Hankel norm approximation for infinite-dimensional systems

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Chapter 8

The non-exponentially stable case

8.1 Introduction

In the previous chapters the sub-optimal Hankel norm approximation problem was solved for various classes of infinite-dimensional systems under the assumption that A generates an *exponentially* stable strongly continuous semigroup. However, there exists an important class of systems with a transfer function $G \in H_\infty(\mathbb{C}^{p\times m})$, for which A does not generate an exponentially stable semigroup (see Examples 3.2.1, 3.2.2 and 3.3.4). In [18], approximating solutions to the optimal Hankel norm approximation problem were obtained without assuming exponential stability, but only for the case that the Hankel operator is nuclear, which is a strong assumption. In this chapter, we will solve the sub-optimal Hankel norm approximation problem for non-exponentially stable infinite-dimensional systems in terms of a solution to the sub-optimal Hankel norm approximation problem for an exponentially stable system which is obtained by shifting the generator of the semigroup of the original system.

8.2 The class of systems

The specific class of systems we consider in this chapter is defined below:

B1. $U = \mathbb{C}^m$, $Y = \mathbb{C}^p$.

B2. 1. $h \in L_2\left([0,\infty), \mathbb{C}^{p\times m}\right)$,

2. $G := \hat{h}$ is continuous and has a unique limit G_∞ at $\pm\infty$.

We remark that 2 implies that $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$. Consequently, from Proposition 8 (page 224, Keulen [89]) it follows that Γ_h is a bounded operator, $G \in H_\infty(\mathbb{C}^{p\times m})$ and $(H_G\hat{u})(i\omega) = \widehat{(\Gamma_h u)}(i\omega)$ for almost all $\omega \in \mathbb{R}$. Furthermore, we note that from 2, by applying Theorem 3.1.1 (in Chapter 3) we obtain that the Hankel operator with symbol G is compact.

B3. There exists a $\delta > 0$, such that for every η satisfying $0 < \eta \leq \delta$,

1. $G(\eta + \cdot) \in H^c_\infty(\mathbb{C}^{p\times m})$ has a compact Hankel operator $\Gamma^{[\eta]}$, with singular values $\sigma_1^{[\eta]} \geq \sigma_2^{[\eta]} \geq \ldots$, and
2. if $\sigma_{l+1}^{[\eta]} < \sigma < \sigma_l^{[l]}$, then $G(\eta + \cdot)$ has a solution to the sub-optimal Hankel norm approximation problem: $K^{[\eta]}(-\cdot) \in H^c_{\infty,l}(\mathbb{C}^{p\times m})$ such that $\left\|G(\eta + i\cdot) + K^{[\eta]}(i\cdot)\right\|_\infty \leq \sigma$.

In Section 8.3 we first prove a useful result which we need for the proof of our main theorem (Theorem 8.5.1). In Section 8.4 we prove a few properties of the class of systems that we consider which will be used in the proving our main Theorem 8.5.1 in Section 8.5.

8.3 A few useful results

The key to the proof of the new result is Corollary 8.3.2 which is an easy consequence of the following lemma.

Lemma 8.3.1 *If $G \in H_\infty(\mathbb{C}^{p\times m})$ and $\omega \mapsto G(i\omega) : \mathbb{R} \to \mathbb{C}^{p\times m}$ is uniformly continuous, then given any $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\sup_{\omega\in\mathbb{R}} \|G(i\omega) - G(\eta + i\omega)\| < \epsilon \text{ for all } \eta \text{ satisfying } 0 \leq \eta \leq \delta.$$

Proof It follows from Theorem 5.18 (page 96, M. Rosenblum and J. Rovnyak [74]) that

$$G(\eta + i\omega) = \frac{\eta}{\pi}\int_{-\infty}^{\infty} \frac{G(it)}{(t-\omega)^2 + \eta^2}dt, \quad \eta > 0.$$

