Hankel norm approximation for infinite-dimensional systems

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Chapter 6

$J-$spectral factorization for the Pritchard-Salamon class

6.1 Introduction

In this chapter, we show the existence of a solution to the key $J-$spectral factorization problem (satisfying S1-6 in Chapter 4) for the smooth Pritchard-Salamon class of exponentially stable, approximately controllable infinite-dimensional systems with finite-dimensional input and output spaces, in terms of the system parameters A, B, C. We list the assumptions below:

P1. $\Sigma(A, B, C)$ is an exponentially stable, smooth Pritchard-Salamon system.

P2. $U = \mathbb{C}^m$, $Y = \mathbb{C}^p$.

P3. $\sigma_{l+1} < \sigma < \sigma_l$.

P4. (A, B) is approximately controllable, that is,
$\mathcal{B}'$ $[\in \mathcal{L}(W', L_2([0,\infty), \mathbb{C}^m))]$ is injective.

We remark that P4 above is equivalent with the condition that $L_B = \mathcal{B}\mathcal{B}'$ $[\in \mathcal{L}(W', W)]$ is injective.

In Curtain and Ran [23] the sub-optimal Hankel norm problem was solved for the Pritchard-Salamon class by finding a solution to the $J-$spectral factorization problem considered in S3 with $G(s) = C(sI - A)^{-1}B$. However, there the starting point was to quote a result from Ball and Helton [6], which states that the sub-optimal Hankel norm approximation problem is equivalent to solving the $J-$spectral factorization problem in S3. Then a solution is constructed from a given realization of G. However, if one looks for the result quoted from [6], one realizes that this is not an obvious corollary of the very abstract and general theory in [6]. This motivated the self-contained proofs of the sub-optimal Hankel norm approximation problem in Chapter 4. Here we use just the results from Chapter 4, but this entails checking a few more properties of the spectral factor, in particular the assumption S6 for which we use the inertia result in Corollary 5.3.2 from Chapter 5.

6.2 A $J-$spectral factor

The sub-optimal Nehari problem is a special case of the sub-optimal Hankel norm approximation problem; one seeks all $K(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$ satisfying $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$ for a given $\sigma > \|\Gamma\|$. In the case that $\sigma > \|\Gamma\|$, a solution Λ to the $J-$spectral factorization problem S3 was constructed in Curtain and Zwart [33] for the Pritchard-Salamon class.

For the sub-optimal Hankel norm problem we have $\sigma_{l+1} < \sigma < \sigma_l$, but nonetheless, the Λ factor is precisely the same. The proof of the following lemma is analogous to that of Lemma 2.11 in Curtain and Zwart [33]:

Lemma 6.2.1 *Suppose that* $\Sigma(A,B,C)$ *satisfies P1-3. Then*

$$\Sigma\left(A', \begin{bmatrix} C' & L_C B \end{bmatrix}, -\frac{1}{\sigma^2}\begin{bmatrix} -CL_B \\ \sigma B' \end{bmatrix} N_\sigma', \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix}\right)$$

is an exponentially stable, smooth Pritchard-Salamon system and its transfer operator is $\Lambda(-\cdot)$ *, where*

$$\Lambda(s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} + \frac{1}{\sigma^2}\begin{bmatrix} -CL_B \\ \sigma B' \end{bmatrix} N_\sigma'(sI + A')^{-1}\begin{bmatrix} C' & L_C B \end{bmatrix}, \tag{6.1}$$

