. It remains to prove that these are all independent relations. For an edge control coefficient which is not a vertex control coefficient we have just one relation like (6.31). Consider the vertex a. Each of the four complex edges incident to a gives us an expression of r_0 as a weighted average. However, the edges e_{ab} and e_{ad} give us the same relation $r_0 = (r_1' + r_1'')/2$, and the edges e_{ap} and e_{aq} both "require" that $r_0 = (r_1 + r_{-1})/2$, where r_{-1} is the control coefficient on the edge e_{ad} "closest" to r_0. One may check that there is a linear relation between these two equations and the four conditions like (6.31) for r_1, r_1', r_{-1} and r_1''. Hence we make keep only one of the two relations for r_0.

Therefore we are in the situation we have claimed, and the stated dimension formula for $S_k^1(\mathcal{M})$ is correct. In particular, for $k = 3$ the dimension is 8. For each triangle there is spline which has non-zero interior control coefficient on it, and has zero control coefficient on other triangles. Following the scheme on the right-hand side of figure 6.16 we give the control coefficient of such a spline f_{abp} for the triangle Ω_{abp} as

			0		0		1		3							3		1		0		0			
0				0		0		5				3	3				1		0		0				0
	0				1		5				5			1				0		0				0	
0		1				3				5		5	5		0				0				0		0
	0		3						3		12			0		0						0		0	
0		1				3				5		5	5		0				0				0		0
	0				1		5				5			1				0		0				0	
0				0		0		5				3	3				1		0		0				0
			0		0		1		3							3		1		0		0			

.

Observe that all the control coefficients on the triangle Ω_{cdq} are zero. We can assign any CG^1 glueing data to the complex edges e_{cd}, e_{cq} and e_{dq} (if only they keep the same CG^1 surround structure at a, b an p), or replace the triangle Ω_{cdq} with any "subcomplex" with at least three boundary edges, and we still have a spline with the same non-zero control coefficients as f_{abq}. This spline reminds the so-called *B-splines* on the triangulation in $\mathbb{R}^2$, see [20].

Similarly we define splines f_{abq}, f_{bcp}, etc. Since we have the same CG^1 glueing data on all complex edges, these spline have the same (up to symmetries of $\mathcal{S}_8$) control coefficients. The realization on the right-hand side of (6.16) is given by the tuple $(g_a - g_c,\ g_b - g_c,\ g_p - g_q)$, where $g_a = f_{abp} + f_{abq} + f_{adp} + f_{adq}$, etc.

Example 6.30 (A "dihedral" sphere) Consider a CG^0 surface complex $\mathcal{S}_{12}$ formed by 12 triangles glued in the same way as the triangular faces of a double pyramid with a hexagonal basis in $\mathbb{R}^3$ do. A glueing scheme is shown on the left-hand side of figure 6.17, the complex combinatorics and topology is demonstrated by the realization of $\mathcal{S}_{12}$ on the right-hand side of the same figure. There are 6 complex vertices $a_1, b_1, c_1, a_2, b_2, c_2$ on the "equator", and two "poles" p and q. The vertices on the "equator" are chosen to be particular. We choose the same triple $(\alpha, \beta, \gamma) = (s+z, -2s, s+z)$ on the edges on the "equator" $a_1b_1c_1a_2b_2c_2$, and we choose the triple $(\alpha, \beta, \gamma) = (s+z, -s, s+z)$ for edges connecting a pole with a vertex on the equator, so that the vertex on the equator corresponds to $s = 0$. One can check that we have a consistent CG^1 surround data for the vertices on the "equator" and for the "poles". The formula (6.30) gives us $\dim S_k^1(\mathcal{S}_{12}) = 6(k^2 - 3\,k + 1)$ for $k \geq 6$. It was computed that $\dim S_3^1(\mathcal{S}_{12}) = 8$. We choose the same triple $(\alpha, \beta, \gamma) = (s+z, -2s, s+z)$ on the edges on the equator $a_1b_1c_1a_2b_2c_2$, and we choose the triple $(\alpha, \beta, \gamma) = (s+z, -s, s+z)$ for edges connecting a pole with a vertex on the equator, so that the vertex on the equator corresponds to $s = 0$. One can check that we have a consistent CG^1 surround data for the vertices on the equator and for the poles.