Since for $\eta > 0$,

$$\frac{\eta}{\pi}\int_{-\infty}^{\infty} \frac{1}{(t-\omega)^2 + \eta^2}dt = 1,$$

and we have

$$\begin{aligned}\|G(i\omega) - G(\eta + i\omega)\| &= \left\| \frac{\eta}{\pi} \int_{-\infty}^{\infty} \frac{G(it)}{(t-\omega)^2 + \eta^2} dt - G(i\omega) \right\| \\ &= \left\| \frac{\eta}{\pi} \int_{-\infty}^{\infty} \frac{G(it) - G(i\omega)}{(t-\omega)^2 + \eta^2} dt \right\|.\end{aligned}$$

Choose a $\zeta > 0$ such that

$$\|G(it) - G(i\omega)\| < \frac{\epsilon}{2} \text{ for all } t \text{ and } \omega \text{ satisfying } |t - \omega| < \zeta.$$

Now choose a $\delta > 0$ such that for any η satisfying $0 \le \eta \le \delta$, we have

$$\left\| \frac{\eta}{\pi} \int_{\mathbb{R} \setminus [\omega - \zeta, \omega + \zeta]} \frac{1}{(t-\omega)^2 + \eta^2} dt \right\| < \frac{\epsilon}{4 \, \|G(i\cdot)\|_\infty}.$$

Thus

$$\begin{aligned}&\|G(i\omega) - G(\eta + i\omega)\| \\ =\ & \left\| \frac{\eta}{\pi} \int_{-\infty}^{\infty} \frac{G(it) - G(i\omega)}{(t-\omega)^2 + \eta^2} dt \right\| \\ \le\ & \frac{\eta}{\pi} \int_{\omega-\zeta}^{\omega+\zeta} \frac{\|G(it) - G(i\omega)\|}{(t-\omega)^2 + \eta^2} dt + \frac{\eta}{\pi} \int_{\mathbb{R} \setminus [\omega - \zeta, \omega + \zeta]} \frac{\|G(it) - G(i\omega)\|}{(t-\omega)^2 + \eta^2} dt \\ \le\ & \frac{\eta}{\pi} \int_{\omega-\zeta}^{\omega+\zeta} \frac{\frac{\epsilon}{2}}{(t-\omega)^2 + \eta^2} dt + \frac{\eta}{\pi} \int_{\mathbb{R} \setminus [\omega - \zeta, \omega + \zeta]} \frac{2\|G(i\cdot)\|_\infty}{(t-\omega)^2 + \eta^2} dt \\ \le\ & \frac{\epsilon}{2} 1 + 2 \, \|G(i\cdot)\|_\infty \, \frac{\epsilon}{4 \, \|G(i\cdot)\|_\infty} = \epsilon.\end{aligned}$$

Since the choice of ω is arbitrary, this completes the proof. ■

Corollary 8.3.2 *If* $G \in H_\infty(\mathbb{C}^{p\times m})$ *and* $G(i\cdot) : \mathbb{R} \to \mathbb{C}^{p\times m}$ *is continuous and has limits* $G_{\pm\infty}$ *at* $\pm\infty$*, then given any* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that* $\sup_{\omega\in\mathbb{R}} \|G(i\omega) - G(\eta + i\omega)\| < \epsilon$ *whenever* $0 \le \eta \le \delta$*.*

Proof Given any $\epsilon > 0$, there exist $M_1 > 0$ and $M_2 > 0$ such that

$$\sup_{\omega\in[M_2,\infty)} \|G(i\omega) - G_\infty\| < \frac{\epsilon}{2}, \text{ and} \tag{8.1}$$

$$\sup_{\omega\in(-\infty,M_1]} \|G(i\omega) - G_{-\infty}\| < \frac{\epsilon}{2}. \tag{8.2}$$

Moreover, since $G(i\cdot) : \mathbb{R} \to \mathbb{C}^{p\times m}$ is continuous, it is uniformly continuous in $[M_1 - 1, M_2 + 1]$, and so given any $\epsilon > 0$, there exists a δ such that $1 > \delta > 0$ and whenever ω_1 and ω_2 lie in $[M_1 - 1, M_2 + 1]$ and $|\omega_1 - \omega_2| < \delta$,

$$\|G(i\omega_1) - G(i\omega_2)\| < \epsilon. \tag{8.3}$$

Thus it follows from (8.1), (8.2) and (8.3) that whenever ω_1 and ω_2 belong to $\mathbb{R}$ and $|\omega_1 - \omega_2| < \delta$, then $\|G(i\omega_1) - G(i\omega_2)\| < \epsilon$. Hence $G(i\cdot) : \mathbb{R} \to \mathbb{C}^{p\times m}$ is uniformly continuous, and so the result follows from Lemma 8.3.1. ■

