Aspects of algorithmic algebra

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Chapter 4

Pull-backs of differential equations

In this chapter we consider differential equations whose differential Galois group is $\mathbb{D}_\infty$ (i.e. the *infinite dihedral group*) or finite, and their pull-backs with respect to finite morphisms. The aim is to obtain differential equations with less complicated differential Galois groups ($\mathbb{G}_m$ or a finite one of smaller order, respectively).

We have already defined the pull-back transformations in chapter 1. In this section we give the general formula for the normalized pull-backs of differential equations $y'' = r\,y$ with $r \in C(t)$, see lemma 4.2.1. In the special case of pull-backs with respect to the cyclic morphisms $x \mapsto t^k$ the corresponding formula (3.3) is given in the previous chapter 3. Such pull-backs with $k = 2$ are the pull-backs which transform second order differential equations with $r \in C[x, x^{-1}]$ and differential Galois group $\mathbb{D}_\infty$ to differential equations with Galois group $\mathbb{G}_m$, see the next chapter.

For differential equations $y'' = r\,y$ with a basis of algebraic solutions (i.e., finite differential Galois group) we occasionally consider more general pull-back to the so-called *Darboux curves*. These are the curves over which the associated Riccati equation has rational solutions. These curves need not to be rational curves isomorphic to $\mathbb{P}^1$. But the considered examples are not complicated. Using pull-backs to the Darboux curves one has alternative way to express the algebraic solutions of differential equations, in comparison with straightforward writing down algebraic equations for these solutions (or the algebraic solutions of the associated Riccati equation). Indeed, if one has written a general solution of the pull-backed differential equation, one essentially has a general solution of the original equation as well. The form of the solutions on Darboux curves is very compact and quite informative.

In a way, the chapter consists of a couple of large examples where we compute the pull-back morphisms to the Darboux curves. In the first section we consider order two linear differential equations with the differential Galois group $\mathbb{D}_\infty$. We will see that a pull-back morphism of degree 2 gives us a differential equation with Galois group $\mathbb{G}_m$. Like in the previous chapter 3, we characterize families of order two differential equations with *two* singular points and Galois group $\mathbb{D}_\infty$. We solve this problem by describing the families of differential equations with two singular points and Galois groups $\mathbb{G}_m$, which are the pull-backs of the differential equations with the Galois group $\mathbb{D}_\infty$.

In the second section we consider second order differential equations with three poles and finite differential Galois group. These are classical differential equations, considered by F. Klein, R. Fuchs, etc. Nowadays we can use the computer algebra tools to prolong their work. We will concentrate on the tetrahedral $A_4^{SL_4}$, octahedral $S_4^{SL_2}$ and icosahedral $A_5^{SL_2}$ differential Galois groups, because the case of the finite dihedral group is not difficult. The pull-backs of the differential equations with these Galois groups has finite cyclic Galois group. We recall the Schwarz classification ([51, 3]) of these differential equations, and their basic transformations. We will show that the differential equations of the same Schwarz type have the same Darboux curve and the same pull-back morphism. Hence there are finitely many different pull-back morphisms needed for the considered differential equations. We give all these pull-back morphisms. The obtained Darboux curves are rational for tetrahedral and octahedral groups. For the icosahedral group we obtain two rational curves and four curves of genus 1.

We note that since the same pull-back morphism works for, let us say, the differential equation $E(1/2, 1/3, 1/3)$ (in the notation below) and $E(79/2, 53/3, 62/3)$, it is possible to give the explicit solutions of all differential equations of the considered type, as long as one can write a solution of their pull-backs. This looks more practical than writing down algebraic equations for the solutions of the differential equations with large differences of local exponents looks very unpractical. It would be very interesting to compare this way of solving differential equations with the methods in [54, 55] which use invariant theory of the corresponding differential Galois groups. See also [59, 64, 58] for the state of the art of finding algebraic solutions of differential equations.

In the last section a differential equation of order three is considered, namely the Hurwitz equation, whose differential Galois group has 168 elements.

4.1 Differential equations with the Galois group $\mathbb{D}_\infty$

Like in the previous chapter, in this section we consider second order differential equations of the form

$$y'' = ry, \quad \text{with } r \in C[x, x^{-1}] \tag{4.1}$$

(where C is an algebraically closed field of characteristic 0). The differential Galois group of equation (4.1) is assumed to be $\mathbb{D}_\infty$. Then the associated Riccati equation

$$u' + u^2 = r, \quad \text{with the same } r \in C[x, x^{-1}], \tag{4.2}$$

has exactly two algebraic (of degree 2) solutions. We may assume that infinity is an irregular singular point, because for equations (4.1) with $r = c/x^2$ the Galois group $\mathbb{D}_\infty$ is not possible. Then applying proposition 6.1 and theorem 8.2 in [57] we conclude:

(a) The local differential Galois group at infinity is $\mathbb{D}_\infty$, and $\text{ord}_\infty(r)$ is odd and ≤ 1.

(b) Zero $x = 0$ is a regular singular point. The local Galois group there is $\mathbb{Z}/4\mathbb{Z}$.

(c) The associated Riccati equation (4.2) has two global solutions defined over the field $C(\sqrt{x})$. These two solutions are conjugate over $C(x)$.

The property (c) suggests that a suitable pull-back of (4.1) with respect to the finite morphism $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ given by $x \mapsto t^2$ should have the differential Galois group $\mathbb{G}_{\rm m}$, because in the extension $C(\sqrt{x}) \supset C(x)$ the two solutions of the Riccati equation (3.2) are rational. This is indeed the case for the normalized pull-back, as we show in the proof of theorem 4.1.1. It also turns out that differential equations (4.1) with the Galois group $\mathbb{D}_\infty$ are determined by their pull-backs with respect to Φ.

Using these facts one can characterize the families of differential equations (4.1) with the differential Galois group $\mathbb{D}_\infty$ by describing the families of their normalized pull-backs. These pull-back equations are defined over the differential field $C(t)$ with the usual derivation, have the same normalized form $Y'' = \tilde{r}\,Y$ with $\tilde{r} \in C(t)$ (due to formula (3.3)) and the differential Galois group $\mathbb{G}_{\rm m}$ (theorem 4.1.1 below). In other words, these pull-backs are subfamilies of the families of differential equations considered in section 3.3 of the previous chapter. Hence the families of equations (4.1) with the Galois group $\mathbb{D}_\infty$ are known if the universal families of their pull-backs inside the algebraic varieties $\mathbf{S}_{n,d_1,d_2}$ of the previous chapter are known. The main problem of this section is to determine when a point on $\mathbf{S}_{n,d_1,d_2}$ represents a pull-back of equation (4.1) with respect to Φ.

Note the algebraic group $\mathbf{G}_{\rm m}$, introduced in chapter 4, acts also on the set of equations (4.1) with the Galois group $\mathbb{D}_\infty$, because the transformations $x \rightharpoonup \beta x$ (with $\beta \in C^*$) does not change singular points and the Galois group of (4.1). In the same way these transformations act on the pull-backs of these differential equations, so the families of these pull-back form $\mathbf{G}_{\rm m}$-orbits in $\mathbf{S}_{n,d_1,d_2}$. Since the algebraic varieties $\mathbf{S}_{n,d_1,d_2}$ consist of a finite number of $\mathbf{G}_{\rm m}$ orbits themselves, it easily follows that a universal family of equations (4.1) with the Galois group $\mathbb{D}_\infty$ (or of their pull-backs with respect to Φ) is a discrete set of $\mathbb{G}_{\rm m}$-orbits. In particular, they are smooth and one-dimensional.

With this set-up this section naturally extends the previous chapter 3. Therefore we use much notation from this chapter, like notation (3.18) for the coordinate functions of the algebraic varieties $\mathbf{S}_{n,d_1,d_2}$. However, the differential equations with the differential Galois group $\mathbb{G}_{\rm m}$ are defined here over the field $C(t)$ (with the usual derivation). Note that we have to distinguish the differential field extension $C(\sqrt{x}) \supset C(x)$ and the field extension $C(t) \supset C(x)$ corresponding to Φ, which are isomorphic as usual fields but have different derivations. For convenience, we use the functional notation with elements of $C(t)$, such as $F_1(t)$ and $F_1(-t)$. In particular, for $f \in C(t)$ the expression $f(\sqrt{x})$ denotes the element of $C(\sqrt{x})$ obtained by replacing t by $\sqrt{x}$.

Our main results are formulated in the following two theorems.

Theorem 4.1.1 *Suppose that the differential equation (4.1) has Galois group $\mathbb{D}_\infty$. Let $\phi : C(x) \to C(t)$ be the field homomorphism with $\phi(x) = t^2$ which defines the covering $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$. Then:*

(i) $\operatorname{ord}_\infty(r) = -2n+1$ *for an integer* $n \geq 0$.

