Hankel norm approximation for infinite-dimensional systems

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Chapter 5

Inertia theorems

5.1 Introduction

In this chapter, we study operator Lyapunov inequalities and equations, for which the infinitesimal generator is not necessarily stable, but it satisfies the spectrum decomposition assumption and it has at most finitely many unstable eigenvalues. Moreover, the input or output operators are not necessarily bounded, but are admissible. We prove a new inertia result: under mild conditions, we show that the number of unstable eigenvalues of the generator is less than or equal to the number of negative eigenvalues of the self-adjoint solution of the operator Lyapunov inequality.

The *inertia* of a square matrix $A \in \mathbb{C}^{n\times n}$ is the triple $(\nu(A), \zeta(A), \pi(A))$ where

$$\begin{aligned} \nu(A) &= \text{number of eigenvalues of } A \text{ in } \mathbb{C}_0^-, \\ \zeta(A) &= \text{number of eigenvalues of } A \text{ on the imaginary axis}, \\ \pi(A) &= \text{number of eigenvalues of } A \text{ in } \mathbb{C}_0^+. \end{aligned}$$

Example 5.1.1 The inertia of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is $(1, 0, 2)$. $\Diamond$

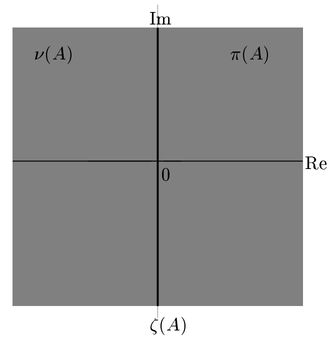

Figure 5.1: Inertia.

Inertia theorems for matrices concern relations between the inertia of Hermitian solutions Q of the Lyapunov equation

$$A^*Q + QA = -C^*C, \tag{5.1}$$

and the matrix A. The fundamental result was by Ostrowski and Schneider [65], and later contributions can be found in Wimmer [97] and Bittanti and Colaneri [10]. We shall generalize the following known theorem (see Theorem 3.3.2 on page 1126, Glover [40]):

Theorem 5.1.2 *Given the matrices* $A \in \mathbb{C}^{n\times n}$ *and* $C \in \mathbb{C}^{n\times p}$ *and a Hermitian solution* Q *to (5.1), if* $\zeta(Q) = 0$*, then* $\pi(A) \leq \nu(Q)$ *and* $\nu(A) \leq \pi(Q)$*.*

There is little known about such inertia theorems for operator Lyapunov equations, and since the operators may have general spectra or infinitely many eigenvalues, it is clear that one can only hope for a partial generalization of the matrix results. In Bunce [12] and Cain [13], they consider the case of A being a bounded linear operator, assuming an exact controllability condition on $\Sigma(A, B, -)$, but these are unrealistic assumptions rarely satisfied by delay or partial differential equation examples.

We now define the notion of the algebraic multiplicity of an isolated eigenvalue of a closed operator on a Hilbert space. Let λ_0 be an eigenvalue of a closed linear operator A on a Hilbert space $\mathcal{H}$. Suppose further that this eigenvalue is isolated; that is, there exists an open set $\mathcal{O}$ containing λ_0 such that $\sigma(A) \cap \mathcal{O} = \{\lambda_0\}$. We say that λ_0 has *order* ν_0 if for every $x \in \mathcal{H}$, $\lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)^{\nu_0}(\lambda I - A)^{-1}x$ exists, but there exists a x_0 such that $\lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)^{\nu_0-1}(\lambda I - A)^{-1}x_0$ does not. If for every $\nu \in \mathbb{N}$ there exists a $x_\nu \in \mathcal{H}$ such that $\lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)^{\nu}(\lambda I - A)^{-1}x_\nu$ does not exist, then the order of λ_0 is infinity. For an isolated eigenvalue λ_0 of finite order ν_0, its *algebraic multiplicity* is defined as $\dim(\ker(\lambda_0 I - A)^{\nu_0})$.

Example 5.1.3 The eigenvalue 1 of the operator

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathcal{L}(\mathbb{C}^2)$$

has order 2. On the other hand, any isolated eigenvalue of a self-adjoint operator $Q \in \mathcal{L}(\mathcal{H})$ has order 1, and in this case, its algebraic multiplicity is equal to its geometric multiplicity[1]. $\Diamond$

Next we define $\pi(A)$ for a closed operator A on a Hilbert space $\mathcal{H}$. Let A be a closed linear operator on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and let $\sigma(A) \cap \mathbb{C}_0^+$ be a bounded set which is isolated from $\sigma(A) \setminus \left(\sigma(A) \cap \mathbb{C}_0^+\right)$ (by which we mean that it is separated from $\sigma(A) \setminus \left(\sigma(A) \cap \mathbb{C}_0^+\right)$ in such a way that a simple, closed, rectifiable curve γ can be drawn so as to enclose an open set containing $\sigma(A) \cap \mathbb{C}_0^+$ in its interior and $\sigma(A) \setminus \left(\sigma(A) \cap \mathbb{C}_0^+\right)$ in its exterior). Let Π denote the spectral projection on $\sigma(A) \cap \mathbb{C}_0^+$. Then $\mathcal{H} = \mathcal{H}^+ \dot{+} \mathcal{H}^-$, where

$$\begin{aligned} \mathcal{H}^+ &:= \Pi\mathcal{H}, \\ \mathcal{H}^- &:= (I - \Pi)\mathcal{H}, \end{aligned} \tag{5.2}$$

and $\dot{+}$ denotes the direct sum of the subspaces $\mathcal{H}^+$ and $\mathcal{H}^-$ (see for example, Theorem 6.17, page 178, Kato [52]). The subspace $\mathcal{H}^+$ is finite-dimensional iff $\sigma(A) \cap \mathbb{C}_0^+$ consists of a finite system of eigenvalues (see Lemma 2.5.7, pages 71-72, Curtain and Zwart [34] and Problem 6.18, page 182, Kato [52]). In this case, the total algebraic multiplicity of the eigenvalues in $\mathbb{C}_0^+$, which we denote by $\pi(A)$, is equal to $\dim(\mathcal{H}^+)$ (see Figure 5.2).