The formula (6.30) gives us $\dim S_k^1(\mathcal{S}_{12}) = 6(k^2 - 3\,k + 1)$ for $k \geq 6$. The author computed that $\dim S_3^1(\mathcal{S}_{12}) = 8$. The generators for $S_3^1(\mathcal{S}_{12})$ can be defined as follows.

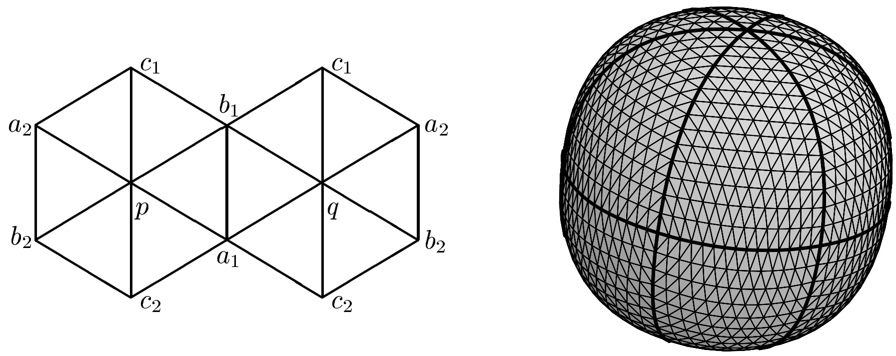

Figure 6.17: A "dihedral" sphere

We define the spline $f_{a_1 b_1 p}$ by the control coefficients in

			0						0			
		0		0				0		0		
	-5		0		5		5		0		-5	
-15		**0**		**0**		15		**0**		**0**		-15
	-27		0		27		3		0		-3	
-25		-22		22		25		-2		2		-25
	-60		0		**60**		**0**		0		**0**	
-25		-22		22		25		-2		2		-25
	-27		0		27		3		0		-3	
-15		**0**		**0**		15		**0**		**0**		-15
	-5		0		5		5		0		-5	
		0		0				0		0		
			0						0			

(here we follow the plane model as follows: the boundary control coefficients for the triangles are typed only once if the triangles are adjacent in the plane model, their interior control coefficients are typed in the bold face). Similarly we define splines $f_{a_1 b_1 q}$, $f_{b_1 c_1 p}$, etc., by applying the symmetries of $\mathcal{S}_{12}$. Note that in this way we get some dependent splines like $f_{a_2 b_2 p} = -f_{a_1 b_1 p}$, etc. We also define the spline $g_{1,p}$ according to

			15						15			
		5		25				25		5		
	0		27		30		30		3		0	
0		**0**		**60**		30		**0**		**0**		0
	0		33		54		6		-3		0	
0		11		55		30		-5		-1		0
	0		33		**60**		**0**		-3		**0**	
0		11		55		30		-5		-1		0
	0		33		54		6		-3		0	
0		**0**		**60**		30		**0**		**0**		0
	0		27		30		30		3		0	
		5		25				25		5		
			15						15			,

and a spline $g_{1,q}$, obtained from $g_{1,p}$ using the symmetry of $\mathcal{S}_{12}$ which permutes the poles and leaves the point on the equator fixed. Then a basis of $S_3^1(\mathcal{S}_{12})$ was computed to be $f_{a_1 b_1 p}$, $f_{a_1 b_1 q}$, $f_{b_1 c_1 p}$, $f_{b_1 c_1 q}$, $f_{c_1 a_2 p}$, $f_{c_1 a_2 q}$, $g_{1,p}$ and $g_{1,q}$. As we see, neither the formula (6.30) holds, nor can we choose the interior control points "on" the triangles freely.