(ii) The normalized pull-back of (4.1) with respect to Φ is the differential equation

$$Y'' = \left(4\,t^2\,\phi(r) + \frac{3}{4\,t^2}\right) Y \qquad (4.3)$$

with the differential Galois group $\mathbb{G}_{\rm m}$.

(iii) *The Riccati equation associated to* (4.3) *has two rational solutions of form*

$$\tilde{u}_1 = v + \frac{s}{t} + \frac{F_1'}{F_1}, \qquad \tilde{u}_2 = -v + \frac{s}{t} + \frac{F_2'}{F_2}, \tag{4.4}$$

where

- F_1 *and* F_2 *are polynomials in* $C[t]$ *of the same degree* d, *without common factors, they satisfy* $F_i(0) = 1$ *and* $F_2(t) = F_1(-t)$.
- v *is a polynomial in* t^2 *of degree* n, *so that* $\deg v = 2n$ *in* $C[t]$;
- $s = -n - d$.

(iv) *The action* $t \rightharpoonup -t$ *of* $\mathbf{G}_{\mathrm{m}}$ *fixes the equation* (4.3), *and permutes the two solutions* (4.4) *of the Riccati equation.*

Proof. The first statement follows from the result (a) cited at the beginning of this section. According to formula (3.3) the equation (4.3) is indeed the normalized pull-back of (4.1) with respect to Φ. Let ϕ also denote the extended field homomorphism $C(\sqrt{x}) \to C(t)$ which sends $\sqrt{x} \mapsto t$. If $u_1, u_2 \in C(\sqrt{x})$ are the two solutions of the Riccati equation (3.2), then the Riccati equation $\tilde{u}' + \tilde{u}^2 = 4\,t^2\,\phi(r) + 3/(4t^2)$ associated to (4.3) has two solutions

$$\tilde{u}_i = 2\,t\,\phi(u_i) - \frac{1}{2\,t}, \qquad \text{for} \quad i = 1, 2. \tag{4.5}$$

They are both rational, because $\phi(u_i) \in C(t)$ for $i = 1, 2$. Since (4.3) has an irregular singular point (at infinity), the differential equation has Galois group $\mathbb{G}_{\mathrm{m}}$.

Further, the functions $\tilde{u}_i$ have the form 3.4) with all the restrictions of theorem 3.2.1. Since $\phi(u_1)$ and $\phi(u_2)$ are permuted under the conjugation $t \mapsto -t$ over $C(x)$, from (4.5) we conclude that $\tilde{u}_2(t) = -\tilde{u}_1(-t)$. This means that the element $t \rightharpoonup -t$ of $\mathbb{G}_{\mathrm{m}}$ exchanges the solutions (4.4). From here the additional restrictions $v \in C[t^2]$ and $F_2(t) = F_1(-t)$ follow. Since $t \rightharpoonup -t$ exchanges the Riccati solutions (4.4), it fixes equation (4.3). □

This theorem justifies the above discussion about families of pull-backs of equations (4.1) with the differential Galois group $\mathbb{G}_{\mathrm{m}}$. Additionally, from (4.3) it is clear that different differential equations $y'' = r\,y$ have different pull-backs with respect to $x \mapsto t^2$. Now the key question is which elements of $\mathbf{S}_{2n,d,d}$ correspond to pull-backs of differential equations with the Galois group $\mathbb{D}_\infty$.

Theorem 4.1.2 *Let* $Q = (v, F_1, F_2, \tilde{r})$ *be an element of* $\mathcal{S}_{2n,d,d}(C)$, *representing a differential equation* $Y'' = \tilde{r}\,Y$ *over* $C(t)$. *The following statements are equivalent:*

- Q *represents the pull-back of a differential equation* (4.1) *with the Galois group* $\mathbb{D}_\infty$;
- *the transformation* $t \rightharpoonup -t$ *in* $\mathbf{G}_{\mathrm{m}}$ *maps* Q *to* $(-v, F_2, F_1, \tilde{r})$;
- v *is a polynomial in* t^2.

If these equivalent statements are satisfied, then the normalized pull-back of the differential equation

$$y'' = \left(\frac{1}{4\,x}\,\tilde{r}(\sqrt{x}) - \frac{3}{16\,x^2} \right) y. \tag{4.6}$$

with respect to $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ given by $x \mapsto t^2$ is $Y'' = \tilde{r}\,Y$. If $\tilde{u}$ is a solution of the Riccati equation $\tilde{u}' + \tilde{u}^2 = \tilde{r}$ in $C(t)$, then $\tilde{u}(\sqrt{x})/(2\sqrt{x}) + 1/(4x)$ is a solution of the Riccati equation associated to (4.6) in $C(\sqrt{x})$. Solutions of the differential equations are related as $y = x^{1/4}\,Y(\sqrt{x})$.

Proof. From the previous theorem it is clear that the first statement implies the other two. Also, the second statement implies the last one. It is enough to prove that the last statement implies the first one. If $v \in C[t^2]$ then the differential equation (4.6) is defined over $C(x)$. Using (4.3) one can compute that the pull-back of it with respect to Φ is $Y'' = \tilde{r}\,Y$. The transformations between solutions of the differential and the Riccati equations can be checked directly. Explicitly, one obtains two solutions of the Riccati equation in $C(\sqrt{x})$ of the following form:

$$u_1 = \frac{v(\sqrt{x})}{2\sqrt{x}} + \frac{s+\frac{1}{4}}{x} + \frac{1}{2\sqrt{x}}\,\frac{F_1'}{F_1}\left(\sqrt{x}\right), \quad u_2 = -\frac{v(\sqrt{x})}{2\sqrt{x}} + \frac{s+\frac{1}{4}}{x} - \frac{1}{2\sqrt{x}}\,\frac{F_1'}{F_1}\left(-\sqrt{x}\right).$$

They are both algebraic of degree 2, because $v(\sqrt{x}) \in C(x)$. There can be no more algebraic solutions of this Riccati equation, because the differential Galois group over $C(x)$ is not finite. Therefore the Galois group of (4.6) is $\mathbb{D}_\infty$. □

We call equation (4.6) the *half pull-back* of the equation $Y'' = r\,Y$ with $r \in C(t^2)$. This equation can by formally obtained by substituting $k = 1/2$ into equation 3.3.

Applying the theorem 4.1.2 we consider families of differential equations (4.1) with Galois group $\mathbb{D}_\infty$ as closed subschemes of $\mathbf{S}_{2n,d,d}$. More generally, one can consider a functor $\mathcal{S}^\infty_{n,d}$ from $\mathcal{R}_C$ to the category of sets, which associate to $R \in \mathcal{R}_C$ the subset of $\mathcal{S}_{2n,d,d}(R)$ of those elements whose first component is a polynomial in t^2. Equivalently, these are precisely those elements of $\mathcal{S}_{2n,d,d}(R)$ on which $t \rightharpoonup -t$ acts in the same way as the involution. The representing scheme $\mathbf{S}^\infty_{n,d}$ of this functor is a closed subschemes of $\mathbf{S}_{2n,d,d}$. Moreover, $\mathbf{S}^\infty_{n,d}$ consists of complete $\mathbb{G}_m$ orbits. Hence it is smooth, reduced, of dimension 1.

To find explicit equations for $\mathbf{S}^\infty_{n,d}$ one can use the same equations (3.8), (3.12) or (3.16-3.17), keeping in mind that v and r are rational functions in t^2, and $F_2(t) = F_1(-t)$. In this way we have considerably less variables, so the obtained equations are very dependent. For example, one of equations (3.16-3.17) is obsolete.

In particular, consider the system of equations (3.23-3.24) derived from (3.12). We keep the same notation for the power series $\Theta = \exp(-2\int v)$ and its coefficients. We have $p_j = (-1)^j q_j$ for free, so we can rewrite these equations as

$$\theta_j + \theta_{j-1}q_1 + \ldots + \theta_0 q_j = (-1)^j q_j, \quad \text{for} \quad j = 1, \ldots, d, \tag{4.7}$$

$$\theta_j + \theta_{j-1}q_1 + \ldots + \theta_{j-d}q_d = 0, \quad \text{for} \quad j = d+1, \ldots, 2n+2d. \tag{4.8}$$

These equations with odd j tells us that the power series $(1+\Theta)F_2 \in C((t))$ have only terms with even degree of t modulo $t^{2n+2d+1}$, and the equations with even j tell us that $(1-\Theta)F_2$ have only terms with odd degree modulo $t^{2n+2d+1}$. Observe that the power series $(1-\Theta)/(1+\Theta)$ have only terms of odd degree, because changing the sign of t transforms $\int v \mapsto -\int v$ and $\Theta \mapsto \Theta^{-1}$. One can easily check that the above equations with even j are consequences of the equations with odd j, by multiplying power series $(1+\Theta)F_2$ by $(1-\Theta)/(1+\Theta)$. It follows that $\mathbf{S}^{\infty}_{n,d}$ is defined by equations

A natural question is how many $\mathbf{G}_{\rm m}$ orbits are contained in $\mathbf{S}^{\infty}_{n,d}$. We conjecture the total weight $\frac{1}{2n+1}\binom{2n+d}{d}$ of $\mathbf{G}_{\rm m}$ orbits in $\mathbf{S}^{\infty}_{n,d}$. This is supported by the examples below. Together with the conjectured formula (3.31) it implies that the total weights of $\mathbf{G}_{\rm m}$ orbits in $\mathbf{S}^{\infty}_{n,d}$ and $\mathbf{S}_{2n,d,0}$ should be equal. In the following examples we see interesting discrete families of differential equations (4.1) which include representatives of orbits in both $\mathbf{S}^{\infty}_{n,d}$ and $\mathbf{S}_{2n,d,0}$. However, coefficients of their v and F_i's have no direct connection in general, as one can check examples with $n=0$ and $n=1$.