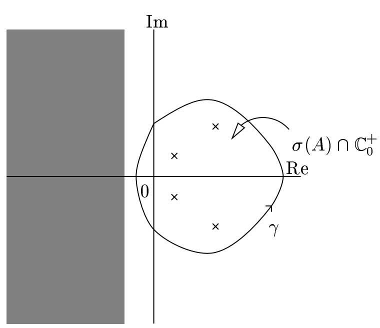

Figure 5.2: $\pi(A)$.

We now state our main result about operator Lyapunov inequalities:

Theorem 5.1.4 *Assume that*

[1] the geometric multiplicity of λ is $\dim(\ker(\lambda I - A))$.

1. *A is a densely defined closed linear operator on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$,*
2. *$\sigma(A) \cap \mathbb{C}_0^+$ is a bounded set which is isolated from $\sigma(A) \setminus \left(\sigma(A) \cap \mathbb{C}_0^+\right)$,*
3. *$\dim(\mathcal{H}^+) < \infty$ (with the notation introduced in (5.2)),*
4. *$Q \in \mathcal{L}(\mathcal{H})$ is a self-adjoint operator such that*

$$\begin{array}{rcl} 0 & \notin & \sigma_p(Q), \\ \sigma(Q) \cap \mathbb{C}_0^- & = & \sigma_p(Q) \cap \mathbb{C}_0^-, \\ \nu(Q) & < & \infty, \end{array}$$

   where $\nu(Q)$ denotes the number of negative eigenvalues of Q and Q satisfies the Lyapunov inequality

$$\langle Qx, Ax\rangle + \langle QAx, x\rangle \leq 0 \text{ for all } x \in D(A). \tag{5.3}$$

Then $\pi(A) \leq \nu(Q)$.

The chapter is organized as follows. First we give some mathematical preliminaries about indefinite inner product spaces and give the proof of our main result. Section 5.3 gives a few elementary corollaries of our main theorem. Finally, we give sufficient conditions for the equality $\pi(A) = \nu(Q)$.

5.2 Preliminaries and main proof

The proof of our main theorem relies on the fact that any self-adjoint solution of the operator Lyapunov inequality gives rise to a natural indefinite inner product space. So we will first state a few preliminaries and results about indefinite inner product spaces which will be used in the proof. We shall tailor the choice of the definitions we present here to the needs of the remainder of this chapter. For more details, see Azizov and Iokhvidov [5].

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. An *indefinite inner product* $[\cdot,\cdot]$ *on* $\mathcal{V}$ is a map $[\cdot,\cdot] : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ satisfying:

1. $[\alpha x_1 + \beta x_2, y] = \alpha[x_1, y] + \beta[x_2, y]$, for all $x_1, x_2, y \in \mathcal{V}$ and for all $\alpha, \beta \in \mathbb{C}$.
2. $[x, y] = \overline{[y, x]}$ for all $x, y \in \mathcal{V}$.

A vector $x \in \mathcal{V}$ is said to be *positive*, *negative* or *neutral* depending on whether $[x, x]$ is > 0, < 0 or $= 0$, respectively. We denote the sets of all positive, negative and neutral vectors of a space by $\mathcal{V}_{++}$, $\mathcal{V}_{--}$ and $\mathcal{V}_0$, respectively, that is

$$\begin{array}{rcl} \mathcal{V}_{++} & = & \{x \mid [x,x] > 0\}, \\ \mathcal{V}_{--} & = & \{x \mid [x,x] < 0\}, \\ \mathcal{V}_0 & = & \{x \mid [x,x] = 0\}. \end{array}$$

We define

$$\mathcal{V}_+ = \mathcal{V}_{++} \cup \mathcal{V}_0, \quad \mathcal{V}_- = \mathcal{V}_{--} \cup \mathcal{V}_0,$$

the sets of all nonnegative and nonpositive vectors in $\mathcal{V}$, respectively. A subspace $\mathcal{W}$ of $\mathcal{V}$ is said to be *nonnegative*, *nonpositive* or *neutral* if

$$\mathcal{W} \subset \mathcal{V}_+, \quad \mathcal{W} \subset \mathcal{V}_-, \quad \text{or} \quad \mathcal{W} \subset \mathcal{V}_0,$$

respectively. A subspace $\mathcal{W}$ of $\mathcal{V}$ is said to be *positive* (*negative*) if

$$\mathcal{W} \subset \mathcal{V}_{++} \cup \{0\} \quad (\mathcal{W} \subset \mathcal{V}_{--} \cup \{0\}).$$

The nonnegative and nonpositive subspaces are said to be *semidefinite*. For a semidefinite subspace $\mathcal{W}$, the following generalization of the Cauchy-Schwarz inequality holds:

$$|[x_1, x_2]|^2 \leq [x_1, x_1]\, [x_2, x_2] \text{ for all } x_1,\ x_2 \in \mathcal{W}. \tag{5.4}$$

First we prove the inequality for nonnegative subspaces. Given $x_1, x_2 \in \mathcal{W}$, we have

$$[\alpha x_1 + \beta x_2, \alpha x_1 + \beta x_2] \geq 0 \text{ for all } \alpha, \beta \in \mathbb{C}.$$

Thus $|\alpha|^2 A - 2\,\mathrm{Re}\left(\alpha\overline{\beta}B\right) + |\beta|^2 C \geq 0$ for all $\alpha, \beta \in \mathbb{C}$, where

$$\begin{array}{rcl} A & = & [x_1, x_1] \geq 0, \\ B & = & [x_1, x_2], \\ C & = & [x_2, x_2] \geq 0. \end{array}$$

As a result, we obtain $AC \geq |B|^2$ which is the desired inequality. In the case of a nonpositive $\mathcal{W}$, we arrive at the same conclusion by considering the indefinite inner product $-[\cdot,\cdot]$.

The following lemma will be crucial in obtaining the inequality in our main inertia theorem.

