Hankel norm approximation for infinite-dimensional systems

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Chapter 4

Characterization of all solutions

4.1 Introduction

In this chapter, first we show the existence of a solution to the sub-optimal Hankel norm approximation problem in terms of a solution to a key $J-$spectral factorization problem. Subsequently, under the same assumptions, we give a characterization of all solutions to the sub-optimal Hankel norm approximation problem. All the proofs in this chapter are based on purely "frequency domain" techniques.

In Chapters 6 and 7 we will give formulas for a spectral factor in terms of the state space parameters A, B and C which satisfies the assumptions in this chapter, hence solving the sub-optimal Hankel norm approximation problem for the Pritchard-Salamon class and the analytic class of infinite-dimensional systems, respectively.

4.2 Existence of a solution

In this section, we prove the existence of a solution to the sub-optimal Hankel norm approximation problem in terms of a solution to a key $J-$spectral factorization problem.

We make the following assumptions:

S1. $G(i\cdot) \in L_\infty(\mathbb{R}, \mathbb{C}^{p\times m})$.

S2. $\sigma_{l+1} < \sigma < \sigma_l$, where σ_k's denote the Hankel singular values of G.

S3. There exists a $\Lambda(-\cdot) \in H_\infty^c\left(\mathbb{C}^{(p+m)\times(p+m)}\right)$ such that W defined by

$$W(s) = \begin{bmatrix} I_p & 0 \\ G^\dagger(s) & I_m \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_p & G(s) \\ 0 & I_m \end{bmatrix} \tag{4.1}$$

has a $J-$spectral factorization

$$W(s) = \Lambda^\dagger(s) \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} \Lambda(s) \text{ for } s = i\omega,\ \omega \in \mathbb{R}, \tag{4.2}$$

where we use the notation $F^\dagger(s) = F(-\overline{s})^*$.

S4. $\Lambda(-\cdot)$ is invertible as an element of $H_\infty^c\left(\mathbb{C}^{(p+m)\times(p+m)}\right)$, that is, there exists a $V(-\cdot) \in H_\infty^c\left(\mathbb{C}^{(p+m)\times(p+m)}\right)$ such that $\Lambda(-s)V(-s) = V(-s)\Lambda(-s) = I_{p+m}$ for all $s \in \overline{\mathbb{C}_0^+}$.

S5. $\lim_{\substack{s\to\infty \\ s\in\overline{\mathbb{C}_0^+}}} \Lambda(-s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix}$.

S6. $\Lambda_{11}(-\cdot)^{-1} \in H_{\infty,l}^c(\mathbb{C}^{p\times p})$. [1]

Remark: It can be shown that S3 implies that the map $\omega \mapsto G(i\omega) : \mathbb{R} \to \mathbb{C}^{p\times m}$ is continuous. Furthermore, from S5 it follows that $\lim_{\omega\to\pm\infty} G(i\omega) = 0$. Thus from Theorem 3.1.1 we have that the Hankel operator is compact.

Although it is known (see Adamjan et al. [2]) that

$$\inf_{K(-\cdot)\in H_{\infty,l}(\mathbb{C}^{p\times m})} \|G(i\cdot) + K(i\cdot)\|_\infty = \sigma_{l+1},$$

we only need the simpler result

$$\inf_{K(-\cdot)\in H_{\infty,l}(\mathbb{C}^{p\times m})} \|G(i\cdot) + K(i\cdot)\|_\infty \geq \sigma_{l+1}.$$

For completeness, we give a simple proof of this in Theorem 4.2.2 below using the following lemma.

Lemma 4.2.1 *Let $S : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact, bounded operator from the Hilbert space $\mathcal{H}_1$ to the Hilbert space $\mathcal{H}_2$ with singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$. If $S_* : \mathcal{H}_1 \to \mathcal{H}_2$ is an arbitrary bounded linear map of rank l, then $\|S - S_*\| \geq \sigma_{l+1}$.*

[1] See the remark following the proof of Theorem 2.1, page 1354, Green et al. [46].

Proof If $\sigma_{l+1} = 0$, then the inequality is trivially satisfied. Let us assume that $\sigma_{l+1} > 0$. Suppose that $\{w_k\}$ is an orthonormal sequence of eigenvectors of the operator SS^*: $SS^* w_k = \sigma_k^2 w_k$, $\|w_k\| = 1$. Define $v_k = \frac{1}{\sigma_k} S^* w_k$. It is easy to verify that the v_k's are orthonormal and $Sv_k = \sigma_k w_k$. Let Π_* be the orthogonal projection from $\mathcal{H}_2$ onto $\mathrm{span}\{w_1, w_2, \ldots, w_{l+1}\}$; then $\|\Pi_*(S - S_*)\| \leq \|S - S_*\|$. Consider the following restriction of $\Pi_* S_*$:

$$\Pi_* S_* : \mathrm{span}\{v_1, v_2, ..., v_{l+1}\} \to \mathrm{span}\{w_1, w_2, ..., w_{l+1}\}$$

which has rank at most l, and hence there exists a $z \in \ker(\Pi_* S_*)$, $\|z\| = 1$. If $z = \sum_{k=1}^{l+1} a_k v_k$, then we have $\sum_{k=1}^{l+1} |a_k|^2 = 1$, and

$$\begin{aligned} \Pi_* S z &= \sum_{k=1}^{l+1} a_k \sigma_k w_k, \\ \|S - S_*\|^2 &\geq \|\Pi_* S z - \Pi_* S_* z\|^2 = \|\Pi_* S z\|^2 \\ &= \sum_{k=1}^{l+1} \sigma_k^2 a_k^2 \geq \sigma_{l+1}^2 \sum_{k=1}^{l+1} a_k^2 = \sigma_{l+1}^2. \end{aligned}$$

Thus $\|S - S_*\| \geq \sigma_{l+1}$. ■

Theorem 4.2.2 *Let $G(i\cdot) \in L_\infty(\mathbb{R}, \mathbb{C}^{p\times m})$ be such that the Hankel operator with symbol G, namely H_G, is compact. If $K(-\cdot) \in H_{\infty,l}(\mathbb{C}^{p\times m})$, then*

$$\inf_{K(-\cdot)\in H_{\infty,l}(\mathbb{C}^{p\times m})} \|G(i\cdot) + K(i\cdot)\|_\infty \geq \sigma_{l+1}.$$

Proof Let $K(\cdot) = G_*(\cdot) + F(\cdot)$, where $G_*(-\cdot)$ has MacMillan degree $\leq l$, and $F(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$. Thus,

$$\|G(i\cdot) + K(i\cdot)\|_\infty \geq \|G + G_* + F\|_H = \|G + G_*\|_H = \|H_G + H_{G_*}\| \geq \sigma_{l+1}(G).$$

The first equality follows easily from Lemma 8.1.2.c and Examples 8.1.3 (page 388, Curtain and Zwart [34]). The last inequality is a consequence of Lemma 4.2.1 above and the classical result that the rank of the Hankel matrix of the transfer function of a finite-dimensional system is equal to its MacMillan degree (see for instance Kalman et al. [51]). ■

In the following corollary, we show that any solution $K_0(-\cdot) \in H^c_{\infty,l}(\mathbb{C}^{p\times m})$ (with *at most* l unstable poles) of the sub-optimal Hankel norm approximation problem in fact has an unstable rational part of MacMillan degree *exactly* l!