$N_\sigma = \left(I - \frac{1}{\sigma^2}L_B L_C\right)^{-1}$ *and* $s \in \overline{\mathbb{C}_0^-}$. *Moreover,*

$$\Sigma\left(A', N_\sigma'\begin{bmatrix} C' & \frac{1}{\sigma}L_C B \end{bmatrix}, \frac{1}{\sigma^2}\begin{bmatrix} -CL_B \\ \sigma B' \end{bmatrix}, \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix}\right)$$

is an exponentially stable, smooth Pritchard-Salamon system and its transfer operator is $V(-\cdot)$ *, where*

$$V(s) = \begin{bmatrix} I_p & 0 \\ 0 & \frac{1}{\sigma}I_m \end{bmatrix} - \frac{1}{\sigma^2}\begin{bmatrix} -CL_B \\ B' \end{bmatrix}(sI+A')^{-1}N_\sigma'\begin{bmatrix} C' & \frac{1}{\sigma}L_C B \end{bmatrix}, \quad s \in \overline{\mathbb{C}_0^-}.$$

Furthermore,

1. $\Lambda(-\cdot) \in \mathcal{M}H_\infty^c$ *is invertible in* $\mathcal{M}H_\infty^c$*, and* $\Lambda(-\cdot)^{-1}$ *is equal to* $V(-\cdot)$.

2. Λ *given by (6.1) is a solution of the* $J-$*spectral factorization problem described by (4.2) and (4.1).*

3. $\lim_{\substack{s\to\infty \\ s\in\overline{\mathbb{C}_0^+}}} \Lambda(-s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix}$.

Remark: Note that while it is algebraically straightforward to verify that $\Lambda(s)$ solves the $J-$spectral factorization problem, it is important to justify the well-posedness of each term at all steps. Use is made of the Lyapunov equations from Lemma 2.4.1, but to make sense of all terms, it was necessary to make use of the smoothing properties of L_B and L_C from Lemma 2.4.1. Of course, these smoothing properties are only known to hold for smooth Pritchard-Salamon systems.

If $\sigma > ||\Gamma||$, it was shown in Curtain and Zwart [33] that $\Lambda_{11}(-\cdot)$

1. belongs to $\mathcal{M}H_\infty^c$,
2. is invertible in $\mathcal{M}H_\infty^c$ and
3. $\Lambda_{11}(-\cdot)^{-1}$ is the transfer operator of an exponentially stable smooth Pritchard-Salamon system.

In the case that $\sigma_{l+1} < \sigma < \sigma_l$, $\Lambda_{11}(-\cdot)^{-1}$ still exists, but it is not stable. We can show under the extra assumption P4 that $\Lambda_{11}(-\cdot)^{-1}$ is the sum of a stable part in $H_\infty^c(\mathbb{C}^{p\times p})$ and an antistable rational part with at most l unstable poles. In order to do so we will use the inertia result in Corollary 5.3.2 from Chapter 5.

For $\sigma_{l+1} < \sigma < \sigma_l$, let $N_\sigma := \left(I - \frac{1}{\sigma^2} L_B L_C\right)^{-1} \in \mathcal{L}(V) \cap \mathcal{L}(W)$, $Q := N_\sigma L_B \in \mathcal{L}(W', W)$ and $Q_* := \left(\iota_W\left(\overline{\iota_Z}\right)^{-1}\right)^{-1} N_\sigma L_B \in \mathcal{L}(W')$, where $\iota_W\left(\overline{\iota_Z}\right)^{-1}$ is the isometry from W' to W. We now show that Q_* is self-adjoint and it has at most l negative eigenvalues.

Lemma 6.2.2 *Suppose that* $\Sigma(A,B,C)$ *satisfies P1-4. Then* $Q \in \mathcal{L}(W', W)$ *is a self-dual operator, and* $Q_* \in \mathcal{L}(W')$ *is self-adjoint with*

$$\begin{array}{rcl} 0 & \notin & \sigma_p(Q_*), \\ \sigma(Q_*) \cap \mathbb{C}_0^- & = & \sigma_p(Q_*) \cap \mathbb{C}_0^-, \quad \textit{and} \\ \nu(Q_*) & \leq & l. \end{array}$$