Lemma 5.2.1 *Let $\mathcal{V}$ be a linear space with an indefinite inner product $[\cdot,\cdot]$ which admits a decomposition into a direct sum $\mathcal{V} = \mathcal{V}_+ \dot{+} \mathcal{V}_-$ of a positive subspace $\mathcal{V}_+$ and a negative subspace $\mathcal{V}_-$. Then the dimension of any nonpositive subspace $\mathcal{W}$ of $\mathcal{V}$ does not exceed the dimension of $\mathcal{V}_-$.*

Proof This follows from Remark 4.4, page 24, Azizov and Iokhvidov [5]. ■

Next we study indefinite inner products that are induced by a bounded self-adjoint operator. Let $\mathcal{H}$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. Let $Q \in \mathcal{L}(\mathcal{H})$ be an arbitrary bounded self-adjoint operator on $\mathcal{H}$. Then $\mathcal{H}$ equipped with the indefinite inner product $[\cdot, \cdot]$ defined by

$$[x, y] = \langle Qx, y \rangle \text{ for all } x \text{ and } y \text{ in } \mathcal{H},$$

is called a $Q-$*space* and Q is called the *Gram operator* of the space $(\mathcal{H}, [\cdot, \cdot])$. It is clear that

$$|[x, y]| \leq \|Q\| \, \|x\| \, \|y\|,$$

where $\|x\| = (\langle x, x \rangle)^{\frac{1}{2}}$, which establishes the continuity of $[\cdot, \cdot] : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$. We denote the sum of two $\langle \cdot, \cdot \rangle-$orthogonal subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ by $\mathcal{W}_1 \langle + \rangle \mathcal{W}_2$, and the sum of two $[\cdot, \cdot]-$orthogonal subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ by $\mathcal{W}_1 \, [+] \, \mathcal{W}_2$.

Example 5.2.2 The *real Minkowski space* $\mathbb{M}^{n+1}$ is defined as the real vector space $\mathbb{R}^{n+1}$ with the indefinite inner product $[x, y] = x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}$, where

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix}.$$

See Figure 5.3.

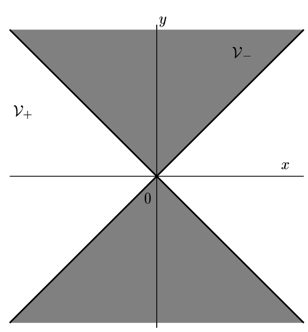

Figure 5.3: Nonpositive and nonnegative subsets of the real Minkowski space $\mathbb{M}^2$.

This is a $Q-$space with $Q = \begin{bmatrix} I_n & 0 \\ 0 & -1 \end{bmatrix}$. ◊

We now prove a useful lemma about $Q-$spaces which will be used in the proof of our main theorem.

Lemma 5.2.3 *Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and let $Q \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then*

1. *The $Q-$space $(\mathcal{H},[\cdot,\cdot])$ admits an $\langle\cdot,\cdot\rangle-$orthogonal direct sum decomposition*
$$\mathcal{H} = \mathcal{H}_{Q_-}\ \langle+\rangle\ \mathcal{H}_{Q_0}\ \langle+\rangle\ \mathcal{H}_{Q_+}, \tag{5.5}$$
where $\mathcal{H}_{Q_-}$ is a negative subspace, $\mathcal{H}_{Q_+}$ is a positive subspace and $\mathcal{H}_{Q_0}$ is a neutral subspace. Furthermore, $\mathcal{H} = \mathcal{H}_{Q_-}[+]\mathcal{H}_{Q_0}[+]\mathcal{H}_{Q_+}$; that is, the decomposition is also $[\cdot,\cdot]-$orthogonal.

2. *If $\sigma(Q)\cap\mathbb{C}_0^- = \sigma_p(Q)\cap\mathbb{C}_0^-$ and $\nu(Q)<\infty$, then $\dim\left(\mathcal{H}_{Q_-}\right) = \nu(Q)$.*

3. *$\mathcal{H}_{Q_0} = 0$ iff $0 \notin \sigma_p(Q)$.*

Proof *1.* Let $\mathcal{E} = \{E(\lambda)\}_{\lambda\in\mathbb{R}}$ be the spectral family of spectral projections $E(\lambda)\in\mathcal{L}(\mathcal{H})$, $\lambda\in\mathbb{R}$, corresponding to the self-adjoint operator Q. Define the projections

$$\begin{aligned} S_- &= \int_{-\infty}^{0-} dE(\lambda) = E(0-), \\ S_0 &= E(0) - E(0-), \text{ and} \\ S_+ &= \int_0^{\infty} dE(\lambda). \end{aligned}$$

These projections are pairwise orthogonal and $I = S_- + S_0 + S_+$, and they generate a $\langle\cdot,\cdot\rangle-$orthogonal decomposition of $\mathcal{H}$ into subspaces $\mathcal{H}_{Q_-}$, $\mathcal{H}_{Q_0}$, $\mathcal{H}_{Q_+}$ (see Figure 5.4), where

$$\begin{aligned} \mathcal{H}_{Q_-} &= \text{ran }(S_-), \\ \mathcal{H}_{Q_0} &= \text{ran }(S_0), \text{ and} \\ \mathcal{H}_{Q_+} &= \text{ran }(S_+). \end{aligned}$$

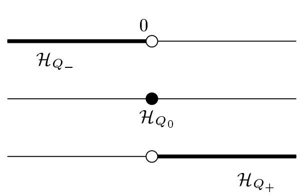

Figure 5.4: The subspaces $\mathcal{H}_{Q_-}$, $\mathcal{H}_{Q_0}$ and $\mathcal{H}_{Q_+}$.

Thus $\mathcal{H} = \mathcal{H}_{Q_-}\ \langle+\rangle\ \mathcal{H}_{Q_0}\ \langle+\rangle\ \mathcal{H}_{Q_+}$.

We first prove that $[x_-, x_-] = \langle Qx_-, x_-\rangle \leq 0$ for all $x_- \in \mathcal{H}_{Q_-}$. We have

$$\begin{aligned} [x_-, x_-] &= \langle Qx_-, x_-\rangle \\ &= \int_{-\infty}^{\infty} \lambda \, d\langle E(\lambda)x_-, x_-\rangle \\ &= \int_{-\infty}^{\infty} \lambda \, d\langle E(\lambda)x_-, S_-x_-\rangle \\ &= \int_{-\infty}^{\infty} \lambda \, d\langle E(\lambda)x_-, E(0-)x_-\rangle \\ &= \int_{-\infty}^{\infty} \lambda \, d\langle E(0-)E(\lambda)x_-, x_-\rangle. \end{aligned}$$

But

$$\langle E(0-)E(\lambda)x_-, x_-\rangle = \begin{cases} \langle E(\lambda)x_-, x_-\rangle & \lambda < 0 \\ \langle E(0-)x_-, x_-\rangle (= \text{a constant}) & \lambda \geq 0 \end{cases},$$

and so

$$\langle Qx_-, x_-\rangle = \int_{-\infty}^{\infty} \lambda d\langle E(0-)E(\lambda)x_-, x_-\rangle = \int_{-\infty}^{0-} \lambda d\langle E(\lambda)x_-, x_-\rangle \leq 0,$$

since $\lambda \mapsto \langle E(\lambda)x_-, x_-\rangle$ is a non-decreasing function and $\lambda < 0$.

