# Hankel norm approximation for infinite-dimensional systems

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# Chapter 7

# $J-$spectral factorization for the analytic class

## 7.1 Introduction

In this chapter, we show the existence of a solution to the key $J-$spectral factorization problem satisfying S1-6 in Chapter 4 for the exponentially stable analytic class of infinite-dimensional systems discussed in Section 2.5 of Chapter 2 with finite-dimensional input and output spaces: throughout this chapter, we consider the class of approximately controllable systems described by the triple $(-A, B, C)$ satisfying A1, A2, A3 with $U = \mathbb{C}^m$, $Y = \mathbb{C}^p$ (see page 28). We list all the assumptions below:

A1. $-A$ is the infinitesimal generator of an exponentially stable analytic semigroup $\{T(t)\}_{t \geq 0}$ on the Hilbert space $Z$.

A2. $B \in \mathcal{L}(\mathbb{C}^m, Z_\alpha)$, where $\alpha$ is a fixed number in $(-1, 0]$.

A3. $C \in \mathcal{L}(Z, \mathbb{C}^p)$.

A4. $\sigma_{l+1} < \sigma < \sigma_l$.

A5. $(A, B)$ is approximately controllable, that is, there exists a $\beta \in \left(\alpha, \frac{1}{2} + \alpha\right)$ such that $\mathcal{B}' \left[\in \mathcal{L}\left(Z'_{-\beta}, L_2([0,\infty), \mathbb{C}^m\right)\right]$ is injective.

We remark that A5 above implies that $\mathcal{B}' \left[\in \mathcal{L}\left(Z'_{-\beta}, L_2([0,\infty), \mathbb{C}^m\right)\right]$ is injective for all $\beta \leq \alpha$. Furthermore, this is equivalent with the condition that $L_B = \mathcal{B}\mathcal{B}' \left[\in \mathcal{L}\left(Z'_{-\beta}, Z_\beta\right)\right]$ is injective for all $\beta \leq \alpha$.

## 7.2 A $J-$spectral factor

We construct a solution $\Lambda$ to the $J-$spectral factorization problem (4.2) which has the properties S3-6. The spectral factor $\Lambda$ is the same one as in Curtain and Ichikawa [20] (in which they consider the Nehari problem for a similar class with some differences; see also Table 2.1), but for the sub-optimal Hankel norm approximation problem we have $\sigma_{l+1} < \sigma < \sigma_l$ and we have to check that $\Lambda_{11}(-\cdot)^{-1} \in H^c_{\infty,l}(\mathbb{C}^{p\times p})$. Extensive use of the results of Section 2.5 is made to show that all the conditions in Chapter 4 are satisfied. The candidate solution is described by the quadruple

$$\left(-A', \left[\begin{array}{cc} C' & L_C B \end{array}\right], \frac{1}{\sigma^2}\left[\begin{array}{c} CL_B \\ -\sigma B' \end{array}\right] N'_\sigma, \left[\begin{array}{cc} I_p & 0 \\ 0 & \sigma I_m \end{array}\right]\right).$$

First we show that this defines a regular linear system (see Section 2.2), and that its transfer function is invertible.

**Lemma 7.2.1** *Let the triple* $(-A,B,C)$ *satisfy A1-4. Let* $\beta$ *be a fixed number satisfying* $-1 < \beta < \alpha$. *Choose any* $\gamma$ *satisfying* $-\beta - \frac{1}{2} < \gamma < \frac{1}{2}$. *Then*

$$\Sigma_1 = \left(-A', \left[\begin{array}{cc} C' & L_C B \end{array}\right], \frac{1}{\sigma^2}\left[\begin{array}{c} CL_B \\ -\sigma B' \end{array}\right] N'_\sigma, 0\right), \tag{7.1}$$

*and*

$$\Sigma_2 = \left(-A', N'_\sigma\left[\begin{array}{cc} C' & \frac{1}{\sigma}L_C B \end{array}\right], \frac{1}{\sigma^2}\left[\begin{array}{c} CL_B \\ -B' \end{array}\right], 0\right) \tag{7.2}$$

*are regular linear systems with state space* $Z'_\gamma$, *input space* $\mathbb{C}^{p+m}$, *and output space* $\mathbb{C}^{p+m}$. *Denote the transfer function of* $\Sigma_1$ *by*

$$G_1(\cdot) = \frac{1}{\sigma^2}\left[\begin{array}{c} CL_B \\ -\sigma B' \end{array}\right] N'_\sigma(\cdot I + A')^{-1}\left[\begin{array}{cc} C' & L_C B \end{array}\right] \in H^c_\infty\left(\mathbb{C}^{(p+m)\times(p+m)}\right). \tag{7.3}$$

*Then we have*

$$\lim_{\substack{s\to\infty \\ s\in\overline{\mathbb{C}_+}}} G_1(s) = 0. \tag{7.4}$$

*Furthermore, if*

$$\Lambda(-s) := \left[\begin{array}{cc} I_p & 0 \\ 0 & \sigma I_m \end{array}\right] + G_1(s) \text{ for all } s \in \overline{\mathbb{C}_0^+}, \tag{7.5}$$

*then* $\Lambda(-\cdot)$ *is an element in* $H^c_\infty\left(\mathbb{C}^{(p+m)\times(p+m)}\right)$, *it is invertible over* $H^c_\infty\left(\mathbb{C}^{(p+m)\times(p+m)}\right)$ *and its inverse is given by* $V(-\cdot)$, *where*

$$V(-\cdot) = \left[\begin{array}{cc} I_p & 0 \\ 0 & \frac{1}{\sigma}I_m \end{array}\right] - G_2(\cdot),$$

*and* $G_2 \in H^c_\infty \left(\mathbb{C}^{(p+m)\times(p+m)}\right)$ *denotes the transfer function of* $\Sigma_2$*:*

$$G_2(s) = \frac{1}{\sigma^2} \begin{bmatrix} CL_B \\ -B' \end{bmatrix} (sI + A')^{-1} N'_\sigma \begin{bmatrix} C' & \frac{1}{\sigma} L_C B \end{bmatrix} \tag{7.6}$$

*for all* $s \in \overline{\mathbb{C}_0^+}$.

**Proof** To make sense of the terms $B'N'_\sigma$ and $L_B N'_\sigma$, we first note by appealing to Lemma 2.5.8 that $N_\sigma \in \mathcal{L}(Z_\beta)$ and $N'_\sigma \in \mathcal{L}\left(Z'_{-\beta}\right)$ are well-defined for any $\beta$ satisfying $-1 < \beta < \alpha$. From Lemma 2.5.1.2, it follows that $L_B \in \mathcal{L}\left(Z'_{-\gamma}, Z\right)$ for $\gamma < 1 + 2\alpha$, and so $L_B N'_\sigma \in \mathcal{L}\left(Z'_{-\beta}, Z\right)$ for $\beta < 1 + 2\alpha$. Here we remark that if $\beta$ satisfies $-1 < \beta < \alpha$, then in particular $\beta < 1 + 2\alpha$. Since $B \in \mathcal{L}(\mathbb{C}^m, Z_\beta)$ for $-1 < \beta < \alpha$, we have $B'N'_\sigma \in \mathcal{L}\left(Z'_{-\beta}, \mathbb{C}^m\right)$. Furthermore, if $-1 < \beta < \alpha$, then using Lemma 2.5.2.2, we obtain that $L_C \in \mathcal{L}(Z_\beta, Z)$, and so $L_C B \in \mathcal{L}(\mathbb{C}^m, Z)$. Consequently, if $-1 < \beta < \alpha$, then we have:

$$\begin{bmatrix} C' & L_C B \end{bmatrix} \in \mathcal{L}\left(\mathbb{C}^{p+m}, Z\right), \quad \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N'_\sigma \in \mathcal{L}\left(Z'_{-\beta}, \mathbb{C}^{p+m}\right).$$

We now fix a $\gamma$ satisfying $-\beta - \frac{1}{2} < \gamma < \frac{1}{2}$. We remark that such a choice is possible, since $-1 < \beta$. The freedom in the choice of $\gamma$ is depicted in Figure 7.1, where we have one of the possible regions depending on the value of $\alpha$.

$$\texttt{Case 1}: -1 < \alpha < -\frac{1}{2}, \quad \texttt{Case 2}: \alpha = -\frac{1}{2}, \quad \texttt{Case 3}: -\frac{1}{2} < \alpha \leq 0.$$

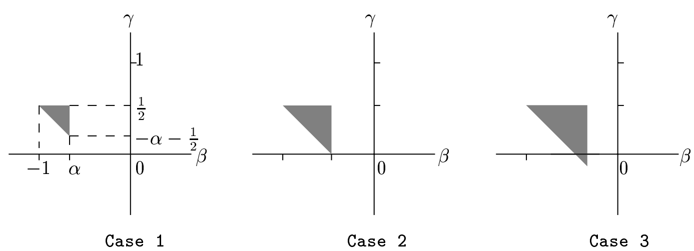


Figure 7.1: The possible cases.