8.4 Properties of the system

Next we prove a few properties of our class of systems which will be used in the sequel. If $0 < \eta \leq \delta$, then the time-domain Hankel operator corresponding to $G(\eta + \cdot) \in H_\infty^c(\mathbb{C}^{p\times m})$, that is, with kernel [1] $h^{[\eta]}(\cdot) = e^{-\eta\cdot}h(\cdot) \in L_1([0,\infty),\mathbb{C}^{p\times m}))$, is denoted by $\Gamma^{[\eta]}$, and the l^{th} Hankel singular value is denoted by $\sigma_l^{[\eta]}$.

Lemma 8.4.1 *If an infinite-dimensional system satisfies the assumptions B1-3, then*

1. $\left[\Gamma^{[\eta]}\right]^* \Gamma^{[\eta]} \to \Gamma^*\Gamma$ *uniformly as* $\eta \to 0$,
2. *Given any* $l \in \mathbb{N}$, $\sigma_l^{[\eta]} \to \sigma_l$ *as* $\eta \to 0$.

Proof 1. Let $S_1^{[\eta]} : L_2([0,\infty),\mathbb{C}^m) \to L_2([0,\infty),\mathbb{C}^m)$ be the multiplication operator by $e^{-\eta t}$: If $u \in L_2([0,\infty),\mathbb{C}^m)$, then

$$\left[S_1^{[\eta]}u\right](t) = e^{-\eta t}u(t) \text{ for all } t \geq 0.$$

Similarly, let $S_2^{[\eta]} : L_2([0,\infty),\mathbb{C}^p) \to L_2([0,\infty),\mathbb{C}^p)$ be the multiplication operator by $e^{-\eta t}$. Then it can be checked that $\Gamma^{[\eta]} = S_2^{[\eta]}\Gamma S_1^{[\eta]}$.

Sublemma 8.4.2 $S_1^{[\eta]} = \left[S_1^{[\eta]}\right]^* \to I_m$ *strongly as* $\eta \to 0$, *and* $S_2^{[\eta]} = \left[S_2^{[\eta]}\right]^* \to I_p$ *strongly as* $\eta \to 0$.

Proof Let $0 \neq u \in L_2([0,\infty),\mathbb{C}^m)$. Given any $\epsilon > 0$, choose a $M > 0$ such that

$$\int_M^\infty \|u(t)\|^2 dt < \frac{\epsilon^2}{2}.$$

Now choose a $\delta > 0$ such that $0 \leq \eta < \delta$ implies that

$$\sup_{t\in[0,M]} \left|e^{-\eta t} - 1\right| < \frac{\epsilon}{\sqrt{2}\,\|u\|_2}.$$

[1] since $h \in L_2([0,\infty),\mathbb{C}^{p\times m}))$ and $e^{-\eta\cdot} \in L_2(0,\infty)$, it follows from the Cauchy-Schwarz inequality that $h^{[\eta]}$ is integrable

Thus

$$
\begin{aligned}
& \left\| S_1^{[\eta]} u - u \right\|^2 \\
= \; & \int_0^\infty \left\| \left[e^{-\eta t} - 1 \right] u(t) \right\|^2 dt \\
\leq \; & \left[\sup_{t\in[0,M]} \left| e^{-\eta t} - 1 \right| \right]^2 \int_0^M \|u(t)\|^2 dt + \int_M^\infty \left| e^{-\eta t} - 1 \right|^2 \|u(t)\|^2 dt \\
\leq \; & \frac{\epsilon^2}{2} + \int_M^\infty \|u(t)\|^2 dt \; \leq \; \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} \; = \epsilon^2 .
\end{aligned}
$$