Similarly it can be checked that $[x_+, x_+] = \langle Qx_+, x_+\rangle \geq 0$, for all $x_+ \in \mathcal{H}_{Q_+}$. Since $\mathcal{H}_{Q_-}$, $\mathcal{H}_{Q_0}$ and $\mathcal{H}_{Q_+}$ are $Q-$invariant, it follows from their $\langle\cdot,\cdot\rangle-$orthogonality that they are $[\cdot,\cdot]-$orthogonal. Finally, to prove that the subspaces $\mathcal{H}_{Q_+}$ and $\mathcal{H}_{Q_-}$ are in fact positive and negative, respectively, we use the generalized Cauchy-Schwarz inequality (5.4): if $x_+ \in \mathcal{H}_{Q_+}$ and $\langle Qx_+, x_+\rangle = 0$, then

$$\begin{aligned} 0 &\leq |\langle Qx_+, Qx_+\rangle|^2 \\ &= |[x_+, Qx_+]|^2 \\ &\leq [x_+, x_+]\,[Qx_+, Qx_+] \\ &= \langle Qx_+, x_+\rangle\,\langle QQx_+, Qx_+\rangle \\ &= 0, \end{aligned}$$

and so $Qx_+ = 0$, that is $x_+ \in \mathcal{H}_{Q_0}$. Consequently, $x_+ \in \mathcal{H}_{Q_+} \cap \mathcal{H}_{Q_0} = \{0\}$.

Similarly, it can be shown that $\mathcal{H}_{Q_-}$ is a negative subspace.

2. If the eigenvalues in $\mathbb{C}_0^-$ are $\lambda_1, \ldots, \lambda_n$, then

$$\begin{aligned} \dim\left(\mathcal{H}_{Q_-}\right) &= \dim\left(E(0-)\right) = \sum_{k=1}^{n} \dim\left(E\left(\lambda_k\right) - E\left(\lambda_k -\right)\right) \\ &= \sum_{k=1}^{n} \dim\left(\ker\ \left(\lambda_k I - Q\right)\right) = \nu(Q). \end{aligned}$$

3. This follows from the fact that $\mathcal{H}_{Q_0} = \ker(Q)$. ■

Remark: We remark that the decomposition in (5.5) is only one amongst several possible ones. So for example, if $[x, x] < 0$, it does not follow that $x \in \mathcal{H}_{Q_-}$: Indeed, with

$$Q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \ (\in \mathcal{L}(\mathbb{C}^2)), \text{ and}$$
$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left(\notin \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \mathcal{H}_{Q_-}\right),$$

we have $[x, x] < 0$.

A linear operator A with an arbitrary domain of definition $D(A)$, operating in a $Q-$space $\mathcal{H}$, is said to be $Q-$*dissipative* if

$$2\,\mathrm{Re}\,([Ax, x]) = \langle QAx, x\rangle + \langle Qx, Ax\rangle \le 0$$

for all $x \in D(A)$. We quote the following crucial result which is an immediate consequence of Theorem 2.21, page 98, Azizov and Iokhvidov [5].

Lemma 5.2.4 *Let $\mathcal{H}$ be a $Q-$space and A be a closed $Q-$dissipative operator on $\mathcal{H}$. Furthermore, assume that σ is a bounded subset of $\sigma(A)$ such that $\sigma \subset \mathbb{C}_+$ and σ is isolated from $\sigma(A) \setminus \sigma$. If Π denotes the spectral projection on σ, then $\Pi\mathcal{H}$ is a nonpositive subspace of the $Q-$space $(\mathcal{H}, [\cdot, \cdot])$.*

We now proceed to give a proof of our main theorem.

Proof (of Theorem 5.1.4)

1. From the Lyapunov inequality, it follows that A is $Q-$dissipative:

   $$2\,\mathrm{Re}\,([Ax, x]) = \langle QAx, x\rangle + \langle Qx, Ax\rangle \le 0 \ \text{ for all } x \in D(A).$$

   Using Lemma 5.2.4, we obtain that $\mathcal{H}^+ = \Pi\mathcal{H}$ (see (5.2)) is a nonpositive subspace of $(\mathcal{H}, [\cdot, \cdot])$. This is a $\pi(A)-$dimensional nonpositive subspace of $(\mathcal{H}, [\cdot, \cdot])$.

2. From Lemma 5.2.3, since $0 \notin \sigma_p(Q)$ it follows that the self-adjoint operator Q induces a $[\cdot, \cdot]-$orthogonal direct sum decomposition $\mathcal{H} = \mathcal{H}_{Q_-} \,[+]\, \mathcal{H}_{Q_+}$, where $\mathcal{H}_{Q_-}$ and $\mathcal{H}_{Q_+}$ are negative and positive subspaces, respectively, in $(\mathcal{H}, [\cdot, \cdot])$, with $\dim\left(\mathcal{H}_{Q_-}\right) = \nu(Q)$.

3. Finally it follows from Lemma 5.2.1, that $\dim(\mathcal{H}^+) \leq \dim\left(\mathcal{H}_{Q_-}\right)$, that is, $\pi(A) \leq \nu(Q)$.

■

5.3 Corollaries

In this section we give a few corollaries of our main theorem applied to Lyapunov equations with possibly unbounded observation operators.

Throughout this section, we assume that X is a Hilbert space and $A : D(A) \to X$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X. Let

$$\begin{aligned} \sigma_+(A) &:= \sigma(A) \cap \overline{\mathbb{C}_0^+}, \\ \sigma_-(A) &:= \sigma(A) \cap \mathbb{C}_0^-. \end{aligned}$$

A satisfies the *spectrum decomposition assumption* if $\sigma_+(A)$ is a bounded set which is separated from $\sigma_-(A)$ in such a way that a rectifiable, simple closed curve, γ, can be drawn so as to enclose an open set containing $\sigma_+(A)$ in its interior and $\sigma_-(A)$ in its exterior (see Figure 5.5).