Corollary 4.2.3 *If $K(-\cdot) \in H_{\infty,l}(\mathbb{C}^{p\times m})$ is such that $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma < \sigma_l$, then $K(\cdot) = G_*(\cdot) + F(\cdot)$, where $G_*(-\cdot)$ has MacMillan degree* **exactly** *l, with all l poles in the open right half-plane and $F(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$, that is, $K(-\cdot) \in H_{\infty,[l]}(\mathbb{C}^{p\times m})$.*

Proof Suppose that $K(\cdot) = G_{*1}(\cdot) + F_1(\cdot)$, where $G_{*1}(-\cdot)$ has MacMillan degree r and all r poles in the open right half-plane, and $F_1(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$. Since $K(-\cdot) \in H_{\infty,l}(\mathbb{C}^{p\times m})$, $r \leq l$. From Theorem 4.2.2, it follows that

$$\sigma_l > \sigma \geq \|G(i\cdot) + K(i\cdot)\|_\infty \geq \inf_{K(-\cdot)\in H_{\infty,r}(\mathbb{C}^{p\times m})} \|G(i\cdot) + K(i\cdot)\|_\infty \geq \sigma_{r+1}.$$

Thus $\sigma_l > \sigma_{r+1}$, which implies that $l < r+1$, and so $l \leq r$. Hence $l = r$. ∎

We now construct a solution to the sub-optimal Hankel norm approximation problem using $K_0(-\cdot) = V_{12}(-\cdot)V_{22}(-\cdot)^{-1}$, where $V(-\cdot) = \Lambda(-\cdot)^{-1}$. A certain block matrix function L plays an important role in the characterization of the solutions.

Lemma 4.2.4 *Under the assumptions S1-6, we have the following:*

1. $V_{22}(-\cdot)$ *is invertible as an element of* $\mathcal{M}H^c_{\infty,\bullet}$, *and* $V_{22}(-\cdot)^{-1} \in H^c_{\infty,l}(\mathbb{C}^{m\times m})$.
2. $V_{12}(-\cdot)V_{22}(-\cdot)^{-1} \in H^c_{\infty,l}(\mathbb{C}^{p\times m})$.
3. *If we define for all* $s \in \overline{\mathbb{C}_0^+}$

$$L(s) = \begin{bmatrix} V_{12}(-s)V_{22}(-s)^{-1} & V_{11}(-s) - V_{12}(-s)V_{22}(-s)^{-1}V_{21}(-s) \\ V_{22}(-s)^{-1} & -V_{22}(-s)^{-1}V_{21}(-s) \end{bmatrix}, \tag{4.3}$$

then

$$L(s) = \begin{bmatrix} -\Lambda_{11}(-s)^{-1}\Lambda_{12}(-s) & \Lambda_{11}(-s)^{-1} \\ \Lambda_{22}(-s) - \Lambda_{21}(-s)\Lambda_{11}(-s)^{-1}\Lambda_{12}(-s) & \Lambda_{21}(-s)\Lambda_{11}(-s)^{-1} \end{bmatrix}$$

and $L \in H^c_{\infty,l}\left(\mathbb{C}^{(p+m)\times(p+m)}\right)$.

Proof *1.* From S6, $\Lambda_{11}(-\cdot)^{-1} \in H^c_{\infty,l}(\mathbb{C}^{p\times p})$. Using $\Lambda(-s)V(-s) = V(-s)\Lambda(-s) = I$, it can be checked that

$$\begin{aligned} & V_{22}(-s)\left(\Lambda_{22}(-s) - \Lambda_{21}(-s)\Lambda_{11}(-s)^{-1}\Lambda_{12}(-s)\right) \\ = & \left(\Lambda_{22}(-s) - \Lambda_{21}(-s)\Lambda_{11}(-s)^{-1}\Lambda_{12}(-s)\right)V_{22}(-s) \\ = & I. \end{aligned}$$

Thus, $V_{22}(-\cdot)$ is invertible as an element of $\mathcal{M}H^c_{\infty,\bullet}$. Moreover, it follows from Lemma 2.6.4 that

$$V_{22}(-\cdot)^{-1} = \Lambda_{22}(-\cdot) - \Lambda_{21}(-\cdot)\Lambda_{11}(-\cdot)^{-1}\Lambda_{12}(-\cdot) \in H^c_{\infty,l}(\mathbb{C}^{m\times m}).$$

2. Since $V_{12}(-\cdot) \in H^c_\infty(\mathbb{C}^{p\times m})$, it follows from part 1 above and Lemma 2.6.4 that $V_{12}(-\cdot)V_{22}(-\cdot)^{-1} \in H^c_{\infty,l}(\mathbb{C}^{p\times m})$.

3. It can be shown that

$$\begin{aligned} V_{11}(-s) - V_{12}(-s)V_{22}(-s)^{-1}V_{21}(-s) &= \Lambda_{11}(-s)^{-1} \\ V_{12}(-s)V_{22}(-s)^{-1} &= -\Lambda_{11}(-s)^{-1}\Lambda_{12}(-s) \\ V_{22}(-s)^{-1}V_{21}(-s) &= -\Lambda_{21}(-s)\Lambda_{11}(-s)^{-1}. \end{aligned}$$

Thus

$$L(s) = \begin{bmatrix} -\Lambda_{11}(-s)^{-1}\Lambda_{12}(-s) & \Lambda_{11}(-s)^{-1} \\ \Lambda_{22}(-s) - \Lambda_{21}(-s)\Lambda_{11}(-s)^{-1}\Lambda_{12}(-s) & \Lambda_{21}(-s)\Lambda_{11}(-s)^{-1} \end{bmatrix}.$$

Now

$$\begin{aligned} L(\cdot) &= \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{22}(-\cdot) \end{bmatrix} + \begin{bmatrix} I \\ \Lambda_{21}(-\cdot) \end{bmatrix} \Lambda_{11}(-\cdot)^{-1} \begin{bmatrix} -\Lambda_{12}(-\cdot) & I \end{bmatrix} \\ &\in H^c_{\infty,l}\left(\mathbb{C}^{(p+m)\times(p+m)}\right), \end{aligned}$$

since

$$\begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{22}(-\cdot) \end{bmatrix}, \begin{bmatrix} I \\ \Lambda_{21}(-\cdot) \end{bmatrix}, \begin{bmatrix} -\Lambda_{12}(-\cdot) & I \end{bmatrix} \in \mathcal{M}H^c_\infty$$

and $\Lambda_{11}(-\cdot)^{-1} \in H^c_{\infty,l}(\mathbb{C}^{p\times p})$ (see Lemma 2.6.4). ■

Theorem 4.2.5 *Under the assumptions S1-6, there exists* $K_0(-\cdot) \in H^c_{\infty,l}(\mathbb{C}^{p\times m})$ *such that* $\|G(i\cdot) + K_0(i\cdot)\|_\infty < \sigma$.