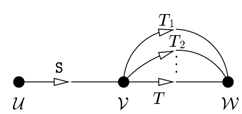
Hence $S_1^{[\eta]} = \left[S_1^{[\eta]} \right]^* \to I_m$ strongly as $\eta \to 0$. Similarly, $S_2^{[\eta]} = \left[S_2^{[\eta]} \right]^* \to I_p$ strongly as $\eta \to 0$ strongly. ■

Since $\left[\Gamma^{[\eta]} \right]^* \Gamma^{[\eta]} = S_1^{[\eta]} \Gamma^* S_2^{[\eta]} \; S_2^{[\eta]} \Gamma S_1^{[\eta]}$, we have

$$
\left[\Gamma^{[\eta]} \right]^* \Gamma^{[\eta]} = S_1^{[\eta]} \left[\Gamma^* S_2^{[\eta]} S_2^{[\eta]} \Gamma - \Gamma^* \Gamma \right] S_1^{[\eta]} + S_1^{[\eta]} \Gamma^* \Gamma S_1^{[\eta]} . \tag{8.4}
$$

Defining $\mathtt{K} = [\Gamma^* \Gamma]^{\frac{1}{2}}$, we have that $\mathtt{K}$ is compact, since $\mathtt{K}^2$ is compact and $\mathtt{K}$ is self-adjoint (using Exercise 18, page 127, Gohberg and Goldberg [43]). We will use the following (Exercise 6.6′, page 136, Weidmann [91]).

Sublemma 8.4.3 *Let $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ be Hilbert spaces. Suppose that T_n, $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ and that $T_n \to T$ strongly. If $\mathtt{S} \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ be compact, then*



$T_n \mathtt{S} \to T \mathtt{S}$ uniformly.

$S_1^{[\eta]} \mathtt{K} \to \mathtt{K}$ uniformly, using the Sublemmas 8.4.2 and 8.4.3. But $\mathtt{K} S_1^{[\eta]} = \left[S_1^{[\eta]} \mathtt{K} \right]^*$, and $\left[S_1^{[\eta]} \mathtt{K} \right]^* \to \mathtt{K}^* \; (= \mathtt{K})$ uniformly, and so $\mathtt{K} S_1^{[\eta]} \to \mathtt{K}$ uniformly. Consequently,

$$
S_1^{[\eta]} \Gamma^* \Gamma S_1^{[\eta]} \to \Gamma^* \Gamma \;\text{ uniformly as }\; \eta \to 0. \tag{8.5}
$$

Using the Sublemmas 8.4.2 and 8.4.3, we obtain that $S_2^{[\eta]} \Gamma \to \Gamma$ uniformly. Thus $\Gamma^* S_2^{[\eta]} = \left[S_2^{[\eta]} \Gamma \right]^* \to \Gamma^*$ uniformly. As a result, $\Gamma^* S_2^{[\eta]} \; S_2^{[\eta]} \Gamma \to \Gamma^* \Gamma$ uniformly, and since $\left\| S_1^{[\eta]} \right\| \leq 1$, we have

$$
S_1^{[\eta]} \left[\Gamma^* S_2^{[\eta]} S_2^{[\eta]} \Gamma - \Gamma^* \Gamma \right] S_1^{[\eta]} \to 0 \;\text{ uniformly as }\; \eta \to 0. \tag{8.6}
$$

From (8.4), (8.6) and (8.5), it follows that $\left[\Gamma^{[\eta]}\right]^* \Gamma^{[\eta]} \to \Gamma^*\Gamma$ uniformly as $\eta \to 0$.

2. Since $\left[\Gamma^{[\eta]}\right]^* \Gamma^{[\eta]} \to \Gamma^*\Gamma$ uniformly as $\eta \to 0$, $\sigma_l^{[\eta]} \to \sigma_l$ as $\eta \to 0$ (using Corollary 4.(a), page 1090, Dunford and Schwartz [37]). ■

8.5 Sub-optimal Hankel norm approximation

In this section, we will prove our main result about the existence of solutions to the sub-optimal Hankel norm approximation problem.

Theorem 8.5.1 *If an infinite-dimensional system satisfies the assumptions B1-3, and if* $\sigma_{l+1} < \sigma < \sigma_l$*, then there exists a* $\eta_0 > 0$ *such that for every* η *satisfying* $0 < \eta < \eta_0$*,* $\left\|G(i\cdot) + K^{[\eta]}(i\cdot)\right\|_\infty \le \sigma$.

