Hankel norm approximation for infinite-dimensional systems

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Chapter 3

Nuclearity and compactness of Hankel operators

3.1 Introduction

In this chapter, we examine the relationships between the exponential (or strong) stability of certain classes of well-posed linear systems and the compactness and nuclearity properties of the Hankel operator. New sufficient conditions for nuclearity are given for exponentially stable, regular, well-posed linear systems with an analytic semigroup. In the last section of this chapter we will derive a bound on the $L_\infty-$error of a sub-optimal Hankel norm approximant of a system with a nuclear Hankel operator.

We consider the Hankel operator of stable matrix-valued functions $G \in H_\infty(\mathbb{C}^{p\times m})$. The Hankel operator with symbol G is the operator $H_G : H_2(\mathbb{C}^m) \to H_2(\mathbb{C}^p)$ given by

$$H_G f = \Pi(\Lambda_G f_-) \quad \text{for } f \in H_2(\mathbb{C}^m), \tag{3.1}$$

where Λ_G is the multiplication map on $L_2(i\mathbb{R}, \mathbb{C}^m)$ induced by G (see Theorem A.6.26, page 647, Curtain and Zwart [34]), Π is the orthogonal projection operator from $L_2(i\mathbb{R}, \mathbb{C}^p)$ onto $H_2(\mathbb{C}^p)$ and $f_-(s) := f(-s)$.

Practical control design is typically based on a reduced-order model of the original system. Many design methodologies utilize a rational approximation of a stable transfer function in the $L_\infty-$norm (for example, see Chapter 9, pages 457-563, Curtain and Zwart [34]). For this to be possible the Hankel operator with symbol $G \in H_\infty(\mathbb{C}^{p\times m})$ should be compact. We quote the following criterion (see for instance Corollary 4.10, page 46, Partington [66]):

Theorem 3.1.1 (Hartman) *$G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$ determines a compact Hankel operator iff $G_-(\cdot) \in H_\infty(\mathbb{C}^{p\times m}) + \mathcal{C}_0(i\mathbb{R}, \mathbb{C}^{p\times m})$, where $\mathcal{C}_0(i\mathbb{R}, \mathbb{C}^{p\times m})$ denotes the space of continuous $p \times m$ complex matrix-valued functions defined on $i\mathbb{R}$, with a (unique) limit at $\pm i\infty$.*

Since most models of infinite-dimensional systems are obtained, not as transfer functions, but as realizations, we are interested in deducing the properties of the Hankel operator from properties of the realization. To do this we introduce the time-domain Hankel operator, which is defined in terms of h, the inverse-Laplace transform of G. If $h \in L_1([0,\infty), \mathbb{C}^{p\times m})$ or $L_2([0,\infty), \mathbb{C}^{p\times m})$, we define the time-domain Hankel operator $\Gamma_h : L_2([0,\infty), \mathbb{C}^m) \to L_2([0,\infty), \mathbb{C}^p)$ by

$$(\Gamma_h u)(t) = \int_0^\infty h(t+s)u(s)ds, \quad t \geq 0, \tag{3.2}$$

for all $u \in L_2([0,\infty), \mathbb{C}^m)$. In the case that $h \in L_1([0,\infty), \mathbb{C}^{p\times m})$, it is well-known that Γ_h is compact (see for example Lemma 8.2.4, page 399, Curtain and Zwart [34]) and so Γ_h has countably many singular values (square roots of the eigenvalues of $\Gamma_h^*\Gamma_h$) $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ and these are also called the *Hankel singular values of* G. If $h \in L_2([0,\infty), \mathbb{C}^{p\times m})$, then (3.2) may not be well-defined. If, however, we also assume that $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$, then (3.2) always defines Γ_h as a bounded operator from $L_2([0,\infty), \mathbb{C}^m)$ to $L_2([0,\infty), \mathbb{C}^p)$ (see Proposition 8, page 224, Keulen [89]). In either of these cases, Γ_h is isomorphic to H_G under the Laplace (or Fourier) transform (see Lemma 8.2.3, page 397, Curtain and Zwart [34] and Keulen [89]).

