Hankel norm approximation for infinite-dimensional systems

Sasane, Amol Jagannath

Publication date:
2001

Chapter 9

Concluding remarks

In this last chapter, we discuss the approach followed in this thesis to solve the sub-optimal Hankel norm approximation problem for infinite-dimensional systems.

In Chapter 4, we gave sufficient frequency domain conditions for the existence of a solution to the sub-optimal Hankel norm approximation problem and furthermore, we also gave a parameterization of all solutions. Subsequently in Chapters 6 and 7, we solved the sub-optimal Hankel norm approximation problem for the exponentially stable smooth Pritchard-Salamon class and the class of exponentially stable analytic systems, by explicitly verifying the frequency domain conditions given in Chapter 4. Finally, in Chapter 8, under certain conditions, we gave a solution to the sub-optimal Hankel norm approximation problem in the case when the infinitesimal generator is not necessarily exponentially stable.

The main advantage of our approach in this thesis is that it is a self-contained, elementary solution to the sub-optimal Hankel norm approximation problem. Moreover, our results in Chapter 4 hold for a class larger than the Wiener class. However, the disadvantage is that we had to show that $\Lambda_{11}(-\cdot)^{-1}$ has at most l unstable poles in the two main classes of systems that we considered in this thesis. This introduced a lot of extra work, and one wonders if it was all necessary and if there was a way around it. However, an interesting byproduct was the birth of new inertia results for operator Lyapunov inequalities and equations. Although our results in Chapter 4 hold for a class larger than the Wiener class, unfortunately, except for the results in Chapter 8, most of our examples have $h \in L_1$!

One might conjecture that an approach similar to the one in this thesis would yield solutions to the sub-optimal Hankel norm approximation problem for more general classes of well-posed linear systems than the ones considered in this thesis. However, we suspect this task to be a formidable one, since we need the smoothness properties of the controllability and observability Gramians to verify the candidate spectral factor formulas given in this thesis.

This thesis also gives alternate state-space formulae for the sub-optimal Hankel norm approximation problem for finite-dimensional systems. We conclude this thesis by explicitly working out an elementary finite-dimensional example, which elucidates the procedure involved and the theorems in Chapters 6 and 7.

Example 9.0.4 *(An elementary finite-dimensional example.)* Consider the rational function

$$G(s) = \frac{1}{\left(s+\frac{1}{2}\right)^2},$$

which has the minimal (and hence controllable) state-space realization of MacMillan degree 2 given by

$$A = \begin{bmatrix} -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The impulse response is $h(t) = Ce^{tA}B = te^{-\frac{1}{2}t}$ for $t \geq 0$ and its controllability and observability Gramians are given by

$$L_B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } L_C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

respectively. We have

$$L_B L_C = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, \quad \sigma\left(L_B L_C\right) = \left\{3+2\sqrt{2}, 3-2\sqrt{2}\right\},$$

and so the Hankel singular values of the system are

$$\sigma_1 = \sqrt{2}+1 > \sigma_2 = \sqrt{2}-1.$$

Let $l = 1$ and $\sigma = 1 \in \left(\sqrt{2}-1, \sqrt{2}+1\right)$. Thus we seek a sub-optimal Hankel norm approximant of G of MacMillan degree at most 1. We have

$$\begin{aligned} V(s) &= \begin{bmatrix} I_p & 0 \\ 0 & \frac{1}{\sigma} I_m \end{bmatrix} - \frac{1}{\sigma^2} \begin{bmatrix} -CL_B \\ B' \end{bmatrix} (sI + A')^{-1} N'_\sigma \begin{bmatrix} C' & \frac{1}{\sigma} L_C B \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s-\frac{1}{2} & 0 \\ 1 & s-\frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2} & -\frac{s-1}{\left(s-\frac{1}{2}\right)^2} \\ \frac{s}{\left(s-\frac{1}{2}\right)^2} & 1 - \frac{\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2} \end{bmatrix}. \end{aligned}$$

With $Q(-s) = 0$, we obtain

$$\begin{aligned} K_0(-s) &= V_{12}(-s)V_{22}(-s)^{-1} \\ &= \frac{s+1}{\left(s+\frac{1}{2}\right)^2} \cdot \left[1 - \frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2}\right]^{-1} \\ &= \frac{s+1}{\left[s+\frac{1}{2}-\frac{1}{\sqrt{2}}\right]\left[s+\frac{1}{2}+\frac{1}{\sqrt{2}}\right]} \\ &= \frac{\frac{1}{2}+\frac{1}{2\sqrt{2}}}{s+\frac{1}{2}-\frac{1}{\sqrt{2}}} + \frac{\frac{1}{2}-\frac{1}{2\sqrt{2}}}{s+\frac{1}{2}+\frac{1}{\sqrt{2}}}. \end{aligned}$$

Thus one sub-optimal Hankel norm approximant of G is

$$G_0(s) = \frac{\frac{1}{2}+\frac{1}{2\sqrt{2}}}{s-\frac{1}{2}+\frac{1}{\sqrt{2}}},$$

and $\|G - G_0\|_H \leq 1$. ◊