Aspects of algorithmic algebra

Vidunas, Raimundas

Chapter 5

Computing bivariate splines

In [7] Billera and Rose represented the module $C^r(\Delta)$ of r-smooth bivariate splines on a triangulation Δ as a kernel of a homomorphism between free modules over $\mathbb{R}[x,y]$. The graded version $C^r(\hat{\Delta})$ is representable by the same construction. We propose to represent these modules as a direct sum of the submodule of global polynomial splines and the kernel of a homomorphism between free modules. The rank of free modules is much smaller than in the Billera-Rose representation. For comparison, if m and n are the numbers of triangular faces and interior edges of Δ respectively, then a natural representation of the proposed kind gives a matrix of size $(n-m+1)\times n$, whereas the matrix in the Billera-Rose representation has size $n\times(m+n)$. Representations of even smaller size will be introduced which use preliminary local computations of splines around some vertices of Δ. We will show that this technique becomes very effective when many interior edges of Δ have the same slopes.

5.1 Introduction

Consider a finite two-dimensional simplicial complex Δ on a region in $\mathbb{R}^2$. Let $C^0(\Delta)$ denote the vector space over $\mathbb{R}$ of continuous polynomial splines on Δ, i.e. continuous piecewise polynomial functions on the support of Δ. In the same way, for a positive integer r let $C^r(\Delta)$ denote the vector space of r-smooth polynomial splines on Δ. Finally, for $r\geq 0$ let $C^r_k(\Delta)$ be the subspace of $C^r(\Delta)$ of splines defined by polynomials of degree at most k. For each k the vector space $C^r_k(\Delta)$ is finite dimensional. The two most important problems for applications are computing the dimensions of vector spaces $C^r_k(\Delta)$ and computing their bases. One may also be interested in how the dimensions of $C^r_k(\Delta)$ depend on k and combinatorial properties of Δ. It is known however, that these dimensions depend also on an embedding of Δ as an abstract simplicial complex into $\mathbb{R}^2$.

By viewing the polynomial ring $\mathbb{R}[x,y]$ as a subspace of global polynomial splines defined by the same polynomial on the whole Δ, we consider $C^r(\Delta)$ as a finitely generated module over $\mathbb{R}[x,y]$. It is even an algebra under the pointwise addition and multiplication of splines as functions. As it was noticed by Billera and Rose in [7], even more important for the formulated problems is the space $C^r(\hat{\Delta})$ of r-smooth (or continuous) splines on a three-dimensional simplicial complex $\hat{\Delta}$. Specifically, $\hat{\Delta}$ is the join of Δ with the origin of $\mathbb{R}^3$, where the ambient $\mathbb{R}^2$ of Δ is embedded into $\mathbb{R}^3$ by $(x, y) \mapsto (x, y, 1)$. The space $C^r(\hat{\Delta})$ is a *graded* module over the polynomial ring $\mathbb{R}[x, y, z]$. The essential characterization of $C^r(\hat{\Delta})$ is that its k-th graded part of $C^r(\hat{\Delta})$ is isomorphic to $C^r_k(\Delta)$ as a real vector space. As a consequence, all information about dimensions of vector spaces $C^r_k(\Delta)$ is encoded in the Hilbert function or Hilbert series of the graded module $C^r(\hat{\Delta})$. Moreover, bases of these vector spaces can be easily computed if a set of generators of $C^r(\hat{\Delta})$ and their syzygies are known. This is especially convenient when this module is free. Then a so-called *reduced* basis of splines exists, introduced by [8], and considered also in [28].

Therefore it is important to compute a set of generators and syzygies of $C^r(\hat{\Delta})$ or its Hilbert series. In the paper [7] Billera and Rose represented (under mild technical conditions on Δ) the module $C^r(\Delta)$ as a kernel of a homomorphism between free $\mathbb{R}[x, y]$-modules. The graded $\mathbb{R}[x, y, z]$-module $C^r(\hat{\Delta})$ is representable by the same construction. Having such a representation one can compute generators or Hilbert series of $C^r(\hat{\Delta})$ using standard Gröbner bases techniques, as it was demonstrated in [6]. The Billera-Rose representation was also exploited in [8, 48] to obtain useful results about $C^r(\hat{\Delta})$.

In this paper we consider representations of $C^r(\hat{\Delta})$ as a direct sum of the submodule of global polynomial splines on Δ and the kernel of a homomorphism between free $\mathbb{R}[x, y, z]$-modules. The rank of these free modules is much smaller than in the Billera-Rose representation. From examples we will see that computations using these representations may cost significantly less time and memory. The main result is a very effective representation of $C^r(\hat{\Delta})$ in the case when Δ has a lot of large classes of interior edges with the same slope. This situation is important in applications.

From here on we assume that given simplicial complexes are connected and *hereditary*. The last property means that any two triangles with a common vertex can be joined by a path of adjacent triangles which all contain the common vertex. We require Δ to be *oriented*. An orientation of Δ is given by an orientation of its edges. For $i = 0, 1, 2$ let Δ_i denote the set of all i-dimensional faces, and let Δ^0_i denote the set of *interior* i-dimensional faces of Δ, i.e. faces which do not lie on the boundary. Let $f_i(\Delta)$ and $f^0_i(\Delta)$ denote the cardinality of Δ_i and Δ^0_i respectively. Let Δ^0_H be the set of holes (or interior boundary components) of Δ. The genus $g(\Delta)$ of Δ is its topological genus, it is equal to the cardinality of Δ^0_H.

An interior edge is called a *pseudoboundary* edge if it is connected to the boundary of Δ by a sequence of edges with the same slope, as it is defined in [48]. We define the *incidence* graph $\Gamma(\Delta)$ between triangular faces and interior edges of Δ so that vertices of $\Gamma(\Delta)$ represent triangles of Δ, and two graph vertices are connected by a graph edge if the corresponding triangles have a common interior edge of Δ. A *star triangulation* is a simplicial complex with a single interior vertex, and such that all its interior edges are incident to that vertex. In particular, for a vertex $v \in \Delta^0_0$ the *star* Δ^*_v of v is the star triangulation formed by triangles of Δ incident to v and their faces.

In the following, R denotes the graded ring $\mathbb{R}[x, y, z]$. If $\mathcal{S}$ is a set of faces or holes of Δ, let $R(\mathcal{S})$ denote the graded R-module of mappings from $\mathcal{S}$ to R. The module $C^r(\hat{\Delta})$ can naturally be viewed as a submodule of $R(\Delta_2)$. Let $\partial_2 : R(\Delta_2) \to R(\Delta^0_1)$

and $\partial_1 : R(\Delta_1^0) \to R(\Delta_0^0)$ be simplicial boundary maps defined as follows. If $w \in R(\Delta_2)$ then $\partial_2(w)$ assigns to each interior edge $e \in \Delta_1^0$ the difference of polynomials assigned by w to the left and right triangles from the oriented e. If $u \in R(\Delta_1^0)$ then $\partial_1(u)$ assigns to an interior vertex $v \in \Delta_0^0$ the sum $\sum \pm u(e)$ over all edges e incident to v, where signs in the sum are determined by orientation of these edges. The defined modules and homomorphisms

$$0 \longrightarrow R(\Delta_2) \xrightarrow{\partial_2} R(\Delta_1^0) \xrightarrow{\partial_1} R(\Delta_0^0) \longrightarrow 0 \tag{5.1}$$

form a complex of R-modules on Δ, as defined in [48]. The matrix of ∂_1 in the obvious free bases of $R(\Delta_1^0)$ and $R(\Delta_0^0)$ is the incidence matrix between interior edges and vertices of Δ. Likewise, the matrix of ∂_2 is the incidence matrix of the properly oriented graph $\Gamma(\Delta)$.

Let $R^{(r)}(\Delta_1^0)$ be the graded submodule of $R(\Delta_1^0)$ of those mappings which assign to each interior edge a polynomial which vanishes on that edge with order $\geq r+1$. If $e \in \Delta_1^0$ let l_e be a homogeneous linear form defining the interior edge e. The module $R^{(r)}(\Delta_1^0)$ is free of rank $f_1^0(\Delta)$. We take for a basis of $R^{(r)}(\Delta_1^0)$ those elements which assign l_e^{r+1} to an edge e and zero polynomial to all other interior edges. A homogeneous element of $R^{(r)}(\Delta_1^0)$ of degree d is a linear combination of the basis elements with coefficients in R of degree $d - r - 1$.