Let us consider two classes of systems; the first was the subject of the book [34] by Curtain and Zwart and the second was the main topic of the recent book by Oostveen [64].

Class 1. $\Sigma(A, B, C)$ where A is the infinitesimal generator of an exponentially stable strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on a separable Hilbert space X, $B \in \mathcal{L}(\mathbb{C}^m, X)$ and $C \in \mathcal{L}(X, \mathbb{C}^p)$.

For this class, it is well-known and easy to verify that the impulse response is $h(t) = CT(t)B$, $h \in L_1([0,\infty), \mathbb{C}^{p\times m})$, and the Hankel operator is compact.

Class 2. $\Sigma(A - BB^*, B, B^*)$, where A generates a contraction semigroup on the Hilbert space X, $B \in \mathcal{L}(\mathbb{C}^m, X)$.

For this class, it is known that $G(s) = B^*(sI - A + BB^*)^{-1}B \in H_\infty(\mathbb{C}^{p\times m})$, $h \in L_2([0,\infty), \mathbb{C}^{p\times m})$, but $h \notin L_1([0,\infty), \mathbb{C}^{p\times m})$ in general (see Curtain and Zwart [35]). So by Proposition 8 (page 224, Keulen [89]) it follows that the time-domain Hankel operator is well-defined by (3.2). $A - BB^*$ in general does not generate an exponentially stable semigroup, but under mild conditions it does generate a *strongly stable semigroup* $\{T_B(t)\}_{t\geq 0}$ (that is, $T_B(t)x \to 0$ as $t \to \infty$ for all $x \in X$).

Since the Hankel operator of Class 1 is compact, one wonders if relaxing exponential stability to strong stability is still sufficient for compactness. Surprisingly, the reverse statement holds if A and A^* are skew-symmetric (that is, $\langle x, Ax\rangle + \langle Ax, x\rangle = 0$ for all $x \in D(A)$ and $\langle x, A^*x\rangle + \langle A^*x, x\rangle = 0$ for all $x \in D(A^*)$). In section 3.2 we prove that if A and A^* are skew-symmetric, $\{T_B(t)\}_{t\geq 0}$ and $\{T_B(t)^*\}_{t\geq 0}$ are strongly stable but not exponentially stable, and X is infinite-dimensional, then Γ_h will never be compact! On the other hand, by means of an example, we show that systems from Class 2 can have a compact Hankel operator even if $\{T_B(t)\}_{t\geq 0}$ is not strongly stable. In other words, the stability of the semigroup says little about the compactness of the Hankel operator.

While compactness of the Hankel operator is essential to obtain a rational approximation in the H_∞−norm, the property of nuclearity is desirable, because it gives a-priori error bounds in terms of the singular values. We recall that a compact Hankel operator is *nuclear* if $\sum_{k=1}^{\infty} \sigma_k < \infty$, where σ_k's are the singular values in decreasing order. (For background material about nuclearity, we refer to Chapter 1 of Partington [66].) In Glover et al. [18] it was shown that if $h \in L_1([0,\infty), \mathbb{C}^{p\times m}) \cap L_2([0,\infty), \mathbb{C}^{p\times m})$, and Γ_h is nuclear, then for a given integer l, there exists a truncated balanced realization $G_{b,l}$ and an optimal Hankel-norm approximation $G_{0,l}$, both of order l, such that

$$\begin{aligned} \|G - G_{b,l}\|_\infty &\leq 2 \cdot \textstyle\sum_{k=l+1}^{\infty} \sigma_k, \text{ and} \\ \|G - G_{0,l}\|_\infty &\leq \textstyle\sum_{k=l+1}^{\infty} \sigma_k. \end{aligned}$$

In the last section of this chapter we will also find a bound on the L_∞−error of any sub-optimal Hankel norm approximant of a transfer function for which the Hankel operator is nuclear.