Example 4.1 $\boxed{n=0.}$ Obviously all points of $\mathbf{S}_{0,d,d}$ are in $\mathbf{S}^{\infty}_{0,d}$ because $v\in C\subset C[t^2]$. Therefore we have exactly one $\mathbf{G}_{\rm m}$ orbit there, of the same weight 1. One can give a representative of this orbit by the half pull-back of the differential equation (3.14) with $d_1=d_2=d$:

$$y''=\left(\frac{1}{16\,x}+\frac{(2d+1)(2d-3)}{16\,x^2}\right)y.$$

The corresponding $v=-1/4\sqrt{x}$ and polynomials F_i are the same as in (3.15), with x replaced by $\sqrt{x}$.

Example 4.2 $\boxed{n=1.}$ From the system of equations (4.7-4.8) we eliminate q_i's and obtain the determinant of a matrix with entries θ_j's or $\theta_0+1=2$. This determinant is homogeneous in α_0, α_2 with respect to the action (3.20) of $\mathbf{G}_{\rm m}$. The diagonal entries of the matrix are $\theta_1,\theta_2,\ldots,\theta_{d+1}$, hence the total weight of the determinant is $(d+1)(d+2)/2$. The determinant completely factors into distinct factor $\alpha_2+\zeta\alpha_0^3$ of weight 3, and (possibly) α_0 of weight 1. It follows that the total weight of orbits in $\mathbf{S}^{\infty}_{n,d}$ is $(d+1)(d+2)/6$.

Example 4.3 $\boxed{d=0.}$ Also here the only $\mathbf{G}_{\rm m}$ orbit of $\mathbf{S}_{2n,0,0}$ lies in $\mathbf{S}^{\infty}_{n,0}$. The half pull-back of the differential equation (3.32) is the same differential equation (3.32) with its n equal to our $n-1/2$.

Example 4.4 $\boxed{d=1.}$ We are looking for $\mathbf{G}_{\rm m}$ orbits in $\mathbf{S}_{2n,1,1}$ fixed under the involution. There is only one orbit with this property, it correspond to $\xi=-1$ (in terms of the corresponding example $d_1=d_2=1$). It is represented by the equation (3.34) with its n equal to our $2n$. Its half pull-back is the following differential equation with the Galois group $\mathbb{D}_\infty$

$$y''=\left(\frac{1}{4}\sum_{k=1}^{2n}k\,x^{2n-k}+\frac{2n+1}{4\,x}+\frac{(2n+1)(2n+5)}{16\,x^2}\right)y.$$

Note how this equation is similar to the $\mathbb{G}_{\rm m}$ example $d_1=1$, $d_2=0$. An algebraic solution of the associated Riccati equation is

$$\frac{1}{2\sqrt{x}}\left(x^n+\ldots+x+1\right)-\frac{2n+1}{2x}+\frac{1}{2\sqrt{x}}\frac{1}{\sqrt{x}-1}.$$

Another one is the conjugate over $C(x)$. A general solution of the differential equation itself can be written as

$$y = C_1 \left(\sqrt{x}-1\right) x^{-\frac{2n+1}{4}}\, e^{\sqrt{x}\left(1+\frac{x}{3}+\ldots+\frac{x^n}{2n+1}\right)} + C_2 \left(\sqrt{x}+1\right) x^{-\frac{2n+1}{4}}\, e^{-\sqrt{x}\left(1+\frac{x}{3}+\ldots+\frac{x^n}{2n+1}\right)}.$$

Example 4.5 $\boxed{d=2.}$ We can take $F_1 = (x-1)(x-\xi)$ and $F_2 = (x+1)(x+\xi)$ for some $\xi \in C \setminus \{0,1,-1\}$. With this choice, replacing ξ by ξ^{-1} will give a $\mathbf{G}_\mathrm{m}$ equivalent equations, as in the example $d_1 = 2, d_2 = 0$ in the previous section. Substitution of these F_i's into (3.8) gives us $2v = \left(c\,x^{2n+4} - 2(1+\xi)(x^2-\xi)\right)/F_1F_2$. This is a polynomial if $c = 2(1-\xi^2)$ and $c\xi^{2n+4} = 2\xi(\xi^2-1)$. It follows that $\xi^{2n+3} = -1$. Because $\xi = -1$ is not allowed, and to each $\mathbf{G}_\mathrm{m}$ orbit there correspond two ξ's, we have $(2n+2)/2 = n+1$ different $\mathbf{G}_\mathrm{m}$ orbits in $\mathbf{S}^{\infty}_{n,2}$. We can compute the component v of the corresponding element of $\mathbf{S}_{2n,2,2}$ as

$$v = \frac{(1-\xi^2)\,t^{2n+4} - (1+\xi)(t^2-\xi)}{(t^2-1)(t^2-\xi^2)} = \sum_{i=0}^{n} (1-\xi^{2k+2})t^{2n-2k} = \sum_{i=0}^{n} (1+\xi^{-2k-1})t^{2k}.$$

Now one can write down solutions of the Riccati equation and the pull-back equation in $\mathcal{S}_{2n,d,d}$ itself:

$$Y'' = \left(\frac{(1-\xi^2)^2(t^{4n+8}-t^2)}{(t^2-1)^2(t^2-\xi^2)} - \frac{(2n+3)(2t^2-1-\xi^2)}{(t^2-1)^2(t^2-\xi^2)} + \frac{(n+2)(n+3)}{t^2} \right) y.$$

The half pull-back of this equation is the equation (3.34) with its n equals to our $n-1/2$, and its ξ — our ξ^2 here.

4.2 Algebraic solutions of differential equations

In this section we consider second order linear differential equations with three singular points, and whose differential Galois group is either tetrahedral $A_4^{SL_2}$, or octahedral $S_4^{SL_2}$, or icosahedral $A_5^{SL_2}$. For completeness we shortly consider also differential equations with a finite dihedral Galois group as an example. The three singular points must be regular singular (because the local Galois groups there must be finite). They can be chosen to be 0, 1 and infinity, and the differential equation can be normalized to

$$y'' = \left(\frac{\lambda_0^2-1}{4\,x^2} + \frac{\lambda_1^2-1}{4\,(x-1)^2} + \frac{\lambda_\infty^2+1-\lambda_0^2-\lambda_1^2}{4\,x\,(x-1)} \right) y. \qquad (4.9)$$

Here $\lambda_0, \lambda_1, \lambda_\infty$ are differences of the local exponents at $0, 1$ and ∞ respectively. We denote such an equation by $E(\lambda_0, \lambda_1, \lambda_\infty | x)$ or $E(\lambda_0, \lambda_1, \lambda_\infty)$.

By a translation the differential equation 4.9 can be transformed to a hypergeometric equation, for instance

$$x\,(x-1)\,y'' + ((2-\lambda_0-\lambda_1)\,x - (1-\lambda_0))\,y' + \frac{(1-\lambda_0-\lambda_1)^2-\lambda_\infty^2}{4}\,y = 0. \qquad (4.10)$$

Klein theorem states that if a second order differential equation has algebraic solutions and its differential Galois group is one of the three named groups, then the differential equation is a pull-back of one of three standard equations, namely $E(1/2, 1/3, 1/3)$ for tetrahedral group, $E(1/2, 1/3, 1/4)$ for octahedral group, and $E(1/2, 1/3, 1/5)$ for icosahedral group.

4.2.1 Basic transformations

As explained in chapter 1, *pull-back transformations* of differential equations can be viewed as homomorphisms between the corresponding skew-polynomial rings $C(x)[\partial_x]$ and $C(t)[\partial_t]$. Moreover, any such homomorphism $\psi : C(x)[\partial_x] \to C(t)[\partial_t]$ is described as

$$x \mapsto f, \quad \partial_x \mapsto \frac{1}{f'}\partial_t + h, \tag{4.11}$$

for two functions $f, h \in C(t)$. The corresponding homomorphism $\phi : C(x) \to C(t)$ of (usual) fields defines a finite morphism $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ given by $x \mapsto f(t)$. This is the *pull-back* morphism for these transformations.