From Theorem 2.5.4 with $\alpha_B = 0$ and $\alpha_C = -\beta$ (which satisfies the condition $\alpha_B \leq \alpha_C < \alpha_B + 1$), it follows that $\Sigma_1$ is a regular linear system on $Z'_\gamma$ with a transfer function $G_1(\cdot)$ given by (7.3), which satisfies $\lim_{\substack{s\to\infty \\ s\in\overline{\mathbb{C}_+}}} G_1(s) = 0$.

Similarly, it can be seen, again using Theorem 2.5.4, that $\Sigma_2$ is a regular linear system on $Z'_\gamma$ with a transfer function $G_2(\cdot)$ given by (7.6).

Finally we have to show that $\Lambda(-\cdot)V(-\cdot) = I_{p+m} = V(-\cdot)\Lambda(-\cdot)$. Consider

$$\begin{aligned}
&\Lambda(-s)V(-s) - \begin{bmatrix} I_p & 0 \\ 0 & I_m \end{bmatrix} \\
= \; & \frac{1}{\sigma^2}\begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N_\sigma' \left(sI + A'\right)^{-1} \begin{bmatrix} C' & \frac{1}{\sigma}L_C B \end{bmatrix} \\
& -\frac{1}{\sigma^2}\begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} \left(sI + A'\right)^{-1} N_\sigma' \begin{bmatrix} C' & \frac{1}{\sigma}L_C B \end{bmatrix} \\
& -\frac{1}{\sigma^4}\begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N_\sigma' \left(sI + A'\right)^{-1} \begin{bmatrix} C' & \frac{1}{\sigma}L_C B \end{bmatrix} \\
& \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} \left(sI - A'\right)^{-1} N_\sigma' \begin{bmatrix} C' & \frac{1}{\sigma}L_C B \end{bmatrix}.
\end{aligned}$$

We have

$$\begin{bmatrix} C' & \frac{1}{\sigma}L_C B \end{bmatrix} \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} = C'CL_B - L_C BB'.$$

Furthermore, it can be verified that for $z \in Z'_{-\beta}$, where $-1 < \beta < \alpha$, there holds

$$\begin{aligned}
C'CL_B z - L_C BB'z &= A'L_C L_B z - L_C L_B A'z \\
&= \sigma^2 \left(N_\sigma'\right)^{-1} \left(sI + A'\right) z - \sigma^2 \left(sI + A'\right) \left(N_\sigma'\right)^{-1} z
\end{aligned}$$

in $Z'_{-\beta-1}$. This can be checked by using the Lyapunov equations (2.16) and (2.24), and Lemmas 2.5.1.2 and 2.5.2.2. Consequently, we obtain

$$\begin{aligned}
\Lambda(-s)V(-s) - \begin{bmatrix} I_p & 0 \\ 0 & I_m \end{bmatrix} &= \frac{1}{\sigma^2}\begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N_\sigma' \left(sI + A'\right)^{-1} \begin{bmatrix} C' & \frac{1}{\sigma}L_C B \end{bmatrix} \\
& -\frac{1}{\sigma^2}\begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} \left(sI + A'\right)^{-1} N_\sigma' \begin{bmatrix} C' & \frac{1}{\sigma}L_C B \end{bmatrix} \\
& -\frac{1}{\sigma^4}\begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N_\sigma' \left(sI + A'\right)^{-1} \\
& \left[\sigma^2 \left(N_\sigma'\right)^{-1} \left(sI + A'\right) - \sigma^2 \left(sI + A'\right) \left(N_\sigma'\right)^{-1}\right] \\
& \left(sI - A'\right)^{-1} N_\sigma' \begin{bmatrix} C' & \frac{1}{\sigma}L_C B \end{bmatrix} \\
&= 0.
\end{aligned}$$

■

**Remark:** In several places in this section we need to verify various identities which, taken algebraically, are easy to verify. However, when dealing with unbounded operators, it is necessary to check at every step that all the terms are well-defined and in the correct spaces. We do this by making extensive use of Lemmas 2.5.1, 2.5.2 and Corollary 2.5.5. We have made this explicit in the first part of this proof of Lemma 7.2.1, but in future verifications we shall omit these detailed explanations.

**Lemma 7.2.2** *Suppose that the triple* $(-A, B, C)$ *satisfies A1-4. Let* $\Lambda$ *be given by (7.5). If* $G(s) = C(sI + A)^{-1}B$*, then* $W$ *defined in (4.1) has a* $J$*−spectral factorization (4.2).*

**Proof** This is proved by a tedious but straightforward substitution, again verifying that all the terms are well-defined at each step. For a step by step calculation bounded $B$ and $C$, see Curtain and Zwart [34]. ∎

In the case that $\sigma_{l+1} < \sigma < \sigma_l$, $\Lambda_{11}(-\cdot)^{-1}$ exists, but it is not stable. We will show that $\Lambda_{11}(-\cdot)^{-1}$ can be written as the sum of a stable part in $H^c_\infty(\mathbb{C}^{p\times p})$ and an antistable rational part with at most $l$ unstable poles. The proof is very technical and long. So we have split it up into a sequence of lemmas.

**Lemma 7.2.3** *Suppose that the triple* $(-A, B, C)$ *satisfies A1-4. Let* $\beta$ *be a fixed number satisfying* $-1 < \beta < \alpha$.

1. $A_1 := -A' - \frac{1}{\sigma^2}C'CL_BN'_\sigma$ *with* $D(A_1)$ *equal to*[1] $Z'_{-\beta}$ *generates an analytic semigroup on* $Z'_{-\beta-1}$.
2. *Let* $\left(Z'_{-\beta-1}\right)_1$ *denote* $D(A_1)$ *with the following norm: Let* $s \in \rho(A_1) \cap \overline{\mathbb{C}_0^+}$. *If* $z \in D(A_1)$*, then* $\|z\|_{\left(Z'_{-\beta-1}\right)_1} = \|(sI - A_1)z\|_{Z'_{-\beta-1}}$. *Then* $\left(Z'_{-\beta-1}\right)_1$ *and* $Z'_{-\beta}$ *are equal as topological spaces, that is, their norms are equivalent.*
3. *For all* $s \in \rho(A_1) \cap \overline{\mathbb{C}_0^+}$, $(sI - A_1)^{-1} \in \mathcal{L}\left(Z'_{-\beta-1}, Z'_{-\beta}\right)$.
4. *For all* $s \in \rho(A_1) \cap \overline{\mathbb{C}_0^+}$, ${}^{11}\vee(s) := I_p - \frac{1}{\sigma^2}CL_BN'_\sigma(sI - A_1)^{-1}C' \in \mathcal{L}(\mathbb{C}^p)$.
5. *The spectrum of* $A_1$ *in the closed right half-plane consists of only eigenvalues, and* $\sigma(A_1) \cap \overline{\mathbb{C}_0^+}$ *is a finite set*[2].

**Proof**

1. $-A'$ generates an analytic semigroup on $Z'_{-\beta-1}$ with domain $Z'_{-\beta}$. We have $-\frac{1}{\sigma^2}C'CL_BN'_\sigma \in \mathcal{L}\left(Z'_{-\beta}, Z'_{-\beta-1}\right)$ and furthermore, it is a compact operator since $C$ has finite rank. Thus it follows from Proposition 2.4.3 (Lunardi [56], page 65) that

$$A_1 := -A' - \frac{1}{\sigma^2}C'CL_BN'_\sigma \text{ with } D(A_1) = Z'_{-\beta}$$

[1] Here we mean set theoretic equality.

