Aspects of algorithmic algebra

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Chapter 1

Introduction

In this thesis two different subjects, namely the symbolic solution of linear differential equations and the algebraic theory of two-dimensional splines, are treated. They have in common the algebraic and algorithmic approach and motivation coming from computer algebra.

1.1 Differential equations

The differential equations considered in this thesis are second order linear differential equations

$$y'' = r\,y, \qquad \text{with} \quad r \in C(x), \tag{1.1}$$

and one third order linear differential equation. Here C is an algebraically closed field of characteristic zero, and $C(x)$ is the differential field of rational functions with the usual derivation. In the next two chapters our main problem is to characterize the set of rational functions r such that the differential equation (1.1) has certain properties. Chapter 4 is devoted to special transformations (so-called *pull-backs*) of differential equations. To formulate these problems and results more precisely we need some tools from *differential Galois theory*. Here is a short exposition of this theory and our problems. The results are summarized at the beginning of the corresponding section.

A *differential field* K is a field with a *derivation*, i.e. a map $D : K \mapsto K$ satisfying $D(a+b) = D(a) + D(b)$ and the Leibnitz rule $D(ab) = D(a)\,b + a\,D(b)$. One usually denotes $D(f)$ by f'. The elements $a \in K$ satisfying $a' = 0$ are the *constants* of K. They form a subfield of K. In our basic example $C(x)$ the derivation is the usual differentiation of rational functions, and the field of constants is C. An *extension* of the differential field K is a differential field L containing K, such that the derivation on L restricted on the elements of K coincides with the derivation on K. For example, an extension of $C(x)$ is the field $C((x))$ of formal Laurent series at $x = 0$ with the derivation $\sum_{n \geq n_0} a_n x^n \mapsto \sum_{n \geq n_0} n a_n x^{n-1}$.

The solutions of the differential equation (1.1) in $C(x)$ or in an extension $K \supset C(x)$ form a linear space over the field of constants, and the dimension of this linear space is at most two, see [57]. There exists a so-called *Picard-Vessiot extension* of (1.1) over

$C(x)$, which is a minimal extension $L \supset C(x)$ of differential fields, with the same field of constants C, and such that the solution space of (1.1) in L has dimension two over C. The Picard-Vessiot extension is unique up to isomorphism. The *differential Galois group* G of (1.1) over $C(x)$ is the group of automorphisms of L which fix $C(x)$ and commute with the derivation on L. The Galois group G acts on the space V of solutions in L, thus we obtain a group homomorphism $G \to \mathrm{GL}(2, C)$. Here $\mathrm{GL}(2, C)$ is the group of the invertible 2×2 matrices over C, which is the group of isomorphisms of V after a choice of basis in V. The homomorphism $G \to \mathrm{GL}(2, C)$ is always injective, and its image is an *algebraic subgroup* of $\mathrm{GL}(2, C)$, see [53]. The differential Galois group G is usually identified with this linear algebraic group. In fact, for differential equations of the form (1.1) the Galois group G is a subgroup of $\mathrm{SL}(2, C)$ — the matrices with determinant 1, see [57]. Similarly one defines the differential Galois group of any linear differential equation over a differential field. As an example, the Picard-Vessiot extension of $y'' = y$ over $C(x)$ is $C(x, e^x)$. A basis of the solutions there is (e^x, e^{-x}). An automorphism σ from the differential Galois group is identity on $C(x)$ and should send e^x to a (non-zero) solution of $y' = y$, thus $\sigma(e^x) = c\, e^x$ for some $c \in C^*$. Hence the differential Galois group of $y'' = y$ over $C(x)$ is the multiplicative group C^* of the constant field. As an algebraic group it is denoted by $\mathbb{G}_{\mathrm{m}}$. The action of σ on the space of solutions in $C(x, e^x)$ in the given basis is $\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$.