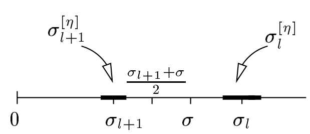
Proof 1. Let $\epsilon = \frac{\sigma - \sigma_{l+1}}{2} > 0$. Choose a $\delta_1 > 0$ small enough so that whenever $0 \le \eta < \delta_1$,

$$\sup_{\omega \in \mathbb{R}} \|G(i\omega) - G(\eta + i\omega)\| < \epsilon.$$

This can be done, owing to assumption B3 and Corollary 8.3.2.

2. Next choose a $\delta_2 > 0$, such that whenever $0 \le \eta < \delta_2$, we have (see Lemma 8.4.1.2)

$$\sigma_{l+1}^{[\eta]} < \frac{\sigma_{l+1} + \sigma}{2} < \sigma_l^{[\eta]}.$$



3. Let $\eta_0 = \min\{\delta_1, \delta_2\}$ and consider any η satisfying $0 < \eta < \eta_0$. From B3, we have the existence of a $K^{[\eta]}(-\cdot) \in H^c_{\infty,[l]}(\mathbb{C}^{p \times m})$ such that

$$\sup_{\omega \in \mathbb{R}} \left\|G(i\omega + \eta) + K^{[\eta]}(i\omega)\right\| \le \frac{\sigma + \sigma_{l+1}}{2}.$$

Thus,

$$\left\|G(i\omega) + K^{[\eta]}(i\omega)\right\| = \left\|G(i\omega) - G(i\omega + \eta) + G(i\omega + \eta) + K^{[\eta]}(i\omega)\right\|$$

$$
\begin{aligned}
&\leq \|G(i\omega) - G(i\omega+\eta)\| + \left\|G(i\omega+\eta) + K^{[\eta]}(i\omega)\right\| \\
&\leq \frac{\sigma-\sigma_{l+1}}{2} + \frac{\sigma_{l+1}+\sigma}{2} \\
&= \sigma.
\end{aligned}
$$

This completes the proof. ■

We now apply the results from Chapters 6 and 7 to obtain explicit formulae for the solutions of the sub-optimal Hankel norm approximation problem in the case when A is not necessarily exponentially stable. We consider the following two classes:

Class A: The triple (A, B, C) satisfies the following assumptions:

1. $\Sigma(A, B, C)$ is a smooth Pritchard-Salamon system with input space $\mathbb{C}^m$ and output space $\mathbb{C}^p$,
2. The impulse response h satisfies B2,
3. For every $\eta > 0$, $A - \eta I$ is the infinitesimal generator of an exponentially stable, strongly continuous semigroup.
4. For every $\eta > 0$, $(A - \eta I, B)$ is approximately controllable, that is, $\left(\mathcal{B}^{[\eta]}\right)'$ $[\in \mathcal{L}(W', L_2([0,\infty), \mathbb{C}^m))]$ [2] is injective.

Remarks:

1. We note that assumption 3 is satisfied if either of the following is true:

   $\mathfrak{A}1$. $\sigma(A) \cap \mathbb{C}_0^+$ is empty and A satisfies the spectrum determined growth assumption[3].

   $\mathfrak{A}2$. A generates a contraction semigroup[4].

   We note that from Exercise 2.4 (page 81, Curtain and Zwart [34]), it follows that if A generates the semigroup $\{T(t)\}_{t\geq 0}$, then $A - \eta I$ is the infinitesimal generator of the strongly continuous semigroup $\{e^{-\eta t}T(t)\}_{t\geq 0}$

[2] Here $\mathcal{B}^{[\eta]}$ denotes the controllability map of the exponentially stable smooth Pritchard-Salamon system $\Sigma(A - \eta I, B, C)$.