Proof Define $J = I - \frac{1}{\sigma^2}\Gamma^*\Gamma \in \mathcal{L}(L_2([0,\infty),\mathbb{C}^m))$ as in Lemma 2.4.3. $0 \notin \sigma(J)$. Consider the spectral decomposition of $L_2([0,\infty),\mathbb{C}^m)$ into L_- and L_+

induced by the self-adjoint operator J. We have $L_2([0,\infty),\mathbb{C}^m) = L_- \oplus L_+$, and if $v_+ \in L_+$, $\langle Jv_+, v_+\rangle \geq 0$. Let $W' = M_- \oplus M_{0+}$ be the spectral decomposition induced by the self-adjoint bounded operator Q_*. We have

$$\begin{aligned}\langle N_\sigma L_B z, z\rangle_{\langle W,W'\rangle} &= \left\langle \left(I - \frac{1}{\sigma^2} L_B L_C\right)^{-1} L_B z, z\right\rangle_{\langle W,W'\rangle} \\ &= \left\langle L_B z, \left(I - \frac{1}{\sigma^2} L_C L_B\right)^{-1} z\right\rangle_{\langle W,W'\rangle} \\ &= \left\langle L_B \left(I - \frac{1}{\sigma^2} L_C L_B\right) z_0, z_0\right\rangle_{\langle W,W'\rangle},\end{aligned}$$

where $z_0 = \left(I - \frac{1}{\sigma^2} L_C L_B\right)^{-1} z$. Thus

$$\begin{aligned}\langle N_\sigma L_B z, z\rangle_{\langle W,W'\rangle} &= \langle \mathcal{B}' z_0, \mathcal{B}' z_0\rangle - \frac{1}{\sigma^2}\langle \mathcal{B}' L_C \mathcal{B}\mathcal{B}' z_0, \mathcal{B}' z_0\rangle \\ &= \langle u, u\rangle - \frac{1}{\sigma^2}\langle \mathcal{B}' L_C \mathcal{B} u, u\rangle,\end{aligned}$$

where $u = \mathcal{B}' z_0$, and so

$$\langle N_\sigma L_B z, z\rangle_{\langle W,W'\rangle} = \left\langle \left(I - \frac{1}{\sigma^2}\Gamma^*\Gamma\right) u, u\right\rangle.$$

Now define $\Upsilon : M_- \to L_-$ as follows: If $z_- \in M_-$, $\Upsilon z_- := \Pi_{L_-} \mathcal{B}' N'_\sigma z_-$, where Π_{L_-} denotes the canonical projection from $L_2([0,\infty),\mathbb{C}^m)$ onto L_-. Υ is clearly linear. Next we show that Υ is injective. For if $\Pi_{L_-}\mathcal{B}' N'_\sigma z_- = 0$, then $v := \mathcal{B}' N'_\sigma z_- \in L_+$ and

$$0 \geq \langle Q_* z_-, z_-\rangle_{W'} = \langle N_\sigma L_B z_-, z_-\rangle_{\langle W,W'\rangle} = \left\langle \left(I - \frac{1}{\sigma^2}\Gamma^*\Gamma\right) v, v\right\rangle \geq 0.$$

Thus, $\langle Jv, v\rangle = 0$. Using $|\langle Jv_+, v'_+\rangle|^2 \leq \langle Jv_+, v_+\rangle \langle Jv'_+, v'_+\rangle$, we obtain $\langle Jv, Jv\rangle = 0$, and so $Jv = 0$. But J is injective, and so $v = 0$, that is, $\mathcal{B}' N'_\sigma z_- = 0$. Since the system is approximately controllable (assumption P4), this implies that $N'_\sigma z_- = 0$, and since N'_σ is invertible, it follows that $z_- = 0$. Thus, $\dim(M_-) \leq \dim(L_-) = l$. Hence the number of negative eigenvalues of Q_* is at most equal to l (from Kato [52], pages 520-521, it follows that since the $\dim(M_-) < \infty$, the essential spectrum in the open left half-plane is empty and so $\dim(M_-)$ is equal to the number of negative eigenvalues of Q_*).