This paper is organized as follows. In section 5.2 we recall the Billera-Rose representation of $C^r(\hat{\Delta})$, and consider two representations of $C^r(\hat{\Delta})$ as a direct sum of the submodule of global polynomial splines on Δ and the kernel of a homomorphism between free R-modules. One of these representations uses preliminary computations of splines on stars of some interior vertices. The speed and required memory of computations with all these representations are compared on the well known Morgan-Scott triangulation (figure 5.1). In section 5.3 we demonstrate how efficiently the module $C^r(\hat{\Delta})$ can be represented if a triangulation Δ has a lot of edges with the same slopes. As an example, if all interior edges of Δ are pseudoboundary, then $C^r(\hat{\Delta})$ is free according to [48]. We will see that a free basis of $C^r(\hat{\Delta})$ in this case can be found using only local computations around all interior vertices of Δ. In the last section we mention a few computation results concerning the Hilbert series of $C^r(\hat{\Delta})$.

5.2 Basic representations of $C^r(\hat{\Delta})$

In this section let $m = f_2(\Delta)$ and $n = f_1^0(\Delta)$. First we recall the Billera-Rose representation introduced in [7]. In this paper L.J.Billera and L.L.Rose have shown that $C^r(\hat{\Delta})$ is isomorphic to the kernel of the map

$$\varphi : R(\Delta_2) \oplus R^{(r)}(\Delta_1^0) \to R(\Delta_1^0) \tag{5.2}$$

defined by $(w, \varepsilon) \mapsto \partial_2(w) + J(\varepsilon)$, and where $J : R^{(r)}(\Delta_1^0) \to R(\Delta_1^0)$ is the inclusion map. Indeed, a spline on Δ is defined by assigning a polynomial for each triangle in Δ_2 with the known restriction (see [7]): if two triangles have a face σ in common, then the difference of polynomials assigned to them should lie in I_σ^{r+1}, where I_σ is the ideal defining σ. Because Δ is hereditary, it is sufficient to take into account only pairs of

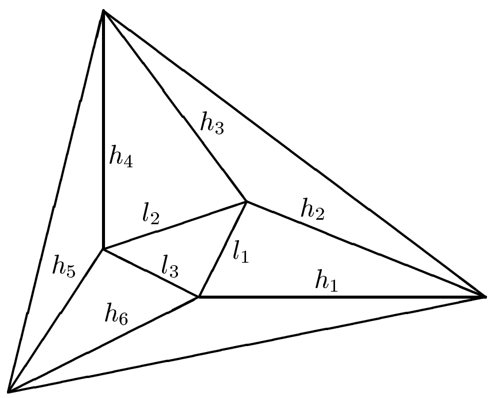

Figure 5.1: The Morgan-Scott triangulation

triangles with a common edge. Therefore a homogeneous element $w \in R(\Delta_2)$ defines a spline on Δ if and only if $\partial_2(w) \in R^{(r)}(\Delta_1^0)$. In this case $\partial_2(w) = -J(\varepsilon)$ for some unique $\varepsilon \in R^{(r)}(\Delta_1^0)$, and w comes from the element (w, ε) of the kernel of φ.

The matrix M of φ has size $n \times (m+n)$ and is very sparse. Each row of M represent a smoothness (or continuity) condition on an interior edge e and has only three non-zero entries in each row: two entries (1 and -1) in the submatrix representing ∂_2, and the entry l_e^{r+1} in the submatrix representing J. The matrix of J is diagonal if bases of $R(\Delta_1^0)$ and $R^{(r)}(\Delta_1^0)$ are ordered according to the same enumeration of interior edges of Δ.

Example 5.1 Let Δ be the Morgan-Scott triangulation (figure 5.1). Choose an orientation of edges on it so that oriented edges l_1, l_2, l_3 form a loop counter-clockwise, and edges h_i are oriented towards the boundary. The matrix which represents $C^r(\hat{\Delta})$ by the recipe of Billera-Rose is:

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & l_1^{r+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & l_2^{r+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & l_3^{r+1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & h_1^{r+1} & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_2^{r+1} & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3^{r+1} & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_4^{r+1} & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_5^{r+1} & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_6^{r+1}
\end{pmatrix}.$$

Our purpose is to look for representations of $C^r(\hat{\Delta})$ which involve free R-modules of smaller rank. It is a well-known fact that $C^r(\hat{\Delta})$ is isomorphic to a direct sum of the module of global polynomial splines and the module of splines which restrict to zero on a fixed triangle. We will represent $C^r(\hat{\Delta})$ rather as the direct sum $R \oplus D^r(\hat{\Delta})$, where R is identified with global polynomial splines, and $D^r(\hat{\Delta})$ is the *image* of $C^r(\hat{\Delta})$ under

$\partial_2 : R(\Delta_2) \to R(\Delta_1^0)$. The isomorphism $C^r(\hat{\Delta}) \cong R \oplus D^r(\hat{\Delta})$ holds, because the kernel of ∂_2 is the summand R of $C^r(\hat{\Delta})$.

Let us define a *path* $\sigma_{i=0}^k$ of triangles on Δ to be a finite sequence $(\sigma_0, \sigma_1, \ldots, \sigma_k)$ of triangular faces of Δ such that consecutive triangles are adjacent. For an element u of $R^{(r)}(\Delta_1^0)$ and a path $\sigma_{i=0}^k$ let $\Sigma(u, \sigma_{i=0}^k)$ denote the sum $\sum_{i=1}^k \pm u(e_i)$, where e_i is the edge between σ_{i-1} and σ_i, and the sign is determined by the orientation of e_i with respect to the pair (σ_{i-1}, σ_i) of adjacent triangles. Then one can identify $D^r(\hat{\Delta})$ with splines vanishing on a fixed triangle σ^0 as follows. If $u \in D^r(\hat{\Delta})$, let us choose for any other triangle σ' a path $\sigma_{i=0}^k$ of triangles connecting σ^0 with σ', and assign $\Sigma(u, \sigma_{i=0}^k)$ to σ'. The obtained element of $R(\Delta_2)$ does not depend on the choice of paths because u is in the image of ∂_2, and it represents a spline because for any interior edge there is a path of triangles "crossing" it. By this construction one can easily compute generators of $C^r(\hat{\Delta})$ from a set of generators of $D^r(\hat{\Delta})$.

The module $D^r(\hat{\Delta})$ is representable as a kernel of a homomorphism between free R-modules, like $C^r(\hat{\Delta})$ is represented by φ in (5.2). The first representation we present is natural and was already used by R.-H. Wang in [62]. What is worth noticing, however, is that the matrix defined by this representation of $D^r(\hat{\Delta})$ is considerably smaller than the one in Billera-Rose representation. We define a homomorphism $\partial_H : R(\Delta_1^0) \to R(\Delta_H^0)$ so that if $u \in R(\Delta_1^0)$ then $\partial_H(u)$ assigns to each hole h of Δ the sum $\sum \pm u(e)$ over all edges e incident to the hole, with signs determined by the orientation of those edges on the boundary of h. Let $\psi : R^{(r)}(\Delta_1^0) \to R(\Delta_0^0) \oplus R(\Delta_H^0)$ be the restriction of $\partial_1 \oplus \partial_H : R(\Delta_1^0) \to R(\Delta_0^0) \oplus R(\Delta_H^0)$ onto $R^{(r)}(\Delta_1^0)$.