[3] A is said to satisfy the *spectrum determined growth assumption* if the growth bound of the semigroup, ω_0 equals $\sup_{\lambda \in \sigma(A)} \mathrm{Re}(\lambda)$

[4] That is, $\|T(t)\| \leq 1$ for all $t \geq 0$

with a growth bound equal to the sum of the growth bound of $\{T(t)\}_{t\geq 0}$ and $-\eta$. Thus if either $\mathfrak{A}1$ or $\mathfrak{A}2$ is satisfied, then for every $\eta > 0$, $A-\eta I$ generates the exponentially stable semigroup $\{e^{-\eta t}T(t)\}_{t\geq 0}$.

2. If the triple (A,B,C) belongs to Class A, then the system $\Sigma(A-\eta I,B,C)$ is an exponentially stable smooth Pritchard-Salamon system. We denote the controllability map of this system by $\mathcal{B}^{[\eta]}$, the observability map by $\mathcal{C}^{[\eta]}$, the Hankel operator by $\Gamma^{[\eta]}$, and the l^{th} Hankel singular value by $\sigma^{[\eta]}$. If $\sigma_{l+1}^{[\eta]} < \sigma < \sigma_l^{[\eta]}$, let $N_\sigma^{[\eta]} := \left(I-\sigma^{-2}L_B^{[\eta]}L_C^{[\eta]}\right)^{-1}$, where $L_B^{[\eta]} := \mathcal{B}^{[\eta]}\left[\mathcal{B}^{[\eta]}\right]'$ and $L_C^{[\eta]} := \left[\mathcal{C}^{[\eta]}\right]'\mathcal{C}^{[\eta]}$.

Class B: The triple (A,B,C) satisfies the following assumptions:

1. $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on the Hilbert space Z, $B\in\mathcal{L}(\mathbb{C}^m,Z_\alpha)$, where $-1<\alpha\leq 0$, and $C\in\mathcal{L}(Z,\mathbb{C}^p)$,
2. The well-posed linear system[5] given by $(-A,B,C)$ has an impulse response h that satisfies B2,
3. For every $\eta>0$, $-A-\eta I$ is the infinitesimal generator of an exponentially stable, strongly continuous semigroup.
4. For every $\eta>0$, $(A-\eta I,B)$ is approximately controllable, that is, given any $\eta>0$, there exists a $\beta_\eta\in\left(\alpha,\frac{1}{2}+\alpha\right)$ such that $\left(\mathcal{B}^{[\eta]}\right)'$ $\left[\in\mathcal{L}(Z'_{-\beta_\eta},L_2([0,\infty),\mathbb{C}^m))\right]$ [6] is injective.

Remarks:

1. We note that assumption 3 is satisfied if either of the following is true:

   $\mathfrak{B}1$. $\sigma(-A)\cap\mathbb{C}_0^+$ is empty and $-A$ satisfies the spectrum determined growth assumption.

   $\mathfrak{B}2$. $-A$ generates a contraction semigroup.

   We note that from Exercise 2.4 (page 81, Curtain and Zwart [34]), it follows that $-A-\eta I$ is the infinitesimal generator of the strongly continuous semigroup $\{e^{-\eta t}T(t)\}_{t\geq 0}$ with a growth bound equal to the sum

[5] The well-posedness follows, for instance, from Theorem 5.7.4.(ii) (page 251, Staffans [81]).
[6] Here $\mathcal{B}^{[\eta]}$ denotes the controllability map of the system given by the triple $(A-\eta I,B,C)$.

of the growth bound of $\{T(t)\}_{t\geq 0}$ and $-\eta$. Thus if either $\mathfrak{B}1$ or $\mathfrak{B}2$ is satisfied, then for every $\eta > 0$, $-A-\eta I$ generates the exponentially stable semigroup $\{e^{-\eta t}T(t)\}_{t\geq 0}$. Furthermore, from Corollary 2.2 (page 81, Pazy [67]) we obtain that in fact $-A-\eta I$ generates an *analytic* semigroup.