[2] that is, it has finitely many elements

generates an analytic semigroup on $Z'_{-\beta-1}$.

2. Let $s \in \rho(A_1) \cap \overline{\mathbb{C}_0^+}$. Since $s \in \overline{\mathbb{C}_0^+}$, it follows that $s \in \rho(-A')$. Consider the identity map $\mathtt{i} : Z'_{-\beta} \to \left(Z'_{-\beta-1}\right)_1$. If $z \in Z'_{-\beta}$ then

$$\begin{aligned}
\|\mathtt{i}(z)\|_{\left(Z'_{-\beta-1}\right)_1} &= \|z\|_{\left(Z'_{-\beta-1}\right)_1} \\
&= \|(sI - A_1)\, z\|_{Z'_{-\beta-1}} \\
&= \left\| \left( sI + A' + \frac{1}{\sigma^2} C'CL_B N'_\sigma \right) z \right\|_{Z'_{-\beta-1}} \\
&\leq \|(sI + A')\, z\|_{Z'_{-\beta-1}} + \left\| \frac{1}{\sigma^2} C'CL_B N'_\sigma z \right\|_{Z'_{-\beta-1}} \\
&\leq \|z\|_{Z'_{-\beta}} + \left\| \frac{1}{\sigma^2} C'CL_B N'_\sigma \right\|_{\mathcal{L}\left(Z'_{-\beta}, Z'_{-\beta-1}\right)} \|z\|_{Z'_{-\beta}},
\end{aligned}$$

and so $\mathtt{i} \in \mathcal{L}\left(Z'_{-\beta}, \left(Z'_{-\beta-1}\right)_1\right)$. Furthermore, $\mathtt{i}$ is one-to-one and onto, and thus it is boundedly invertible, that is, $\mathtt{i}^{-1} \in \mathcal{L}\left(\left(Z'_{-\beta-1}\right)_1, Z'_{-\beta}\right)$. Consequently, $\left(Z'_{-\beta-1}\right)_1$ and $Z'_{-\beta}$ are equal as topological spaces.

3. We will show that $(sI - A_1)^{-1} \in \mathcal{L}\left(Z'_{-\beta-1}, \left(Z'_{-\beta-1}\right)_1\right)$, and then from 2 above, it follows that $(sI - A_1)^{-1} \in \mathcal{L}\left(Z'_{-\beta-1}, Z'_{-\beta}\right)$. If $z \in Z'_{-\beta-1}$, then

$$\left\| (sI - A_1)^{-1} z \right\|_{\left(Z'_{-\beta-1}\right)_1} = \left\| (sI - A_1)(sI - A_1)^{-1} z \right\|_{Z'_{-\beta-1}} = \|z\|_{Z'_{-\beta-1}}.$$

Thus $(sI - A_1)^{-1} \in \mathcal{L}\left(Z'_{-\beta-1}, Z'_{-\beta}\right)$.

4. This is a consequence of 3 above.

5. *Step 1.* $\sigma(A_1) \cap \overline{\mathbb{C}_0^+} = \sigma_p(A_1) \cap \overline{\mathbb{C}_0^+}$.

For $s \in \overline{\mathbb{C}_0^+}$, we have

$$\begin{aligned}
sI - A_1 &= sI + A' + \tfrac{1}{\sigma^2} C'CL_B N'_\sigma \\
&= \left[ I + \tfrac{1}{\sigma^2} C'CL_B N'_\sigma (sI + A')^{-1} \right] (sI + A').
\end{aligned} \tag{7.7}$$

So $s \in \rho(A_1)$ iff $\kappa_0 := I + \frac{1}{\sigma^2} C'CL_B N'_\sigma (sI + A')^{-1}$ is invertible. But $\frac{1}{\sigma^2} C'CL_B N'_\sigma (sI + A')^{-1} \in \mathcal{L}\left(Z'_{-\beta-1}\right)$, and furthermore, it is compact since $C$ is compact. So its spectrum consists of only eigenvalues except possibly for 0. As a consequence, we have that the spectrum of $\kappa_0$ consists of only eigenvalues except possibly for $1$. Thus $\kappa_0$ is invertible iff $0$ is not an eigenvalue. But $0$ is an eigenvalue of $\kappa_0$ (with a corresponding eigenvector $z_0$) iff $(sI+A')^{-1}z_0$ is an eigenvector of $A_1$ with eigenvalue $s$. This follows from (7.7) above. So we arrive at the conclusion that $s \notin \rho(A_1) \cap \overline{\mathbb{C}_0^+}$ iff $s$ is an eigenvalue of $A_1$, and so $\sigma(A) \cap \overline{\mathbb{C}_0^+} = \sigma_p(A) \cap \overline{\mathbb{C}_0^+}$.

*Step 2. If* $\kappa_1 := I + \frac{1}{\sigma^2} C L_B N'_\sigma (sI+A')^{-1} C'$ *is invertible in* $\mathcal{L}(\mathbb{C}^p)$, *then* $s \in \rho(A_1) \cap \overline{\mathbb{C}_0^+}$.

We have

$$I = (sI - A_1)(sI+A')^{-1} - \frac{1}{\sigma^2} C' C L_B N'_\sigma (sI+A')^{-1}, \tag{7.8}$$

and so $C' = (sI - A_1)(sI+A')^{-1} C' - \frac{1}{\sigma^2} C' C L_B N'_\sigma (sI+A')^{-1} C'$, and hence we obtain the following identity for $s \in \overline{\mathbb{C}_0^+}$:

$$C'\kappa_1 = (sI - A_1)(sI+A')^{-1} C'. \tag{7.9}$$

If $\kappa_1$ is invertible, then from (7.9), we obtain

$$C' = (sI - A_1)(sI+A')^{-1} C' \kappa_1^{-1}$$

and substituting this in the second term in (7.8) gives

$$I = (sI - A_1)(sI+A')^{-1} \cdot \left\{ I - \frac{1}{\sigma^2} C' \kappa_1^{-1} C L_B N'_\sigma (sI+A')^{-1} \right\}.$$

On the other hand, we have

$$\begin{aligned} & (sI+A')^{-1} \left\{ I - \frac{1}{\sigma^2} C' \kappa_1^{-1} C L_B N'_\sigma (sI+A')^{-1} \right\} \cdot (sI - A_1) \\ = \; & (sI+A')^{-1} \cdot \\ & \left\{ sI - A_1 - \frac{1}{\sigma^2} C' \kappa_1^{-1} \cdot C L_B N'_\sigma \cdot \left[ I + \frac{1}{\sigma^2} (sI+A')^{-1} C' C L_B N'_\sigma \right] \right\}, \end{aligned}$$

using

$$(sI+A')^{-1}(sI - A_1) = I_{Z'_{-\beta}} + \frac{1}{\sigma^2}(sI+A')^{-1} C' C L_B N'_\sigma.$$

Thus we have

$$\begin{aligned} & (sI+A')^{-1} \left\{ I - \frac{1}{\sigma^2} C' \kappa_1^{-1} C L_B N'_\sigma (sI+A')^{-1} \right\} \cdot (sI - A_1) \\ = \; & (sI+A')^{-1} \left\{ sI - A_1 - \frac{1}{\sigma^2} C' \kappa_1^{-1} \cdot \kappa_1 \cdot C L_B N'_\sigma \right\} \\ = \; & (sI+A')^{-1} \left\{ sI - A_1 - \frac{1}{\sigma^2} C' C L_B N'_\sigma \right\} \\ = \; & I_{Z'_{-\beta}} \\ = \; & I_{D(A_1)}. \end{aligned}$$

The linear operator $(sI+A')^{-1}\left\{I-\frac{1}{\sigma^2}C'\kappa_1^{-1}CL_BN'_\sigma(sI+A')^{-1}\right\}$ is bounded, and so $s\in\rho(A_1)$. Hence if $s\in\sigma(A_1)\cap\overline{\mathbb{C}_0^+}$, then $\det(\kappa_1)=0$.