Proof Define $K_0(-\cdot) := V_{12}(-\cdot)V_{22}(-\cdot)^{-1} \in H^c_{\infty,l}(\mathbb{C}^{p\times m})$. We know from Lemma 4.2.4.1 that $V_{22}(-\cdot)^{-1} = G_u + G_s$, where $G_s \in \mathcal{M}H^c_\infty$, and G_u is a strictly proper rational transfer matrix of a system with all its poles in the open right half-plane. Thus, $V_{22}(i\omega)^{-1}$ is defined for all $\omega \in \mathbb{R}$. Hence,

$$\begin{aligned} \begin{bmatrix} G(i\omega) + K_0(i\omega) \\ I_m \end{bmatrix} &= \begin{bmatrix} I_p & G(i\omega) \\ 0 & I_m \end{bmatrix} \begin{bmatrix} K_0(i\omega) \\ I_m \end{bmatrix} \\ &= \begin{bmatrix} I_p & G(i\omega) \\ 0 & I_m \end{bmatrix} V(i\omega) \begin{bmatrix} 0 \\ V_{22}(i\omega)^{-1} \end{bmatrix}, \end{aligned}$$

with $\omega \in \mathbb{R}$, and so we have [2]

$$(G + K_0)^\dagger(G + K_0) - \sigma^2 I_m$$

[2]for notational convenience, we restrict writing out the argument $i\omega$

$$= \begin{bmatrix} G+K_0 \\ I_m \end{bmatrix}^\dagger \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} G+K_0 \\ I_m \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix}^\dagger V^\dagger \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix}^\dagger \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix} V \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix}^\dagger \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix},$$

where we have used S3, the definition of V and Lemma 4.2.4.1. Thus it follows that $\|(G+K_0)(i\omega)u\|^2 - \sigma^2\|u\|^2 = -\left\|V_{22}(i\omega)^{-1}u\right\|^2$ for $u \in \mathbb{C}^m$ and $\omega \in \mathbb{R}$. Since $V_{22}(i\omega)V_{22}(i\omega)^{-1} = I$,

$$\|u\| \leq \|V_{22}(i\omega)\| \left\|V_{22}(i\omega)^{-1}u\right\|.$$

Since $V_{22}(-\cdot) \in \mathcal{M}H^c_\infty$, there exists a constant $M > 0$ such that $\|V_{22}(-s)\| \leq M$ for all $s \in \overline{\mathbb{C}_0^+}$. We have

$$\|u\| \leq M \left\|V_{22}(i\omega)^{-1}u\right\|$$

for all $u \in \mathbb{C}^m$ and for all $\omega \in \mathbb{R}$. Hence it follows that $\|G(i\cdot)+K_0(i\cdot)\|_\infty < \sigma$. ■

In the above theorem, we have constructed a solution $K_0(-\cdot) \in H^c_{\infty,l}(\mathbb{C}^{p\times m})$ with *at most* l unstable poles. In fact, it follows from Corollary 4.2.3 that $K_0(-\cdot)$ has an unstable rational part of MacMillan degree *exactly* l.

Finally we collect more precise information concerning our constructed solution $K_0(-\cdot) = V_{12}(-\cdot)V_{22}(-\cdot)^{-1}$ to the sub-optimal Hankel norm problem from Theorem 4.2.5.

Corollary 4.2.6 *Suppose that S1-6 hold. Then* $K_0(-\cdot) = V_{12}(-\cdot)V_{22}(-\cdot)^{-1}$ *has the following properties*

1. $K_0(-\cdot) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$.
2. $(V_{12}(-\cdot), V_{22}(-\cdot))$ *is a right coprime factorization over* $\mathcal{M}H^c_\infty$ *of* $K_0(-\cdot) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$.
3. $\det(V_{22}(-\cdot))$ *has no zeros on the imaginary axis, and exactly* l *zeros in* $\mathbb{C}_0^+$.

Proof *1.* From Lemma 4.2.4.2, we know that $K_0(-\cdot) = V_{12}(-\cdot)V_{22}(-\cdot)^{-1} \in H^c_{\infty,l}(\mathbb{C}^{p\times m})$, and from the proof of Theorem 4.2.5, we know that it satisfies $\|G(i\cdot) + K_0(i\cdot)\|_\infty < \sigma < \sigma_l$. Thus from Corollary 4.2.3 above, $V_{12}(-\cdot)V_{22}(-\cdot)^{-1} \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$.

2. $V_{12}(-\cdot)V_{22}(-\cdot)^{-1}$ belongs to $H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$, and $V_{12}(-\cdot)$ and $V_{22}(-\cdot)$ belong to $\mathcal{M}H^c_\infty$. Moreover,

$$\Lambda_{22}(-s)V_{22}(-s) - (-\Lambda_{21}(-s))\,V_{12}(-s) = I,$$

where $\Lambda_{22}(-\cdot)$ and $-\Lambda_{21}(-\cdot)$ belong to $\mathcal{M}H^c_\infty$. Hence it follows that $(V_{12}(-\cdot), V_{22}(-\cdot))$ is a right coprime factorization of $V_{12}(-\cdot)V_{22}(-\cdot)^{-1}$.

3. This follows from the above and Lemma 2.6.6. ∎

We prove a few properties that will be used in the proof of the characterization theorem.

Lemma 4.2.7 *Under the assumptions S1-S6, we have the following:*

1. $\Lambda_{21}(-\cdot)$ *is strictly proper.*
2. $\det(\Lambda_{22}(-\cdot))$ *has a nonzero limit at infinity in* $\overline{\mathbb{C}_0^+}$.
3. $\det(V_{22}(-\cdot))$ *has a nonzero limit at infinity in* $\overline{\mathbb{C}_0^+}$.
4. $V_{21}(-\cdot)$ *is strictly proper.*
5. $\left\|V_{22}(i\cdot)^{-1}V_{21}(i\cdot)\right\|_\infty < 1$, *where* $\|\cdot\|_\infty$ *denotes the* L_∞*−norm.*

Proof Parts 1 and 2 follow from assumption S5.

3. From $\Lambda(-s)V(-s) = I_{p+m}$, we obtain that $\Lambda_{21}(-s)V_{12}(-s) + \Lambda_{22}(-s)V_{22}(-s) = I_m$. From part 1 above, it follows that $\lim_{\substack{s\to\infty\\ s\in\overline{\mathbb{C}_0^+}}} \Lambda_{22}(-s)V_{22}(-s) = I_m$. Since the map $\det : \mathbb{C}^{m\times m} \to \mathbb{C}$ is continuous, it follows that

$$\lim_{\substack{s\to\infty\\ s\in\overline{\mathbb{C}_0^+}}} \det(\Lambda_{22}(-s))\,\det(V_{22}(-s)) = 1.$$

From part 2 above, we know that $\lim_{\substack{s\to\infty\\ s\in\overline{\mathbb{C}_0^+}}} \det(\Lambda_{22}(-s))$ exists and is nonzero. Consequently, we obtain that $\lim_{\substack{s\to\infty\\ s\in\overline{\mathbb{C}_0^+}}} \det(V_{22}(-s))$ exists and is nonzero.

4. From $V(-s)\Lambda(-s) = I_{p+m}$, we obtain that $V_{21}(-s)\Lambda_{11}(-s) + V_{22}(-s)\Lambda_{21}(-s) = 0$, and since $V_{22}(-\cdot) \in \mathcal{M}H^c_\infty$ and $\Lambda_{21}(-\cdot)$ is strictly proper, it follows that $V_{21}(-\cdot)\Lambda_{11}(-\cdot)$ is strictly proper too. Furthermore, since $\Lambda_{11}(-\cdot) \in \mathcal{M}H^c_\infty$, we have $\mathrm{adj}(\Lambda_{11}(-\cdot)) \in \mathcal{M}H^c_\infty$, and so

$$V_{21}(-\cdot)\,\det(\Lambda_{11}(-\cdot)) = V_{21}(-\cdot)\,\Lambda_{11}(-\cdot)\,\mathrm{adj}(\Lambda_{11}(-\cdot))$$

is also strictly proper. Owing to S5, we know that $\Lambda_{11}(-\cdot)$ has the limit I_p as $s \to \infty$ in $\overline{\mathbb{C}_0^+}$. Thus $\lim_{\substack{s\to\infty \\ s\in\overline{\mathbb{C}_0^+}}} \det(\Lambda_{11}(-s)) = 1$. Hence $V_{21}(-\cdot)$ is strictly proper.