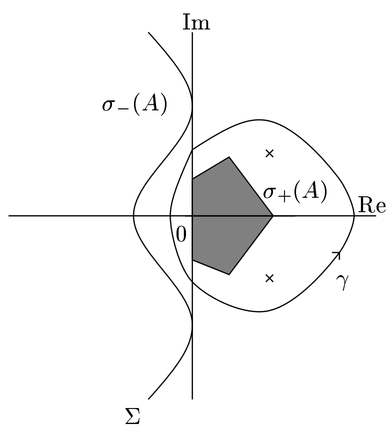

Figure 5.5: The spectrum decomposition: Here $\sigma_+(A)$ comprises the shaded region together with the crosses, and $\sigma_-(A)$ is contained in the region to the left of the curve Σ.

The decomposition of the spectrum in this way induces a corresponding direct sum decomposition of the state space X:

$$X = X^+ \dot{+} X^-, \quad X^+ = \Pi X, \quad X^- = (I - \Pi)X, \tag{5.6}$$

where Π is the spectral projection on $\sigma_+(A)$:

$$\Pi x = \frac{1}{2\pi i}\int_\gamma (\lambda I - A)^{-1} x d\lambda \text{ for all } x \in X,$$

and γ is traversed once in the positive direction (counterclockwise).

Let us denote by Z_{-1} is the completion of X with respect to the norm

$$\|x\|_{-1} = \left\|(\gamma I - A^*)^{-1} z\right\|,$$

where $\gamma \in \rho(A^*)$ is fixed. If X is the pivot space (that is, if we identify X with X^*), then it follows that $Z_{-1}^* = X_1$.

We now consider the operator *Lyapunov equation*

$$A^*Qx + QAx = -C^*Cx \text{ for all } x \in D(A), \tag{5.7}$$

with values in Z_{-1}, where $C \in \mathcal{L}(X_1, Y)$ and Y is a Hilbert space. We say that (5.7) has a *self-adjoint solution* $Q = Q^* \in \mathcal{L}(X)$ if (5.7) holds. For the theory of such Lyapunov equations with self-adjoint nonnegative definite solutions $Q \in \mathcal{L}(X)$, see Hansen and Weiss [49] and Grabowski [45].

Corollary 5.3.1 *If*

1. *A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on the Hilbert space X,*
2. *$C \in \mathcal{L}(X_1, Y)$,*
3. *A satisfies the spectrum decomposition assumption,*
4. *$\dim(X^+) < \infty$ (with the notation used in (5.6)),*
5. *$Q \in \mathcal{L}(X)$ is a self-adjoint solution of (5.7) such that*

$$\begin{array}{rcl} 0 & \notin & \sigma_p(Q), \\ \sigma(Q) \cap \mathbb{C}_0^- & = & \sigma_p(Q) \cap \mathbb{C}_0^-, \text{ and} \\ \nu(Q) & < & \infty, \end{array}$$

then $\pi(A) \leq \nu(Q)$.

Proof We observe that

$$2\,\mathrm{Re}\,([Ax, x]) = \langle QAx, x\rangle + \langle Qx, Ax\rangle = -\langle Cx, Cx\rangle \leq 0 \text{ for all } x \in D(A),$$

that is, A is $Q-$dissipative. An application of Theorem 5.1.4 yields the desired inequality. ■

We now appeal to known sufficient conditions for A to satisfy the spectrum decomposition assumption.

Corollary 5.3.2 *If*

1. $C \in \mathcal{L}(X,Y)$ *has finite rank,*
2. $\Sigma(A,-,C)$ *is exponentially detectable, and*
3. $Q \in \mathcal{L}(X)$ *is a self-adjoint solution of (5.7),*

then A *satisfies the spectrum decomposition assumption, in the closed right half-plane* A *has a pure point spectrum* $\left(\sigma(A) \cap \overline{\mathbb{C}_0^+} = \sigma_p(A) \cap \overline{\mathbb{C}_0^+}\right)$ *and* $\zeta(A) = 0$.

Furthermore, if

$$\begin{array}{rcl} 0 & \notin & \sigma_p(Q), \\ \sigma(Q) \cap \mathbb{C}_0^- & = & \sigma_p(Q) \cap \mathbb{C}_0^-, \quad \textit{and} \\ \nu(Q) & < & \infty, \end{array}$$

then $\pi(A) \leq \nu(Q)$.

Proof From Theorem 5.2.7 (page 235, Curtain and Zwart [34]), it follows that A satisfies the spectrum decomposition assumption, and X^+ is finite-dimensional. So from Problem 6.18 (page 182, Kato [52]) we conclude that $\sigma_+(A)$ comprises finitely many eigenvalues of finite algebraic multiplicity.

Next, we show that A has no eigenvalues on the imaginary axis. Assume the contrary; that is, suppose that there exists a $\omega_0 \in \mathbb{R}$ and a $x_0\ (\neq 0) \in X$ such that $Ax_0 = i\omega_0 x_0$. From (5.7), we obtain that

$$\begin{aligned} -\|Cx_0\|^2 = -\langle Cx_0, Cx_0\rangle &= \langle Ax_0, Qx_0\rangle + \langle x_0, QAx_0\rangle \\ &= i\omega_0\langle Qx_0, x_0\rangle - i\omega_0\langle Qx_0, x_0\rangle = 0, \end{aligned}$$

and so $Cx_0 = 0$. Thus, $x_0 \in \ker(C)$. But since $\Sigma(A,-,C)$ is exponentially detectable with a finite-rank C, we have

$$\ker(sI - A) \cap \ker(C) = \{0\} \text{ for all } s \in \overline{\mathbb{C}_0^+}, \tag{5.8}$$

(see Theorem 5.2.11, pages 240-241, Curtain and Zwart [34]), and so we arrive at a contradiction.

If $0 \notin \sigma_p(Q)$, $\sigma(Q) \cap \mathbb{C}_0^- = \sigma_p(Q) \cap \mathbb{C}_0^-$, $\nu(Q) < \infty$, then using Corollary 5.3.1 above, we obtain that $\pi(A) \leq \nu(Q)$. ∎

Remark: For the case that C is admissible and finite-rank, Corollary 5.3.2 still holds, if for *exponential detectability* we use the following definition:

There exists a $L \in \mathcal{L}(Y, X_1)$ such that $A^{X_1} + LC$ generates an exponentially stable semigroup on W, where A^{X_1} denotes the restriction of A to $D\left(A^{X_1}\right) = \{x \in D(A) \mid Ax \in D(A)\}$.