The possible differential Galois groups for equation (1.1) are well-known, see [37, 57]. In this thesis we are interested in differential equations whose differential Galois groups is (up to a conjugation):

- The *Borel group* $\mathbb{B}$, i.e. the group of the matrices $\begin{pmatrix} c & b \\ 0 & c^{-1} \end{pmatrix}$, where $c \in C^*$ and $b \in C$. In the next chapter we try to characterize the set of differential equations (1.1) with $r \in C[x]$ and this Galois group.
- The multiplicative group $\mathbb{G}_{\mathrm{m}}$. Especially we are interested in the case $r \in C[x, x^{-1}]$ in chapter 3.
- The *infinite dihedral group* $\mathbb{D}_\infty$ of matrices $\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & -c \\ c^{-1} & 0 \end{pmatrix}$ with $c \in C^*$.
- A finite group.

From the differential Galois group one can decide whether the considered differential equation has solutions in "closed form" or not. Specifically, an extension $L \supset C(x)$ of differential fields is a *Liouvillian extension* if it is obtained in (possibly) several steps by a sequence of intermediate fields, so that each step is either an algebraic extension, or adjoining of an integral (that is, a solution of $y' = f$) or an exponential of an integral (a solution of $y' = f\,y$) of an element f of the extension constructed in the previous step. Solutions of a linear differential equation in an Liouvillian extension are called *Liouvillian* and they can be written in "closed form". It is known that the Picard-Vessiot extension of equation (1.1) is Liouvillian (so all solutions there are Liouvillian) if and only if the differential Galois group is a proper subgroup of $\mathrm{SL}(2, C)$, see [57, 37]. Otherwise the differential equation has no Liouvillian solutions.

The Kovacic algorithm (see [37]) determines the differential Galois group and a basis of Liouvillian solutions of (1.1) (or any other second order linear differential equation). This algorithm uses the fact that the differential Galois group is determined by the rational and algebraic (over $C(x)$) solutions of the *Riccati equation* associated to (1.1):

$$u' + u^2 = r, \qquad \text{with the same } r \in C(x), \tag{1.2}$$

see [53, 37]. One can check that if y is a solution of the differential equation (1.1) then the logarithmic derivative $u = y'/y$ is a solution of the Riccati equation. If $u \in C(x)$ is a rational solution of the Riccati equation, then we have one-dimensional linear space of Liouvillian solutions of (1.1) given by $y' = u\,y$. Such a linear space is invariant under action of the differential Galois group G of (1.1), and that makes the differential equation "reducible". (More precisely, the differential operator $\partial^2 - r$ associated to (1.1) is reducible since it has the right-hand factor $\partial - u$, see [31].) On the other hand, a one-dimensional G-invariant space of solutions in the Picard-Vessiot extension gives rise to a rational solution of the Riccati equation, see [57, 53]. In particular, if the differential Galois group G is isomorphic to the Borel group, then there is a G-invariant space in the space of solutions, hence there is a rational solution of the Riccati equation (1.2). Similarly, $G \cong \mathbb{G}_{\rm m}$ implies that there are two rational solutions of the Riccati equation.

An important step in the Kovacic algorithm is to construct a rational solution of the Riccati equation from "local" solutions at the "singular" points of the differential equation (1.1). In geometric terminology, the elements of $C(x)$ are rational functions on the projective line $\mathbb{P}^1$ over C. As a point set, $\mathbb{P}^1$ is $C \cup \{\infty\}$. For a point $\alpha \in \mathbb{P}^1$ let $z = x - \alpha$ if $\alpha \in C$, or $z = x^{-1}$ if $\alpha = \infty$. This is a *local parameter* at α. To consider equation (1.1) locally at α means to consider it over the differential field $C((z))$ of formal Laurent series at α. The derivation is defined by $z' = 1$ for $\alpha \in C$ and $z' = -z^{-2}$ for $\alpha = \infty$. The *differential Galois group* of (1.1) at α is the differential Galois group of (1.1) over $C((z))$, and a *local solutions* of (1.1) or the Riccati equation (1.2) is a solution in $C((z))$ or its extension. In this context the differential Galois group over $C(x)$ and the corresponding solutions are called *global*.