5. Λ satisfies S3; and so taking inverses, we obtain

$$V(s)\begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} V^\dagger(s) = \begin{bmatrix} I_p & G(s) \\ 0 & -I_m \end{bmatrix}^{-1} \begin{bmatrix} I_p & 0 \\ 0 & -\frac{1}{\sigma^2} I_m \end{bmatrix} \begin{bmatrix} I_p & 0 \\ G^\dagger(s) & -I_m \end{bmatrix}^{-1}$$

where $s = i\omega$, $\omega \in \mathbb{R}$. Considering the $(2,2)-$block of the above gives $V_{21}(s)V_{21}^\dagger(s) - V_{22}(s)V_{22}^\dagger(s) = -\frac{1}{\sigma^2} I_m$ where $s = i\omega$, $\omega \in \mathbb{R}$. Lemma 4.2.4.1 shows that $V_{22}(i\cdot)$ is invertible as an element of L_∞, and thus $\left\|V_{22}(i\omega)^{-1}V_{21}(i\omega)u\right\|^2 - \|u\|^2 = -\frac{1}{\sigma^2}\left\|V_{22}(i\omega)^{-1}u\right\|^2$. Choosing a $M > \frac{1}{\sigma}$ such that $\|V_{22}(i\omega)\| \leq M$ for all $\omega \in \mathbb{R}$, we obtain $\|u\|^2 \leq \|V_{22}(i\omega)\|^2 \left\|V_{22}(i\omega)^{-1}u\right\|^2 \leq M^2 \left\|V_{22}(i\omega)^{-1}u\right\|^2$. Hence

$$\left\|V_{22}(i\omega)^{-1}V_{21}(i\omega)\right\|^2 \leq 1 - \frac{1}{\sigma^2 M^2} < 1 \text{ for all } \omega \in \mathbb{R},$$

and so we have $\left\|V_{22}(i\cdot)^{-1}V_{21}(i\cdot)\right\|_\infty < 1$. ■

4.3 Characterization of solutions

In this section we obtain a nice parameterization of all solutions to the sub-optimal Hankel norm approximation problem under the assumptions S1-S6 listed in the previous section.

Theorem 4.3.1 *Suppose that S1-6 hold. If* $Q(-\cdot) \in H_\infty^c(\mathbb{C}^{p\times m})$ *satisfies* $\|Q(i\cdot)\|_\infty \leq 1$, *and* $K(-\cdot) := R_1(-\cdot)R_2(-\cdot)^{-1}$, *where*

$$\begin{bmatrix} R_1(-\cdot) \\ R_2(-\cdot) \end{bmatrix} := \Lambda(-\cdot)^{-1} \begin{bmatrix} Q(-\cdot) \\ I_m \end{bmatrix}, \tag{4.4}$$

then $K(-\cdot) \in H_{\infty,[l]}^c(\mathbb{C}^{p\times m})$ *and* $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$.

Proof *Step 1: We show that* $\det(V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))$ *has exactly* l *zeros in* $\overline{\mathbb{C}_0^+}$, *and they are contained in the open right half-plane.*

By Corollary 4.2.6.3, we know that $s \mapsto \det(V_{22}(-s))$ has no zeros on the imaginary axis, and exactly l zeros in $\mathbb{C}_0^+$. So there exists an $\epsilon > 0$ such that all its zeros are contained in the half-plane $\mathbb{C}_\epsilon^+$.

From Lemmas 4.2.4.1 and 2.6.4, it follows that $V_{22}(-\cdot)^{-1}V_{21}(-\cdot) \in H^c_{\infty,l}(\mathbb{C}^{p\times m})$. Thus, we have from Lemma 4.2.7.5 and Lemma 2.6.7 applied to $V_{22}(-\cdot)^{-1}V_{21}(-\cdot)$, that there exists a δ, $0<\delta<\epsilon$, such that whenever $0\leq\zeta\leq\delta$,

$$\sup_{\omega\in\mathbb{R}} \left\| V_{22}(-\zeta-i\omega)^{-1}V_{21}(-\zeta-i\omega)\right\| < 1.$$

Fix such a $\zeta>0$. Consider

$$\phi(\alpha,s) := \det\left(\alpha V_{21}(-\zeta-s)Q(-\zeta-s)+V_{22}(-\zeta-s)\right),$$

where $\alpha\in[0,1]$. The maps $\phi(0,\cdot)$ and $\phi(1,\cdot)$ are meromorphic (actually analytic in $\mathbb{C}^+_{-\frac{\zeta}{2}}$) on an open set containing $\overline{\mathbb{C}^+_0}$ with nonzero limits at infinity in $\overline{\mathbb{C}^+_0}$ (using Lemma 4.2.7). We have:

1. $(\alpha,s)\mapsto\phi(\alpha,s):[0,1]\times i\mathbb{R}\to\mathbb{C}$ is a continuous function.
2. $\phi(0,i\omega)=\det(V_{22}(-\zeta-i\omega))$, and
   $\phi(1,i\omega)=\det(V_{21}(-\zeta-i\omega)Q(-\zeta-i\omega)+V_{22}(-\zeta-i\omega))$.
3. $I_m+\alpha V_{22}(-\zeta-i\omega)^{-1}V_{21}(-\zeta-i\omega)Q(-\zeta-i\omega)$ is invertible, since

$$\begin{aligned} & \left\|\alpha V_{22}(-\zeta-i\omega)^{-1}V_{21}(-\zeta-i\omega)Q(-\zeta-i\omega)\right\| \\ \leq\ & \alpha\left\|V_{22}(-\zeta-i\omega)^{-1}V_{21}(-\zeta-i\omega)\right\|\left\|Q(-\zeta-i\omega)\right\| \\ <\ & 1. \end{aligned}$$

   Moreover, since $s\mapsto\det(V_{22}(-\zeta-s))$ has no zeros on the imaginary axis, it follows that $\phi(\alpha,i\omega)$ is nonzero for all $\alpha\in[0,1]$ and $\omega\in\mathbb{R}$.
4. $\phi(\alpha,\infty)$ is nonzero for all $\alpha\in[0,1]$, since

$$\lim_{\substack{s\to\infty\\ s\in\mathbb{C}^+_0}} \det(\alpha V_{21}(-s-\zeta)Q(-s-\zeta)+V_{22}(-s-\zeta)) = \frac{1}{\sigma^m}\neq 0.$$

So the assumptions of Lemma A.1.18 (Curtain and Zwart [34], page 570) are satisfied by ϕ and so the Nyquist indices of $\phi(0,\cdot)$ and $\phi(1,\cdot)$ are the same. Consequently, the number of zeros are the same (the number of poles in each case is zero, since $\phi(0,\cdot)$, $\phi(1,\cdot)$ are analytic in $\mathbb{C}^+_{-\frac{\zeta}{2}}$). We already know from Corollary 4.2.6.3 that when $\alpha=0$, $\det(V_{22}(-\cdot-\zeta))$ has l zeros in the closed right half-plane $\overline{\mathbb{C}^+_0}$. Thus the number of zeros of $\phi(1,\cdot)$ is l. But since ζ can be chosen arbitrarily small, $\det(V_{21}(-\cdot)Q(-\cdot)+V_{22}(-\cdot))$ has exactly l zeros in $\mathbb{C}^+_0$.