There exist various known conditions for nuclearity in terms of the transfer functions.

Theorem 3.1.2 (Coifman and Rochberg [14].) *If* $G \in H_\infty(\mathbb{C}^{p\times m})$, *then* H_G *is nuclear iff*

$$\int_{-\infty}^{\infty}\int_0^{\infty} \|G''(x+iy)\| \; dxdy < \infty. \tag{3.3}$$

Theorem 3.1.3 (Coifman and Rochberg [14]) *The Hankel operator given by (3.2) is nuclear iff* G *possesses on* $\mathbb{C}_0^+$ *an expansion of the form*

$$G(s) = \sum_{n=1}^{\infty} \frac{1}{s - a_n} \mathtt{G}_n \tag{3.4}$$

where $\mathtt{G}_n \in \mathbb{C}^{p\times m}$ *and* $a_n \in \mathbb{C}_0^-$ *are such that*

$$\sum_{n=1}^{\infty} \frac{\|\mathtt{G}_n\|}{|Re(a_n)|} < \infty. \tag{3.5}$$

These theorems made it possible to obtain sufficient conditions for the nuclearity of transfer functions of delay systems (see Zwart et al. [19] and Glover et al. [41]). See also the recent results on fractional transfer functions in Bonnet and Partington [11]. In this chapter we give sufficient conditions for nuclearity in terms of the state-space realizations.

Our new results on nuclearity in Section 3.3 are

1. Class 1 has a nuclear Hankel operator (Theorem 3.3.1).
2. A class of regular well-posed linear systems with an exponentially stable analytic semigroup and unbounded B and C has a nuclear Hankel operator (Theorem 3.3.3).

Analytic semigroups are generated by parabolic partial differential operators and hyperbolic partial differential operators with structural damping (see Pazy [67]). Consequently, Theorem 3.3.3 has important consequences for model reduction of distributed systems with an analytic semigroup with unbounded sensing and control.

3.2 Compactness of Hankel operators

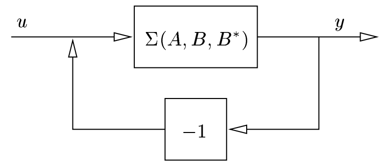
In Section 3.1 we recalled that linear systems with bounded, finite-rank input and output operators, and an exponentially stable semigroup always have a compact Hankel operator. One might hope that this would also be the case if we only have strong stability. However, by means of two examples, we show that the property of strong stability does not imply the compactness of the Hankel operator.

Let us consider Class 2. This class arises by stabilizing the open-loop system $\Sigma(A, B, B^*)$ with transfer function $G_0(s) = B^*(sI - A)^{-1}B$, via the static output feedback $u = -y$,

which results in the closed-loop system $\Sigma(A - BB^*, B, B^*)$ with transfer function

$$G(s) = G_0(s)\,(I + G_0(s))^{-1} = B^*\,(sI - A + BB^*)^{-1}\,B.$$

This closed-loop system has several nice properties (see Curtain and Zwart [35]):



C1. $A - BB^*$ generates a contraction semigroup $\{T_B(t)\}_{t\geq 0}$.

C2. $\int_0^\infty \|B^*T_B(t)x\|^2\, dt \leq \frac{1}{2}\|x\|^2$.

C3. $\int_0^\infty \|B^*T_B(t)^*x\|^2\, dt \leq \frac{1}{2}\|x\|^2$.

C4. $G(s) = B^*\left(sI - A + BB^*\right)^{-1} B \in H_\infty\left(\mathcal{L}(U)\right)$.