*Step 3. The set* $\sigma(A_1)\cap\overline{\mathbb{C}_0^+}$ *is finite.*

Consider the function $f(s)=\det\left(I+\frac{1}{\sigma^2}CL_BN'_\sigma(sI+A')^{-1}C'\right)=\det(\kappa_1)$. Since $-A'$ generates an exponentially stable semigroup, there exists an $\epsilon>0$ such that $f$ is analytic in $\mathbb{C}^+_{-\epsilon}$. We now prove that $f$ is not identically zero in $\mathbb{C}^+_{-\epsilon}$. Since $-A'$ generates an exponentially stable analytic semigroup on $Z'_{-\beta-1}$, there exists an $M$ such that

$$\left\|(sI+A')^{-1}\right\|_{\mathcal{L}\left(Z'_{-\beta-1}\right)}\leq\frac{M}{|s|}.$$

So there exists $R>0$ such that if $s\in B(0,R):=\{s\in\overline{\mathbb{C}_0^+}\mid|s|\geq R\}$, then

$$\left\|\frac{1}{\sigma^2}CL_BN'_\sigma(sI+A')^{-1}C'\right\|_{\mathcal{L}(\mathbb{C}^p)}\leq\frac{1}{2}. \tag{7.10}$$

Thus $\kappa_1$ is invertible for every $s\in\overline{\mathbb{C}_0^+}\setminus B(0,R)$. Hence $f$ is not identically zero, and $f$ has no zeros in $\overline{\mathbb{C}_0^+}\setminus B(0,R)$. Inside the compact set

$$\overline{B(0,R)\cap\overline{\mathbb{C}_0^+}},$$

the analytic function $f$ can have at most finitely many zeros, and so $\sigma(A_1)\cap\overline{\mathbb{C}_0^+}$ contains at most finitely many points. ■

We shall see that there is a nice relationship between the number of unstable eigenvalues [3] of $A_1$ and the negative eigenvalues of a self-adjoint operator $Q_*$ constructed from the solution $Q$ of a Riccati equation.

**Lemma 7.2.4** *Suppose that the triple* $(-A,B,C)$ *satisfies A1-4. If* $-1<\beta<\alpha$ *then*

$$Q:=L_BN'_\sigma\in\mathcal{L}\left(Z'_{-\beta},Z_\beta\right)\cap\mathcal{L}\left(Z'_{-\beta-1},Z_\beta\right)\cap\mathcal{L}\left(Z'_{-\beta},Z_{\beta+1}\right)$$

*is a solution of the Riccati equation*

$$AQz+QA'z+\frac{1}{\sigma^2}QC'CQz-N_\sigma BB'N'_\sigma z=0 \tag{7.11}$$

*in* $Z_\beta$ *for all* $z\in Z'_{-\beta}$.

[3] We mean counting eigenvalues by taking their algebraic multiplicity into account. For the definition of the algebraic multiplicity of an eigenvalue, see Chapter 5.

**Proof** We have for $z \in Z'_{-\beta+1}$,

$$\begin{aligned}
& \frac{1}{\sigma^2} QC'CQz - N_\sigma BB'N'_\sigma z \\
= & \frac{1}{\sigma^2} N_\sigma L_B C'CL_B N'_\sigma z - N_\sigma BB'N'_\sigma z \\
= & N_\sigma \left[ \frac{1}{\sigma^2} L_B C'CL_B - BB' \right] N'_\sigma z \\
= & N_\sigma \left[ \frac{1}{\sigma^2} L_B \left( A'L_C + L_C A \right) L_B - AL_B - L_B A' \right] N'_\sigma z \\
= & N_\sigma \left[ L_B A' \left( \frac{1}{\sigma^2} L_C L_B - I \right) + \left( \frac{1}{\sigma^2} L_B L_C - I \right) AL_B \right] N'_\sigma z \\
= & N_\sigma \left[ -L_B A' \left( N_\sigma \right)^{-1} - N_\sigma^{-1} AL_B \right] N'_\sigma z \\
= & -N_\sigma L_B A'z - AL_B N'_\sigma z \\
= & -QA'z - AQz.
\end{aligned}$$

But we know that

$$\begin{array}{rcll}
AQ & \in & \mathcal{L}\left( Z'_{-\beta}, Z_\beta \right) & \text{since} \quad Z'_{-\beta} \xrightarrow{Q} Z_{\beta+1} \xrightarrow{A} Z_\beta \\
QA' & \in & \mathcal{L}\left( Z'_{-\beta}, Z_\beta \right) & \text{since} \quad Z'_{-\beta} \xrightarrow{-A'} Z'_{-\beta-1} \xrightarrow{Q} Z_\beta \\
\frac{1}{\sigma^2} QC'CQ & \in & \mathcal{L}\left( Z'_{-\beta}, Z_\beta \right) & \text{since} \quad Z'_{-\beta} \xrightarrow{Q} Z_{\beta+1} \xrightarrow{C'C} Z'_{-\beta-1} \xrightarrow{Q} Z_\beta \\
N_\sigma BB'N'_\sigma & \in & \mathcal{L}\left( Z'_{-\beta}, Z_\beta \right) & \text{since} \quad Z'_{-\beta} \xrightarrow{N'_\sigma} Z'_{-\beta} \xrightarrow{BB'} Z_\beta \xrightarrow{N_\sigma} Z_\beta,
\end{array}$$

and $Z'_{-\beta+1}$ is dense in $Z'_{-\beta}$. Hence the result follows. ∎

**Lemma 7.2.5** *Suppose that the triple $(-A, B, C)$ satisfies A1-4. If $-1 < \beta < \alpha$ then $Q := L_B N'_\sigma \in \mathcal{L}\left( Z'_{-\beta}, Z_\beta \right)$ is a self-dual operator, that is,*

$$\left( L_B N'_\sigma = \right) \;\; Q = Q' \;\; \left( = N_\sigma L_B \right) \;\; \in \mathcal{L}\left( Z'_{-\beta}, Z_\beta \right).$$

**Proof** We know that $N'_\sigma \in \mathcal{L}(Z'_{-\beta})$. Moreover, $L_B \in \mathcal{L}(Z'_{-\beta}, Z_{\beta+1})$ and so $L_B \in \mathcal{L}(Z'_{-\beta}, Z_\beta)$. As a result, we have $Q = L_B N_\sigma \in \mathcal{L}(Z'_{-\beta}, Z_\beta)$. Furthermore, we know that $N_\sigma \in \mathcal{L}(Z_{\beta+1})$ and so $N_\sigma L_B \in \mathcal{L}(Z'_{-\beta}, Z_{\beta+1})$. Consequently, $N_\sigma L_B \in \mathcal{L}(Z'_{-\beta}, Z_\beta)$. Finally, the proof is completed by observing that

$$N_\sigma L_B = \left( I - \frac{1}{\sigma^2} L_B L_C \right)^{-1} L_B = L_B \left( I - \frac{1}{\sigma^2} L_C L_B \right)^{-1} = L_B N'_\sigma.$$

∎

**Lemma 7.2.6** *Suppose that the triple $(-A,B,C)$ satisfies A1-4. If $-1<\beta<\alpha$ then*

1. $Q := L_B N'_\sigma \in \mathcal{L}\left(Z'_{-\beta}, Z_\beta\right) \cap \mathcal{L}\left(Z'_{-\beta-1}, Z_\beta\right) \cap \mathcal{L}\left(Z'_{-\beta}, Z_{\beta+1}\right)$ *is a solution of the Riccati equation*

$$\begin{aligned} & \langle A_1 z_1, Q z_2\rangle_{\langle Z'_{-\beta-1}, Z_{\beta+1}\rangle} + \langle Q z_1, A_1 z_2\rangle_{\langle Z_{\beta+1}, Z'_{-\beta-1}\rangle} \\ = & -\frac{1}{\sigma^2}\langle CQz_1, CQz_2\rangle_{\mathbb{C}^p} - \langle B'N'_\sigma z_1, B'N'_\sigma z_2\rangle_{\mathbb{C}^m}, \end{aligned} \tag{7.12}$$

   *for all* $z_1, z_2 \in Z'_{-\beta}$.