Finally, $\det(V_{21}(-\cdot)Q(-\cdot)+V_{22}(-\cdot))$ has no zeros on the imaginary axis, since $\det\left(I+V_{22}(-i\omega)^{-1}V_{21}(-i\omega)Q(-i\omega)\right)\neq 0$ for all $\omega\in\mathbb{R}$.

Step 2: We show that

$K(-\cdot) := (V_{11}(-\cdot)Q(-\cdot) + V_{12}(-\cdot))(V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))^{-1} \in H^c_{\infty,l}(\mathbb{C}^{p\times m})$.

$\det(V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))$ has a nonzero limit at infinity in $\overline{\mathbb{C}_0^+}$, since $Q(-\cdot) \in \mathcal{M}H^c_\infty$ is proper, $V_{21}(-\cdot)$ is strictly proper, $V_{22}(-\cdot)$ is proper in $\overline{\mathbb{C}_0^+}$ and $\det(V_{22}(-\cdot))$ has a nonzero limit at infinity in $\overline{\mathbb{C}_0^+}$ (see Lemma 4.2.7). So by Lemma 2.6.3 and using Step 1 above, it follows that $K(-\cdot) = (V_{11}(-\cdot)Q(-\cdot) + V_{12}(-\cdot))\,(V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))^{-1}$ is a well-defined element of $H^c_{\infty,l}(\mathbb{C}^{p\times m})$.

Step 3: We show that $\|G(i\cdot) + K(i\cdot)\|_\infty \le \sigma$.

$\det(V_{22}(-\cdot))$ has no zeros on the imaginary axis and from Lemma 4.2.7.5, it follows that $\left\|V_{22}(i\omega)^{-1}V_{21}(i\omega)Q(i\omega)\right\| < 1$ for all $\omega \in \mathbb{R}$. Thus $R_2(i\omega) = V_{21}(i\omega)Q(i\omega) + V_{22}(i\omega)$ is invertible for every $\omega \in \mathbb{R}$. Consequently with $s = i\omega$, for all $\omega \in \mathbb{R}$,

$$\begin{aligned}
&(G+K)^\dagger(G+K) - \sigma^2 I_m \\
&= (R_2^{-1})^\dagger \begin{bmatrix} Q^\dagger & I_m \end{bmatrix} V^\dagger \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix}^\dagger \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix} V \begin{bmatrix} Q^\dagger \\ I_m \end{bmatrix} R_2^{-1} \\
&= (R_2^{-1})^\dagger (Q^\dagger Q - I_m) R_2^{-1},
\end{aligned}$$

where we have used (4.1) and (4.2). Thus $\|G(i\cdot) + K(i\cdot)\|_\infty \le \sigma$.

Step 4: We show that $K(-\cdot) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$.

Finally, from Corollary 4.2.3, and Steps 2 and 3 above, it follows that $K(-\cdot) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$. ∎

Remarks:

1. The previous solution to the sub-optimal Hankel norm approximation problem given in Theorem 4.2.5 corresponds to the case when $Q = 0$.
2. A version of Theorem 4.3.1 holds under the weaker assumption $Q(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$ (that is, without the continuity assumption). We have:

Corollary 4.3.2 *Suppose that S1-6 hold. If* $Q(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$ *satisfies* $\|Q(i\cdot)\|_\infty \le 1$, *and* $K(-\cdot) := R_1(-\cdot)R_2(-\cdot)^{-1}$, *where*

$$\begin{bmatrix} R_1(-\cdot) \\ R_2(-\cdot) \end{bmatrix} := \Lambda(-\cdot)^{-1} \begin{bmatrix} Q(-\cdot) \\ I_m \end{bmatrix}, \tag{4.5}$$

then $K(-\cdot) \in H_{\infty,[l]}(\mathbb{C}^{p\times m})$ *and* $\|G(i\cdot) + K(i\cdot)\|_\infty \le \sigma$.

Proof Proceeding as in Step 1 of the proof of Theorem 4.3.1, it can be shown that $s \mapsto \det(V_{21}(-s)Q(-s)+V_{22}(-s))$ has exactly l zeros in $\mathbb{C}_0^+$.

Step 2: We now show that $K(-\cdot) \in H_{\infty,l}(\mathbb{C}^{p\times m})$.

$s \mapsto \det(V_{21}(-s)Q(-s)+V_{22}(-s))$ has a nonzero limit at infinity in $\mathbb{C}_0^+$, since $Q(-\cdot) \in \mathcal{M}H_\infty$ is proper, $V_{21}(-\cdot)$ is strictly proper, $V_{22}(-\cdot)$ is proper in $\mathbb{C}_0^+$ and $\det(V_{22}(-\cdot))$ has a nonzero limit at infinity in $\mathbb{C}_0^+$. Since $\det(V_{21}(-\cdot)Q(-\cdot)+V_{22}(-\cdot))$ and $\det(V_{22}(-\cdot))$ have a finite number zeros in $\mathbb{C}_0^+$, there exists a $\epsilon > 0$ such that $\det(V_{21}(-\cdot)Q(-\cdot)+V_{22}(-\cdot))$ and $\det(V_{22}(-\cdot))$ have no zeros in the strip $\{s \in \mathbb{C}_0^+ \mid 0 < \mathrm{Re}(s) \le \epsilon\}$. Since $\left\|V_{22}(i\cdot)^{-1}V_{21}(i\cdot)\right\|_\infty < 1$, there exists a $r > 0$ such that $\left\|V_{22}(i\cdot)^{-1}V_{21}(i\cdot)\right\|_\infty = 1-r$. Next we choose a $\delta > 0$ such that $\delta < \epsilon$ and $\left\|V_{22}(-s)^{-1}V_{21}(-s)\right\| < 1-\frac{r}{2}$ for $0 \le \mathrm{Re}(s) \le \delta$ (using Lemma 2.6.7). Thus we have for $0 \le \mathrm{Re}(s) \le \delta$ that $\left\|V_{22}(-s)^{-1}V_{21}(-s)Q(-s)\right\| \le 1-\frac{r}{2}$, and so

$$\left\|\left(I+V_{22}(-s)^{-1}V_{21}(-s)Q(-s)\right)^{-1}\right\| \le \frac{1}{1-\left(1-\frac{r}{2}\right)} = \frac{2}{r}.$$

Since $V_{22}(-\cdot)^{-1} \in \mathcal{M}H^c_{\infty,l}$, and $\det(V_{22}(-\cdot))$ has no zeros in the strip $\{s \in \mathbb{C}_0^+ \mid 0 \le \mathrm{Re}(s) \le \delta\}$ it follows that $\|(V_{22}(-\cdot)+V_{21}(-\cdot)Q(-\cdot))^{-1}\|$ is bounded in the strip $0 \le \mathrm{Re}(s) \le \delta$. Thus the modulus of each entry of $(V_{22}(-\cdot)+V_{21}(-\cdot)Q(-\cdot))^{-1}$ is bounded in $0 \le \mathrm{Re}(s) \le \delta$ and since $\det\left((V_{21}(-s)Q(-s)+V_{22}(-s))^{-1}\right)$ is a certain sum of products of entries of $(V_{22}(-s)+V_{21}(-s)Q(-s))^{-1}$, it follows that

$$\frac{1}{|\det(V_{21}(-\cdot)Q(-\cdot)+V_{22}(-\cdot))|}$$

is bounded in $0 \le \mathrm{Re}(s) \le \delta$. So by Lemma 2.6.3 and using Step 1 above, it follows that

$$K(-\cdot) = (V_{11}(-\cdot)Q(-\cdot)+V_{12}(-\cdot))\,(V_{21}(-\cdot)Q(-\cdot)+V_{22}(-\cdot))^{-1}$$

is a well-defined element of $H_{\infty,l}(\mathbb{C}^{p\times m})$.