We are interested in the *normalized pull-backs*, i.e., in homomorphisms which send a differential operator of the form $\partial_x^2 - r(x)$ to a differential operator of the form $a(t)\,\partial_t^2 + b(t)$. Such homomorphisms are characterized as follows.

Lemma 4.2.1 *Let $f \in C(t)$, and let $g = C_0/\sqrt{f'}$ for some constant $C_0 \in C$. Then a homomorphism $\psi : C(x)[\partial_x] \to C(t)[\partial_t]$ defined by*

$$x \mapsto f, \quad \partial_x \mapsto \frac{1}{f'}\left(\partial_t - \frac{g'}{g}\right)$$

sends the differential operator $L_x = \partial_x^2 - r(x)$ to $1/(f')^2\, L_t$, where

$$L_t = \partial_t^2 - \left((f')^2\, \varphi(r)\,(f) + \frac{g''}{g}\right).$$

Moreover, $y \in C(x)$ is in the kernel of L_x if and only if $g\,\phi(y)$ is in the kernel of L_t.

Any normalized pull-back is of the above form.

Proof. One can check these statements by direct computations. □

For finding singular points of a pull-backed equation and differences of local exponents at them we observe the following. If E_2 is a pull-back of E_1 on a curve C_1 by a finite morphism $\Phi : C_2 \to C_1$, then the difference of local exponents of E_2 at $P \in C_2$ is equal to the difference of local exponents at $\Phi(P)$ multiplied by the ramification order of Φ at P. Recall that for a second order differential equation E_1 a point on C_1 is non-singular, if the local exponents of E_1 at this point are 0 and 1. If the difference of local exponents at some point is 1, then this point becomes non-singular after a translation $\partial_x \to \partial_x + h$. It follows that E_2 may have singular points above non-singular points of E_1 and otherwise, and the singular points of the pull-backed equation can be found by considering differences of local exponents at the branching points and at the points above the singularities of E_1 (and, eventually, the poles of h).

We also use the Gauss contiguity relations, which are usually formulated for the classical hypergeometric equations (see [32],[4]). These relations imply that if not all differences of local exponents λ_0, λ_1 and λ_∞ are integers in (4.9), then the differential Galois group of a hypergeometric equation (4.10) does not change if local exponents are transformed as

$$\lambda_i \mapsto \pm\lambda_i + k_i \ (k_i \in \mathbb{Z}) \text{ for } i \in \{0,1,\infty\}, \quad k_1 + k_2 + k_3 \text{ is even.} \tag{4.12}$$

We call also the equations (4.9) whose differences of local exponents are related in this way to be *contiguous*. From the explicit relations between the solutions of the contiguous hypergeometric equations we derive the following relations between contiguous equations (4.9). If $y(x)$ satisfies $E(\lambda_0, \lambda_1, \lambda_\infty)$, then

$$\begin{aligned}
\frac{(\lambda_0+\lambda_1+2)\,x-\lambda_0-1}{2\sqrt{x\,(x-1)}}y(x)-\sqrt{x\,(x-1)}\,y'(x) \quad &\text{satisfies} \quad E(\lambda_0+1,\lambda_1+1,\lambda_\infty),\\
\frac{(\lambda_\infty-1)\,x+\lambda_0+1}{2\sqrt{x}}y(x)+\sqrt{x}\,(x-1)\,y'(x) \quad &\text{satisfies} \quad E(\lambda_0+1,\lambda_1,\lambda_\infty+1),\\
\frac{(\lambda_\infty-1)\,x-\lambda_1-\lambda_\infty}{2\sqrt{x-1}}y(x)+x\sqrt{x-1}\,y'(x) \quad &\text{satisfies} \quad E(\lambda_0,\lambda_1+1,\lambda_\infty+1).
\end{aligned}$$

To decrease differences of local exponents one can use the same formulas with signs of λ_0, λ_1 or λ_∞ changed.

If all local exponents are real numbers, then an equivalence class of differential equations (4.9) with respect to the transformations (4.12) can be represented by $E(\lambda_0, \lambda_1, \lambda_\infty)$ such that the differences of local exponents lie in the interval $[0, 1]$. Using transformations like $(\lambda_0, \lambda_1, \lambda_\infty) \mapsto (1-\lambda_0, 1-\lambda_1, \lambda_\infty)$ we can further require that the sum of any two λ_i, λ_j be less or equal 1.

One can check that the solutions of the Riccati equations for contiguous differential equations (4.9) are defined over the same field. For instance, if $r(x)$ is a solutions of the Riccati equation for $E(\lambda_0, \lambda_1, \lambda_\infty)$, then

$$\frac{((\lambda_\infty+2)x+\lambda_0)\;r(x)+2\,x\,(x-1)\left(\frac{\lambda_0^2+\lambda_0}{4\,x^2}+\frac{\lambda_1^2-1}{4\,(x-1)^2}+\frac{\lambda_\infty^2+\lambda_\infty+1-\lambda_0^2-\lambda_0-\lambda_1^2}{4\,x\,(x-1)}\right)}{2\,x\,(x-1)\,r(x)+((\lambda_\infty-1)x+\lambda_0+1)}$$

is a solution of the Riccati equation for $E(\lambda_0+1, \lambda_1, \lambda_\infty+1)$. It follows that if the differential Galois group of (4.9) is tetrahedral, octahedral or icosahedral, then the contiguous differential equations will also have the same Galois group, because they are determined by the smallest degree of the algebraic solutions of the Riccati equation.

Example 4.6 For the sake of completeness we consider here the example of differential equations (4.9) with the dihedral Galois groups. Here we consider a differential equation $E(n+1/2, \mu, m+1/2)$, where n and m are non-negative integers, and $\mu \in \mathbb{R} \setminus \mathbb{Z}$. This differential equation is contiguous to $E(1/2, \tilde{\mu}, 1/2)$ for some $\tilde{\mu} \in (0, 1)$.

We consider the pull-back of $E(1/2, \tilde{\mu}, 1/2)$ with respect to $x \mapsto t^2$, which transforms solutions as $y(x) \mapsto \frac{1}{\sqrt{t}}\, y(t^2)$, compare with (3.3) with $k = 2$. This gives us the differential equation

$$Y(t)'' = -\frac{1-\tilde{\mu}^2}{(t^2-1)^2}\, Y(t). \tag{4.13}$$

This is a differential equation with two singular points $x = 1$ and $x = -1$. Its Riccati equation has two rational solutions, so the Galois group and a general solution

$$Y(t) = C_1 (t-1)^{\frac{1+\mu}{2}} (t+1)^{\frac{1-\mu}{2}} + C_2 (t+1)^{\frac{1+\mu}{2}} (t-1)^{\frac{1-\mu}{2}},$$

can be easily computed. The Galois group is $\mathbb{G}_{\mathrm{m}}$ if $\tilde{\mu} \notin \mathbb{Q}$, and it is a finite cyclic group if $\tilde{\mu} \in \mathbb{Q}$. Since the Riccati equation for $E(1/2, \tilde{\mu}, 1/2)$ does not have rational solutions (because $\tilde{\mu} \notin \mathbb{Z}$), it follows that the Galois group of $E(1/2, \tilde{\mu}, 1/2)$ is the infinite dihedral group $\mathbb{D}_\infty$ if $\tilde{\mu} \notin \mathbb{Q}$, and it is a finite dihedral group otherwise.

Consider now the same normalized pull-back with respect to $x \mapsto t^2$ of the equation $E(n+1/2, \mu, m+1/2)$:

$$Y(t)'' = \left(\frac{n^2+n}{t^2} + \frac{\mu^2-1}{(t^2-1)^2} + \frac{m^2+m-n^2-n}{t^2-1} \right) Y(t). \tag{4.14}$$

The Riccati equations for $E(n+1/2, \mu, m+1/2)$ and (4.14) have the same kind of solutions as the ones for $E(1/2, \tilde{\mu}, 1/2)$ and (4.13). Moreover, μ and $\tilde{m}u$ are either both rational or both irrational numbers, and if they are both rational then their denominators are equal. It follows that the differential Galois groups of $E(n+1/2, \mu, m+1/2)$ and $E(1/2, \tilde{\mu}, 1/2)$ are the same, i.e. it is $\mathbb{D}_\infty$ if $\mu \notin \mathbb{Q}$, and it is a finite dihedral group if $\mu \in \mathbb{Q} \setminus \mathbb{Z}$.