Finally we prove that $0 \notin \sigma_p(Q_*)$. Suppose, on the contrary, that there exists a nonzero eigenvector $w_1 \in W'$ corresponding to the eigenvalue 0, that is, $Q_* w_1 = 0$. Then we have $N_\sigma L_B w_1 = 0$ and hence $L_B w_1 = 0$. But since L_B is injective, it follows that $w_1 = 0$. ■

We now relate the number of negative eigenvalues of Q_* to the unstable part of $\Lambda_{11}(-\cdot)^{-1}$.

Lemma 6.2.3 *Suppose that* $\Sigma(A,B,C)$ *satisfies P1-4. If* Λ *is given by (6.1), then* $\Lambda_{11}(-\cdot)^{-1} \in H^c_{\infty,l}(\mathbb{C}^{p\times p})$.

Proof We have $\Lambda_{11}(-s) = I_p + \frac{1}{\sigma^2} CL_B N'_\sigma (sI - A')^{-1} C'$ for all $s \in \overline{\mathbb{C}^+_0}$. We claim that $\Lambda_{11}(-\cdot)$ has the inverse

$$^{11}\vee(-s) = I_p - \frac{1}{\sigma^2} CL_B N'_\sigma \left(sI - A' + \frac{1}{\sigma^2} C'CL_B N'_\sigma \right)^{-1} C'.$$

It follows from Property 2 (on page 23) that $A - \frac{1}{\sigma^2} N_\sigma L_B C'C$ is the infinitesimal generator of a strongly continuous semigroup on V, W and Z, since $F = -\frac{1}{\sigma^2} N_\sigma L_B C' \in \mathcal{L}(\mathbb{C}^p, W)$ is a Pritchard-Salamon admissible input operator. The operator C is a Pritchard-Salamon admissible output operator and so

$$\Sigma\left(A - \frac{1}{\sigma^2} N_\sigma L_B C'C, -\frac{1}{\sigma^2} N L_B C', C, I_p \right)$$

is a smooth Pritchard-Salamon system by Property 2. Thus it follows from Property 4 (on page 24) that

$$\Sigma\left(A' - \frac{1}{\sigma^2} C'CL_B N'_\sigma, -\frac{1}{\sigma^2} CL_B N'_\sigma, C', I_p \right)$$

is a smooth Pritchard-Salamon system and $^{11}\vee(-\cdot)$ is its transfer operator.

We calculate

$$\begin{aligned}
& {}^{11}\vee(-s)\, \Lambda_{11}(-s) \\
= & \left[I_p - \frac{1}{\sigma^2} CL_B N'_\sigma \left(sI - A' + \frac{1}{\sigma^2} C'CL_B N'_\sigma \right)^{-1} C' \right] \\
& \left[I_p + \frac{1}{\sigma^2} CL_B N'_\sigma (sI - A')^{-1} C' \right] \\
= & I_p + \frac{1}{\sigma^2} CL_B N'_\sigma \left(sI - A' + \frac{1}{\sigma^2} C'CL_B N'_\sigma \right)^{-1} \\
& \left[-(sI - A') + \left(sI - A' + \frac{1}{\sigma^2} C'CL_B N'_\sigma \right) - \frac{1}{\sigma^2} C'CL_B N'_\sigma \right] \\
& (sI - A')^{-1} C',
\end{aligned}$$

where we note that $C' : Y \to W'$, $(sI - A')^{-1} C' \in D\left((A')^{W'} \right)$ and $C'CL_B N'_\sigma : W' \to W'$. Thus $^{11}\vee(-s)\, \Lambda_{11}(-s) = I_p$.