For an element $f \neq 0$ of $C(x)$ and a point $\alpha \in \mathbb{P}^1$ the *order* $\operatorname{ord}_\alpha(f)$ of f at α is defined as follows. Let z be the chosen local parameter at α. One can write $f = a_k z^k + a_{k+1} z^{k+1} + \ldots$ as an element of $C((z)) \supset C(x)$ with $a_k \neq 0$, by the inclusion $C(x) \subset C((z))$. Then $\operatorname{ord}_\alpha(f) = k \in \mathbb{Z}$. For example, $\operatorname{ord}_\infty(x) = -1$. We also use the module of the *differential forms* of $C(x)$ over C. This is a $C(x)$-linear space generated by symbols df with $f \in C(x)$, which satisfy the rules $d(f\,g) = f\,dg + g\,df$, $d(f+g) = df + dg$ and $dc = 0$ for $c \in C$. One can check that $df = f'\,dx$, hence this vector space is one-dimensional, generated by dx. For a differential form $\omega \neq 0$ the *order* $\operatorname{ord}_\alpha(\omega)$ and the *residue* $\operatorname{Res}_\alpha(\omega)$ at α are defined as follows. One can write $\omega = f\,dz$ for some $f \in C(x)$. Then $\operatorname{ord}_\alpha(\omega) := \operatorname{ord}_\alpha(f)$. Further, one can write again $f = a_k z^k + a_{k+1} z^{k+1} + \ldots$ with $k = \operatorname{ord}_\alpha(\omega)$. Then $\operatorname{Res}_\alpha(\omega) := a_{-1}$ (with the agreement that $a_{-1} = 0$ if $k \geq 0$). As an example, $\operatorname{ord}_\alpha(dx) = 0$ for $\alpha \in C$, and $\operatorname{ord}_\alpha(dx) = -2$ for $\alpha = \infty$. Also recall that for a non-zero $f \in C(x)$ we have $\operatorname{Res}_\alpha(\frac{1}{f}df) = \operatorname{ord}_\alpha(f)$.

From the inclusion $C(x) \subset C((z))$ one can deduce that a local differential Galois group can be embedded into the global Galois group. One of our problems is to determine the possible collection of local differential Galois groups when the global one is $\mathbb{G}_{\rm m}$, let

us say. The determination of a local differential Galois group is rather easy. The possibilities are summarized in [57], propositions 6.1 and 6.2. We reformulate them using the following terminology. A point $\alpha \in \mathbb{P}^1$ is called:
(a) *regular*, if $\mathrm{ord}_\alpha(r) + 2\,\mathrm{ord}_\alpha(dx) \geq 0$;
(b) *singular*, if $\mathrm{ord}_\alpha(r) + 2\,\mathrm{ord}_\alpha(dx) < 0$;
(c) *regular singular*, if $\mathrm{ord}_\alpha(r) + 2\,\mathrm{ord}_\alpha(dx) \geq -2$;
(d) *irregular (singular)*, if $\mathrm{ord}_\alpha(r) + 2\,\mathrm{ord}_\alpha(dx) < -2$.
In this terminology "regular" is a special case of "regular singular". At a regular singular point α we define the *local exponents* of equation (1.1) as follows. Let z be the local parameter at α. If $\alpha \in C$ then we may write $r = \sum_{n\geq -2} b_n z^n$; otherwise $r = \sum_{n\geq 2} b_n z^n$. The local exponents are the roots of the polynomial $\lambda(\lambda-1) - b_{-2}$ (for $\alpha \in C$) or $\lambda(\lambda+1) - b_2$ (for $\alpha = \infty$). If $\lambda \in C$ is a local exponent of (1.1) at α, then the differential equation usually has a local solution of form $z^\lambda \sum_{n\geq 0} a_n z^n$ with $a_0 \neq 0$.

The set of local exponents determines almost completely the local differential Galois group at a regular singular point, where the possibilities are: the multiplicative group $\mathbb{G}_{\mathrm{m}}$, a finite cyclic group, the additive group $\mathbb{G}_{\mathrm{a}} \cong C$, or $\{\pm 1\} \times \mathbb{G}_{\mathrm{a}}$, see [57]. Additionally, the differential Galois group at a regular point is trivial, and at an irregular singular point it is either $\mathbb{G}_{\mathrm{m}}$ or $\mathbb{D}_\infty$.