Theorem 5.2.1 *The graded R-module $D^r(\hat{\Delta})$ is isomorphic to the kernel of $\psi : R^{(r)}(\Delta_1^0) \to R(\Delta_0^0) \oplus R(\Delta_H^0)$.*

Proof. We will use the notion of a path and the notation $\Sigma(u, \sigma_{i=0}^k)$. As it was mentioned, $w \in R(\Delta_2)$ is in $C^r(\hat{\Delta})$ if and only if $\partial_2(w) \in R^{(r)}(\Delta_1^0)$, thus $D^r(\hat{\Delta})$ is the submodule of $R^{(r)}(\Delta_1^0)$ of all those elements which are mapped to zero by ∂_2. That means that $u \in R^{(r)}(\Delta_1^0)$ is in $D^r(\hat{\Delta})$ if and only if sums $\Sigma(u, \sigma_{i=0}^k)$ are the same for paths with the same starting and final triangles σ_0 and σ_k. Equivalently, for each *closed* path $\sigma_{i=0}^k$ of triangles (i.e. one with $\sigma_k = \sigma_0$) the sum $\Sigma(u, \sigma_{i=0}^k)$ must be zero. Note that $\psi(u)$ assigns to an interior vertex v precisely the sum $\Sigma(u, \sigma_{(v)})$, where $\sigma_{(v)}$ is a closed path of all (and different) triangles incident to v, with a suitable orientation. Similarly, $\psi(u)$ assigns to a hole h the sum $\Sigma(u, \sigma_{(h)})$, where $\sigma_{(h)}$ is a closed path of all (and different) triangles which have at least a common vertex with the boundary of h. It follows that if $u \in D^r(\hat{\Delta})$ then $\psi(u) = 0$, that is $D^r(\hat{\Delta}) \subset \ker \psi$.

It remains to prove the opposite inclusion $\ker \psi \subset D^r(\hat{\Delta})$. Let $u \in R^{(r)}(\Delta_1^0)$ be in the kernel of ψ. We would like to see that for any closed path $\sigma_{i=0}^k$ we have $\Sigma(u, \sigma_{i=0}^k) = 0$. A path of triangles on Δ represents a cycle in the incidence graph $\Gamma(\Delta)$. From basic algebraic topology ([41]) we know that cycles of $\Gamma(\Delta)$ form the first homology group $H_1(\Gamma(\Delta))$ of the graph. This homology group is free and abelian. Its rank is equal to $n - m + 1$, since $\Gamma(\Delta)$ is connected, and has m vertices and n edges. There is a well defined homomorphism of abelian groups $\omega_u : H_1(\Gamma(\Delta)) \to R$ which sends a cycle ρ to $\Sigma(u, \sigma_{i=1}^k)$, where $\sigma_{i=1}^k$ is a closed path of triangles representing the cycle ρ. Consider $f_0^0(\Delta)$ cycles of $\Gamma(\Delta)$ represented by closed paths $\sigma_{(v)}$ around each interior vertex $v \in \Delta_0^0$, and $g(\Delta)$ cycles represented by closed paths $\sigma_{(h)}$ around each hole h. These $f_0^0(\Delta)+g(\Delta)$ cycles are sent to zero by ω_u because u is in the kernel of ψ. They are also independent in $H_1(\Gamma(\Delta))$ because each of them loops around a different vertex or hole of Δ. From Euler formulas $f_0(\Delta) - f_1(\Delta) + (m+1) = 2 - g(\Delta)$ and $f_0(\Delta) - f_0^0(\Delta) = f_1(\Delta) - n$ it follows that $f_0^0(\Delta) + g(\Delta) = n - m + 1$. Hence the chosen $f_0^0(\Delta) + g(\Delta)$ cycles form the basis of $H_1(\Gamma(\Delta))$, and ω_u sends all cycles to zero. We conclude that $\Sigma(u, \sigma_{i=0}^k) = 0$ for any closed path $\sigma_{i=0}^k$, thus $\ker\psi \subset D^r(\hat{\Delta})$. The claim of the theorem follows. □

The matrix M of this representation has size $(n-m+1) \times n$, since $f_0^0(\Delta) + g(\Delta) = n - m + 1$. It is much smaller then the one in Billera-Rose representation. Columns of M can be put in a correspondence with interior edges, and its rows — with interior vertices and holes of Δ. An entry of M corresponding to an edge e is $\pm l_e^{r+1}$ if e is incident to the corresponding vertex or hole, and is zero otherwise.

Example 5.2 Consider the Morgan-Scott triangulation (figure 5.1) as Δ again. The proposed representation of $D^r(\hat{\Delta})$ is given by the matrix

$$\begin{pmatrix} -l_1^{r+1} & l_2^{r+1} & 0 & 0 & h_2^{r+1} & h_3^{r+1} & 0 & 0 & 0 \\ 0 & -l_2^{r+1} & l_3^{r+1} & 0 & 0 & 0 & h_4^{r+1} & h_5^{r+1} & 0 \\ l_1^{r+1} & 0 & -l_3^{r+1} & h_1^{r+1} & 0 & 0 & 0 & 0 & h_6^{r+1} \end{pmatrix}.$$

There are two simple types of triangulations for which the module $D^r(\hat{\Delta})$ is very easy to compute. Suppose that a triangulation Δ has no holes (i.e. $g(\Delta) = 0$) and no interior vertices. This is a special case of the stacked complex as defined in [28]. Then and only then ψ is the zero map and $D^r(\hat{\Delta}) \cong R^{(r)}(\Delta_1^0)$. Note that in this situation any interior edge has its both end-vertices on the exterior boundary. An edge with this property in a general triangulation Δ is called a *cross-cut* edge. The presence of cross-cut edges in a general Δ allows us to simplify computations with $D^r(\hat{\Delta})$ as follows. A cross-cut edge e splits Δ into two connected hereditary triangulations Δ_L and Δ_R which have only edge e in common, and one can express $D^r(\hat{\Delta})$ as a direct sum of graded R-modules:

$$D^r(\hat{\Delta}) \cong D^r(\hat{\Delta}_L) \oplus l_e^{r+1}R \oplus D^r(\hat{\Delta}_R). \tag{5.3}$$

Here summands $D^r(\hat{\Delta}_L)$ and $D^r(\hat{\Delta}_R)$ assign non-zero polynomials only to interior edges of Δ_L and Δ_R respectively, and $l_e^{r+1}R$ assigns a multiple of l_e^{r+1} to e and zero to all other edges of Δ. An injection $D^r(\hat{\Delta}) \hookrightarrow D^r(\hat{\Delta}_L) \oplus l_e^{r+1}R \oplus D^r(\hat{\Delta}_R)$ is obvious, surjectivity follows from theorem 5.2.1.

The next easiest case is a star triangulation Δ^*. Let $h_1, \ldots, h_k$ be linear forms in R defining interior edges of Δ^*. It was first proved by L.L. Schumaker in [50] that our $D^r(\hat{\Delta}^*)$ is isomorphic to the kernel of the matrix

$$M = \begin{pmatrix} h_1^{r+1} & h_2^{r+1} & \ldots & h_k^{r+1} \end{pmatrix}. \tag{5.4}$$

It is of course an easy consequence of theorem (5.2.1). We may move by an affine transformation of $\mathbb{R}^2$ the interior vertex of Δ^* to the origin $(0,0)$. Then all h_i become homogeneous linear forms in $\mathbb{R}[x,y]$, and $\ker M$ may be considered as a graded $\mathbb{R}[x,y]$-module. From an easy homological proposition 3.4 in [8] and $D^r(\hat{\Delta}^*) \cong \ker M \otimes_{\mathbb{R}[x,y]} \mathbb{R}[x,y,z]$ it follows that $D^r(\hat{\Delta}^*)$ is a free R-module. According to [6] the rank of $C^r(\hat{\Delta})$ must be k, so the rank of $D^r(\hat{\Delta})$ must be equal to $k-1$.

Here we present another representation of $D^r(\hat{\Delta})$. It uses preliminary easy computations of bases of $D^r(\hat{\Delta}^*_v)$ for stars of some vertices $v \in \Delta^0_0$, and the representation homomorphism links these local modules into the global module $D^r(\hat{\Delta})$. Note that no linking is needed when $g(\Delta) = 0$ and no two interior vertices are connected by an edge. Then cross-cut edges split Δ into a set of isolated stars and single triangles, hence $D^r(\hat{\Delta})$ is a free module generated by bases of $D^r(\hat{\Delta}^*)$ for all stars in Δ and elements coming from cross-cut edges. The new representation is based on the following theorem.