2. If the triple (A,B,C) belongs to Class B, then it is easy to see that the system given by the triple $(-A-\eta I,B,C)$ is a regular well-posed linear system satisfying the assumptions A1, A2, A3 listed in Section 2.5 of Chapter 2. We denote the controllability map of this system by $\mathcal{B}^{[\eta]}$, the observability map by $\mathcal{C}^{[\eta]}$, the Hankel operator by $\Gamma^{[\eta]}$, and the l^{th} Hankel singular value by $\sigma^{[\eta]}$. If $\sigma^{[\eta]}_{l+1} < \sigma < \sigma^{[\eta]}_{l}$, let $N^{[\eta]}_{\sigma} := \left(I-\sigma^{-2}L^{[\eta]}_{B}L^{[\eta]}_{C}\right)^{-1}$, where $L^{[\eta]}_{B} := \mathcal{B}^{[\eta]}\left[\mathcal{B}^{[\eta]}\right]'$ and $L^{[\eta]}_{C} := \left[\mathcal{C}^{[\eta]}\right]'\mathcal{C}^{[\eta]}$.

Theorem 8.5.2 *Suppose that the triple (A,B,C) is in Class* A. *Let $\sigma_{l+1} < \sigma < \sigma_l$. If $Q(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$, and $\|Q(i\cdot)\|_\infty \leq 1$, then there exists a $\delta > 0$ such that for every η satisfying $0<\eta<\delta$,*

$$K^{[\eta]}(-\cdot) = R^{[\eta]}_1(-\cdot)\left[R^{[\eta]}_2(-\cdot)\right]^{-1},$$

where

$$\begin{bmatrix} R^{[\eta]}_1(-\cdot) \\ R^{[\eta]}_2(-\cdot) \end{bmatrix} = \left[\Lambda^{[\eta]}(-\cdot)\right]^{-1}\begin{bmatrix} Q(-\cdot-\eta) \\ I_m \end{bmatrix}$$

and

$$\Lambda^{[\eta]}(\cdot) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} + \sigma^{-2}\begin{bmatrix} -CL^{[\eta]}_B \\ \sigma B' \end{bmatrix}\left[N^{[\eta]}_\sigma\right]'(\cdot I + A' - \eta I)^{-1}\begin{bmatrix} C' & L^{[\eta]}_C B \end{bmatrix}$$

is such that $K^{[\eta]}(-\cdot) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$ and $\left\|G(i\cdot)+K^{[\eta]}(i\cdot)\right\|_\infty \leq \sigma$.

Proof This is a consequence of Theorem 6.2.4 and Theorem 8.5.1. ■

Theorem 8.5.3 *Suppose that the triple (A,B,C) is in Class* B. *Let $\sigma_{l+1} < \sigma < \sigma_l$. If $Q(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$, and $\|Q(i\cdot)\|_\infty \leq 1$, then there exists a $\delta > 0$ such that for every η satisfying $0<\eta<\delta$,*

$$K^{[\eta]}(-\cdot) = R^{[\eta]}_1(-\cdot)\left[R^{[\eta]}_2(-\cdot)\right]^{-1},$$

where

$$\begin{bmatrix} R^{[\eta]}_1(-\cdot) \\ R^{[\eta]}_2(-\cdot) \end{bmatrix} = \left[\Lambda^{[\eta]}(-\cdot)\right]^{-1}\begin{bmatrix} Q(-\cdot-\eta) \\ I_m \end{bmatrix}$$

and

$$\Lambda^{[\eta]}(\cdot)=\begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix}+\sigma^{-2}\begin{bmatrix} -CL_B^{[\eta]} \\ \sigma B' \end{bmatrix}\left[N_\sigma^{[\eta]}\right]'(\cdot I - A' - \eta I)^{-1}\begin{bmatrix} C' & L_C^{[\eta]}B \end{bmatrix}$$

is such that $K^{[\eta]}(-\cdot) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$ *and* $\left\|G(i\cdot)+K^{[\eta]}(i\cdot)\right\|_\infty \leq \sigma.$

Proof This is a consequence of Theorem 7.2.11 and Theorem 8.5.1. ∎

Example 3.2.2 (continued) We revisit Example 3.2.2 from Chapter 3. The pair (A, B) is not approximately controllable, since the vector

$$\begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

is orthogonal to $B, AB, A^2B, \ldots$ (see for instance Exercise 4.6.e, page 194, Curtain and Zwart [34]). So we seek another realization of the transfer function G_0, which is approximately controllable.