To write down explicit solutions of $E(n+1/2, \mu, m+1/2)$, recall theorem 4.1, corollary 4.2 and the further discussion in the previous chapter 3 about the solutions of differential equations (or their Riccati equations) with Galois group $\mathbb{G}_{\mathrm{m}}$. The same holds for differential equations with finite cyclic Galois group of order > 2. The equation (4.14) has such a Galois group. Its singular points are $x = 1$, $x = -1$ (with the pair $(\frac{1+\mu}{2}, \frac{1-\mu}{2})$ of local exponents at them), and possibly, $x = 0$ (the local exponents are $-n$ and $n+1$) and infinity (the local exponents are $-m$ and $m+1$). The two global solutions in $C(t)$ of the Riccati equation for (4.14) should be permuted after changing the sign of μ. From here one obtains the only possible form for these two Riccati solutions. The corresponding basis for the space of solutions of (4.14) is given by

$$Y(t) = C_1 F_1 (t-1)^{\frac{1+\mu}{2}} (t+1)^{\frac{1-\mu}{2}} + C_2 F_2 (t+1)^{\frac{1+\mu}{2}} (t-1)^{\frac{1-\mu}{2}},$$

where F_1, F_2 are polynomials in $C[t]$ of degree $n+m$, such that the rational function F_1/F_2 simultaneously approximates $(1+t)^\mu/(1-t)^\mu$ at zero up to order $2n$, and at infinity — up to order $2m$ in the Padé sense. The general solution of $E(n+1/2, \mu, m+1/2)$ is $x^{1/4}\, Y(\sqrt{x})$.

4.2.2 Darboux curves of Fucshian equations with three poles

Suppose that a second order differential equation E has tetrahedral, octahedral or icosahedral Galois group G. Let $C(C_0) \supset C(x)$ be its Piccard-Vessiot extension, it is a function field of an algebraic curve C_0. The finite covering $\varphi_0 : C_0 \to \mathbb{P}^1$ is Galois, of degree

24, 48 and 120 respectively. From results in [58] we know that the genus of C_0 is at least 2 for equations with three singular points. Each of the named finite groups has a center $\{\pm 1\}$, so we can factor φ_0 as $C_0 \to C_1 \to \mathbb{P}^1$, with $C_0 \to C_1$ being a Galois covering of degree 2, and $\varphi_1 : C_1 \to \mathbb{P}^1$ being a Galois covering with Galois group $A_4^{SL_2}/\{\pm 1\}$, $S_4^{SL_2}/\{\pm 1\}$, or $A_5^{SL_2}/\{\pm 1\}$ respectively. It is known from classical literature that genus of C_1 is zero for the three *standard* equations. This is not the case in general.

Let y be an algebraic solution of E. Then $u = y'/y$ is an algebraic solution of the Riccati equation of E. The function field $C(x, u)$ is the function field of some algebraic curve C_2, so that $C(x, u) = C(C_2)$. We will call C_2 a *Darboux curve* of the differential equation E. It is known ([54],[55]) that if u is of degree m over $C(x)$, then the differential Galois group $G \subset GL(2)$ has a so-called *1-reducible* subgroup of index m. For the considered groups ($A_4^{SL_2}$, $S_4^{SL_2}$ and $A_5^{SL_2}$) the minimal possible index for a 1-reducible subgroup is 4, 6 and 12 respectively. It is also known that the field extension $C(x, y) \supset C(x, u)$ is a cyclic Galois extension. It is a classical result ([45, 21, 9]) that for a second order differential equation E the degree of a solution y is always the largest possible degree, namely the order of G. That means that $C(C_0) = C(x, y)$ for any solution y of E. It also follows that an 1-reducible subgroup $N \subset G$ of minimal index is a cyclic group of order 6, 8 and 10 for considered G respectively. We can factor the finite morphism $\varphi_0 : C_0 \to \mathbb{P}^1$ as follows:

$$\varphi_0 : C_0 \xrightarrow{1:2} C_1 \xrightarrow{\varphi_2} C_2 \xrightarrow{\Phi} \mathbb{P}^1. \tag{4.15}$$

Here $\varphi_2 : C_1 \to C_2$ is a cyclic Galois covering of degree 3, 4 or 5 respectively (so that $C_0 \to C_2$ is a cyclic covering with Galois group N), and $\Phi : C_2 \to \mathbb{P}^1$ is a finite morphism of degree 4, 6 or 12 respectively. Further, let $\tilde{E}$ be the pull-back of E with respect to $\Phi : C_2 \to \mathbb{P}^1$. Then the Riccati equation of $\tilde{E}$ has a rational solution, and the differential Galois group of $\tilde{E}$ is N.

An equation for the Darboux curve can be found by writing down the minimal polynomial for a solution of the Riccati equation of E of minimal degree. This polynomial is determined by the corresponding solution of the Riccati equation of the 4th (respectively, 6th or 12th) symmetric power of E. There is a well known recurrence relation between coefficients of this polynomial. However, this minimal polynomial is very large. It would be more practical to define the field $C(C_2)$ by a simpler polynomial, and express solutions of E (or its Riccati equation) as a radical expression in this field. It would be interesting to find out whether finding algebraic solutions of differential equations through pull-backs is substantially independent from the methods of invariant theory of differential Galois groups, see [54, 55, 64].

Particularly we are interested in the case when the Darboux curve C_2 has genus 0. Then the function field $C(C_2)$ is isomorphic to the field $C(x)$ of rational functions. Once we know a parametrization of C_2 we can easily write down the pull-backed equation $\tilde{E}$. Since the Galois group of $\tilde{N}$ is a cyclic finite group, there are two rational solution of the Riccati equation for $\tilde{E}$, so we can easily solve $\tilde{E}$. Knowing a general solution of $\tilde{E}$ one can use lemma (4.2.1) and write easily a general solution of the original equation E in a quite satisfactory form. We will mention how one can use the knowledge of all differential equations with rational Darboux curves also in the general case when the genus of (all) Darboux curves of given E is positive.

Note that Darboux curves of the three standard equations of type (4.9) are rational, because C_1 is rational for them. Also note that equivalent equations of form (4.9) with respect to Gauss transformations (4.12) have isomorphic Darboux curves. We can see this from transformations of contiguous functions given above. One can get a fractional-linear relation (with coefficients in $C(x)$) between solutions of equivalent Riccati equations, hence solutions of equivalent Riccati equations lie in the same field. It follows that one can use the same pull-back morphism to pull-back E to an equation with cyclic Galois group.

In the following, we will write down explicitly pull-back morphisms (of minimal degree $4, 6$ or 12) from a Darboux curve to $\mathbb{P}^1$ for (almost) all equivalence classes of equations of form (4.9) with three poles. We have classified them in the previous section in such a way that differential equations of the same type have isomorphic Darboux curves and the same pull-back morphism. The genus of obtained Darboux curves is zero, except for types $T[3,3,5], T_1[3,5,5], T_2[3,5,5]$ and $T[5,5,5]$, where the Darboux curve has genus 1.

If the genus of all Darboux curves is positive, then one can find an equation for it as follows. Let $\Psi : \mathbb{P}^1 \to \mathbb{P}^1$ be a finite morphism such that the given equation E is the pull-back (with respect to Ψ) of an equation E_0 with the same Galois group, and such that a Darboux curve D_0 of E_0 is rational. Then the Darboux curve of E is the fiber product of $\mathbb{P}^1$ and D_0 with respect to Ψ and the known pull-back morphism $D_0 \to \mathbb{P}^1$. Using this construction one can write down a much simpler equation for a Darboux curve of E than the minimal polynomial for a solution of the Riccati equation of E. We will demonstrate this by computing non-rational Darboux curves for equations of form (4.9).

Note that we can always find the required morphism $\Psi : \mathbb{P}^1 \to \mathbb{P}^1$ by taking E_0 to be one of the three standard equations, because of the mentioned Klein theorem, and the fact that Darboux curves for them are rational. (It is known that such a pull-back morphism from a standard equation is unique.) However, the pull-back morphism from a standard equation may have very large degree. As we will see form examples like $E(1/3, 2/5, 3/5)$, there could be a pull-back morphism of another equation with rational Darboux curve of much smaller degree.

4.2.3 Classification

All differential equations with three singular points and with considered Galois groups were first classified by Schwarz. Here we classify such differential equations in such a way, that the differential equations in the same "class" have the same pull-back morphism to their Darboux curves, as we will show in the next subsection. To show that they indeed have required Galois groups, we give necessary pull-back morphisms ([34], [35], Brioschi). We also compare this classification with the fifteen type of Schwarz, and the branching types of Fucshian equations introduced by M. van der Put and F. Ulmer in [58].

- Tetrahedral type $T[2,3,3]$ of equations equivalent to $E(1/2, 1/3, 1/3)$. Note that all differential equations whose differences of local exponents have (pure) denominators $2,3,3$ at $0,1$ and infinity are in this class, because $E(1/2, a, b)$ is equivalent to

$E(1/2, 1-a, b)$, etc. Branching types from the list in [58] are $[3,3,4], [3,4,6], [4,6,6]$. The Schwarz type is II.