Theorem 5.2.2 *Let $\mathcal{S}$ be a set of interior vertices of a triangulation Δ such that any two vertices from $\mathcal{S}$ are not connected by an interior edge. Let N be the kernel of $\pi_{\mathcal{S}} \circ \psi : R^{(r)}(\Delta^0_1) \to R(\mathcal{S})$, where $\pi_{\mathcal{S}}$ is the projection $R(\Delta^0_0) \oplus R(\Delta^0_H) \to R(\mathcal{S})$. Then N is a free module of rank $n - \#\mathcal{S}$, and the module $D^r(\hat{\Delta})$ is isomorphic to the kernel of $\Psi : N \to R(\Delta^0_0 \setminus \mathcal{S}) \oplus R(\Delta^0_H)$, the restriction of ψ onto N.*

Proof. It is clear that $D^r(\hat{\Delta})$ is a submodule of N, and that the image of N under ψ lies in the direct complement of $R(\mathcal{S})$ in $R(\Delta^0_0) \oplus R(\Delta^0_H)$, which can be identified with $R(\Delta^0_0 \setminus \mathcal{S}) \oplus R(\Delta^0_H)$. It remains to prove that N is a free module of required rank. Let $R^{(r)}(\Delta^0_1 : \mathcal{S})$ be the submodule of $R^{(r)}(\Delta^0_1)$ of elements which assign zero to all edges incident to some $v \in \mathcal{S}$. We claim that

$$N \cong \left\{ \oplus_{v \in \mathcal{S}} D^r(\hat{\Delta}^*_v) \right\} \oplus R^{(r)}(\Delta^0_1 : \mathcal{S}). \tag{5.5}$$

A homomorphism from N to the direct sum on the right is clear: N is projected into a summand $D^r(\hat{\Delta}^*_v)$ or $R^{(r)}(\Delta^0_1 : \mathcal{S})$ by restricting assignments of elements of N only to edges involved in the direct summand. This homomorphism is injective because each interior edge is involved in a summand on the right of (5.5). It is also surjective because polynomials to each edge are assigned only by one summand. Hence isomorphism (5.5) holds. Recall that the module $D^r(\hat{\Delta}^*_v)$ is free of rank $f^0_1(\Delta^*_v) - 1$. It follows that N is a free R-module of rank $f^0_1(\Delta) - \#\mathcal{S} = n - \#\mathcal{S}$. □

The above proposition allows us to represent $D^r(\hat{\Delta})$ as the kernel of $\Psi : N \to R(\Delta^0_0 \setminus \mathcal{S}) \oplus R(\Delta^0_H)$, a homomorphism between free modules of rank $n - \#\mathcal{S}$ and $f^0_0(\Delta) - \#\mathcal{S} + g(\Delta)$, whereas the rank of free modules of its representation by ψ is n and $f^0_0(\Delta) + g(\Delta)$. In order to find explicitly the matrix of Ψ one needs to proceed in three steps. **At first** a set $\mathcal{S}$ with required restrictions must be chosen. The set $\mathcal{S}$ should be large, because each element of $\mathcal{S}$ decreases rank of involved modules. Finding the largest possible set $\mathcal{S}$ however might be a very hard combinatorial problem. In section 5.3 we give a more general formulation of theorem 5.2.2, and describe a heuristic algorithm which after easy adaptations would give a large set $\mathcal{S}$ in our situation. In **the second step** one should compute for each vertex $v \in \mathcal{S}$ a basis of the module $D^r(\hat{\Delta}^*_v)$. To simplify computations one can use affine transformations to move temporarily vertices in $\mathcal{S}$ into the origin of $\mathbb{R}^2$ and use two homogeneous variables, as described in the example above. In the **third step** one writes down the matrix M of Ψ. It has size

$(n - m - \#\mathcal{S} + 1) \times (n - \#\mathcal{S})$. Each column of M corresponds either to a generator of some $D^r(\hat{\Delta}_v^*)$ for $v \in \mathcal{S}$, or to an interior edge not incident to any $v \in \mathcal{S}$. Each row of M correspond either to an interior vertex not in $\mathcal{S}$, or to a hole of Δ. The submatrix of M representing $R^{(r)}(\Delta_1^0 : \mathcal{S}) \to R(\Delta_0^0 \setminus \mathcal{S}) \oplus R(\Delta_H^0)$ has the same form as the matrix of ψ. Let u be a generator of $D^r(\hat{\Delta}_v^*)$ for some $v \in \mathcal{S}$. The entry of M corresponding to u and to a vertex $v' \notin \mathcal{S}$ is $\pm u(e)$ if there is an edge e connecting v and v', and zero otherwise. The entry corresponding to u and a hole in Δ is the sum $\sum \pm u(e)$ over all interior edges e which are incident to v and to the hole.

Computational complexity of these preliminary steps should be insignificant compared with computations of generators of $D^r(\hat{\Delta})$ using the map Ψ, unless we are able to include almost all vertices into $\mathcal{S}$ and $g(\Delta)$ is small. In any case these preliminary steps and the obtained representation together should give an effective alternative to a straightforward application of theorem 5.2.1 or Billera-Rose representation. Computations of generators or Hilbert series of $C^r(\hat{\Delta})$ using considered representations are usually based on Gröbner bases, so it is difficult to estimate their worst case complexity. Recall that the kernel of a homomorphism between free R-modules is the module of first syzygies between columns of its matrix. We may expect that for general triangulation the numbers n, m and $\#\mathcal{S}$ are proportional. That means that dimensions of the matrix M of Ψ are proportionally smaller than dimensions of matrices of ψ and φ. In particular, the smaller number of columns in M may substantially decrease the number of syzygies we need to compute.

Example 5.3 Consider the Morgan-Scott triangulation (figure 5.1). We can include at most one interior vertex into $\mathcal{S}$. Let it be the common vertex v of edges l_1 and l_2. As the next preliminary step we find a basis of $D^r(\hat{\Delta}_v^*)$ by computing the kernel of matrix $(-l_1^{r+1}\ l_2^{r+1}\ h_2^{r+1}\ h_3^{r+1})$. Let $\{u_1, u_2, u_3\}$ be a basis of $D^r(\hat{\Delta}_v^*)$. Then the matrix of Ψ is

$$\begin{pmatrix} u_1(l_1) & u_2(l_1) & u_3(l_1) & -l_3^{r+1} & h_1^{r+1} & 0 & 0 & h_6^{r+1} \\ -u_1(l_2) & -u_2(l_2) & -u_3(l_2) & l_3^{r+1} & 0 & h_4^{r+1} & h_5^{r+1} & 0 \end{pmatrix}.$$

As another example, consider the triangulation on figure 5.2. We orientate it as follows: edges on the line PQ are oriented towards Q, other pseudoboundary edges are oriented towards the boundary, and the remaining edges are oriented towards B. Let l_{AB} be a linear form defining the edge AB and so on. We take $\mathcal{S} = \{C, E, F\}$, one can check that this is the largest possible set of vertices we can choose. For each vertex $v \in \mathcal{S}$ let u^v denote a basis of $D^r(\hat{\Delta}_v^*)$. Note that the set u^C should have four elements, and sets u^E and u^F have three elements each. The matrix of Ψ has size 3×17 and looks like

$$\begin{pmatrix} u^C(AC) & 0 & -u^F(AF) & l_{AP}^{r+1} & l_{AR}^{r+1} & l_{AB}^{r+1} & 0 & 0 & 0 & 0 \\ -u^C(BC) & 0 & -u^F(BF) & 0 & 0 & -l_{AB}^{r+1} & l_{BS}^{r+1} & -l_{BD}^{r+1} & 0 & 0 \\ -u^C(CD) & u^D(DE) & 0 & 0 & 0 & 0 & 0 & l_{BD}^{r+1} & l_{DS}^{r+1} & l_{DT}^{r+1} \end{pmatrix},$$

where the first "column" above represents four columns, one for each generator in u^C, and the next two "columns" represent three columns each. In each of these columns a generator is evaluated at the given edges. For comparison, the matrix of Billera-Rose