Let $\mathcal{R}$ denote the closed linear span of $B, AB, A^2B, \ldots$ in $\ell_2(\mathbb{Z})$. This is an $A-$invariant subspace of $\ell_2(\mathbb{Z})$. Let A_1 denote the restriction of A to $\mathcal{R}$. Let $B_1 = B \in \mathcal{L}(\mathbb{C}, \mathcal{R})$, and $C_1 = B_1^* = \langle\cdot, B\rangle \in \mathcal{L}(\mathcal{R}, \mathbb{C})$.

1. Using Lemma 2.5.6 (page 70, Curtain and Zwart [34]), it is easy to see that $(sI - A)^{-1}B \in \mathcal{R}$ for all s in $\mathbb{C} \setminus [-i, i]$ (which is the maximal connected component of $\rho(A)$ that contains an interval $[r, \infty)$), and so
$$(sI - A_1)\,(sI - A)^{-1}B = (sI - A)(sI - A)^{-1}B = B.$$
Since $\rho(A) \subset \rho\,(A_1)$, from the above we have for $s \in \rho(A)$
$$G_0(s) = B^*(sI - A)^{-1}B = C_1\,(sI - A_1)^{-1}\,B_1.$$
Thus the triple (A_1, B_1, C_1) realizes G_0 and so $\Sigma\,(A_1 - B_1B_1^*, B_1, B_1^*)$ realizes the transfer function $G(s) = \frac{1}{1+\sqrt{s^2+1}}$.
2. $A_1 - B_1B_1^*$, B_1 and $C_1 := B_1^*$ are all bounded, and so $\Sigma(A_1 - B_1B_1^*, B_1, C_1)$ is a smooth Pritchard-Salamon system with $W = V = \mathcal{R}$.
3. Since $A_1 - B_1B_1^*$ is bounded, it generates the analytic semigroup $\left\{e^{A_1 - B_1B_1^*t}\right\}_{t\geq 0}$. Moreover, $B_1 \in \mathcal{L}(\mathbb{C}^m, Z_\alpha)$ with $m = 1$ and $\alpha = 0$.

4. Since $A_1 + A_1^* = 0$, it follows from Corollary 2.2.3 (page 33, Curtain and Zwart [34]) that A_1 generates a contraction semigroup. Thus from Lemma 2.2.6 (page 19, Oostveen [64]), it follows that the system given by the triple $(A_1 - B_1B_1^*, B_1, B_1^*)$ is output stable, that is, the map $z \mapsto B_1^* e^{\cdot(A_1 - B_1B_1^*)} z \in \mathcal{L}(\mathcal{R}, L_2(0,\infty))$. So in particular, with $z = B_1 \in \mathcal{R}$, we obtain $h(\cdot) = B_1^* e^{\cdot(A_1 - B_1B_1^*)} B_1 \in L_2(0,\infty)$.

5. Since $A_1 - B_1B_1^*$ generates a contraction semigroup (see Lemma 2.2.6.P1, page 19, Oostveen [64]), it follows that $A_1 - B_1B_1^* - \eta I$ generates and exponentially stable, strongly continuous semigroup for every $\eta > 0$.

6. Furthermore, we have already verified in Chapter 3 that G is continuous on the imaginary axis with the unique limit 1 at $\pm i\infty$.

7. Finally, since the pair (A_1, B_1) is approximately controllable (see Exercise 4.6.e, page 194, Curtain and Zwart [34]), it follows from the proofs[7] of the parts a and b of Lemma 4.1.6 (pages 146-147, Curtain and Zwart [34]) that the pair $(A_1 - B_1B_1^* - \eta I, B_1)$ is approximately controllable for every $\eta > 0$.

Hence this example fits in both classes A and B considered in the previous section.
$\Diamond$

[7] In Lemma 4.1.6, approximate controllability on an interval $[0, \tau]$ is considered and so we cannot use this lemma directly. However, in the proof, it is shown that the ranges of the controllability maps of the three pairs (A, B), $(A + BF, B)$ and $(\mu I + A, B)$ are the same! Now we use the definition of approximate controllability (see Definition 4.1.17 on page 157 and Theorem 4.1.22.a on page 160 of [34]).