- Tetrahedral $T[3,3,3]$ of equations equivalent to $E(1/3, 1/3, 2/3)$. One can pull back the standard $E(1/2, 1/3, 1/3|z)$ to this equation (in x) by the degree 2 covering ramified over 0 and ∞, given by $z = (2x-1)^2$. Branching types are $[3,3,6]$ and $[6,6,6]$. Schwarz type III.
- Octahedral $T[2,3,4]$ of equations equivalent to $E(1/2, 1/3, 1/4)$. All equations with denominators (of differences of local exponents) are in this class. Branching types are $[3,4,8]$ and $[4,6,8]$. Schwarz type IV.
- Octahedral $T[3,4,4]$ of equations equivalent to $E(2/3, 1/4, 1/4)$. One can pull back $E(1/2, 1/3, 1/4|z)$ to this equation by the degree 2 covering $z/(z-1) = (x-2)^2/x^2$, ramified over 0 and 1. Branching types are $[3,8,8]$ and $[6,8,8]$. Schwarz type V.
- Icosahedral $T[2,3,5]$ of equations equivalent to $E(1/2, 1/3, 1/5)$ or to $E(1/2, 1/3, 2/5)$. Again, all equations with these denominators are in this class. One can pull back to $E(1/2, 1/3, 2/5|x)$ from the standard $E(1/2, 1/3, 1/5|z)$ by the degree 7 covering $z - 1 = (x-1)(2916x^2 - 3375x - 3125)^3/(189x - 125)^5$. Branching types are $[3,4,5], [3,4,10], [3,5,6], [4,6,10]$. Schwarz types VI and VII.
- Icosahedral $T[2,5,5]$ of equations equivalent to $E(1/2, 1/5, 2/5)$. One can pull back $E(1/2, 1/3, 1/5|z)$ to this equation by the degree 3 covering $z-1 = -(4x-3)^3/27(x-1)$. Branching types are $[3,4,5], [3,4,10], [3,5,6], [4,6,10]$. Schwarz type VIII.
- Icosahedral $T[3,3,5]$ of equations equivalent to $E(1/3, 1/3, 2/5)$ or to $E(1/3, 2/3, 1/5)$. The standard equation $E(1/3, 1/3, 1/5|z)$ pull-backs to $E(1/3, 1/3, 2/5|x)$ by the degree 2 covering $z = (2x+1)^2$, and it pull-backs to $E(1/3, 2/3, 1/5|x)$ by degree 6 covering $z - 1 = -27x^2(x-1)(3x+125)^3/4(9x-25)^5$. Branching types are $[3,3,10], [3,5,6], [6,6,10]$. Schwarz types IX and X.
- Icosahedral $T_1[3,5,5]$ of equations equivalent to $E(2/3, 1/5, 1/5)$ and $E(1/3, 2/5, 3/5)$. One can pull-back the standard $E(1/2, 1/3, 1/5|z)$ to $E(2/3, 1/5, 1/5|x)$ by the degree 2 covering $z/(z-1) = (x-2)^2/x^2$. One can pull-back $E(1/2, 1/3, 2/5|z)$ to $E(2/3, 2/5, 2/5|x)$ equivalent to $E(1/3, 2/5, 3/5)$ by the same degree 2 morphism; one needs a morphism of degree at least 10 to pull-back the standard equation to $E(1/3, 2/5, 3/5)$. Branching types are $[3,5,10], [5,5,6], [6,10,10]$. Schwarz types XI and XII.
- Icosahedral $T_2[3,5,5]$ of equations equivalent to $E(1/3, 1/5, 3/5)$. One can pull-back the standard equation to it by degree 4 covering $z - 1 = -x(9x-8)^3/64(x-1)$. Branching types are $[3,5,5], [3,10,10], [5,6,10]$. Schwarz type XIII.
- Icosahedral $T[5,5,5]$ of equations equivalent to $E(1/5, 1/5, 4/5)$ and $E(2/5, 2/5, 2/5)$. One can pull-back $E(1/2, 1/5, 2/5|z)$ to $E(1/5, 1/5, 4/5|x)$ by the degree 2 covering $z = (2x+1)^2$, and using the same morphism one can pull-back $E(1/2, 2/5, 1/5)$ to $E(2/5, 2/5, 2/5)$. One needs degree 6 coverings to pull-back the standard $E(1/2, 1/3, 1/5)$ to the two representatives of this type. Branching types are $[5,5,10], [10,10,10]$. Schwarz types XIV and XV.

Comparing this list with the one of Schwartz or van der Put-Ulmer, we see that it is complete (up to permutations of $\lambda_0, \lambda_1, \lambda_\infty$).

4.2.4 Pull-back morphisms

4.2.4.1 Tetrahedral type $T[2,3,3]$

A pull-back morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is given by

$$x = \frac{(t^2-3)^2}{4\,(2\,t+3)} = 1 + \frac{(t+1)^3\,(t-3)}{4\,(2\,t+3)}. \tag{4.16}$$

One can check that the pull-back of the standard equation $E(1/2, 1/3, 1/3)$ by this morphism is

$$y(t)'' = -\frac{18}{(t-3)^2\,(2\,t+3)^2}\,y(t).$$

This differential equation has only two singular points and finite cyclic differential Galois group. Its general solution is

$$y(t) = C_1\,(t-3)^{1/3}\,(2\,t+3)^{2/3} + C_2\,(t-3)^{2/3}\,(2\,t+3)^{1/3}.$$

Hence we can use the same pull-back morphism for all equations of this type.

To find the pull-backed equation explicitly one can use lemma 4.2.1 with

$$f(t) = \frac{(t^2-3)^2}{4\,(2\,t+3)}, \quad g(t) = \frac{2\,t+3}{(t+1)\,\sqrt{t^2-3}}.$$

Knowing a general solution $y(t)$ of the pull-back, a general solution of the original equation can be written as

$$y(x) = \frac{(t+1)\,\sqrt{t^2-3}}{2\,t+3}\,y(t), \quad \text{where } t \text{ satisfies } (t^2-3)^2 - 4\,x\,(2\,t+3) = 0.$$

In particular, a general solution of $E(1/2, 1/3, 1/3)$ is

$$\begin{aligned} y(x) \;=\; & C_1\,x^{1/4}\,(t+1)\,(t-3)^{1/3}\,(2\,t+3)^{-1/12} \\ & +C_2\,x^{1/4}\,(t+1)\,(t-3)^{2/3}\,(2\,t+3)^{-5/12}. \end{aligned}$$

Note that the finite morphism (4.16) is ramified above $0, 1$ and infinity in such a way that only two points ($t = 3$ above $x = 1$, and $t = -3/2$ above $x = \infty$) are singular and differences of local exponents at them are not integers. Other possible singular points of the pull-backed equation are $t = \sqrt{3}, -\sqrt{3}, -1$ or $t = \infty$. One can move three of the possible 6 singular points to favorite positions (like $0, 1, \infty$) by a fractional-linear transformation of t. All pull-back morphisms (from Darboux curves to $\mathbb{P}^1$) we will consider ramify only over singular points of the original equation in quite uniform manner, in the same way over points whose differences of local exponents have the same denominators. Possibly one can prove this observation by local considerations.

The expression (4.16) for x determines a parametrization of the Darboux curve, when it is written as an algebraic equation for a solution of the Riccati equation of degree 4. (There are two such equations in tetrahedral case).

4.2.4.2 Tetrahedral type $T[3,3,3]$

The Darboux curve is rational and a pull-back morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is given by

$$x = \frac{t^3\,(t+2)}{2\,t+1} = 1 + \frac{(t+1)^3\,(t-1)}{2\,t+1}. \tag{4.17}$$

This finite morphism is ramified above $0, 1$ and infinity in such a way that only three points ($t = -2, 1$ or $t = -1/2$) are singular and differences of local exponents at them are not integers. Other possible singular points of the pull-backed equation are $t = 0, -1$ or $t = \infty$.

Knowing a general solution $y(t)$ of the pull-back, a general solution of the original equation can be written as

$$y(x) = \frac{t\,(t+1)}{2\,t+1} y(t).$$

We need to show that the equation $E(1/3, 1/3, 2/3)$ pull-backs to an equation with a cyclic Galois group with respect to (4.17). One can directly compute that the pull-back is

$$y(t)'' = 3\,\frac{t^4 + 2\,t^3 - 3\,t^2 - 4\,t - 2}{(t+2)^2\,(t-1)^2\,(2\,t+1)^2}\,y(t).$$

A general solution of this equation is

$$y(t) = C_1\,(t+2)^{1/3}(t-1)^{1/3}(2\,t+1)^{5/6} + C_2\,(t+2)^{2/3}(t-1)^{2/3}(2\,t+1)^{1/6}.$$

4.2.4.3 Octahedral type $T[2,3,4]$

The Darboux curve is rational and a pull-back morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is given by

$$x = -\frac{t^2\,(t^2-18)^2}{108\,(t-4)\,(t+4)} = 1 - \frac{(t^2-12)^3}{108\,(t-4)\,(t+4)}. \tag{4.18}$$

This finite morphism is ramified above $0, 1$ and infinity in such a way that only two points ($t = 4$ and $t = -4$) are singular and differences of local exponents at them are not integers. Other possible singular points of the pull-backed equation are $t = 0, 3\sqrt{2}, -3\sqrt{2}, 2\sqrt{3}, -2\sqrt{3}$ or $t = \infty$.