C2 and C3 show that the system has an impulse response $h(\cdot) = B^*T_B(\cdot)B \in L_2([0,\infty), \mathcal{L}(U))$ and bounded observability and controllability maps $\mathcal{C}$ and $\mathcal{B}$ defined as follows:

1. $\mathcal{B} : L_2([0,\infty), U) \to X$ is defined by

$$\mathcal{B}u = \int_0^\infty T_B(t)Bu(t)dt \text{ for all } u \in L_2([0,\infty), U). \tag{3.6}$$

2. $\mathcal{C} : X \to L_2([0,\infty), Y)$ is defined by

$$(\mathcal{C}x)(t) = B^*T_B(t)x \text{ for all } t \geq 0 \text{ and all } x \in X. \tag{3.7}$$

The semigroup $\{T_B(t)\}_{t\geq 0}$ is not necessarily strongly stable. Sufficient conditions for $\{T_B(t)\}_{t\geq 0}$ and $\{T_B(t)^*\}_{t\geq 0}$ to be strongly stable can be found in Arendt and Batty [3]:

N1. The intersection of the spectrum of A with the imaginary axis is at most countable.

OR

N2. $\{x \in X \mid B^*T(t)x = 0,\ \|T(t)^*x\| = \|x\| = \|T(t)x\|\ \forall\, t \geq 0\} = \{0\}$.

Note that in N2, $B^*T(t)x = 0$ can be replaced by $B^*T(t)^*x = 0$, and if $\Sigma(A, B, B^*)$ is approximately controllable or observable, then N2 holds.

The Hankel operator Γ_h is equal to $\mathcal{CB}$. The controllability Gramian $L_B = \mathcal{BB}^*$ and the the observability Gramian $L_C = \mathcal{C}^*\mathcal{C}$ always satisfy their respective Lyapunov equations:

N3. $(A - BB^*)L_B x + L_B(A - BB^*)^* x = -BB^* x$ for all $x \in D(A)$.

N4. $(A - BB^*)^* L_C x + L_C(A - BB^*)x = -BB^* x$ for all $x \in D(A^*)$.

If $\{T_B(t)^*\}_{t\geq 0}$ is strongly stable, N3 has the unique solution L_B and if $\{T_B(t)\}_{t\geq 0}$ is strongly stable, N4 has the unique solution L_C. Suppose now that A and A^* are skew-symmetric and $\{T_B(t)\}_{t\geq 0}$ and $\{T_B(t)^*\}_{t\geq 0}$ are strongly stable, but not exponentially stable. Then N3 and N4 have the unique solution $\frac{1}{2}I$ and we now prove that Γ_h will not be compact. We have

$$\Gamma_h^* \Gamma_h = \mathcal{B}^* \mathcal{C}^* \mathcal{C} \mathcal{B} = \mathcal{B}^* L_C \mathcal{B} = \mathcal{B}^* \frac{1}{2} I \mathcal{B} = \frac{1}{2} \mathcal{B}^* \mathcal{B}.$$

If Γ_h is compact, then $\Gamma_h^* \Gamma_h$ is also compact and it follows from the above that $\mathcal{B}$ must be compact (the compactness of $\mathcal{B}$ follows, for example, from Theorem 6.4.(c), page 131, Weidmann [91]). Consequently we obtain $\mathcal{B}\mathcal{B}^* = L_B = \frac{1}{2}I_X$ must be compact, a contradiction, since we assume that X is an infinite-dimensional. So if A and A^* are skew-symmetric and $\{T_B(t)\}_{t\geq 0}$ and $\{T_B(t)^*\}_{t\geq 0}$ are strongly stable but not exponentially stable, then Γ_h can never be compact.

Many examples of partial differential equation systems of the Class 2 structure with a skew-symmetric operator A can be found in Oostveen [63]. Here we give a simple example which can be readily analyzed and which captures the salient features of the partial differential equation examples.