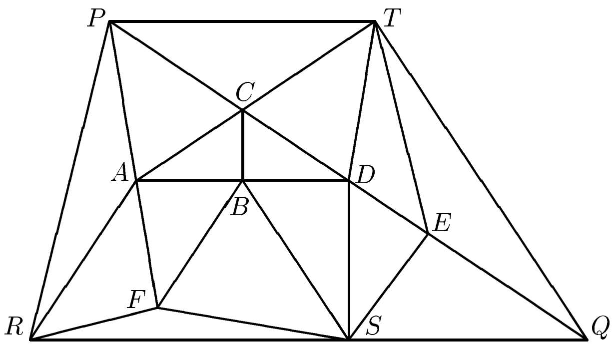

Figure 5.2: An example

r	Billera-Rose		With ψ		With Ψ	
8	17"	950	6"	400	3"	350
10	47"	1700	16"	700	8"	525
12	1′ 44"	2775	35"	1075	18"	775
14	3′ 38"	4350	1′ 14"	1650	37"	1150
16	6′ 2"	6850	2′ 13"	2600	1′ 6"	1700
18	11′ 6"	9575	3′ 57"	3525	1′ 55"	2350
20	18′ 43"	14275	6′ 35"	5350	3′ 11"	3475
22	28′ 6"	18750	10′ 23"	6925	5′ 2"	4550
24	45′ 9"	25600	16′ 2"	9575	7′ 38"	6125
25	56′ 3"	30150	18′ 53"	11150	9′ 12"	7175

Table 5.1: Comparison of computations with representations of $C^r(\hat{\Delta})$

representation for the triangulation in figure (5.2) would have size 20×35, and the matrix for ψ would have size 6×20.

All three representations of $C^r(\hat{\Delta})$ considered in this section were implemented in Macaulay 3.0 with Δ being the Morgan-Scott triangulation. The speed and required memory of computing generators of $C^r(\hat{\Delta})$ using these representation is compared in table 5.1. For each representation and several r the elapsed time (in minutes and seconds) and required memory (in kilobytes) of computations is given. Computing generators of $C^r(\hat{\Delta})$ from generators of $D^r(\hat{\Delta})$ is expected to be very fast.

We mention a few details of performed computations with Macaulay 3.0. Because of well known its limitations, the finite field $K = \mathbf{F}_{31991}$ was used instead of $\mathbb{R}$. While working with Billera-Rose representation only the $R(\Delta_2)$ components of the kernel of φ were computed using "modulo" command. Direct computations of the kernel of φ require $40 - 60\%$ more time and $\approx 25\%$ more memory. The results shown in table 5.1

were obtained using the default monomial ordering on free $K[x,y,z]$-modules of Macaulay 3.0. The author tried the option "c", which forces an ordering of generators of a free module to be more important than ordering of monomials in $K[x,y,z]$. Computations with "c" option were 4-5 times slower. First of all, computations with this option run up to degree $3(r+1)$ whereas default computations — up to degree $2r+1$. Even after forcing both computations run up to degree $2r+1$ the "c" option remains 2-3 times slower. It was also noticed that a proper ordering of components of involved free modules may speed up computations up to 15%. It is advisable to let the generators of $R^{(r)}(\Delta_1^0)$ corresponding to edges closer to the exterior boundary to be greater. Computations with Ψ become slightly faster if we let generators of modules $D^r(\hat{\Delta}_v^*)$ to be greater than other generators of N. Preliminary computations of bases of modules $D^r(\hat{\Delta}_v^*)$ cost less than 0.5% of used time.

5.3 Computing splines on special triangulations

In this section we suppose that a given triangulation Δ has a lot of edges with the same slopes. Such triangulations are important in applications, also because the modules $C^r(\hat{\Delta})$ and $D^r(\hat{\Delta})$ usually have generators of smaller degree and less syzygies in this case. We will reformulate the theorem 5.2.2 in a more general way, so that the non-connectedness restriction on vertices in $\mathcal{S}$ may become much weaker. The new representation uses the orientation of Δ extensively, in fact the set $\mathcal{S}$ is practically defined by the orientation. Considered examples will show that many edges with the same slopes may allow us to enlarge the set $\mathcal{S}$ considerably. Then preliminary local computations around more vertices should substantially decrease the size of the global representation of $D^r(\hat{\Delta})$.

First we specify an equivalence class of edges "with the same slope". An *edge-sequence* is a sequence $(e_1,\ldots,e_k)$ of interior edges, all lying on the same line in $\mathbb{R}^2$, and such that consecutive edges are incident either to the same interior vertex or to the same hole of Δ. An edge-sequence is said to be oriented *uniformly* if all its edges are oriented to the same direction. An *edge-series* is an edge-sequence oriented uniformly towards e_k. We define two interior edges to be *equivalent* if they are elements in the same edge-series; this equivalence relation slightly resembles the one defined in [48].

For the new representation of $D^r(\hat{\Delta})$ we need another basis of $R^{(r)}(\Delta_1^0)$. If e is an interior edge, let $\bar{w}_e$ be the element of $R^{(r)}(\Delta_1^0)$ which assigns the polynomial l_e^{r+1} to all edges in the edge-series which starts with e and continues following the direction of e as long as possible, and $\bar{w}_e$ assigns zero to the remaining edges. It is clear that $\bar{w}_e$'s of all interior edges e form a basis of $D^r(\hat{\Delta})$. It is useful to see how the matrix M of $\varphi : R^{(r)}(\Delta_1^0) \to R(\Delta_0^0) \oplus R(\Delta_H^0)$ changes in this basis of $D^r(\hat{\Delta})$. Let v be an interior vertex, and e be an edge incident to v and oriented towards it. If there is an edge in the star of v with the same slope as e and oriented away from it, then the entry $-l_e^{r+1}$ corresponding to v and e will become zero. Otherwise new non-zero entries occur on the row corresponding to v if there are other edges equivalent to e. On the rows corresponding to holes of Δ similar changes occur.

We call an interior vertex v *regular* if each edge incident to v and oriented towards it has a successor in an edge series, i.e. there is an edge with the same slope, incident to v and oriented away from it. A *star complex* is a complex of two-dimensional cones which have a common vertex, cover all $\mathbb{R}^2$, and intersect each other either in a half-line or in the common vertex. Cones in a star complex are not required to be convex. For a star complex $\Delta^\star$ we define $D^r(\hat{\Delta}^\star)$ to be the graded module whose elements assign to each half-line ℓ of $\Delta^\star$ a polynomial in R vanishing on ℓ with order $r+1$, and these assignments sum up to zero. The module $D^r(\hat{\Delta}^\star)$ can be represented in the same way as the analogous module for a star triangulation, namely as the kernel of the matrix in (5.4), with $h_1, \ldots, h_k$ being linear forms defining all half-lines of $\Delta^\star$. It is a free R-module of rank $k-1$.

Here we present a modification of theorem 5.2.2 on which the new representation is based.