2. $\sigma(A_1) \cap i\mathbb{R}$ *is empty.*

**Proof** *1.* This follows from Lemma 7.2.4.

*2.* From (7.12) we obtain, for $\omega_0 \in \mathbb{R}$,

$$\begin{aligned} & \langle (i\omega_0 - A_1) z_1, Q z_2\rangle_{\langle Z'_{-\beta-1}, Z_{\beta+1}\rangle} + \langle Q z_1, (i\omega_0 - A_1) z_2\rangle_{\langle Z_{\beta+1}, Z'_{-\beta-1}\rangle} \\ = & \; i\omega_0\left[\langle z_1, Qz_2\rangle_{\langle Z'_{-\beta-1}, Z_{\beta+1}\rangle} - \langle Qz_1, z_2\rangle_{\langle Z_{\beta+1}, Z'_{-\beta-1}\rangle}\right] \\ & + \frac{1}{\sigma^2}\langle CQz_1, CQz_2\rangle_{\mathbb{C}^p} + \langle B'N'_\sigma z_1, B'N'_\sigma z_2\rangle_{\mathbb{C}^m} \\ = & \; i\omega_0\left[\langle z_1, Qz_2\rangle_{\langle Z'_{-\beta}, Z_{\beta}\rangle} - \langle Qz_1, z_2\rangle_{\langle Z_{\beta}, Z'_{-\beta}\rangle}\right] \\ & + \frac{1}{\sigma^2}\langle CQz_1, CQz_2\rangle_{\mathbb{C}^p} + \langle B'N'_\sigma z_1, B'N'_\sigma z_2\rangle_{\mathbb{C}^m} \\ = & \; \frac{1}{\sigma^2}\langle CQz_1, CQz_2\rangle_{\mathbb{C}^p} + \langle B'N'_\sigma z_1, B'N'_\sigma z_2\rangle_{\mathbb{C}^m}. \end{aligned} \tag{7.13}$$

Thus if $\omega_0 \in \mathbb{R}$ is such that $i\omega_0 \in \sigma(A_1) \cap i\mathbb{R} = \sigma_p(A_1) \cap i\mathbb{R}$, with a corresponding eigenvector $z_0 \in Z'_{-\beta}$, then $CL_BN'_\sigma z_0 = 0$ and $B'N'_\sigma z_0 = 0$. In particular, $\frac{1}{\sigma^2}C'CL_BN'_\sigma z_0 = 0$, and so $-A'z_0 = -A'z_0 - \frac{1}{\sigma^2}C'CL_BN'_\sigma z_0 = A_1 z_0 = i\omega_0 z_0$, contradicting the fact that $-A'$ generates an exponentially stable semigroup on $Z'_{-\beta-1}$. Thus $\sigma(A_1) \cap i\mathbb{R}$ is empty. ∎

Thus we have shown that $A_1$ satisfies the spectrum decomposition assumption and has no spectrum on the imaginary axis. Let $\pi_{\sigma(A_1)\cap\overline{\mathbb{C}_0^+}}$ denote the spectral projection with respect to $\sigma(A_1) \cap \overline{\mathbb{C}_0^+}$ on $Z'_{-\beta-1}$, and let

$$\begin{aligned} Z^+ &:= \pi_{\sigma(A_1)\cap\overline{\mathbb{C}_0^+}} Z'_{-\beta-1}, \quad \text{and} \\ Z^- &:= \left[I_{Z'_{-\beta-1}} - \pi_{\sigma(A_1)\cap\overline{\mathbb{C}_0^+}}\right] Z'_{-\beta-1}. \end{aligned}$$

**Lemma 7.2.7** *Suppose that the triple $(-A, B, C)$ satisfies A1-4.*

1. *$s \in \rho(A_1) \cap \overline{\mathbb{C}_0^+}$ iff $\kappa_1 = I + \frac{1}{\sigma^2} C L_B N'_\sigma (sI + A')^{-1} C'$ is invertible in $\mathcal{L}(\mathbb{C}^p)$.*
2. *The restriction of $A_1$ to $Z^-$, say $A_1^- : D(A_1) \cap Z^- \to Z^-$ is the infinitesimal generator of an exponentially stable semigroup in $Z^-$.*

**Proof** *1.* In step 2 of the proof of part 5 of Lemma 7.2.3 we had shown the "if" part. Here we show the "only if" part: If $s \in \rho(A_1) \cap \overline{\mathbb{C}_0^+}$, then (7.9) yields

$$(sI - A_1)^{-1} C' \kappa_1 = (sI + A')^{-1} C' \tag{7.14}$$

and so

$$\frac{1}{\sigma^2} C L_B N'_\sigma (sI - A_1)^{-1} C' \kappa_1 = \frac{1}{\sigma^2} C L_B N'_\sigma (sI + A')^{-1} C'. \tag{7.15}$$

Suppose now that $\kappa_1$ is not invertible in $\mathcal{L}(\mathbb{C}^p)$. Then there exists an eigenvector $y_0 \neq 0$ such that $\kappa_1 y_0 = 0$. Substituting this in (7.14), we obtain $C' y_0 = 0$. Now from $\kappa_1 y_0 = 0$ and $C' y_0 = 0$, we have $y_0 = 0$, a contradiction.

*2.* From Lemma 2.5.7.c (pages 71-72, Curtain and Zwart [34]), we know that $A_1^-$ generates a strongly continuous semigroup $\{T_1^-(t)\}_{t \geq 0}$ on $Z^-$, and $(sI - A_1)^{-1}|_{Z^-} = \left(sI - A_1^-\right)^{-1}$. Furthermore, since $\sigma\left(A_1^-\right) \subset \mathbb{C}_0^-$, $\left(sI - A_1^-\right)^{-1}$ is analytic in $\mathbb{C}_0^+$. We now proceed to show that $\left(sI - A_1^-\right)^{-1} z$ is in $H_2\left(Z'_{-\beta-1}\right)$ for every $z \in Z'_{-\beta-1}$. From (1) above, we have that $\kappa_1$ is invertible in $\rho(A_1) \cap \overline{\mathbb{C}_0^+}$, and using (7.14) we obtain

$$(sI - A_1)^{-1} C' = (sI + A')^{-1} C' \kappa_1^{-1}. \tag{7.16}$$

Using the properties of the spectral projection from Lemma 2.5.7 (pages 71-72, Curtain and Zwart [34]), we obtain

$$\begin{aligned}
& \left(sI - A_1^-\right)^{-1} \left[I_{Z'_{-\beta-1}} - \pi_{\sigma(A_1) \cap \overline{\mathbb{C}_0^+}}\right] C' \\
= \;& (sI - A_1)^{-1} \left[I_{Z'_{-\beta-1}} - \pi_{\sigma(A_1) \cap \overline{\mathbb{C}_0^+}}\right] C' && (7.17) \\
= \;& \left[I_{Z'_{-\beta-1}} - \pi_{\sigma(A_1) \cap \overline{\mathbb{C}_0^+}}\right] (sI - A_1)^{-1} C' \\
= \;& \left[I_{Z'_{-\beta-1}} - \pi_{\sigma(A_1) \cap \overline{\mathbb{C}_0^+}}\right] (sI + A')^{-1} C' \kappa_1^{-1}, && (7.18)
\end{aligned}$$

where we have used (7.16). The expression in (7.17) above is analytic in $\mathbb{C}_0^+$, since $A_1^-$ has no spectrum there. In addition, $(sI + A')^{-1} C'$ is analytic in $\mathbb{C}_0^+$, and from (7.10), for $s \in \mathbb{C}_0^+ \setminus B(0, R)$. Thus, for sufficiently large $R$,