It is readily verified that $Q = N_\sigma L_B \in \mathcal{L}(W', W)$ is a self-dual solution of the following Riccati equation .

$$\langle Qz_2, A'z_1 \rangle_{\langle W,W' \rangle} + \langle A'z_2, Qz_1 \rangle_{\langle W',W \rangle} = \frac{1}{\sigma^2} \langle CQz_2, CQz_1 \rangle_{\mathbb{C}^p} - \langle B'N'_\sigma z_2, B'N'_\sigma z_1 \rangle_{\mathbb{C}^m}$$

for all z_1 and z_2 in $D\left((A')^{W'}\right)$. Thus if $A_1 = A' - \frac{1}{\sigma^2}C'CL_BN'_\sigma$, then $Q = N_\sigma L_B \in \mathcal{L}(W', W)$ is a self-dual solution of the Riccati equation

$$\begin{aligned} & \langle Qz_2, A_1z_1\rangle_{\langle W,W'\rangle} + \langle A_1z_2, Qz_1\rangle_{\langle W',W\rangle} \\ = \quad & -\frac{1}{\sigma^2}\langle CQz_2, CQz_1\rangle_{\mathbb{C}^p} - \langle B'N'_\sigma z_2, B'N'_\sigma z_1\rangle_{\mathbb{C}^m}. \end{aligned}$$

We now express this Riccati equation in a "weaker" form: suppose that Z is a pivot space such that

$$\begin{array}{rcl} W \hookrightarrow & Z & \hookrightarrow V, \\ D\left((A')^{W'}\right) \hookrightarrow V' \hookrightarrow & Z' & \hookrightarrow W', \end{array}$$

and we identify the duals of $\mathbb{C}^m$ and $\mathbb{C}^p$ with themselves. The isometry from W' to W is given by $c := \iota_W\left(\overline{\iota_Z}\right)^{-1}$. Then $Q_* := c^{-1}Q \in \mathcal{L}(W')$ is self-adjoint and satisfies the Riccati equation

$$\begin{aligned} & \langle Q_*z_2, A_1z_1\rangle_{W'} + \langle A_1z_2, Q_*z_1\rangle_{W'} \\ = \quad & -\frac{1}{\sigma^2}\langle CcQ_*z_2, CcQ_*z_1\rangle_{\mathbb{C}^p} - \langle B'N'_\sigma z_2, B'N'_\sigma z_1\rangle_{\mathbb{C}^m}, \end{aligned}$$

for all z_1 and z_2 in $D\left((A')^{W'}\right)$. Thus $Q_* \in \mathcal{L}(W')$ is a self-adjoint solution of the Lyapunov equation

$$Q_*A_1z + A_1^*Q_*z = -\left[B'N'_\sigma\right]^* B'N'_\sigma z - \left[\frac{1}{\sigma}CcQ_*\right]^*\left[\frac{1}{\sigma}CcQ_*\right]z$$

for all $z \in D(A_1)$, where $\cdot^*$ denotes the adjoint in the Hilbert space W'. We have

$$Q_*A_1z + A_1^*Q_*z = -\begin{bmatrix} B'N'_\sigma \\ \frac{1}{\sigma}CcQ_* \end{bmatrix}^*\begin{bmatrix} B'N'_\sigma \\ \frac{1}{\sigma}CcQ_* \end{bmatrix}z \text{ for all } z \in D(A_1),$$

and $\begin{bmatrix} B'N'_\sigma \\ \frac{1}{\sigma}CcQ_* \end{bmatrix} \in \mathcal{L}(W', \mathbb{C}^m \oplus \mathbb{C}^p)$. Moreover, the pair

$$\left(A_1, \begin{bmatrix} B'N'_\sigma \\ \frac{1}{\sigma}CL_BN'_\sigma \end{bmatrix}\right) = \left(A' - \frac{1}{\sigma^2}C'CL_BN'_\sigma, \begin{bmatrix} B'N'_\sigma \\ \frac{1}{\sigma}CL_BN'_\sigma \end{bmatrix}\right)$$