Summarizing, the Kovacic algorithm uses (some parts of) the local solutions of the Riccati equation (1.2) at the singular points to construct global rational solutions of it. The algebraic solutions of (1.2) are usually determined from rational solutions of the Riccati equations associated to some higher order linear differential equations determined by (1.1), so-called *symmetric powers* of it, see [53, 37, 64]. The global Galois group is finally determined by the set of global algebraic solutions of the Riccati equation.

The transformations "pull-backs" of differential equations mentioned at the beginning of this introduction can be properly defined in terms of homomorphisms of (non-commutative) rings of differential operators. The ring of differential operators over $C(x)$ is the *skew polynomial* ring $C(x)[\partial]$ with the multiplication given by "Leibnitz rule" $\partial\, x = x\,\partial + 1$, see [31]. In explicit terms, a *differential operator* is a finite expression $a_n\partial^n + a_{n-1}\partial^{n-1} + \ldots + a_1\partial + a_0$ with $a_k \in C(x)$ for $k = 0, 1, \ldots, n$. Considering $C(x)[\partial]$ as a $C(x)$-linear space with basis $1, \partial, \partial^2, \ldots$ we have the addition and the multiplication *from the left* of the differential operators. The multiplication is determined by the Leibnitz rule. In particular, $\partial\, f = f\,\partial + f'$. To a differential operator L one associates a linear differential equation $L(y) = 0$ by interpreting the symbol ∂ as the derivation operator $y \mapsto y'$ on $C(x)$. Properties of a differential operator and the corresponding linear differential equation (and vice versa) are closely related.

Let $C(x)[\partial_x]$ and $C(t)[\partial_t]$ be two (isomorphic) rings of differential operators defined as above. It is not a difficult exercise (see lemma 6.1 in [31]) to check that a homomorphism $C(x)[\partial_x] \to C(t)[\partial_t]$ of these skew polynomial rings is given by

$$x \mapsto f, \qquad \partial_x \mapsto \frac{1}{f'}\,\partial_t + h \tag{1.3}$$

for some f and h in $C(t)$. Here we keep the standard derivations on both fields $C(x)$ and $C(t)$ (so the restriction $C(x) \to C(t)$ of this map is *not* a homomorphism of differential fields).

The "pull-back" is a geometrical term. Let $\phi : C(x) \to C(t)$ be a homomorphism of fields. This homomorphism defines a *finite covering* $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ of the projective line with the rational parameter x, i.e. a non-constant map (a *finite morphism*, see [30] for details) from the projective line with the parameter t. Consider a differential equation $L\,y = 0$ over $C(x)$, where $L \in C(x)[\partial_x]$. A *pull-back (equation)* of this differential equation with respect to the finite covering Φ is a differential equation $\tilde{L}\,\tilde{y} = 0$ over $C(t)$, where $\tilde{L} = b\,\psi(L)$ with $\psi : C(x)[\partial_x] \to C(t)[\partial_t]$ being a homomorphism of the skew polynomial rings such that $\psi(x) = \phi(x)$, and $b \in C(t)$. In the terms of expression (1.3) the condition on ψ is $f = \psi(t)$. Note that a pull-back equation has the same order. The morphism $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ is the *pull-back morphism* for such a transformation. Further, a *normalized pull-back* of a differential equation (1.1) is a pull-back equation of the same form, i.e. $\tilde{y}'' = \tilde{r}\,\tilde{y}$ (with $\tilde{r} \in C(t)$). The normalized pull-back exists and is unique, because a second order differential equation $y'' + a\,y' + b\,y = 0$ over $C(t)$ can be (uniquely) transformed to this form by the automorphism $\partial_t \to \partial_t - a/2$ of $C(t)[\partial_t]$. As an example, a pull-back of the Airy equation $y'' - x\,y = 0$ with respect to the finite covering given by $\phi(x) = t^2$ is $\tilde{y}'' - \frac{1}{t}\,\tilde{y}' - 4\,t^4\,\tilde{y} = 0$, by taking $\psi(\partial_x) = \frac{1}{2t}\partial_t$ and $b = 4\,t^2$. The normalized pull-back in this case is $\tilde{y}'' = \left(4\,t^4 + \frac{3}{4\,t^2}\right)\,\tilde{y}$.