The Steps 3 and 4 are similar to those in the proof of Theorem 4.3.1. ∎

Next we show that any continuous solution $K(-\cdot) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$ to the sub-optimal Hankel norm approximation problem has the form (4.4).

Theorem 4.3.3 *Suppose that S1-6 hold. If* $K(-\cdot) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$ *and* $\|G(i\cdot)+K(i\cdot)\|_\infty \le \sigma$*, then* $K(-\cdot) = R_1(-\cdot)R_2(-\cdot)^{-1}$*, where*

$$\begin{bmatrix} R_1(-\cdot) \\ R_2(-\cdot) \end{bmatrix} = \Lambda(-\cdot)^{-1}\begin{bmatrix} Q(-\cdot) \\ I_m \end{bmatrix}$$

for some $Q(-\cdot) \in H^c_\infty(\mathbb{C}^{p\times m})$ *satisfying* $\|Q(i\cdot)\|_\infty \le 1$.

Proof Let $K(-\cdot) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$ satisfy $\|G(i\cdot)+K(i\cdot)\|_\infty \leq \sigma$ and suppose it has the coprime factorization $K(-\cdot) = NM^{-1}$ over $\mathcal{M}H^c_\infty$, where N and M are in $\mathcal{M}H^c_\infty$, M is rational, and $\det(M) \in \mathcal{R}_\infty$ has exactly l zeros in $\overline{\mathbb{C}^+_0}$ and none on the imaginary axis.

Define

$$\begin{bmatrix} U_1(-\cdot) \\ U_2(-\cdot) \end{bmatrix} = \Lambda(-\cdot)\begin{bmatrix} K(-\cdot) \\ I_m \end{bmatrix} = \begin{bmatrix} \Lambda_{11}(-\cdot)K(-\cdot)+\Lambda_{12}(-\cdot) \\ \Lambda_{21}(-\cdot)K(-\cdot)+\Lambda_{22}(-\cdot) \end{bmatrix}, \tag{4.6}$$

and

$$\begin{bmatrix} \overline{U}_1(-\cdot) \\ \overline{U}_2(-\cdot) \end{bmatrix} = \begin{bmatrix} U_1(-\cdot) \\ U_2(-\cdot) \end{bmatrix} M(\cdot) = \begin{bmatrix} \Lambda_{11}(-\cdot)N(\cdot)+\Lambda_{12}(-\cdot)M(\cdot) \\ \Lambda_{21}(-\cdot)N(\cdot)+\Lambda_{22}(-\cdot)M(\cdot) \end{bmatrix}. \tag{4.7}$$

Step 1: We show that $U_2(i\omega)$ is invertible with $U_1(i\cdot)U_2(i\cdot)^{-1} \in L_\infty(\mathbb{R},\mathbb{C}^{p\times m})$ and $\left\|U_1(i\cdot)U_2(i\cdot)^{-1}\right\|_\infty \leq 1$.

From (4.6) we have

$$\begin{bmatrix} U_1(i\omega) \\ U_2(i\omega) \end{bmatrix} = \Lambda(i\omega)\begin{bmatrix} I_p & G(i\omega) \\ 0 & I_m \end{bmatrix}^{-1}\begin{bmatrix} G(i\omega)+K(i\omega) \\ I_m \end{bmatrix}, \quad \omega \in \mathbb{R}.$$

For $s = i\omega$, $\omega \in \mathbb{R}$, we have

$$U_1^\dagger U_1 - U_2^\dagger U_2 = \begin{bmatrix} U_1^\dagger & U_2^\dagger \end{bmatrix}\begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix},$$

and appealing to S3, we see that for $s = i\omega$

$$\begin{bmatrix} I_p & 0 \\ G^\dagger & I_m \end{bmatrix}^{-1}\Lambda^\dagger\begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix}\Lambda\begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix}^{-1} = \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix}.$$

Thus for $s = i\omega$,

$$U_1^\dagger U_1 - U_2^\dagger U_2 = \begin{bmatrix} G+K \\ I_m \end{bmatrix}^\dagger\begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix}\begin{bmatrix} G+K \\ I_m \end{bmatrix} \leq 0. \tag{4.8}$$

Hence for all $u \in \mathbb{C}^m$ and all $\omega \in \mathbb{R}$, we have from equation (4.8) that

$$\|U_1(i\omega)u\|^2 - \|U_2(i\omega)u\|^2 = \|(G(i\omega)+K(i\omega))u\|^2 - \sigma^2\|u\|^2 \leq 0,$$

and so

$$\|U_1(i\omega)u\| \leq \|U_2(i\omega)u\|. \tag{4.9}$$

Since $V(-\cdot) = \Lambda(-\cdot)^{-1}$, from (4.6) we obtain

$$V(-\cdot)\begin{bmatrix} U_1(-\cdot) \\ U_2(-\cdot) \end{bmatrix} = \begin{bmatrix} K(-\cdot) \\ I_m \end{bmatrix}, \tag{4.10}$$

and so

$$V_{21}(-\cdot)U_1(-\cdot) + V_{22}(-\cdot)U_2(-\cdot) = I_m. \tag{4.11}$$

We claim that $\ker(U_2(-i\omega)) = \{0\}$ for all $\omega \in \mathbb{R}$. Suppose on the contrary that there exists $x \neq 0$ such that $U_2(-i\omega_0)x = 0$. Then from (4.9), we obtain $U_1(-i\omega_0)x = 0$, which violates (4.11).

Concluding, we have that $\det(U_2(i\omega)) \neq 0$ for all $\omega \in \mathbb{R}$, and so $U_2(i\omega)^{-1}$ exists for all $\omega \in \mathbb{R}$. From (4.8), we deduce that

$$\left\|U_1(i\omega)U_2(i\omega)^{-1}y\right\|^2 \leq \|y\|^2 \text{ for all } \omega \in \mathbb{R},$$

and so $U_1(i\cdot)U_2(i\cdot)^{-1} \in L_\infty(\mathbb{R}, \mathbb{C}^{p\times m})$ satisfies $\left\|U_1(i\cdot)U_2(i\cdot)^{-1}\right\|_\infty \leq 1$.

Step 2: We now construct a $Q(-\cdot) \in H^c_{\infty,\bullet}(\mathbb{C}^{p\times m})$ *such that* $\|Q(i\cdot)\|_\infty \leq 1$ *and* $K(-\cdot) = (V_{11}(-\cdot)Q(-\cdot) + V_{12}(-\cdot))\,(V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))^{-1}$.