is exponentially detectable, since $\begin{bmatrix} 0 & \frac{1}{\sigma}C' \end{bmatrix} \in \mathcal{L}(\mathbb{C}^m \oplus \mathbb{C}^p, W')$ and

$$\begin{aligned} & A' - \frac{1}{\sigma^2}C'CL_BN'_\sigma + \begin{bmatrix} 0 & \frac{1}{\sigma}C' \end{bmatrix}\begin{bmatrix} B'N'_\sigma \\ \frac{1}{\sigma}CL_BN'_\sigma \end{bmatrix} \\ = \quad & A' - \frac{1}{\sigma^2}C'CL_BN'_\sigma + \frac{1}{\sigma^2}C'CL_BN'_\sigma \\ = \quad & A' \end{aligned}$$

generates an exponentially stable semigroup on W'. Thus applying Corollary 5.3.2, we obtain that $\pi(A_1) \leq \nu(Q_*) \leq l$. Hence,

$$W' = M_+ \dot{+} M_-, \;\; A_1 = \begin{bmatrix} A_1^+ & 0 \\ 0 & A_1^- \end{bmatrix};$$

where $A_1^+ : M_+ \to M_+$, $\dim(M_+) = r \leq l$, and A_1^+ has all its r eigenvalues in the open right half-plane. The operator $A_1^- : D(A_1) \cap M_- \to M_-$ is the infinitesimal generator of an exponentially stable semigroup on M_-. Thus, $\Lambda_{11}(-\cdot)^{-1}$ can be written as a sum of the transfer function of a system with MacMillan degree at most l with all poles in the open right half-plane and a function in $\mathcal{M}H_\infty^c$. ■

Finally we give the main result of this chapter.

Theorem 6.2.4 *Suppose that $\Sigma(A,B,C)$ satisfies P1-4. Let Λ be given by (6.1). Then we have:*

$K(-\cdot) \in H_{\infty,l}^c(\mathbb{C}^{p\times m})$ satisfies $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$ iff $K(-\cdot) = R_1(-\cdot)R_2(-\cdot)^{-1}$, where

$$\begin{bmatrix} R_1(-\cdot) \\ R_2(-\cdot) \end{bmatrix} = \Lambda(-\cdot)^{-1} \begin{bmatrix} Q(-\cdot) \\ I_m \end{bmatrix}$$

for some $Q(-\cdot) \in H_\infty^c(\mathbb{C}^{p\times m})$ satisfying $\|Q(i\cdot)\|_\infty \leq 1$.

Proof The assumptions S1-6 in Chapter 4 hold. S1 is satisfied since G is the transfer operator of an exponentially stable smooth Pritchard-Salamon system (see 2.11) and S2 holds. Lemma 6.2.1 shows that S3-5 are satisfied, and Lemma 6.2.3 shows that S6 also holds. Now the result follows from Theorem 4.3.4. ■

Example 2.4.4 (continued) All the results in this chapter apply to the Example 2.4.4 considered earlier in Chapter 2: we show that the system is approximately controllable, that is, it satisfies the assumption P4.

From Theorem 4.2.10 (page 171, Curtain and Zwart [34]), it follows that the smooth Pritchard-Salamon system with state-space $Z = \mathbb{C} \times L_2(-\tau, 0)$ is approximately controllable. But $\Sigma(A,B,C)$ is also a smooth Pritchard-Salamon system with

$$\mathcal{W} = Z = W^{1,2} \hookrightarrow V = \mathbb{C} \times L_2(-\tau, 0),$$

in which case B is an *unbounded* control operator and C is a *bounded* observation operator (see for example, page 804, Curtain and Salamon [24] or page 123, Salamon [77]. In the latter, he considers the more general case of a delay system with a delay h_u in the input, which is simply 0 in our case.) From Corollary 4.2.10 (pages 130-131, Salamon [77]), it follows that the system is also approximately controllable with state space $\mathcal{W}$. ◊