Theorem 5.3.1 *Let $\mathcal{S}$ be a set of regular vertices of Δ. Let $\tilde{N}$ be the kernel of $\pi_{\mathcal{S}} \circ \psi : R^{(r)}(\Delta_1^0) \to R(\mathcal{S})$, where $\pi_{\mathcal{S}} : R(\Delta_0^0) \oplus R(\Delta_H^0) \to R(\mathcal{S})$ is the projection. Then $\tilde{N}$ is a free module of rank $n - \#\mathcal{S}$, and the module $D^r(\hat{\Delta})$ is isomorphic to the kernel of $\tilde{\Psi} : \tilde{N} \to R(\Delta_0^0 \setminus \mathcal{S}) \oplus R(\Delta_H^0)$, the restriction of ψ onto $\tilde{N}$.*

Proof. As in the theorem 5.2.2, it is clear that $D^r(\hat{\Delta})$ is a submodule of $\tilde{N}$ and that the image of $\tilde{N}$ under ψ lies in the direct complement $R(\Delta_0^0 \setminus \mathcal{S}) \oplus R(\Delta_H^0)$ of $R(\mathcal{S})$ in $R(\Delta_0^0) \oplus R(\Delta_H^0)$. It remains to prove that $\tilde{N}$ is a free module of required rank. Let $R^{(r)}(\Delta_1^0 : \mathcal{S})$ be the submodule of $R^{(r)}(\Delta_1^0)$ generated by $\bar{w}_e$ for all e which are not oriented away from a vertex in $\mathcal{S}$. For a vertex $v \in \mathcal{S}$ let $\Delta_v^\star$ be the star complex defined by those edges incident to v which are oriented away from it. We claim that

$$\tilde{N} \cong \left\{ \oplus_{v \in \mathcal{S}} D^r(\hat{\Delta}_v^\star) \right\} \oplus R^{(r)}(\Delta_1^0 : \mathcal{S}) \tag{5.6}$$

Let us define a projection $\pi_v : \tilde{N} \to D^r(\hat{\Delta}_v^\star)$ as follows. Let $u \in \tilde{N}$, let ℓ be a half-line of $\Delta_v^\star$, and let e be an edge incident to v lying on ℓ. If e has a predecessor e' in an edge series then we let $\pi_v(u)$ assign $u(e) - u(e')$ to ℓ. Otherwise $\pi_v(u)(\ell) = u(e)$. Further, we project $\pi_{\bar{w}} : \tilde{N} \to R^{(r)}(\Delta_1^0 : \mathcal{S})$ by sending $u \in \tilde{N}$ to the linear combination $\sum c_e \bar{w}_e$ of basis elements of $R^{(r)}(\Delta_1^0 : \mathcal{S})$ such that $c_e \bar{w}_e$ assigns $u(e) - u(e')$ to e if it has a predecessor e' in an edge-series, and $c_e \bar{w}_e$ assigns $u(e)$ to e otherwise. These projections define a homomorphism θ from $\tilde{N}$ to the direct sum in (5.6). To see that θ is injective, consider an edge-series $(e_1, \ldots, e_k)$ which can not be extended at the beginning. If $u \in \tilde{N}$ is in the kernel of θ, then $u(e_1) = 0$ because only one projection π_v (if e_1 is incident to a vertex $v \in \mathcal{S}$ and is oriented away from it) or $\pi_{\bar{w}}$ (otherwise) uses the assignment $u(e_1)$. By induction, if $u(e_1) = \ldots = u(e_{j-1}) = 0$, then only one summand can possibly assign a non-zero polynomial to e_j, hence all of them should assign zero to e_j. We conclude that $u = 0$, thus θ is injective. To see that θ is surjective it is enough to show that a set of generators of the direct sum in (5.6) are in the image of θ. If $\tilde{u} \in D^r(\hat{\Delta}_v^\star)$ for some $v \in \mathcal{S}$, we define $u \in \tilde{N}$ by letting $u(e) = \tilde{u}(\ell)$ if e is in an edge series starting at v and ℓ is the half-line of $\Delta^\star$ on which e lies, and letting $u(e) = 0$ otherwise. Then $\pi_v(u) = \tilde{u}$ and u is projected to zero on the other summands. Also note that generators $\bar{w}_e$ of $R^{(r)}(\Delta_1^0 : \mathcal{S})$ are mapped to zero by projections π_v for all $v \in \mathcal{S}$. Thus θ is surjective as well.

It follows that isomorphism (5.6) holds, hence $\tilde{N}$ is a free R-module. Its rank equals to $n - \#\mathcal{S}$ by the same computations as in theorem 5.2.2. □

In applications we can take $\mathcal{S}$ to be the set of all regular vertices with respect to a chosen orientation of Δ. The matrix of the new representation has size $(n - m - \#\mathcal{S} + 1) \times (n - \#\mathcal{S})$, like of Φ in theorem 5.2.2. But the cardinality of $\mathcal{S}$ can be much larger for some Δ if we choose a very good orientation. To compare these two theorems, note that if we have a set $\mathcal{S}$ with the restriction of theorem 5.2.2, then one can choose an orientation of Δ so that vertices in $\mathcal{S}$ become regular. This shows that theorem 5.3.1 is more general.

Before discussing the best choice of the orientation of Δ we describe the matrix $\tilde{M}$ of $\tilde{\Psi}$ and consider a few examples. Similarly as for the matrix of Ψ in theorem 5.2.2, columns of $\tilde{M}$ can be put into a correspondence with generators of modules $D^r(\hat{\Delta}^\star_v)$ and with interior edges emanating not from a vertex in $\mathcal{S}$, and rows — with interior vertices not in $\mathcal{S}$ and holes of Δ. (Note that computing splines on star complexes does not differ at all from computing splines on star triangulations). Let u be a generator of $D^r(\hat{\Delta}^\star_v)$ for some $v \in \mathcal{S}$. Then the entry of $\tilde{M}$ corresponding to u and a vertex $v' \notin \mathcal{S}$ is $-u(\ell)$ if there is an edge series starting at v (and laying on ℓ) such that it ends up at v' and cannot be made longer, and this entry is zero otherwise. Similarly, the entry corresponding to u and a hole is $\sum -u(\ell)$ over all edge series starting at v (and laying on ℓ) such that the last edge in it is oriented towards the hole and it can not be made longer. Further, let e be an edge emanating from $v' \notin \mathcal{S}$. Then all entries on the column corresponding to e are zero except possibly one entry l_e^{r+1} (if v' is an interior vertex or it lies on the boundary of a hole) and one entry $-l_e^{r+1}$ (if the longest edge series starting with e does not end up on the exterior boundary of Δ).

Example 5.4 Consider for Δ the triangulation on figure 5.2 with the same orientation which was described in the previous example. There are several edge-sequences in this triangulation with more than one edge, that is they lay on lines PQ, AT, FP and AD. Note that all interior vertices are regular with respect to the chosen orientation except B. We can include into $\mathcal{S}$ the remaining vertices, so that $\mathcal{S} = \{A, C, D, E, F\}$. Then the matrix $\tilde{M}$ of $\tilde{\Psi}$ should have size 1×15. For $v \in \mathcal{S}$ let u^v be a basis of $D^r(\hat{\Delta}^\star_v)$. Note that sets u^A, u^D, u^F have three elements, and sets u^C and u^E have two elements. The matrix $\tilde{M}$ is

$$\begin{pmatrix} -u^A(AB) & -u^C(BC) & -u^D(BD) & 0^E & -u^F(BF) & 0 & l_{BS}^{r+1} \end{pmatrix}$$

where the first five "columns" correspond to generators of modules $D^r(\hat{\Delta}^\star_v)$ for $v \in \mathcal{S}$, and 0^E denotes two zero columns, and the last zero column corresponds to the edge CP. The zero columns mean that $D^r(\hat{\Delta}^\star_E)$ and the submodule generated by $\bar{w}_{CP}$ are direct summands of $D^r(\hat{\Delta})$.

It is true for a general triangulation Δ that if an interior vertex v is regular, and all longest edge-series starting at v end-up on the exterior boundary, then $D^r(\hat{\Delta}^\star_v)$ is a direct summand of $D^r(\hat{\Delta})$. For a proof one needs only to note that columns of the matrix of $\tilde{\Psi}$ corresponding to generators of this $D^r(\hat{\Delta}^\star_v)$ contain only zero entries. Further, recall that a *cross-cut* is an edge-sequence whose edges have two end-vertices on the exterior

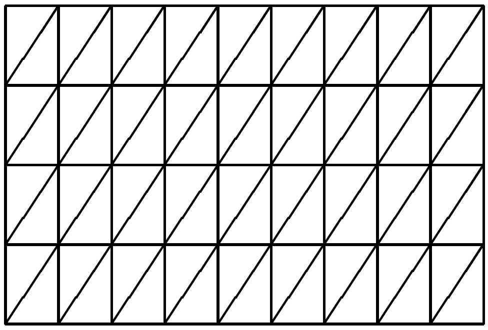

Figure 5.3: A cross-cut grid partition

boundary in total. Suppose that Δ has a cross-cut. Then elements of $D^r(\hat{\Delta})$ supported only on it form a direct summand, because if the cross-cut is orientated uniformly the matrix of $\tilde{\Psi}$ has a zero column corresponding to the first edge on it.