$\kappa_1^{-1}$ is uniformly bounded in norm in $\overline{\mathbb{C}_0^+} \setminus B(0,R)$, and inside the semi-disc $B(0,R) \cap \mathbb{C}_0^+$, it has finitely many poles. However, the equality of (7.17) and (7.18) reveals that the expression in (7.18) cannot have any poles in $\overline{\mathbb{C}_0^+}$. Thus

$$\left(sI - A_1^-\right)^{-1} \left[I_{Z'_{-\beta-1}} - \pi_{\sigma(A_1)\cap\overline{\mathbb{C}_0^+}}\right] C'$$

is uniformly bounded in norm in $\overline{\mathbb{C}_0^+}$. For $z \in Z^-$ and $s \in \mathbb{C}_0^+$, from Lemma 2.5.7 (pages 71-72, Curtain and Zwart [34]) we obtain

$$\begin{aligned}
& \left(sI - A_1^-\right)^{-1} z \\
= \ & (sI - A_1)^{-1} \left[I_{Z'_{-\beta-1}} - \pi_{\sigma(A_1)\cap\overline{\mathbb{C}_0^+}}\right] z \\
= \ & \left[I_{Z'_{-\beta-1}} - \pi_{\sigma(A_1)\cap\overline{\mathbb{C}_0^+}}\right] (sI - A_1)^{-1} z \\
= \ & \left[I_{Z'_{-\beta-1}} - \pi_{\sigma(A_1)\cap\overline{\mathbb{C}_0^+}}\right] \cdot \\
& \left[(sI + A')^{-1} - (sI - A_1)^{-1} \frac{1}{\sigma^2} C'CL_B N'_\sigma (sI + A')^{-1}\right] z \ \text{ (using (7.8))} \\
= \ & \left[I_{Z'_{-\beta-1}} - \pi_{\sigma(A_1)\cap\overline{\mathbb{C}_0^+}}\right] (sI + A')^{-1} z \\
& - \left(sI - A_1^-\right)^{-1} \left[I_{Z'_{-\beta-1}} - \pi_{\sigma(A_1)\cap\overline{\mathbb{C}_0^+}}\right] \frac{1}{\sigma^2} C'CL_B N'_\sigma (sI + A')^{-1} z.
\end{aligned}$$

Now from the fact that $-A'$ generates an exponentially stable semigroup Lemma 5.1.2 (page 215, Curtain and Zwart [34]) and the Payley-Wiener theorem (Theorem A.6.21, page 645, Curtain and Zwart [34]), it follows that $(sI + A')^{-1} z \in H_2\left(Z'_{-\beta-1}\right)$. Notice that we have already shown that

$$\left(sI - A_1^-\right)^{-1} \left[I_{Z'_{-\beta-1}} - \pi_{\sigma(A_1)\cap\overline{\mathbb{C}_0^+}}\right] C'$$

is uniformly bounded in norm in $\overline{\mathbb{C}_0^+}$. Thus for any $z \in Z^-$, $\left(sI - A_1^-\right)^{-1} z \in H_2\left(Z'_{-\beta-1}\right)$ (see Theorem A.6.26.b, page 647, Curtain and Zwart [34]) as claimed. Corollary A.6.23 (page 646, Curtain and Zwart [34]) then implies that

$$\int_0^\infty \left\|T_1^-(t)z\right\|^2_{Z'_{-\beta-1}} dt < \infty$$

and Lemma 5.1.2 (page 215, Curtain and Zwart [34]) shows that $\left\{T_1^-(t)\right\}_{t\geq 0}$ is exponentially stable. ■

**Lemma 7.2.8** *$Z^+ \subset Z'_{-\beta}$, and for all $z \in Z^+$, $\langle Qz, z\rangle_{\langle Z_\beta, Z'_{-\beta}\rangle} \leq 0$.*

**Proof** From Lemma 2.5.7.b,c (pages 71-72, Curtain and Zwart [34]), we know that $Z^+ \subset D(A_1) = Z'_{-\beta}$, $Z^+$ is $A_1-$invariant and the restriction of $A_1$ to $Z^+$, say $A_1^+$, is a bounded linear operator on $Z^+$ (where $Z^+$ is equipped with the subspace topology induced by the topology of $Z'_{-\beta-1}$) and $\sigma\left(A_1^+\right) = \sigma(A_1) \cap \overline{\mathbb{C}_0^+}$. For $R > 0$ large enough and for $z \in Z^+$, integration

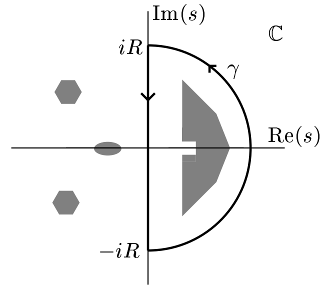


Figure 7.2: The contour $\gamma$.

along the semicircular contour $\gamma$ (see Figure 7.2) yields

$$\begin{aligned} z &= \pi_{\sigma(A_1)\cap\overline{\mathbb{C}_0^+}} z \\ &= \frac{1}{2\pi i}\int_{[iR,-iR]} \left(sI - A_1^+\right)^{-1} z ds + \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(I - R^{-1}e^{-i\theta}A_1^+\right)^{-1} z d\theta. \end{aligned} \tag{7.19}$$

For any $\epsilon > 0$, there exists a $R > 0$ such that

$$1 - \epsilon < \left\| \left(I - R^{-1}e^{-i\theta}A_1^+\right)^{-1} \right\|_{\mathcal{L}(Z^+)} < 1 + \epsilon,$$

and hence the second integral in (7.19) converges (in $Z^+$, and hence in $Z'_{-\beta-1}$) to $\frac{1}{2}z$ as $R \to \infty$. Therefore, the first integral also converges to $\frac{1}{2}z$ as $R \to \infty$.

From (7.13), we have, for $\omega \in \mathbb{R}$, and $z \in Z^+$,

$$\mathrm{Re}\left(\left\langle \left(i\omega - A_1^+\right) z, Qz\right\rangle_{\left\langle Z'_{-\beta-1}, Z_{\beta+1}\right\rangle}\right) \geq 0.$$

But since $i\omega \in \rho\left(A_1^+\right)$, it follows that for every $z \in Z^+$,

$$\mathrm{Re}\left(\left\langle z, Q\left(i\omega - A_1^+\right)^{-1} z\right\rangle_{\left\langle Z'_{-\beta-1}, Z_{\beta+1}\right\rangle}\right) \geq 0,$$

that is,

$$\mathrm{Re}\left(\left\langle Qz, \left(i\omega - A_1^+\right)^{-1} z\right\rangle_{\left\langle Z_{\beta}, Z'_{-\beta}\right\rangle}\right) \geq 0.$$

Thus integrating over the segment $[iR,-iR]$, we obtain

$$\operatorname{Re}\left(\left\langle Qz, \frac{1}{2\pi i}\int_{[iR,-iR]}\left(sI-A_1^+\right)^{-1} z ds\right\rangle_{\left\langle Z_\beta, Z'_{-\beta}\right\rangle}\right) \le 0.$$

But

$$z_R := \frac{1}{2\pi i}\int_{[iR,-iR]}\left(sI-A_1^+\right)^{-1} dsz$$

is an element in $Z^+$ and $z_R \to \frac{1}{2}z$ in $Z^+$ as $R\to\infty$. Since $A_1^+ \in \mathcal{L}(Z^+)$, we have $A_1^+ z_R \to \frac{1}{2}A_1^+ z$ in $Z^+$. So we have

$$\begin{array}{lcl} z_R & \longrightarrow & \frac{1}{2}z \\ A_1 z_R & \longrightarrow & \frac{1}{2}A_1 z \end{array} \quad \text{in } Z'_{-\beta-1} \text{ as } R\to\infty.$$

Consequently, $z_R \to \frac{1}{2}z$ in $Z'_{-\beta}$. Thus we obtain $\langle Qz, z\rangle_{\left\langle Z_\beta, Z'_{-\beta}\right\rangle} = \operatorname{Re}\left(\langle Qz, z\rangle_{\left\langle Z_\beta, Z'_{-\beta}\right\rangle}\right) \le 0.$ ■

We now define the operator $Q_*$ to be the self-adjoint operator $\iota_{Z'_{-\beta}} jQ \in \mathcal{L}\left(Z'_{-\beta}\right)$, where $\iota_{Z'_{-\beta}}$ denotes the canonical isometry from $\mathcal{L}\left(Z'_{-\beta}, \mathbb{C}\right)$ to $Z'_{-\beta}$, and $j$ is an isometry from $Z_\beta$ to $\mathcal{L}\left(Z'_{-\beta}, \mathbb{C}\right)$. For the sake of simplicity of notation, we denote $Z'_{-\beta}$ by $W$ in the following.