In chapter 4 we use pull-back morphsms to transform differential equations to the equations with an easier differential Galois group. However, the pull-backs are also useful in the other two chapters as well. In particular, let $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ be an automorphism of the projective line, so we identify here $C(x)$ and $C(t)$ (and the skew polynomial rings). The corresponding automorphism $\phi : C(x) \to C(x)$ of the rational function field is given by

$$\phi(x) = \frac{\beta_1\,x + \gamma_1}{\beta_2\,x + \gamma_2}, \qquad \beta_1, \beta_2, \gamma_1, \gamma_2 \in C, \quad \beta_1\,\gamma_2 - \beta_2\,\gamma_1 \neq 0. \tag{1.4}$$

Then the normalized pull-back of equation (1.1) is

$$\tilde{y}'' = \frac{(\beta_1\gamma_2 - \beta_2\gamma_1)^2}{(\beta_2\,x + \gamma_2)^4}\,\phi(r)\,\tilde{y}. \tag{1.5}$$

One can easily check that if $y \in C(x)$ is a solution of (1.1) then $(\beta_2\,x + \gamma_2)\,\phi(y)$ is a solution of the pull-back 1.5. (In the functional notation, $\phi(y) = y \circ \Phi$.) It follows that the (global) differential Galois group of the pull-back equation is the same as of (1.1). Moreover, the singular points of the pull-back are precisely $\Phi(\alpha)$, where α is a singular point of (1.1), and the local Galois group (and other "local properties") of (1.5) at $\Phi(\alpha)$ is the same as the one of (1.1) at α. For example, by considering in chapter 2 the differential equations with $r \in C[x]$ (i.e., one singular point $\alpha = \infty$) and the global Galois group $\mathbb{B}$, we essentially consider the equations with this Galois group and a single singularity at any other location $\alpha \in C$ due to the automorphisms $\Phi : \mathbb{P}^1 \to \mathbb{P}^1$ with $\Phi(\infty) = \alpha$ and their inverses.

Now we are ready to present the results of this thesis on differential equations. In the next two chapters we consider the sets (or geometrically speaking, "families") of differential equations (1.1) with fixed (global) differential Galois group and location of the singular points. The considered families are determined by the variation of the coefficients of $r \in C(x)$ of the differential equations (1.1) in the family, so they can be interpreted as differential equations *with parameters*. There are discrete parameters like the number and the order of the poles of a solution u of the Riccati equation (1.2) (associated to a differential equation in the family). After fixing the discrete parameters we are left with "continuous" families, where the parameter space has a structure of an *algebraic variety* (see [30]). In this thesis the algebraic structure is given in terms of *representable functors*, see chapter 2. In particular, there is a "universal family" of differential equations with the fixed properties, such that any family of the same kind is a specialization of the universal one.

In chapter 2 families of differential equations (1.1) with $r \in C[x]$ and the differential Galois group $\mathbb{B}$ are considered. We study the basic properties (such as dimension, smoothness, irreducibility) of the algebraic varieties above which the universal families are defined. In chapter 3 we consider mainly equations (1.1) with $r \in C[x, x^{-1}]$ and the global differential Galois group $\mathbb{G}_{\mathrm{m}}$. It turns out that there are very few possibilities for the combination of the local differential Galois groups at the singular points, and that (up to automorphisms of $\mathbb{P}^1$ given by $\phi(x) = \beta_2\, x$ in formula (1.4) there are only discrete families of these differential equations. The constraints on the local Galois groups are considered, more generally, for $r \in C(x)$ (and the same global Galois group). In chapter 4 we apply pull-back transformations in order to obtain differential equations with a simpler differential Galois group. In particular, the equations (1.1) with $r \in C[x, x^{-1}]$ and the Galois group $\mathbb{D}_\infty$ are transformed to equations with the differential Galois group $\mathbb{G}_{\mathrm{m}}$. Pull-backs are also applied to differential equations with finite Galois group, in particular, to equations (1.1) with three poles, and one equation of order three.