Consider $\overline{U}_2(-\cdot) \in \mathcal{M}H^c_\infty$. We know that $\Lambda_{21}(-\cdot)$ is strictly proper and both $\Lambda_{22}(-\cdot)$ and $M(\cdot)$ are proper with a nonzero limit at infinity in $\overline{\mathbb{C}_0^+}$. So there exists a $R > 0$ such that for every $s \in \overline{\mathbb{C}_0^+}$, with $|s| > R$, $\det\left(\overline{U}_2(-s)\right) \neq 0$. Since $\det\left(\overline{U}_2(-\cdot)\right)$ is analytic in $\mathbb{C}_0^+$, it follows that if its zeros have an accumulation point in the compact set $\left\{s \in \overline{\mathbb{C}_0^+} \mid |s| \leq R\right\}$, then it must lie on the imaginary axis. But $\det\left(\overline{U}_2(-s)\right) \neq 0$ on the imaginary axis, since $\det\left(\overline{U}_2(-s)\right) = \det(U_2(-s))\ \det(M(s))$, and neither $\det(U_2(-\cdot))$ nor $\det(M(\cdot))$ have any zeros on the imaginary axis. Thus $\det\left(\overline{U}_2(-\cdot)\right)$ has only finitely many zeros in $\overline{\mathbb{C}_0^+}$, and they are all contained in the open right half-plane. So $\det(\overline{U}_2(-\cdot)) \in \mathcal{S}$ and it follows from Lemma 2.6.3 that $\overline{U}_2(-\cdot)^{-1}$ is an element of $\mathcal{M}H^c_{\infty,\bullet}$ and $Q(-\cdot) := \overline{U}_1(-\cdot)\overline{U}_2(-\cdot)^{-1}$ is a well-defined element of $\mathcal{M}H^c_{\infty,\bullet}$. We also note that

$$U_2(-\cdot) \quad \left[= \overline{U}_2(-\cdot)M(\cdot)^{-1}\right]$$

is invertible as an element of $\mathcal{M}H^c_{\infty,\bullet}$ and $Q(-\cdot) := U_1(-\cdot)U_2(-\cdot)^{-1}$. From Step 1 we see that $\|Q(i\cdot)\|_\infty \leq 1$. Now from (4.6) we obtain

$$\begin{bmatrix} K(-\cdot) \\ I_m \end{bmatrix} = \Lambda(-\cdot)^{-1}\begin{bmatrix} U_1(-\cdot) \\ U_2(-\cdot) \end{bmatrix} = V(-\cdot)\begin{bmatrix} U_1(-\cdot) \\ U_2(-\cdot) \end{bmatrix} \tag{4.12}$$

and so

$$\begin{aligned} K(-\cdot) &= V_{11}(-\cdot)U_1(-\cdot) + V_{12}(-\cdot)U_2(-\cdot) \\ I_m &= V_{21}(-\cdot)U_1(-\cdot) + V_{22}(-\cdot)U_2(-\cdot). \end{aligned}$$

Thus

$$\begin{aligned} K(-\cdot) &= (V_{11}(-\cdot)Q(-\cdot) + V_{12}(-\cdot))\,U_2(-\cdot) \\ &= (V_{11}(-\cdot)Q(-\cdot) + V_{12}(-\cdot))\,(V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))^{-1}, \end{aligned}$$

as claimed.

Step 3: We show that $\overline{U}_1(-\cdot)$ and $\overline{U}_2(-\cdot)$ *are right coprime over* $\mathcal{M}H_\infty^c$.

Since M and N are right coprime, there exist R and S in $\mathcal{M}H_\infty^c$ such that $RM - SN = I$. Now it is readily verified using $V(-\cdot)\Lambda(-\cdot) = I$ that

$$\begin{aligned} V_{11}(-\cdot)\overline{U}_1(-\cdot) + V_{12}(-\cdot)\overline{U}_2(-\cdot) &= K(-\cdot)M(\cdot) = N(\cdot) && (4.13)\\ V_{21}(-\cdot)\overline{U}_1(-\cdot) + V_{22}(-\cdot)\overline{U}_2(-\cdot) &= M(\cdot). && (4.14)\end{aligned}$$

Consider now

$$\begin{aligned} & \left(-S(\cdot)V_{12}(-\cdot) + R(\cdot)V_{22}(-\cdot)\right)\overline{U}_2(-\cdot)\\ & -\left(S(\cdot)V_{11}(-\cdot) - R(\cdot)V_{21}(-\cdot)\right)\overline{U}_1(-\cdot)\\ = \;& R(\cdot)\left(V_{21}(-\cdot)\overline{U}_1(-\cdot) + V_{22}(-\cdot)\overline{U}_2(-\cdot)\right)\\ & -S(\cdot)\left(V_{11}(-\cdot)\overline{U}_1(-\cdot) + V_{12}(-\cdot)\overline{U}_2(-\cdot)\right)\\ = \;& R(\cdot)M(\cdot) - S(\cdot)N(\cdot)\\ = \;& I.\end{aligned}$$

Thus $\overline{U}_1(-\cdot)$ and $\overline{U}_2(-\cdot)$ are right coprime over $\mathcal{M}H_\infty^c$.

Step 4: $Q(-\cdot) \in H_\infty^c(\mathbb{C}^{p\times m})$.

The zeros of $\det(V_{22}(-\cdot))$, $\det(M(\cdot))$ and $\det\left(\overline{U}_2(-\cdot)\right)$ are contained in some half-plane $\mathbb{C}_\epsilon^+$, where $\epsilon > 0$. Since $\left\|V_{22}(i\cdot)^{-1}V_{21}(i\cdot)\right\|_\infty < 1$, there exists a $r > 0$ such that $\left\|V_{22}(i\cdot)^{-1}V_{21}(i\cdot)\right\|_\infty = 1 - r$. It follows from Lemma 2.6.7 that there exists a $\delta_1 > 0$ such that $\delta_1 < \epsilon$ and for any ζ satisfying $0 < \zeta < \delta_1$, $\left\|V_{22}(-\zeta - i\cdot)^{-1}V_{21}(-\zeta - i\cdot)\right\|_\infty \le 1 - \frac{r}{2}$. Similarly it follows from Lemma 2.6.7 that there exists a $\delta_2 > 0$ such that $\delta_2 < \epsilon$ and for any ζ satisfying $0 < \zeta < \delta_2$,

$$\|Q(-\zeta - i\cdot)\|_\infty \le 1 + \frac{\frac{r}{4}}{1 - \frac{r}{4}} = \frac{1}{1 - \frac{r}{4}}.$$

Let $\delta := \min\{\delta_1, \delta_2\}$, and fix a ζ satisfying $0 < \zeta < \delta$. Let

$$\phi(\alpha, s) = \det\left(\alpha V_{21}(-s-\zeta)\overline{U}_1(-s-\zeta) + V_{22}(-s-\zeta)\overline{U}_2(-s-\zeta)\right),$$

where $\alpha \in [0, 1]$.

1. We know that

$$\begin{aligned}\phi(0,\cdot) &= \det\left(V_{22}(-\cdot-\zeta)\overline{U}_1(-\cdot-\zeta)\right) \quad \text{and}\\ \phi(1,\cdot) &= \det\left(V_{21}(-\cdot-\zeta)\overline{U}_1(-\cdot-\zeta) + V_{22}(-\cdot-\zeta)\overline{U}_2(-\cdot-\zeta)\right)\end{aligned}$$

are meromorphic in $\mathbb{C}^+_{-\zeta/2}$.