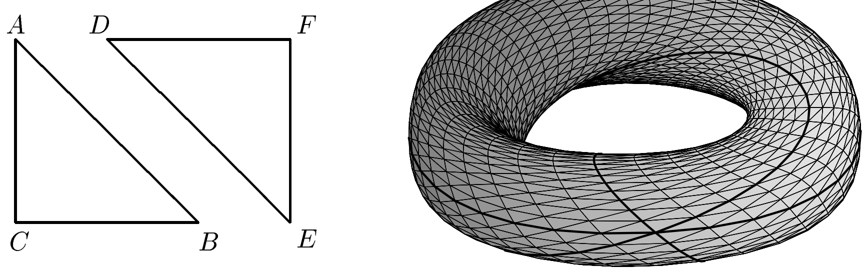

Figure 6.18: A geometrically continuous torus

Example 6.31 (The CG^1 torus) For a CG^1 surface complex $\mathcal{T}$, which topologically represents the torus, one can usually take rational CG^1 glueing data (μ, Θ) of degree zero on all its interior edges, see example 6.16. Then theorem 6.4.6 implies that

$$\dim S_k^1(\mathcal{T}) = (k-1)(k-2)\,\#\mathcal{V} + \#\mathcal{V}_{\mathrm{prt}} \tag{6.32}$$

for $k \geq 5$. The following example shows that this formula may not be true for $k = 4$. Hence the main result of this chapter is quite sharp in general.

Consider the Bézier complex $\mathcal{T}_2$ formed by two triangles ABC and DEF, and the following pairs of polygonal edges being identified (with the orientation being also specified): AB and DE, BC and FD, AC and FE, see the left-hand side of figure 6.18. (For a comparison, recall example 6.23 and figure 6.9.) The torus has one complex vertex surrounded by the six angles A, D, C, E, B and F in the given order. To all three complex edges we assign the rational CG^1 glueing data $(\alpha, \beta, \gamma) = (1, -1, 1)$, of degree zero. This indeed defines consistent CG^1 surround glueing data at the complex vertex.

Formula (6.32) tells us that $\dim S_k^1(\mathcal{T}_2) = (k-1)(k-2)$ for $k \geq 5$. However, the dimension of $S_4^1(\mathcal{T}_2)$ turns out to be 7. It is not difficult to point out 7 independent generators. Let c_{BC} and s_{BC} be two splines represented schematically by, respectively,

$$\begin{pmatrix} 0 & & & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & & & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & & & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & & & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & & & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & & & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & & & 0 \end{pmatrix}.$$

Further, one can define functions c_{AC}, s_{AC} and c_{AB}, s_{AB} by applying the symmetries of the CG^1 surface complex on the given two functions, that is, the permutation $(A, B, C, D, E, F) \mapsto (C, A, B, F, D, E)$ of the vertices and the inverse permutation. Then one can check that a basis for $S_4^1(\mathcal{T}_2)$ is the set containing these 6 splines and a constant (non-zero) spline. The right-hand side of figure 6.18 depicts us the realization of $\mathcal{T}_2$ given by the tuple $(r\,c_{AC} + c_{BC},\ r\,s_{AC} + s_{BC},\ s_{AB})$ with $r = 3$.

At the end we recall the CG^1 surface complex introduced in chapter 1, see figure 1.1. This complex has 32 triangles, 48 edges and 16 vertices. Let $\mathcal{T}_{4\times 4}$ denote the CG^1 complex with the easiest CG^1 glueing data $(\alpha, \beta, \gamma) = (1, -1, 1)$ on all complex edges. It turns out that $\dim S_2^1(\mathcal{T}_{4\times 4}) = 10$. In this space there are functions "like" $c_{AC}, s_{AC}, \ldots$ above, and by the "same map" as above one gets the realization on figure 1.1.