This is a bounded concept of detectability on the state space X_1 and as in the proof of Corollary 5.3.2, we conclude that A^{X_1} satisfies the spectrum decomposition assumption on X_1. But the spectra of A and its restriction A^{X_1} are the same (see Curtain et al. [21]) and so A satisfies the spectrum decomposition on X and X^+ is finite-dimensional. Similarly, we can argue that (5.8) holds. Finally, we remark that this concept of exponential detectability is equivalent to the existence of an "admissible control" operator $L \in \mathcal{L}(Y, X)$ such that $A + LC$ generates an exponentially stable semigroup on X (see Curtain et al. [21]). The more general concepts of detectability in the literature do not imply that A satisfies the spectrum decomposition assumption, even if C has finite rank (see Rebarber [72]).

Finally, we remark that similar theorems can be proved for admissible control operators $B \in \mathcal{L}(U, X_{-1})$, where the input space U is a Hilbert space (see Hansen and Weiss [49]).

5.4 The equality $\pi(A) = \nu(Q)$

In this section we prove stronger inertia results than ones established so far under the extra assumption that the infinitesimal generator A is normal: we obtain $\pi(A) = \nu(Q)$ instead of $\pi(A) \leq \nu(Q)$. Furthermore, we provide a direct proof of these inertia results, that is, without appealing to the theory of indefinite inner product spaces.

A closed, densely defined operator A on the Hilbert space $\mathcal{H}$ is said to be *normal* if $A^*A = AA^*$. The implication of this relation is complicated on account of the domain relations involved. In particular, the above relation implies that

$$D(A) = D(A^*) \supset D(A^*A) = D(AA^*)$$

and $\|Ax\| = \|A^*x\|$ for all $x \in D(A)$. Self-adjoint operators are special cases of normal operators.

Suppose now that $\sigma(A) \cap \mathbb{C}_0^+$ is a bounded set which is isolated from $\sigma(A) \setminus \left(\sigma(A) \cap \mathbb{C}_0^+\right)$, and let Π denote the spectral projection on $\sigma(A) \cap \mathbb{C}_0^+$. Then Π is normal, and so Π is an orthogonal projection. Thus, we have that

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

that is $\mathcal{H}$ is the **orthogonal** direct sum of the subspaces $\mathcal{H}^+ = \Pi\mathcal{H}$ and $\mathcal{H}^- = (I - \Pi)\mathcal{H}$ (see also (5.2)).

If A is the infinitesimal generator of a strongly continuous semigroup on the Hilbert space X, and if A is normal and satisfies the spectrum decomposition assumption, then the decomposition of the state space X (5.6) induces the corresponding block decomposition

$$A = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}, \tag{5.9}$$

where A_+ and A_- are the infinitesimal generators of strongly continuous semigroups on the subspaces X^+ and X^-, respectively (see Lemma 2.5.7, page 71, Curtain and Zwart [34] and page 232 of [34]). If $C \in \mathcal{L}(X,Y)$, then we have

$$C = \begin{bmatrix} C_+ & C_- \end{bmatrix}, \tag{5.10}$$

where $C_+ = C\Pi \in \mathcal{L}(X^+, Y)$ and $C_- = C(I - \Pi) \in \mathcal{L}(X^-, Y)$.

Our main result in this section is that the equality $\pi(A) = \nu(Q)$ in Corollary 5.3.2 holds under the extra assumption that the generator is normal. But before we state and prove our main result, we give the following result, which gives sufficient conditions for the existence of a unique self-adjoint solution $Q \in \mathcal{L}(X)$ of the Lyapunov equation (5.7).

Theorem 5.4.1 *Suppose that A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on the Hilbert space X and $C \in \mathcal{L}(X,Y)$. Then the following conditions are sufficient for (5.7) to have a unique self-adjoint solution $Q \in \mathcal{L}(X)$.*

1. *A is normal,*
2. *A satisfies the spectrum decomposition assumption,*
3. *A_- is the infinitesimal generator of a strongly stable semigroup $\{T_-(t)\}_{t\geq 0}$,*
4. *there exists a $\beta > 0$ such that for all $x_- \in X^-$,*

$$\int_0^\infty \|C_- T_-(t) x_-\|^2 dt < \beta \|x_-\|^2, \tag{5.11}$$

5. *$\sigma(A) \cap \overline{\mathbb{C}_0^+}$ comprises finitely many eigenvalues $\lambda_1, \ldots, \lambda_k$,*
6. *$-\overline{\lambda_1}, \ldots, -\overline{\lambda_k} \notin \sigma(A)$.*

Remark: From 5 and 6 above, it follows that $\lambda_1, \ldots, \lambda_k$ have *positive* real parts.

Proof

1. Since $\sigma(A_+)$ comprises finitely many eigenvalues with positive real part, $-A_+$ generates the exponentially stable semigroup $\left\{e^{-A_+ t}\right\}_{t\geq 0}$ on X^+. It is

well-known (see for example Theorem 4.1.23, page 160, Curtain and Zwart [34]) that there exists a unique self-adjoint $\tilde{Q} \geq 0$ such that

$$\tilde{Q}(-A_+)x_+ + (-A_+)^* Q x_+ + C_+^* C_+ x_+ = 0,$$

for all $x_+ \in X^+$. Thus $Q_+ := -\tilde{Q} \leq 0$ is bounded, self-adjoint and

$$Q_+ A_+ + A_+^* Q_+ x_+ + C_+^* C_+ x_+ = 0, \tag{5.12}$$

for all $x_+ \in X^+$.

2. The inequality (5.11) shows that $\mathcal{C}_- : X^- \to L_2([0,\infty), Y)$ defined by $\mathcal{C}_- x_- = C_- T_-(t) x_-$ for all $x_- \in X^-$ is a bounded linear operator. From Grabowski [45], the observability Gramian of $\Sigma(A_-, -, C_-)$, $Q_- := \mathcal{C}_-^* \mathcal{C}_-$, satisfies

$$Q_- A_- x_- + A_-^* Q_- x_- + C_-^* C_- x_- = 0, \tag{5.13}$$

for all $x_- \in D(A_-)$, and since $\{T_-(t)\}_{t \geq 0}$ is strongly stable, this solution is unique.