1.2 Splines

A *one-dimensional spline* on a closed interval $[A, B] \subset \mathbb{R}^2$ is a piecewise polynomial function on it, i.e., it is given by subdividing $[A, B]$ into several small closed intervals and prescribing a polynomial function on every small interval as its restriction to this interval. If the obtained function on $[A, B]$ is r times continuously differentiable then it is a C^r spline. Using these splines one can approximate functions on $[A, B]$, or curves $[A, B] \to \mathbb{R}^3$, by prescribing polynomial functions of rather small degree on the smaller intervals. The subject of study in the last two chapters of this thesis is *two-dimensional splines*. They are widely used in CAGD (Computer Aided Geometric Design) for the approximation and the construction of "smooth enough" surfaces.

In chapter 5 we consider a triangulation Δ of a region in the plane $\mathbb{R}^2$. A *bivariate* C^r spline on Δ is a real-valued function on the triangulated region with two properties: it is r times differentiable, and its restriction to each triangle is a polynomial function in the Cartesian coordinates x and y. The C^r splines on Δ form a real vector space $C^r(\Delta)$. In particular, their subspaces $C^r_k(\Delta)$ of splines defined by polynomials of degree at most k are finite-dimensional. For applications it is important to compute the dimension of such a vector space and to give an algorithm for the construction of a basis for it.

As in the work of L.J. Billera and L.L. Rose (see [6, 7]) we use algebraic methods to study these two problems. The space $C^r(\Delta)$ is also a module over the polynomial ring $\mathbb{R}[x, y]$. An important step is to homogenize the situation, i.e., to represent the restriction of a spline in $C^r_k(\Delta)$ to each triangle of Δ as a homogeneous polynomial in the graded ring $\mathbb{R}[x, y, z]$, by interpreting the variable z as the constant function 1. In

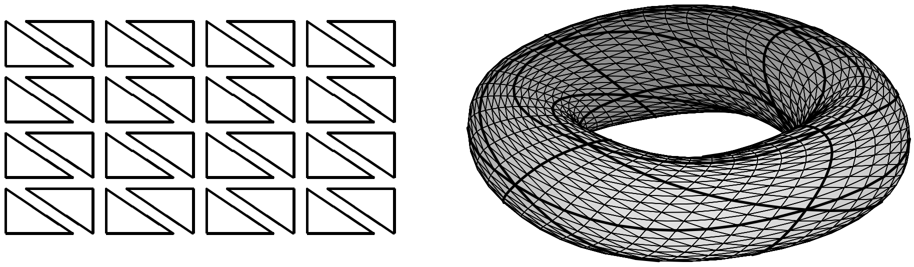

Figure 1.1: A CG^1 surface complex with a realization

this way one obtains the *graded* module $C^r(\hat{\Delta})$ over $\mathbb{R}[x,y,z]$, such that its kth graded part is isomorphic to $C^r_k(\Delta)$ as a real vector space via the substitution $z \mapsto 1$.

With the algebraic approach the dimension and bases of the spaces $C^r_k(\Delta)$ can often be easily computed from the generators (and their relations) of the graded module $C^r(\hat{\Delta})$. In [6] Billera and Rose represented the module $C^r(\hat{\Delta})$ as a kernel of a homomorphism between free graded $\mathbb{R}[x,y,z]$-modules. From this representation one can compute the generators and the relations of $C^r(\hat{\Delta})$ using Gröbner bases techniques. In chapter 5 of this thesis two representations of the same kind are proposed, where homomorphisms between free $\mathbb{R}[x,y,z]$-modules of much smaller rank are used. The more effective representation uses preliminary "local" computations of splines defined on triangles surrounding some vertices of Δ. The algorithm producing this representation is further modified in such a way that for some special triangulations often used in practice (in particular, they have a lot of edges with the same slope) it gives a map between free $\mathbb{R}[x,y,z]$-modules of (relatively) very small rank. We compare the efficiency of computations with the Billera-Rose and the proposed representations. The computations with the new representations turn out to be several times faster on a test example. For the sake of curiosity, the results of some computations (that is, the dimension series of some spline spaces) are briefly discussed.