**Lemma 7.2.9** *Suppose that the triple $(-A,B,C)$ satisfies A1-5. The operator $Q_* \in \mathcal{L}(W)$ is self-adjoint,*

$$\begin{array}{rcl} 0 & \notin & \sigma_p(Q_*), \\ \sigma(Q_*)\cap\mathbb{C}_0^- & = & \sigma_p(Q_*)\cap\mathbb{C}_0^- \quad \textit{and} \\ \nu(Q_*) & \le & l. \end{array}$$

**Proof** It follows from Lemma 7.2.5 that $Q_*$ is self-adjoint. Define $J = I - \frac{1}{\sigma^2}\Gamma^*\Gamma \in \mathcal{L}(L_2([0,\infty),\mathbb{C}^m)$ as in Lemma 2.5.9. $0 \notin \sigma(J)$. Consider the spectral decomposition of $L_2([0,\infty),\mathbb{C}^m)$ into $L_-$ and $L_+$ induced by the self-adjoint operator $J$. We have $L_2([0,\infty),\mathbb{C}^m) = L_- \oplus L_+$, and if $v_+ \in L_+$, $\langle Jv_+, v_+\rangle \ge 0$. Let $W = W_- \oplus W_{0+}$ be the spectral decomposition induced by the self-adjoint bounded operator $Q_*$. We have

$$\begin{aligned} \langle Q_* w, w\rangle_W &= \langle L_B N'_\sigma w, w\rangle_{\left\langle Z_\beta, Z'_{-\beta}\right\rangle} \\ &= \left\langle L_B\left(I - \frac{1}{\sigma^2}L_C L_B\right)^{-1} w, w\right\rangle_{\left\langle Z_\beta, Z'_{-\beta}\right\rangle} \\ &= \left\langle L_B w_0, \left(I - \frac{1}{\sigma^2}L_C L_B\right) w_0\right\rangle_{\left\langle Z_\beta, Z'_{-\beta}\right\rangle}, \end{aligned}$$

where $w_0 = \left(I - \frac{1}{\sigma^2} L_C L_B\right)^{-1} w$. Thus

$$\begin{aligned}\langle Q_* w, w\rangle_W &= \langle \mathcal{B}\mathcal{B}' w_0, w_0\rangle_{\langle Z_\beta, Z'_{-\beta}\rangle} - \frac{1}{\sigma^2}\langle \mathcal{B}\mathcal{B}' w_0, L_C \mathcal{B}\mathcal{B}' w_0\rangle_{\langle Z_\beta, Z'_{-\beta}\rangle} \\ &= \langle \mathcal{B}' w_0, \mathcal{B}' w_0\rangle_{L_2([0,\infty),\mathbb{C}^m)} - \frac{1}{\sigma^2}\langle \mathcal{B}' L_C \mathcal{B}\mathcal{B}' w_0, \mathcal{B}' w_0\rangle_{L_2([0,\infty),\mathbb{C}^m)}, \\ &= \langle u, u\rangle_{L_2([0,\infty),\mathbb{C}^m)} - \frac{1}{\sigma^2}\langle \mathcal{B}' L_C \mathcal{B} u, u\rangle_{L_2([0,\infty),\mathbb{C}^m)},\end{aligned}$$

where $u = \mathcal{B}' w_0$, and so

$$\langle Q_* w, w\rangle_W = \left\langle \left(I - \frac{1}{\sigma^2}\Gamma^*\Gamma\right) u, u \right\rangle_{L_2([0,\infty),\mathbb{C}^m)}.$$

Now define $\Upsilon : W_- \to L_-$ as follows: If $w_- \in W_-$, $\Upsilon w_- := \Pi_{L_-}\mathcal{B}' N'_\sigma w_-$, where $\Pi_{L_-}$ denotes the canonical projection from $L_2([0,\infty),\mathbb{C}^m)$ onto $L_-$. $\Upsilon$ is clearly linear. Next we show that $\Upsilon$ is injective. For if $\Pi_{L_-}\mathcal{B}' N'_\sigma w_- = 0$, then $v := \mathcal{B}' N'_\sigma w_- \in L_+$ and

$$\begin{aligned}0 \geq \langle Q_* w_-, w_-\rangle_W &= \langle L_B N'_\sigma w_-, w_-\rangle_{\langle Z_\beta, Z'_{-\beta}\rangle} \\ &= \left\langle \left(I - \frac{1}{\sigma^2}\Gamma^*\Gamma\right) v, v\right\rangle_{L_2([0,\infty),\mathbb{C}^m)} \geq 0.\end{aligned}$$

Thus, $\langle Jv, v\rangle = 0$. Using $|\langle Jv_+, v'_+\rangle|^2 \leq \langle Jv_+, v_+\rangle\langle Jv'_+, v'_+\rangle$, we obtain $\langle Jv, Jv\rangle = 0$, and so $Jv = 0$. But $J$ is injective, and so $v = 0$, that is, $\mathcal{B}' N'_\sigma w_- = 0$. Since the system is approximately controllable (assumption A5), this implies that $N'_\sigma w_- = 0$, and since $N'_\sigma$ is invertible, it follows that $w_- = 0$. Thus

$$\dim(W_-) \leq \dim(L_-) = l.$$

Hence the number of negative eigenvalues of $Q_*$ is at most equal to $l$ (from Kato [52], pages 520-521, it follows that since the $\dim(W_-) < \infty$, the essential spectrum in the open left half-plane is empty and so $\dim(W_-)$ is equal to the number of negative eigenvalues of $Q_*$).

Finally we prove that $0 \notin \sigma_p(Q_*)$. Suppose, on the contrary, that there exists a nonzero eigenvector $w_1 \in W$ corresponding to the eigenvalue 0, that is $Q_* w_1 = 0$. This implies that $L_B N'_\sigma w_1 = 0$. But since $L_B$ is injective and $N'_\sigma$ is invertible, it follows that $w_1 = 0$. ■

The Lemmas 5.2.1 and 5.2.3 from Chapter 5 will be crucial in obtaining the inequality between the number of unstable eigenvalues of $A_1$ and the number of negative eigenvalues of $Q_*$. Applying these two lemmas to $Q_*$ in Lemma 7.2.9 gives us information about $\sigma(A_1) \cap \overline{\mathbb{C}_0^+}$.

**Theorem 7.2.10** *Suppose that the triple $(-A, B, C)$ satisfies A1-5.*

1. *$A_1$ satisfies the spectrum decomposition assumption, and has no spectrum on the imaginary axis. The set $\sigma(A_1)\cap\overline{\mathbb{C}_0^+}$ is the same as the set $\sigma_p(A_1)\cap\overline{\mathbb{C}_0^+}$, and it contains at most finitely many points and the number of unstable eigenvalues of $A_1$ is at most $l$.*

2. *$\Lambda_{11}(-\cdot)^{-1}\in H^c_{\infty,l}(\mathbb{C}^{p\times p})$.*

**Proof** *1.* That $A_1$ satisfies the spectrum decomposition assumption, has no spectrum on the imaginary axis, the set $\sigma(A_1)\cap\overline{\mathbb{C}_0^+}$ is the same as the set $\sigma_p(A_1)\cap\overline{\mathbb{C}_0^+}$ and it contains at most finitely many points was established in Lemmas 7.2.3 and 7.2.6.

Applying Lemma 5.2.3 to the self-adjoint operator $Q_*$ in Lemma 7.2.9, we conclude that $W_{Q_{*0}}=0$ and $\dim\left(W_{Q_{*-}}\right)$ is equal to the number of negative eigenvalues of $Q_*$, which is at most equal to $l$. So from Lemma 5.2.1 we know that the dimension of any nonpositive subspace of $(W,[\cdot,\cdot]_{Q_*})$ cannot exceed $\dim\left(W_{Q_{*-}}\right)\leq l$. Furthermore, from Lemma 7.2.8, it follows that $Z^+$ is a nonpositive subspace of $(W,[\cdot,\cdot]_{Q_*})$:

$$[w,w]_{Q_*}=\langle Q_*w,w\rangle_W=\langle Qw,w\rangle_{\langle Z_\beta,Z'_{-\beta}\rangle}\leq 0,$$

for all $w\in Z^+$. Consequently, $\dim(Z^+)\leq l$. Thus the total number of eigenvalues (counting algebraic multiplicity) of $A_1$ in the open right half-plane is at most $l$ (see Problem 6.18, page 182, Kato [52]).