2. $\phi(0,\cdot)$ has a nonzero limit at infinity in $\overline{\mathbb{C}_0^+}$: $\det(V_{22}(-\cdot))$ has a nonzero limit at infinity in $\overline{\mathbb{C}_0^+}$ and $\det\left(\overline{U}_2(-\cdot)\right)$ has a nonzero limit at infinity in $\overline{\mathbb{C}_0^+}$, since $\det\left(\overline{U}_2(-\cdot)\right) \in \mathcal{S}$. $\phi(1,\cdot)$ has a nonzero limit at infinity in $\overline{\mathbb{C}_0^+}$, since $V_{21}(-\cdot)$ is strictly proper, $\overline{U}_1(-\cdot)$ is proper in $\overline{\mathbb{C}_0^+}$, and the above.

3. $(\alpha,s) \mapsto \phi(\alpha,s) : [0,1]\times i\mathbb{R} \to \mathbb{C}$ is a continuous function, and

$$\begin{aligned}
\phi(0,i\omega) &= \det\left(V_{22}(-\zeta-i\omega)\overline{U}_1(-\zeta-i\omega)\right)\\
&= \det(V_{22}(-\zeta-i\omega))\ \det\left(\overline{U}_2(-\zeta-i\omega)\right), \text{ and}\\
\phi(1,i\omega) &= \det\left(V_{21}(-\zeta-i\omega)\overline{U}_1(-\zeta-i\omega)+V_{22}(-\zeta-i\omega)\overline{U}_2(-\zeta-i\omega)\right).
\end{aligned}$$

4. We have

$$\begin{aligned}
&\phi(\alpha,i\omega)\\
=\ &\det(V_{22}(-\zeta-i\omega))\ \det\left(\overline{U}_2(-\zeta-i\omega)\right)\\
&\det\left(I+\alpha V_{22}(-\zeta-i\omega)^{-1}V_{21}(-\zeta-i\omega)\overline{U}_1(-\zeta-i\omega)\overline{U}_2(-\zeta-i\omega)^{-1}\right)\\
\neq\ &0,
\end{aligned}$$

since

$$\begin{aligned}
&\left\|\alpha V_{22}(-\zeta-i\cdot)^{-1}V_{21}(-\zeta-i\cdot)\overline{U}_1(-\zeta-i\cdot)\overline{U}_2(-\zeta-i\cdot)^{-1}\right\|_\infty\\
\leq\ &1\ \left\|V_{22}(-\zeta-i\cdot)^{-1}V_{21}(-\zeta-i\cdot)\right\|_\infty \left\|\overline{U}_1(-\zeta-i\cdot)\overline{U}_2(-\zeta-i\cdot)^{-1}\right\|_\infty\\
\leq\ &\left[1-\frac{r}{2}\right]\ \|Q(-\zeta-i\cdot)\|_\infty\\
\leq\ &\left[1-\frac{r}{2}\right]\ \frac{1}{1-\frac{r}{4}}\ <1,
\end{aligned}$$

and $\det\left(\overline{U}_2(-\zeta-i\omega)\right)\neq 0$.

5. $\phi(\alpha,\infty)\neq 0$, since $V_{21}(-\cdot)$ is strictly proper, $\overline{U}_1(-\cdot)$ is proper in $\overline{\mathbb{C}_0^+}$, and $\det(V_{22}(-\cdot))\ \det\left(\overline{U}_2(-\cdot)\right)$ has a nonzero limit at infinity in $\overline{\mathbb{C}_0^+}$.

Thus the assumptions in Lemma A.1.18 (Curtain and Zwart [34], page 570) are satisfied by ϕ, and hence it follows that the Nyquist indices of $\phi(0,\cdot)$ and $\phi(1,\cdot)$ are the same. Consequently, the number of zeros are the same (the number of poles is zero, as $\phi(0,\cdot)$, $\phi(1,\cdot)$ are analytic in $\mathbb{C}^+_{-\frac{\delta}{2}}$) and so the sum of the number of zeros of $s\mapsto\det(V_{22}(-\zeta-s))$ in $\overline{\mathbb{C}_0^+}$ and the number of zeros of $s\mapsto\det\left(\overline{U}_2(-\zeta-s)\right)$ in $\overline{\mathbb{C}_0^+}$ equals the number of zeros of $s\mapsto\det\left(V_{21}(-\zeta-s)\overline{U}_1(-\zeta-s)+V_{22}(-\zeta-s)\overline{U}_2(-\zeta-s)\right)$ (= $\det(M(\zeta+s)$, using 4.14) in $\overline{\mathbb{C}_0^+}$, that is, $l+\mathtt{den}(Q)=l$, where $\mathtt{den}(Q)$ denotes the number of zeros of $s\mapsto\det\left(\overline{U}_2(-\zeta-s)\right)$ in $\overline{\mathbb{C}_0^+}$. Thus $Q(-\cdot)$ has no poles in $\overline{\mathbb{C}_\zeta^+}$. But

$\zeta > 0$ can be chosen arbitrarily small, which implies that $Q(-\cdot) \in H^c_\infty(\mathbb{C}^{p\times m})$.
■

Remark: For the Nehari problem ($l = 0$), it is possible to establish a sharper version of Theorem 4.3.3, namely under the weaker hypothesis $K(-\cdot) \in H_{\infty,[l]}(\mathbb{C}^{p\times m})$; that is, $K(-\cdot)$ is not required to be continuous on the imaginary axis. However, for the case $l > 0$, we have only been able to show that $Q(-i\cdot) \in L_\infty(\mathbb{R}, \mathbb{C}^{p\times m})$ satisfies $\|Q(i\cdot)\|_\infty \leq 1$ and it has finitely many poles in $\mathbb{C}_0^+$. Consequently, our main result in this chapter is the following:

Theorem 4.3.4 *Suppose that S1-6 hold. Then* $K(-\cdot) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$ *and* $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$ *iff* $K(-\cdot) = R_1(-\cdot)R_2(-\cdot)^{-1}$, *where*

$$\begin{bmatrix} R_1(-\cdot) \\ R_2(-\cdot) \end{bmatrix} = \Lambda(-\cdot)^{-1} \begin{bmatrix} Q(-\cdot) \\ I_m \end{bmatrix}$$

for some $Q(-\cdot) \in H^c_\infty(\mathbb{C}^{p\times m})$ *satisfying* $\|Q(i\cdot)\|_\infty \leq 1$.

As a corollary we derive an equivalent representation of $K(-\cdot)$ in terms of a linear fractional transformation using the matrix function L from Lemma 4.2.4.

Corollary 4.3.5 *Suppose that S1-6 hold. Then* $K(-\cdot) \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$ *and* $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$ *iff*

$$K(-\cdot) := L_{11}(\cdot) + L_{12}(\cdot)Q(-\cdot)(I - L_{22}(\cdot)Q(-\cdot))^{-1}L_{21}(\cdot),$$

where L *is given by (4.3), for some* $Q(-\cdot) \in H^c_\infty(\mathbb{C}^{p\times m})$ *satisfying* $\|Q(-i\cdot)\|_\infty \leq 1$.

Proof This follows from Theorems 4.3.1, 4.3.3 above and Lemma 8.3.14 in Curtain and Zwart [34]. ■