3. Let $\lambda \in \mathbb{C}_0^-$. Then $-\lambda \in \rho(A_-)$, the resolvent set of A_-, and $\lambda \in \rho(A_+^*)$, the resolvent set of A_+^*. Thus, $\left(\lambda I - A_+^*\right)^{-1}$ and $(\lambda I + A_-)^{-1}$ are bounded operators. Now we prove that $\left(\lambda I - A_+^*\right)^{-1}$ and $(\lambda I + A_-)^{-1}$ have disjoint spectra. It can be shown that

a. $\mu \in \rho\left(\left(\lambda I - A_+^*\right)^{-1}\right)$ iff $\mu\lambda - 1 \in \rho\left(\mu A_+^*\right)$, and

b. $\mu \in \rho\left((\lambda I + A_-)^{-1}\right)$ iff $\mu\lambda - 1 \in \rho\left(-\mu A_-\right)$.

If $\mu \in \sigma\left(\left(\lambda I - A_+^*\right)^{-1}\right) \cap \sigma\left((\lambda I + A_-)^{-1}\right)$, then $\mu\lambda - 1 \in \sigma\left(\mu A_+^*\right) \cap \sigma\left(-\mu A_-\right)$. $\mu \neq 0$, since otherwise $-1 \in \sigma(0)$. Using $\sigma\left(\mu A_+^*\right) = \mu\sigma\left(A_+^*\right)$, and $\sigma\left(-\mu A_-\right) = -\mu\sigma\left(A_-\right)$, we obtain $-\overline{\lambda_j} \in \sigma\left(A_-\right)$ for some $j \in \{1, \ldots, k\}$, which contradicts our hypothesis. Thus $\sigma\left(\left(\lambda I - A_+^*\right)^{-1}\right) \cap \sigma\left((\lambda I + A_-)^{-1}\right)$ is empty. From Gohberg, Goldberg and Kaashoek [44] (page 17, Theorem 4.1), there exists a unique solution $Q_0 \in \mathcal{L}\left(X^-, X^+\right)$ of

$$\left(\lambda I - A_+^*\right)^{-1} Q_0 - Q_0 \left(\lambda I + A_-\right)^{-1} = -\left(\lambda I - A_+^*\right)^{-1} C_+^* C_- \left(\lambda I + A_-\right)^{-1}.$$

Moreover, Q_0^* is the unique solution of

$$\begin{aligned} & Q_0^* \left[\left(\lambda I - A_+^*\right)^{-1}\right]^* - \left[\left(\lambda I + A_-\right)^{-1}\right]^* Q_0^* \\ = & -\left[\left(\lambda I + A_-\right)^{-1}\right]^* C_-^* C_+ \left[\left(\lambda I - A_+^*\right)^{-1}\right]^*. \end{aligned}$$

Using $\left[\left(\lambda I - A_+^*\right)^{-1}\right]^* = \left(\overline{\lambda} I - A_+\right)^{-1}$ and $\left[\left(\lambda I + A_-\right)^{-1}\right]^* = \left(\overline{\lambda} I + A_-^*\right)^{-1}$, we get that Q_0^* is the unique solution of

$$Q_0^* \left(\overline{\lambda} I - A_+\right)^{-1} - \left(\overline{\lambda} I + A_-^*\right)^{-1} Q_0^* = -\left(\overline{\lambda} I + A_-^*\right)^{-1} C_-^* C_+ \left(\overline{\lambda} I - A_+\right)^{-1}.$$

If Λ is a closed operator, and $\alpha \in \rho(\Lambda)$, then

$$\begin{aligned} (\alpha I - \Lambda)^{-1}(\alpha I - \Lambda)x = x \qquad & \text{for all } x \in D(\Lambda), \\ (\alpha I - \Lambda)(\alpha I - \Lambda)^{-1}x = x \qquad & \text{for all } x \in X. \end{aligned}$$

Using these equations for A_+^*, A_+, $-A_-^*$, and $-A_-$ in place of Λ we have from the above that

$$Q_0 A_- x_- + A_+^* Q_0 x_- + C_+^* C_- x_- = 0 \tag{5.14}$$

for all $x_- \in D(A_-)$, and

$$A_-^* Q_0^* x_+ + Q_0^* A_+ x_+ + C_-^* C_+ x_+ = 0 \tag{5.15}$$

for all $x_+ \in X^+$.

4. From the above steps, we conclude that there exists a unique (self-adjoint)

$$Q := \begin{bmatrix} Q_+ & Q_0 \\ Q_0^* & Q_- \end{bmatrix} \in \mathcal{L}(X),$$

such that $QAx + A^*Qx + C^*Cx = 0$ for all $x \in D(A)$. This completes the proof. ■

Theorem 5.4.2 *Under the assumptions of Theorem 5.4.1, if* (A_+, C_+) *is observable*[2]*, then the unique solution* Q *of the Lyapunov equation (5.7) is such that* $\sigma(Q) \cap \mathbb{C}_0^- = \sigma_p(Q) \cap \mathbb{C}_0^-$ *and the number of eigenvalues in the open left half-plane equals the number of eigenvalues of* A *in the open right half-plane, that is,* $\nu(Q) = \pi(A)$.

Proof Corresponding to the decomposition 5.6 of the state space we have

$$Q = \begin{bmatrix} Q_+ & Q_0 \\ Q_0^* & Q_- \end{bmatrix}, \quad T(t) = \begin{bmatrix} T_+(t) & 0 \\ 0 & T_-(t) \end{bmatrix}, \tag{5.16}$$

where A_- is the infinitesimal generator of the strongly stable semigroup $\{T_-(t)\}_{t\geq 0}$, and $T_+(t) = e^{A_+ t}$. Thus equations (5.12), (5.13), (5.14) and (5.15) hold. Since X^+ is finite-dimensional, without loss of generality, we can

[2]This is equivalent to $\ker\left(\begin{bmatrix} C_+ \\ \lambda I - A_+ \end{bmatrix}\right) = 0$ for all $\lambda \in \mathbb{C}$ (see for instance, Trentelman et al. [88]).