In chapter 6 more general two-dimensional splines are considered. They are defined on so-called CG^1 (or *geometrically continuous*) *surface complexes*. Such an object is a collection of closed polygons (say, triangles) in the real plane, which are not fitted into a region in $\mathbb{R}^2$ as in chapter 5, but are given prescribed identification of polygonal edges and some glueing data (μ, Θ) for the pairs of identified edges. Specifically, μ is a homeomorphism between the identified edges, and Θ is an isomorphism of the two-dimensional "tangent" vector bundles on them. This data is enough to define CG^1 *functions* and *splines* (by assigning C^1 functions or polynomials to the polygons of the complex, with certain constraints given by the glueing data) with usual properties of C^1 functions on a C^1 surface. In particular, a triple of such CG^1 functions would give a map, defined on every polygon of the complex, to $\mathbb{R}^3$ such that generically the images of these polygons fit together into a C^1 surface in $\mathbb{R}^3$.

The study of such CG^1 splines naturally extends the concept of *geometric continuity* in CAGD, which deals with the most general situations when several *polygonal patches* (i.e. maps from a closed polygonal domain in $\mathbb{R}^2$ to an Euclidean space $\mathbb{R}^m$) form a C^r

surface in $\mathbb{R}^m$ in the sense of differential geometry. The necessary theory was developed in [29, 18] et al. In our setting the conditions for CG^1 *joining* of two polygonal patches are replaced by the similar "one-dimensional" conditions for CG^1 continuity of splines. We suggest to model a "geometrically continuous" surface in $\mathbb{R}^m$ as a *realization* of a CG^1 surface complex (that is, a suitable injective map from the complex to $\mathbb{R}^m$ defined by CG^1 functions).

For example, consider 32 triangles in $\mathbb{R}^2$ as on the left-hand side of figure 1.1. One can define such an identification of edges of the triangles that the obtained topological space is homeomorphic to the torus. As an intermediate step one identifies the edges of distinct triangles which are drawn close to each other on figure 1.1, and gets a rectangle. By identifying linearly the opposite sides of the rectangle in the classical way (see [41], or example 6.16 of this thesis) one obtains the torus. One can choose the CG^1 glueing data which copies the CG^1 glueing data of a suitable triangulation of a rectangle in $\mathbb{R}^2$ by a few vertical, horizontal and diagonal lines, and where the identification of the opposite edges is given by a horizontal or vertical translation. Then a realization of the obtained CG^1 surface complex can be constructed by approximating a known parametrization of a torus in $\mathbb{R}^3$ by the triangulated rectangle (by the CG^1 functions, of course). A realization of this kind is depicted on the right-hand side of figure 1.1. In the terminology of J.M. Hahn [29] it is a CG^1 surface patch complex. In this particular realization all 32 triangular patches have degree two, see example 6.31.

The first part of chapter 6 refines the definitions of geometrically continuous patching known in CAGD to our setting, basically following [29]. For instance, the definition of J.M. Hahn of CG^1 joining of several patches at a common vertex is translated into consistency restrictions on the glueing data for the pairs of polygonal edges incident to the "common vertex", see also [27]. It is shown that the topological space $\mathcal{X}$ (without boundary), determined by a CG^1 surface complex, has a unique structure of a C^1 surface such that the CG^1 functions on the complex correspond precisely to C^1 functions on $\mathcal{X}$.

In the second part of chapter 6 we restrict ourselves to special CG^1 surface complexes $\mathcal{M}$, named *Bézier complexes*, which are formed exclusively by triangles, and in each CG^1 glueing data (μ, Θ) the homeomorphism μ is linear, and Θ is expressed in rational functions in the common linear parameter of the identified edges, see definition 6.20. Similarly as in chapter 5 we are interested in finding the dimension and bases of the spaces of CG^1 splines on $\mathcal{M}$ defined by polynomials of bounded degree k. These spline spaces are denoted by $S^1_k(\mathcal{M})$. The situation is homogenized using the *barycentric coordinates* on each triangle. The main result (theorem 6.4.6) of chapter 6 is a formula for the dimension of $S^1_k(\mathcal{M})$ for sufficiently large k. A few algorithms are sketched for the construction of CG^1 surface complexes and for finding bases for $S^1_k(\mathcal{M})$ in chapter 6. The definitions and results are illustrated by a series of examples, which give us a number of realizations of the topological sphere and the torus.