Knowing a general solution $y(t)$ of the pull-back, a general solution of the original equation can be written as

$$y(x) = \frac{(t^2-12)\,\sqrt{t\,(t^2-18)}}{(t-4)\,(t+4)}\,y(t).$$

We need to show that the equation $E(1/2, 1/3, 1/4)$ pull-backs to an equation with a cyclic Galois group with respect to (4.18). One can directly compute that the pull-back is

$$y(t)'' = -\frac{15}{(t-4)^2\,(t+4)^2}\,y(t).$$

A general solution of this equation is

$$y(t) = C_1\,(t-4)^{3/8}\,(t+4)^{5/8} + C_2\,(t-4)^{5/8}\,(t-4)^{3/8}.$$

4.2.4.4 Octahedral type $T[3,4,4]$

The Darboux curve is rational and a pull-back morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is given by

$$x = \frac{(t^2-3)^3}{27\,(t-1)\,(t+1)}, \quad = 1 + \frac{t^4\,(t-3)\,(t+3)}{27\,(t-1)\,(t+1)}. \tag{4.19}$$

This finite morphism is ramified above $0, 1$ and infinity in such a way that four points ($t = 1, -1, 3$ and $t = -3$) are singular and differences of local exponents at them are not integers. Other possible singular points of the pull-backed equation are $t = 0, \sqrt{3}, -\sqrt{3}$ or $t = \infty$.

Knowing a general solution $y(t)$ of the pull-back, a general solution of the original equation can be written as

$$y(x) = \frac{t^{3/2}\,(t^2-3)}{(t-1)\,(t+1)}\,y(t).$$

We need to show that an equation $E(1/3, 1/4, 3/4)$ (equivalent to $E(2/3, 1/3, 1/4)$) pull-backs to an equation with a cyclic Galois group with respect to (4.19). One can directly compute that the pull-back is

$$y(t)'' = 2\,\frac{t^6 - 15\,t^4 + 27\,t^2 - 27}{(t-1)^2\,(t+1)^2\,(t-3)^2\,(t+3)^2}\,y(t).$$

A general solution of this equation is

$$y(t) = C_1(t-1)^{1/8}(t+1)^{7/8}(t-3)^{5/8}(t+3)^{3/8} + C_2(t-1)^{7/8}(t+1)^{1/8}(t-3)^{3/8}(t+3)^{5/8}.$$

4.2.4.5 Icosahedral type $T[2,3,5]$

The Darboux curve is rational and a pull-back morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is given by

$$x = \frac{(t^2+1)^2\,(t^4+18\,t^3+74\,t^2-18\,t+1)^2}{1728\,t^5\,(t^2+11\,t-1)} = 1 + \frac{(t^4+12\,t^3+14\,t^2-12\,t+1)^3}{1728\,t^5\,(t^2+11\,t-1)}. \tag{4.20}$$

Let $\alpha_{1,2} = (-11 \pm 5\sqrt{5})/2$. The finite morphism (4.20) is ramified above $0, 1$ and infinity in such a way that only two points ($t = \alpha_1$ and $t = \alpha_2$) are singular and differences of local exponents at them are not integers.

Knowing a general solution $y(t)$ of the pull-back, a general solution of the original equation can be written as

$$y(x) = \frac{(t^4+12\,t^3+14\,t^2-12\,t+1)\,\sqrt{(t^2+1)\,(t^4+18\,t^3+74\,t^2-18\,t+1)}}{t^3\,(t^2+11\,t-1)}\,y(t)$$

We need to show that equation $E(1/2, 1/3, 1/5)$ and $E(1/2, 1/3, 2/5)$ pull-back to equations with a cyclic Galois group with respect to (4.20). One can directly compute that the pull-back of $E(1/2, 1/3, 1/5)$ is

$$y(t)'' = -\frac{30}{(t^2+11\,t-1)^2}\,y(t). \tag{4.21}$$

A general solution of this equation is given by

$$y(t) = C_1\ (t-\alpha_1)^{2/5}\,(t-\alpha_2)^{3/5} + C_2\ (t-\alpha_1)^{3/5}\,(t-\alpha_2)^{2/5}\,.$$

One can also compute that the pull-back of $E(1/2, 1/3, 2/5)$ is

$$y(t)'' = \frac{3}{4}\,\frac{t^4+12\,t^3-26\,t^2-12\,t+1}{t^2\,(t^2+11\,t-1)^2}\,y(t). \tag{4.22}$$

A general solution of this equation is

$$y(t) = C_1\frac{5\,t-2\,\alpha_1-1}{\sqrt{t}}\,(t-\alpha_1)^{3/10}\,(t-\alpha_2)^{7/10}+\ldots$$

4.2.4.6 Icosahedral type $T[2,5,5]$

The Darboux curve is rational and a pull-back morphism $\mathbb{P}^1 \to \mathbb{P}^1$ is given by

$$x = \frac{(t^2+1)^2\,(t^4-2\,t^3-6\,t^2+2\,t+1)^2}{64\,t^5\,(t^2+t-1)} = 1 + \frac{(t-1)^5\,(t+1)^5\,(t^2-4\,t-1)}{64\,t^5\,(t^2+t-1)}. \tag{4.23}$$

Let $\beta_{1,2} = 2 \pm \sqrt{5}$ and $\gamma_{1,2} = (-1 \pm \sqrt{5})/2$. The finite morphism (4.23) is ramified above $0, 1$ and infinity in such a way that only points ($t = \beta_1, \beta_2, \gamma_1$ and $t = \gamma_2$) are singular and differences of local exponents at them are not integers.

Knowing a general solution $y(t)$ of the pull-back, a general solution of the original equation can be written as

$$y(x) = \frac{(t-1)^2\,(t+1)^2\,\sqrt{(t^2+1)\,(t^4-2\,t^3-6\,t^2+2\,t+1)}}{t^3\,(t^2+t-1)}\,y(t)$$

We need to show that the equation $E(1/2, 1/5, 2/5)$ pull-backs to an equation with a cyclic Galois group with respect to (4.23). One can directly compute that the pull-back is

$$y(t)'' = \frac{3}{4}\,\frac{t^8-4\,t^7-12\,t^6+12\,t^5-10\,t^4-12\,t^3-12\,t^2+4\,t+1}{t^2\,(t^2+t-1)^2\,(t^2-4\,t-1)^2}\,y(t).$$

A general solution of this equation is given by

$$y(t) = C_1 t^{-1/2}\,(t-\beta_1)^{3/5}\,(t-\beta_2)^{2/5}\,(t-\gamma_1)^{3/10}\,(t-\gamma_2)^{7/10}+\ldots$$

4.2.4.7 Icosahedral type $T[3,3,5]$

The Darboux curve has genus 1 in this case. To find its equation and a pull-back morphism let us recall the degree two covering $z \mapsto (2x-1)^2$ by which $E(1/2, 1/3, 1/5|z)$ is pull-backed to $E(1/3, 1/3, 2/5|x)$, and the pull-back morphism (4.20, variable z instead of x) to the Darboux curve of $E(1/2, 1/3, 1/5|z)$. Eliminating z we get the following equation for the Darboux curve

$$(2\,x-1)^2 = \frac{(t^2+1)^2\,(t^4+18\,t^3+74\,t^2-18\,t+1)^2}{1728\,t^5\,(t^2+11\,t-1)}.$$

We easily see that the Darboux curve is isomorphic to the genus 1 curve

$$D_1 : w^2 = 3\,t\,(t^2 + 11\,t - 1).$$

The pull-back morphism $D_1 \to \mathbb{P}^1$ can be written in the following way, so that we can see ramification points easily:

$$x = \frac{(w\,t + w + 2\,t^2 + 9\,t - 1)^3}{48\,w\,t^2\,(3\,w + 18\,t - 1)}, \quad (x - 1 = -\mathsf{subs}(w \mapsto -w, x)) .$$

It is ramified in such a way that only points $(t, w) = (\gamma_1, 0)$ and $(\gamma_2, 0)$ are singular with non-integer difference of local exponents. Other possible singular points are $(w, t) = (0, 0)$, defined by $t^4 + 12t^3 + 14t^2 - 12t + 1 = 0$, and the point at infinity.