consider (5.12) as a matrix Lyapunov equation. From (5.12), it follows that if A_+ has an eigenvalue on the imaginary axis, then the corresponding eigenvector must lie in $\ker(C_+)$. But since the pair (A_+, C_+) is observable, this is impossible. Consequently $\zeta(A_+) = 0$. Thus $\nu(A_+) + \pi(A_+) = n$. Since $\nu(Q_+) + \zeta(Q_+) + \pi(Q_+) = n$, applying Theorem 5.1.2, we now obtain

$$\begin{aligned} \pi\left(Q_+\right) &= \nu\left(A_+\right) = 0, \\ \nu\left(Q_+\right) &= \pi\left(A_+\right) \text{ and} \\ \zeta\left(Q_+\right) &= 0. \end{aligned}$$

Thus $Q_+ < 0$ and it is invertible. From part 2 of the proof of Theorem 5.4.1, $Q_- = \mathcal{C}_-^*\mathcal{C} \geq 0$. Thus $\nu\left(Q_-\right) = 0$. Next we transform Q into a block diagonal form as follows:

$$TQT^* = \begin{bmatrix} Q_+ & 0 \\ 0 & Q_- - Q_0^* Q_+^{-1} Q_0 \end{bmatrix}, \tag{5.17}$$

where $T = \begin{bmatrix} I & 0 \\ -Q_0^* Q_+^{-1} & I \end{bmatrix}$ is boundedly invertible. The proof is now completed in three steps:

Step 1: $\nu\left(TQT^*\right) = \pi(A)$. $\nu\left(TQT^*\right) = \nu\left(Q_+\right) + \nu\left(Q_- - Q_0^* Q_+^{-1} Q_0\right)$. Now $Q_- \geq 0$ and $Q_+ < 0$, shows that $Q_- - Q_0^* Q_+^{-1} Q_0 \geq 0$, and thus $\nu\left(Q_- - Q_0^* Q_+^{-1} Q_0\right) = 0$. Thus TQT^* has a pure point spectrum in $\mathbb{C}_0^-$, that is, $\sigma\left(TQT^*\right) \cap \mathbb{C}_0^- = \sigma_p\left(TQT^*\right) \cap \mathbb{C}_0^-$ and

$$\nu\left(TQT^*\right) = \nu\left(Q_+\right) = \pi\left(A_+\right) = \pi(A).$$

Step 2: $\nu\left(TQT^*\right) \geq \nu(Q)$. Let the spectral decompositions of the bounded self-adjoint operators Q and TQT^* induce the orthogonal decompositions of X into the subspaces $\mathcal{V}_-$, $\mathcal{V}_0$, $\mathcal{V}_+$, and $\mathcal{W}_-$, $\mathcal{W}_{0+}$, respectively (see Lemma 5.2.3). We now prove that $\mathcal{V}_-$ is injectively embedded in $\mathcal{W}_-$. Note that in Step 1, we have shown that $\mathcal{W}_-$ is finite dimensional. Let $\pi_- : X \to \mathcal{W}_-$ be the orthogonal projection onto the closed subspace $\mathcal{W}_-$ and define $\Theta : \mathcal{V}_- \to \mathcal{W}_-$ by $\Theta v_- = \pi_-(T^*)^{-1} v_-$ for $v_- \in \mathcal{V}_-$. Clearly, Θ is bounded, and we claim it is one-to-one; for if $\Theta v_- = 0$, then $(T^*)^{-1} v_- = w_{0+} \in \mathcal{W}_{0+}$. Thus $v_- = T^* w_{0+}$. Now

$$0 \leq \langle TQT^* w_{0+}, w_{0+} \rangle = \langle QT^* w_{0+}, T^* w_{0+} \rangle = \langle Qv_-, v_- \rangle \leq 0.$$

Thus $\langle Qv_-, v_- \rangle = 0$. Hence, using the inequality (5.4), we obtain $Qv_- = 0$. So $v_- \in \ker(Q) \subset \mathcal{V}_0$, and $v_- = 0$. From the above, we obtain

$$\dim\left(\mathcal{V}_-\right) \leq \dim\left(\mathcal{W}_-\right) = \nu\left(TQT^*\right),$$

and since $\nu(Q) \leq \dim\left(\mathcal{V}_-\right)$, we get $\nu(Q) \leq \nu\left(TQT^*\right)$.

Step 3: $\nu\left(TQT^*\right) \leq \nu(Q)$. Using the finite dimensionality of $\mathcal{V}_-$ (shown in step 2), and Remark 1.11 (Kato [52], pages 520-521), it can be shown that Q has only eigenvalues of finite multiplicities in $\mathbb{C}_0^-$, that is $\sigma(Q)\cap\mathbb{C}_0^- = \sigma_p(Q)\cap\mathbb{C}_0^-$, and $\nu(Q)<\infty$. Thus the spectrum of Q has finitely many negative eigenvalues in $\mathbb{C}_0^-$, just as TQT^*. Since T is invertible, we can interchange the roles of Q and TQT^* in the above to obtain

$$\nu\left(TQT^*\right) \leq \nu\left(T^{-1}TQT^*\left(T^{-1}\right)^*\right) = \nu(Q). \tag{5.18}$$

So we conclude that

$$\nu(Q) = \nu\left(TQT^*\right) = \pi(A),$$

and this completes the proof of the theorem. ■

A special case of a system $\Sigma(A,-,C)$ which satisfies the assumptions of the above theorem is when C has finite rank and $\Sigma(A,-,C)$ is exponentially detectable.

Corollary 5.4.3 *Suppose that A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on the Hilbert space X and $C\in\mathcal{L}(X,Y)$. If*

1. *Q is a self-adjoint solution of (5.7),*
2. *A is normal,*
3. *C has finite rank and*
4. *$\Sigma(A,-,C)$ is exponentially detectable,*

then Q is such that $\sigma(Q)\cap\mathbb{C}_0^- = \sigma_p(Q)\cap\mathbb{C}_0^-$ and $\nu(Q)=\pi(A)$.

Proof Since $\Sigma(A,-,C)$ is exponentially detectable and C has finite rank, we have that A satisfies the spectrum decomposition assumption, X^+ is finite-dimensional, $\{T_-(t)\}_{t\geq 0}$ is exponentially stable, and $\Sigma(A_+,-,C_+)$ is observable (see Curtain and Zwart [34], page 235, Theorem 5.2.7). Since we already assumed the existence of a self-adjoint bounded solution to (5.7), the steps in the proof of Theorem 5.4.2 go through to yield $\nu(Q)=\pi(A)$. ■