*2.* We have

$$\begin{aligned} Z'_{-\beta-1} &= Z^+\dot{+}Z^-, \quad \text{and} \\ A_1 &= \begin{bmatrix} A_1^+ & 0 \\ 0 & A_1^- \end{bmatrix}, \end{aligned}$$

where $A_1^+:Z^+\to Z^+$, $\dim(Z^+)\leq l$, and $A_1^+$ has all its eigenvalues in the open right half-plane. $A_1^-:D(A_1)\cap Z^-\to Z^-$ is the infinitesimal generator of an exponentially stable semigroup on $Z^-$. Thus, ${}^{11}\vee(s)$ (see item 4 of Lemma 7.2.3) can be written as a sum of the transfer function of a system with MacMillan degree at most $l$, with all its poles in the open right half-plane, and a function in $\mathcal{M}H^c_\infty$.

Now we will prove this by simply checking that ${}^{11}\vee(s)$ is the inverse of $\Lambda_{11}(-s)$ for all $s\in\rho(A_1)\cap\rho(-A')$. We have

$$\begin{aligned} &\Lambda_{11}(-s)\ {}^{11}\vee(s)-I_p \\ =\ &\frac{1}{\sigma^2}CL_BN'_\sigma(sI+A')^{-1}C'-\frac{1}{\sigma^2}CL_BN'_\sigma(sI-A_1)^{-1}C' \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{\sigma^4} C L_B N_\sigma' \left(sI + A'\right)^{-1} C' C L_B N_\sigma' \left(sI - A_1\right)^{-1} C' \\
= \; & \frac{1}{\sigma^4} C L_B N_\sigma' \left(sI + A'\right)^{-1} \left[\sigma^2 \left(sI - A_1\right) \left(I - \frac{1}{\sigma^2} L_C L_B\right)\right. \\
& \left. -\sigma^2 \left(sI + A'\right) \left(I - \frac{1}{\sigma^2} L_C L_B\right) - C' C L_B\right] N_\sigma' \left(sI - A_1\right)^{-1} C' \\
= \; & \frac{1}{\sigma^4} C L_B N_\sigma' \left(sI + A'\right)^{-1} \cdot \\
& \left[\sigma^2 \left(-A_1 + A'\right) + A_1 L_C L_B - A' L_C L_B - C' C L_B\right] N_\sigma' \left(sI - A_1\right)^{-1} C' \\
= \; & \frac{1}{\sigma^4} C L_B N_\sigma' \left(sI + A'\right)^{-1} \cdot \\
& \left[C' C L_B N_\sigma' - \frac{1}{\sigma^2} C' C L_B N_\sigma' L_C L_B - C' C L_B\right] N_\sigma' \left(sI - A_1\right)^{-1} C' \\
= \; & 0,
\end{aligned}
$$

where in the above we have verified that all the terms are well-defined at each step. ■

Finally we give the main result of this chapter.

**Theorem 7.2.11** *Suppose that* $(-A, B, C)$ *is a triple satisfying the assumptions A1-5 listed at the beginning of this chapter. Let* $\Lambda$ *be given by (7.5).*

$K(-\cdot) \in H^c_{\infty,l}(\mathbb{C}^{p\times m})$ *satisfies* $\|G(i\cdot) + K(i\cdot)\|_\infty \le \sigma$ *iff* $K(-\cdot) = R_1(-\cdot)R_2(-\cdot)^{-1}$, *where*

$$
\begin{bmatrix} R_1(-\cdot) \\ R_2(-\cdot) \end{bmatrix} = \Lambda(-\cdot)^{-1} \begin{bmatrix} Q(-\cdot) \\ I_m \end{bmatrix}
$$

*for some* $Q(-\cdot) \in H^c_\infty(\mathbb{C}^{p\times m})$ *satisfying* $\|Q(i\cdot)\|_\infty \le 1$.

**Proof** The assumptions S1-6 in Chapter 4 hold. S1 is satisfied (see Theorem 2.5.4) and S2 holds. That S3, S4 and S5 hold follows from Lemma 7.2.1 and Lemma 7.2.2. Lemma 7.2.10.2 shows that S6 also holds. The result now follows from Theorem 4.3.4. ■

**Example 2.5.10** (continued) We remark that all the results in this chapter apply to the Example 2.5.10 considered earlier in Chapter 2: we show that $(A, B)$ is approximately controllable, that is, it satisfies the assumption A5.

In fact, we will show that for this example, $\mathcal{B}' \in \mathcal{L}\left(Z'_{-\beta}, L_2(0,\infty)\right)$ is injective for all $\beta < \frac{1}{2} + \alpha$. It can be verified that $B'\ \left[\in \mathcal{L}\left(Z'_{-\alpha}, \mathbb{C}\right)\right]$ is given by

$$Bf = \int_0^\infty (1+x)^{-\alpha} x^{-\frac{m}{2}} f(x)dx \text{ for all } f \in Z'_{-\alpha} = D\left((A')^{-\alpha}\right) = D\left(A^{-\alpha}\right).$$

From the proof of Lemma 2.5.1.3, we know that

$$\left(\mathcal{B}'f\right)(t) = B'T(t)'f = \int_0^\infty (1+x)^{-\alpha} x^{-\frac{m}{2}} e^{-(1+x)t} f(x)dx, \tag{7.20}$$

for $t \geq 0$ and $f \in Z'_{-\beta}$. For any $t > 0$, we have

$$\begin{aligned} &\left[\int_0^\infty \left|(1+x)^{-\alpha} x^{-\frac{m}{2}} e^{-(1+x)t} f(x)\right| dx\right]^2 \\ \leq\ & \int_0^\infty (1+x)^{2\beta-\alpha} x^{-m} e^{-2(1+x)t} dx \cdot \int_0^\infty (1+x)^{-\alpha-2\beta} |f(x)|^2 dx \\ =\ & \|f\|^2_{Z'_{-\beta}} \cdot \int_0^\infty (1+x)^{2\beta-\alpha} x^{-m} e^{-2(1+x)t} dx. \end{aligned}$$

But since

$$e^{2(1+x)t} = \sum_{k=0}^\infty \frac{2^k t^k}{k!}(1+x)^k \geq \frac{2^n t^n}{n!}(1+x)^n$$

for all $n \in \mathbb{N}$, we have

$$e^{-2(1+x)t} \leq \frac{n!}{2^n t^n}(1+x)^{-n}$$

for all $n \in \mathbb{N}$. If we choose $n > 2\beta - \alpha - m + 1$ (which is possible owing to the archimedean property of $\mathbb{R}$), then using (2.31), we obtain

$$\int_0^\infty (1+x)^{2\beta-\alpha} x^{-m} e^{-2(1+x)t} dx \leq \frac{n!}{2^n t^n} \int_0^\infty (1+x)^{2\beta-\alpha-k} x^{-m} dx < \infty.$$

Hence $x \mapsto e^{(1+x)t} x^{-\frac{m}{2}} (1+x)^{-\alpha} f(x) \in L_1(0,\infty)$. Defining $g(x) = x^{-\frac{m}{2}}(1+x)^{-\alpha} f(x)$, we have $e^{-t} e^{-\cdot t} g(\cdot) \in L_1(0,\infty)$ and so $e^{-\cdot t} g(\cdot) \in L_1(0,\infty)$. Consequently, the Laplace transform of $g$ exists in $\overline{\mathbb{C}^+_\epsilon}$, for any $\epsilon > 0$, and in particular, in $\overline{\mathbb{C}^+_1}$ (see Definition A.6.1, page 635, Curtain and Zwart [34]). If $\left(\mathcal{B}'f\right)(t) = 0$ for $t \geq 0$, then from (7.20) we have

$$e^{-t} \int_0^\infty e^{-xt} g(x)dx = 0 \text{ for } t \geq 0$$

and so

$$\int_0^\infty e^{-xt} g(x)dx = 0 \text{ for } t \geq 0.$$

Thus the Laplace transform of $g$ vanishes on the half-line $[1,\infty)$. Since the Laplace transform is analytic in $\mathbb{C}_1^+$ (see Property A.6.2.a, page 636, Curtain and Zwart [34]), it follows that it must be identically equal to zero. Finally from the injectivity of the Laplace transform (Property A.6.2.b, page 636, Curtain and Zwart [34]), we have $g=0$. This means that $x^{-\frac{m}{2}}(1+x)^{-\alpha}f(x)=0$ for almost all $x\in(0,\infty)$ and so $f=0$. $\Diamond$