As a special case, assume that $g(\Delta) = 0$ and all interior edges are pseudoboundary. It was proved by H.Schenck and M.Stillman in [48] that the module $C^r(\hat{\Delta})$ is free. (Complementary, they also proved that if there is at least one interior edge which is not pseudoboundary, then $C^r(\hat{\Delta})$ is not free for large enough r, and gave effective estimates for such r.) Here we are able to give a basis of $D^r(\hat{\Delta})$ almost explicitly. We need to orientate all edge-sequences of pseudoboundary edges towards the boundary. Additionally, we may orientate all edges in each cross-cut sequence to the same direction. Then for a basis of $D^r(\hat{\Delta})$ one can take generators of $D^r(\hat{\Delta}_v^\star)$ for all interior vertices, and append generators coming from cross-cuts. In particular, in this way we can easily compute a basis of $D^r(\hat{\Delta})$ for so-called *cross-cut grid partitions* like in figure 5.3, quite extensively studied in the literature ([12, 16]). Of course, one has to do some more work to compute a more application-friendly basis of $C^r(\hat{\Delta})$, whose elements are as much locally supported as possible.

It remains to discuss how to choose an orientation of Δ so that we would get quite a lot of regular vertices. This step is surely not straightforward. Choosing the optimal orientation with the largest possible set $\mathcal{S}$ might be a very hard computational problem. An obvious recommendation is to orientate all pseudoboundary edges uniformly towards the boundary (if possible, towards the exterior boundary), because that does not give any restrictions on regularity of interior vertices. In practice this should be done immediately. It may seem that one should orientate all edge-sequences uniformly. However, it is not true that the optimal orientation satisfies this condition. For example, consider the triangulation in figure 5.2. If we orientate edges AB and BD in the same way, then at least two vertices would not be regular, namely A or D, and B or C. But the orientation we used gives only one non-regular vertex.

Nevertheless, it is easy to see that it is always worth to orientate an edge-sequence towards one vertex on it. If possible, this vertex should be on the boundary of Δ.

Otherwise one vertex on the edge-sequence must be non-regular. In general, whenever it is clear that a vertex v will not be regular in the orientation we are choosing, it is surely advisable to orientate edges in all edge-sequence passing through v towards it.

In the following we describe a heuristic algorithm which chooses a good orientation. This algorithm keeps track of how candidate vertices are connected with each other by means of a graph. In each iteration we implicitly append a vertex to $\mathcal{S}$, and check whether its neighbouring vertices are already non-regular with respect to current orientations. More specifically, interior vertices which are connected with just one other candidate are immediately declared regular, because such an action alone does not make the largest possible set $\mathcal{S}$ smaller. Vertices which become isolated from other candidates are already regular by that time. In general situation we apply the heuristic idea that one should get more regular vertices if we implicitly choose more of them close to the exterior boundary of Δ. We avoid considering too many different situations by using the following simple lemma.

Lemma 5.3.2 *Let Γ' be an outerplanar graph, i.e. a planar graph whose all vertices lie on its boundary. Then there is a vertex of Γ' connected with at most two neighbouring vertices.*

Proof. Let N be the number of vertices in Γ'. If $N \leq 3$ the statement is trivial. If $N > 3$ and a vertex v has more than two neighbours, then one of edges incident to v cuts the graph into two smaller outerplanar graphs. The statement follows by induction on N. □

Here is a sketch of the heuristic algorithm. It orientates Δ so that the set of regular vertices is expectedly large.

(i) Orientate uniformly all pseudoboundary edges of Δ towards the boundary. Let Γ be the planar graph whose vertices are interior vertices of Δ, and edges of Γ are those interior edges of Δ which are not pseudoboundary.

(ii) If Γ has no edges, exit.

(iii) If there is a vertex in Γ of degree one, let v be such a vertex. Otherwise let v be a vertex on the boundary of Γ which is connected with no more than two other vertices on the boundary of Γ. (Such a vertex exists, according to lemma 5.3.2 applied to the graph defined by boundary vertices of Γ and edges between them.)

(iv) For each edge e of Γ incident to v do the following. Orientate e (as an edge of Δ) away from v. Let $v' \in \Gamma$ be the other end-vertex of e. If all other edges of Γ incident to v' have different slope from e, then in all edge-sequences which contain an edge in Γ incident to v', orientate all edges towards v'. If necessary, override previous orientations.

(v) Remove all edges oriented in the previous step from Γ, and go to step 2.

To see that the algorithm is correct, note that whenever we are at step 2, edges of Γ are precisely those edges of Δ which are not oriented yet. Hence we exit when we have oriented all interior edges of Δ. At step 3 we implicitly declare the chosen vertex v regular. At step 4 we find some neighbouring vertices of v which become non-regular. (Here the algorithms may fail to recognize that a vertex is already non-regular if that vertex lies on an edge-sequence in between two vertices declared regular by then. For simplicity we ignore this special situation.) If we recognize a non-regular vertex v', then we orientate all edge-sequences which pass through it and are not oriented at it, towards v'. Re-orientation of some edges is useful because it allows us to orientate the whole edge-sequence at once whenever we noticed the first possibility, and it does not change regularity of other vertices.

The complexity of the heuristic algorithm is polynomial in the number of edges and vertices of Δ. To adopt this algorithm for applications of the theorem 5.2.2, in step 1 one should orientate only edges which have a common vertex with the boundary of Δ, and step 4 should be replaced by the instruction "For each vertex v' neighbouring v in Γ, orientate all edges incident to it in Γ towards v'".

In step 1 of the heuristic algorithm we may additionally require that an edge-sequence which connects the exterior boundary with a hole should be oriented towards the exterior boundary. That does not influence regularity of vertices, but may increase the number of zero entries in the matrix of $\tilde{\Psi}$, namely on its rows corresponding to holes. Also note the following modification of step 1, which gives a decomposition of $D^r(\hat{\Delta})$ into direct summands. Let Γ be the graph whose vertices are all vertices of Γ which do not lie on the exterior boundary, and edges of Γ are all edges of Δ except those which lie on the exterior boundary or are connected to it by an edge-sequence. Then every connected component of Γ gives us a direct summand of $D^r(\hat{\Delta})$. For a proof consider the homomorphism $\tilde{\Phi}$ in theorem 5.3.1 with Δ oriented so that interior edges not in Γ are oriented uniformly towards the exterior boundary; then the matrix of $\tilde{\Psi}$ has block structure which mimics the decomposition of Γ into connected components. In particular, an isolated vertex v of Γ represents a direct summand $D^r(\hat{\Delta}^\star_v)$, which itself is a free module. In this decomposition of $D^r(\hat{\Delta})$ one should not forget direct summands coming from cross-cuts.

5.4 Computation results

We have mentioned in the introduction that the vector space $C^r_k(\Delta)$ is isomorphic to the kth graded part of the graded module $C^r(\hat{\Delta})$. Hence the function $H_{\Delta,r}(k) := \dim_{\mathbb{R}} C^r_k(\Delta)$ is the Hilbert function of $C^r(\hat{\Delta})$, and the power series

$$h_{\Delta,r}(t) = \sum_{k=0}^{\infty} \dim_{\mathbb{R}} C^r_k(\Delta)\, t^k. \tag{5.7}$$

are the *Hilbert series* of the graded R-module $C^r(\hat{\Delta})$. From basic commutative algebra [19] we know that $H_{\Delta,r}(k)$ is a polynomial function for large enough k. It is also known that the Hilbert series can be written as the power series of a rational function, namely (as power series)

$$h_{\Delta,r}(t) = \frac{p_{\Delta,r}(t)}{(1-t)^3}, \tag{5.8}$$

where $p_{\Delta_r}(t)$ is a polynomial. The polynomial $p_{\Delta,r}(t)$ is determined by the the number of generators and syzygies of the the graded module $C^r(\hat{\Delta})$ in each degree. The function $H_{\Delta,r}(k)$ can be easily computed from the polynomial $h_{\Delta,r}(t)$. Using their representation of $C^r(\hat{\Delta})$ L.J.Billera and L.L.Rose have proved in [6] the following properties of the polynomial $p_{\Delta,r}(t)$ in (5.8):

$$p_{\Delta,r}(1) = f_2(\Delta), \qquad \frac{d\,p_{\Delta,r}}{d\,t}(1) = (r+1)\, f_1^0(\Delta). \tag{5.9}$$

In [5] a few more properties of $p_{\Delta,r}(t)$ for continuous or C^1 splines on a generically embedded Δ are expressed in its combinatorial characteristics. However, the explicit conditions for genericity of simplicial embeddings are not understood (or even known) well.