A derivation in the differential field $C(D_1)$ is defined by $t' = 1, w' = 3(3\,t^2 + 22\,t - 1)/2w$. Knowing a general solution $y(t)$ of the pull-back, a general solution of the original equation can be written as

$$y(x) = \frac{t^4 + 12\,t^3 + 14\,t^2 - 12\,t + 1}{w^{3/2}\,t}\,y(w, t).$$

One can check that the pull-back of $E(1/3, 1/3, 2/5)$ is defined over the field $C(t)$ and is the same as (4.21). Thus we know the solution of this equation. Instead of $E(1/3, 2/3, 1/5)$ we can take an equivalent equation $E(1/3, 1/3, 4/5)$. The pull back of $E(1/3, 1/3, 4/5)$ is also defined over the field $C(t)$ and is the same as (4.22).

4.2.4.8 Icosahedral type $T_1[3, 5, 5]$

The Darboux curve has genus 1. Analogously as for the type $T[3, 3, 5]$, we use the degree 2 pull-back morphism $z/(z-1) = (x-2)^2/x^2$ of $E(1/2, 1/3, 1/5)$ to $E(2/3, 1/5, 1/5)$, and the pull-back morphism of $E(1/2, 1/3, 1/5)$ to its Darboux curve, to obtain the following equation for the Darboux curve of $E(2/3, 1/5, 1/5)$:

$$\frac{(x-2)^2}{x^2} = \frac{(t^2 + 1)^2\,(t^4 + 18\,t^3 + 74\,t^2 - 18\,t + 1)^2}{(t^4 + 12\,t^3 + 14\,t^2 - 12\,t + 1)^3}.$$

Hence the Darboux curve for equations equivalent to $E(2/3, 1/5, 1/5)$ is isomorphic to:

$$D_2 : w^2 = t^4 + 12\,t^3 + 14\,t^2 - 12\,t + 1.$$

or to $\eta^2 = \xi\,(\xi^2 + 5\,\xi - 5)$ by $t = (\eta - 3\xi)/(\xi - 5)$. A nice pull-back homomorphism probably can be computed using automorphisms of the obtained genus 1 curve.

To show that equations equivalent to $E(1/3, 2/5, 3/5)$ have the same Darboux curve we recall that a representative $E(2/3, 2/5, 2/5)$ is the pull-back of $E(1/2, 1/3, 2/5)$ with respect to the same degree 2 morphism.

4.2.4.9 Icosahedral type $T_2[3, 5, 5]$

The Darboux curve has genus 1. We use degree 4 morphism $z - 1 = -x(9x - 8)^3/(x - 1)$ which pull-backs $E(1/2, 1/3, 1/5)$ to $E(1/3, 1/5, 3/5)$, and the pull-back morphism of

$E(1/2,1/3,1/5)$ to its Darboux curve. An equation for the Darboux curve for differential equations of this type is:

$$\frac{(t^4+12\,t^3+14\,t^2-12\,t+1)^3}{1728\,t^5\,(t^2+11\,t-1)}+\frac{x\,(9\,x-8)^3}{x-1}=0.$$

This curve is isomorphic to $\eta^2=(\xi-3)\,(\xi-4)\,(\xi+12)$. An isomorphism is given by

$$x=\frac{(\xi\,\eta-12\,\eta+\xi^2-24\,\xi+48)^3}{864\,(\xi-4)^2\,(\eta+3\,\xi-4)\,(\eta+5\,\xi-20)},\qquad t=\frac{8\,(\eta+3\,\xi-12)}{\xi\,(\xi-4)^2}.$$

4.2.4.10 Icosahedral type $T[5,5,5]$

The Darboux curve has genus 1. We use degree 2 morphism $z=(2x-1)^2$ which pull-backs $E(1/2,1/5,2/5)$ and $E(1/2,2/5,1/5)$ to $E(1/2,1/5,4/5)$ and $E(2/5,2/5,2/5)$ respectively, and the pull-back morphism of type $T[2,5,5]$ to obtain the equation:

$$(2x-1)^2=\frac{(t^2+1)^2\,(t^4-2\,t^3-6\,t^2+2\,t+1)^2}{64\,t^5\,(t^2+t-1)}.$$

It is clear that the Darboux curve for equations of this type can be given by

$$w^2=t\,(t^2+t-1).$$

4.3 An order three example

Consider the Hurwitz equation

$$y'''+\frac{7\,x-4}{x\,(x-1)}\,y''+\frac{2592\,x^2-2963\,x+560}{252\,x^2\,(x-1)^2}\,y'+\frac{57024\,x-40805}{24696\,x^2\,(x-1)^2}\,y=0. \qquad (4.24)$$

The differential Galois group of this equation is known to be G_{168}, the finite group of 168 elements isomorphic to $\mathrm{PSL}(2,\mathbb{F}_7)$. It is known that the Riccati equation associated to the 21st symmetric power of (4.24) has a rational solution, see [55, 64, 59]. In other words, there are algebraic solutions of the Riccati equation associated to (4.24) of degree 21.

Our aim is to find a covering of $\Phi : D_H\mathbb{P}^1$ of degree 21 corresponding to such an algebraic solution. The Darboux curve D_H turns out to be rational. Recall that the local exponents of (4.24) at zero are $(-2/3,-1/3,0)$, at one — $(-1/2,0,1/2)$, and at infinity — $(8/7,9/7,11/7)$. The differential Galois group of the pull-back equation should not have of order dividing 21. Therefore the corresponding covering of Φ should have branching points of order 3 above zero and branching points of order 7 above infinity. The obtained covering has 8 simple branching points above $x=1$ and no other branching points, so that the genus of D_H is zero by the Hurwitz formula. Actually there are no other reasonable possibilities, because if we take 10 branching points above $x=1$, we would have only one singular point of the pull-backed equation with local Galois group $\mathbb{Z}/2\mathbb{Z}$, whereas its global Galois group should have at least $168/21=8$ elements.

The covering $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ was obtained after lengthy calculations with Maple. It is given by:

$$\Phi : x \mapsto \phi(t) = \frac{t^3\,(3\,t^2-7)^3\,(2\,t^2-7\,t+7)^3\,(11\,t^2-35\,t+28)^3}{1728\,(t^3-7\,t+7)^7}. \tag{4.25}$$

By factoring $\phi(t)-1$ one can check that Φ has simple branching points given by $3\,t^4 - 14\,t^3+14\,t^2+14\,t-21=0$ and $9\,t^4-42\,t^3+77\,t^2-70\,t+28=0$ above $x=1$. Another five points above $x=1$ are $t=3/2$ and the roots of $4\,t^4-21\,t^2+28=0$. These will be singular points of the pull-backed equation with the local Galois group $\mathbb{Z}/2\mathbb{Z}$.

To specify a pull-back, we take the homomorphism of the (skew symmetric) rings of differential operators given by:

$$\partial_x \mapsto \frac{1}{\phi'(t)}\left(\partial_t - \frac{\phi''(t)}{\phi'(t)} - \frac{1}{2\,t-3} - \frac{8\,t^3-21\,t}{4\,t^4-21\,t^2+28}\right). \tag{4.26}$$

Then we obtain the following pull-back of (4.24):

$$\begin{aligned}(2\,t-3)\,(4\,t^4-21\,t^2+28)\,(t^3-7\,t+7)\,y'''+&\\ (36\,t^7-36\,t^6-427\,t^5+840\,t^4+441\,t^3-1953\,t^2+1127\,t)\,y''+&\\ (18\,t^6-9\,t^5-238\,t^4+525\,t^3-231\,t^2-252\,t+196)\,y'+&\\ (-3\,t^5+49\,t^3-105\,t^2+63\,t)\,y &= 0.\end{aligned} \tag{4.27}$$

One can check that the local exponents at $t=3/2$ and at the roots of $4\,t^4-21\,t^2+28$ are $(0,1/2,1)$, at the roots of $t^3-7\,t+7$ they are $(0,1,3)$, and at infinity — $(-1/2,1/2,3/2)$. At other (non-singular) points the local exponents are $(0,1,2)$.

Suppose that u is a solution of the Riccati equation for (4.27) in $C(t)$. Then the residue of u at each point of $\mathbb{P}^1$ is a local exponent of the equation (4.27). Since the sum of the residues of u must be zero, and the only negative local exponent is $-1/2$ at infinity, the function $u \in C(t)$ has exactly two non-zero residues, namely $-1/2$ at infinity, and $1/2$ at one of the singular points above $x=1$. The possible candidates for u are $1/(2\,t-3)$ and $1/2(x-\zeta)$, where ζ satisfies $4\,\zeta^4-21\,\zeta^2+28=0$. One can check that $(2\,t-3)\partial_t-1$ is indeed a right-hand divisor of the differential operator in (4.27), whereas $2(t-\zeta)\partial_t-1$ are not. It follows that $1/(2\,t-3)$ is a solution of the corresponding Riccati equation, and $\sqrt{2\,t-3}$ is a solution of (4.27). Recalling (4.26) we conclude that $\phi'(t)\,\sqrt{4\,t^4-21\,t^2+28}\,(2\,t-3)$ is a solution of the Hurwitz equation (4.24), where t is the root of $x-\phi(t)=0$.