The purpose of performed computations was to look for more patterns and properties of the polynomial $p_{\Delta,r}(t)$ which determines the dimension series of the spline spaces $C^r_k(\Delta)$. The considered triangulation was the Morgan-Scott triangulation Δ in figure 5.1, where the edges and vertices are determined by the following linear forms:

$$\begin{array}{lll} l_1 = y - 2x\,, & l_2 = x - 3y + 5z\,, & l_3 = x + 2y\,, \\ h_1 = y\,, & h_2 = 2x + 5y - 12z\,, & h_3 = 4x + 3y - 10z\,, \\ h_4 = x + 2z\,, & h_5 = 3x - 2y + 8z\,, & h_6 = x - 2y\,. \end{array}$$

These linear forms are expected to be general for the Morgan-Scott triangulation. We computed the Hilbert series of the graded R-modules $C^r(\hat{\Delta})$ for different r. Also the similar spaces on the complex $\Delta^\star$, defined by the same configuration of edges, but with a hole in the place of the middle triangle (i.e., the one with edges l_1, l_2 and l_3), were considered.

The numerators $p_{\Delta,r}$ in (5.8) of the Hilbert series of $C^r(\hat{\Delta})$ we computed up to degree $r \leq 50$. Some of the results are shown in the table 5.2. The computations were performed using Macaulay computer package, actually over the finite field $\mathbf{F}_{31991}$ because of the mentioned limitations of Macaulay. The computations were performed using Macaulay computer package, actually over the finite field $\mathbf{F}_{31991}$ because of the mentioned features of Macaulay.

For the triangulation $\Delta^\star$, obtained from the Morgan-Scott triangulation (figure 5.1) by considering the triangle in the middle as a hole in the triangulation, the module $C^r(\hat{\Delta}^\star)$ is never free, because $g(\Delta^\star) > 0$ (see [48]). As a simplicial complex $\Delta^\star$ has six triangles, six interior edges and no interior vertices. The module $D^r(\hat{\Delta}^\star)$ is isomorphic to the kernel of the matrix

$$\left(\begin{array}{cccccc} h_1^{r+1} & h_2^{r+1} & h_3^{r+1} & h_4^{r+1} & h_5^{r+1} & h_6^{r+1} \end{array}\right).$$

The Hilbert series of $C^r(\hat{\Delta}^\star)$ were also computed up to degree $r \leq 50$. Numerators of the first few of them are

$$\begin{array}{ll} p_{\Delta^\star,0} = 1 + 3\,t + 3\,t^2 - t^3, & p_{\Delta^\star,5} = 1 + 5\,t^9 + 9\,t^{10} - 9\,t^{11}, \\ p_{\Delta^\star,1} = 1 + 8\,t^3 - 3\,t^4, & p_{\Delta^\star,6} = 1 + 12\,t^{11} - t^{12} - 6\,t^{13}, \\ p_{\Delta^\star,2} = 1 + 3\,t^4 + 6\,t^5 - 4\,t^6, & p_{\Delta^\star,7} = 1 + 21\,t^{13} - 15\,t^{14} - t^{15}, \\ p_{\Delta^\star,3} = 1 + 8\,t^6 - 3\,t^8, & p_{\Delta^\star,8} = 1 + 6\,t^{14} + 14\,t^{15} - 15\,t^{16}, \\ p_{\Delta^\star,4} = 1 + 15\,t^8 - 10\,t^9, & p_{\Delta^\star,9} = 1 + 15\,t^{16} - 10\,t^{18}. \end{array}$$

The author noticed that all Hilbert series of $C^r(\hat{\Delta}^\star)$ for computed r have a particular form, namely their numerators as in (5.8) can be expressed as a difference of two

r	$p_{\Delta,r}(t)$	r	$p_{\Delta,r}(t)$
0	$1+3\,t+3\,t^2$	15	$1+27\,t^{28}-18\,t^{29}-3\,t^{30}$
1	$1+6\,t^3$	16	$1+3\,t^{29}+27\,t^{30}-24\,t^{31}$
2	$1+9\,t^5-3\,t^6$	17	$1+9\,t^{31}+18\,t^{32}-21\,t^{33}$
3	$1+3\,t^6+6\,t^7-3\,t^8$	18	$1+18\,t^{33}+3\,t^{34}-15\,t^{35}$
4	$1+6\,t^8+3\,t^9-3\,t^{10}$	19	$1+27\,t^{35}-12\,t^{36}-9\,t^{37}$
5	$1+9\,t^{10}-3\,t^{12}$	20	$1+36\,t^{37}-27\,t^{38}-3\,t^{39}$
6	$1+15\,t^{12}-9\,t^{13}$	21	$1+6\,t^{38}+30\,t^{39}-30\,t^{40}$
7	$1+3\,t^{13}+12\,t^{14}-9\,t^{15}$	22	$1+15\,t^{40}+15\,t^{41}-24\,t^{42}$
8	$1+6\,t^{15}+9\,t^{16}-9\,t^{17}$	23	$1+24\,t^{42}-18\,t^{44}$
9	$1+12\,t^{17}-6\,t^{19}$	24	$1+36\,t^{44}-21\,t^{45}-9\,t^{46}$
10	$1+18\,t^{19}-9\,t^{20}-3\,t^{21}$	25	$1+48\,t^{46}-42\,t^{47}$
11	$1+24\,t^{21}-18\,t^{22}$	26	$1+9\,t^{47}+33\,t^{48}-36\,t^{49}$
12	$1+6\,t^{22}+15\,t^{23}-15\,t^{24}$	27	$1+21\,t^{49}+12\,t^{50}-27\,t^{51}$
13	$1+12\,t^{24}+6\,t^{25}-12\,t^{26}$	28	$1+33\,t^{51}-9\,t^{52}-18\,t^{53}$
14	$1+18\,t^{26}-3\,t^{27}-9\,t^{28}$	29	$1+45\,t^{53}-30\,t^{54}-9\,t^{55}$

Table 5.2: Numerators of Hilbert series h_r

polynomials:

$$h_r^*(t) \;=\; \frac{P_r(t)-Q_r(t)}{(1-t)^3},$$

where

$$P_r(t) \;=\; \begin{cases} 1+6\,t^{5k-1}+42\,t^{5k}+6\,t^{5k+1}, & \text{if} \quad r \;=\; 3k-1, \\ 1+18\,t^{5k\pm1}+36\,t^{5k\pm2}, & \text{if} \quad r+1 \;=\; 3k\pm1 \end{cases}$$

and

$$Q_r(t) \;=\; \begin{cases} 6\,t^{12k-1}+37\,t^{12k}+6\,t^{12k+1}, & \text{if} \quad r \;=\; 7k-1, \\ 15\,t^{12k\pm1}+33\,t^{12k\pm2}+t^{12k\pm3}, & \text{if} \quad r+1 \;=\; 7k\pm1, \\ 28\,t^{12k\pm3}+21\,t^{12k\pm4}, & \text{if} \quad r+1 \;=\; 7k\pm2, \\ 3\,t^{12k\pm4}+36\,t^{12k\pm5}+10\,t^{12k\pm6}, & \text{if} \quad r+1 \;=\; 7k\pm3. \end{cases}$$

We conjecture that the Hilbert series of $C^r(\hat{\Delta}^\star)$ always have this form. At least they would satisfy the necessary conditions (5.9), since polynomials $P(t)$ and $Q(t)$ satisfy

$$P(1) \;=\; 55, \quad P'(1) \;=\; 90\,(r+1), \qquad Q(1) \;=\; 49, \quad Q'(1) \;=\; 84\,(r+1).$$