Example 3.2.1 Let $A : D(A)\,(\subset \ell_2(\mathbb{N})) \to \ell_2(\mathbb{N})$ be the operator given by

$$A = \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 0 & 2 \\ -2 & 0 \end{matrix}} & & \\ & & \boxed{\begin{matrix} 0 & 3 \\ -3 & 0 \end{matrix}} & \\ & & & \ddots \end{bmatrix},$$

with

$$D(A) = \left\{ x \in \ell_2(\mathbb{N}) \mid \sum_{k=1}^{\infty} \left(|-k\langle x, e_{2k-1}\rangle|^2 + |k\langle x, e_{2k}\rangle|^2 \right) < \infty \right\},$$

where $\{e_n\}_{n\geq 1}$ denotes the standard orthonormal basis for $\ell_2(\mathbb{N})$. We first show that A is closed and densely defined. Clearly, all elements x in $\ell_2(\mathbb{N})$ with $\langle x, e_n\rangle = 0$ for all sufficiently large n lie in $D(A)$ and form a dense set in $\ell_2(\mathbb{N})$. So A is densely defined.

Let $\{x_m\}_{m\geq 1}$ be a sequence in $D(A)$ and let $x_m \to x_0$ and $Ax_m \to y_0$ as $m \to \infty$. Since the sequence $\{Ax_m\}_{m\geq 1}$ is bounded, there exists a $M > 0$ such that

$$\sum_{k=1}^{\infty} \left(\left|-k\langle x_m, e_{2k-1}\rangle\right|^2 + \left|k\langle x_m, e_{2k}\rangle\right|^2 \right) < M, \quad \text{for all } m \geq 1.$$

Consequently, for any $N \in \mathbb{N}$,

$$\sum_{k=1}^{N} \left(\left|-k\langle x_m, e_{2k-1}\rangle\right|^2 + \left|k\langle x_m, e_{2k}\rangle\right|^2 \right) < M, \quad \text{for all } m \geq 1.$$

Owing to the continuity of the inner product and the fact that $x_m \to x_0$, we obtain

$$\sum_{k=1}^{N} \left(\left|-k\langle x_0, e_{2k-1}\rangle\right|^2 + \left|k\langle x_0, e_{2k}\rangle\right|^2 \right) \leq M.$$

Since the choice of N was arbitrary, it follows that

$$\sum_{k=1}^{\infty} \left(\left|-k\langle x_0, e_{2k-1}\rangle\right|^2 + \left|k\langle x_0, e_{2k}\rangle\right|^2 \right) \leq M.$$

Consequently, $x_0 \in D(A)$ with $Ax_0 = y_0$ and so A is closed.

In fact it can be easily checked that A is a Riesz spectral operator with the (totally disconnected) set of simple unstable eigenvalues $\{\pm ni\}_{n\in\mathbb{N}}$ (see Figure 3.1) and the corresponding (orthogonal) Riesz basis of eigenvectors $\left\{\frac{1}{\sqrt{2}}(e_n \pm ie_{n+1})\right\}_{n\in\mathbb{N}}$.

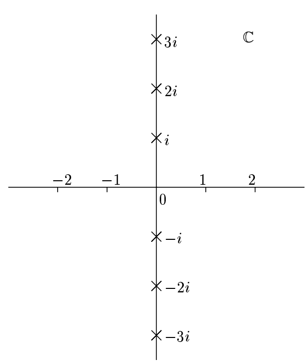


Figure 3.1: $\sigma(A)$.

Since $Ax + A^*x = 0$ for all $x \in D(A) = D(A^*)$, it follows that A is the infinitesimal generator of a contraction strongly continuous semigroup on the

Hilbert space $\ell_2(\mathbb{N})$. We have $0 \in \rho(A)$ and

$$A^{-1} = \begin{bmatrix} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{matrix}} & & \\ & & \boxed{\begin{matrix} 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 \end{matrix}} & \\ & & & \ddots \end{bmatrix} \in \mathcal{L}(\ell_2(\mathbb{N})).$$

It can be easily seen that $\mathtt{R}_n \to A^{-1}$ as $n \to \infty$ in the uniform operator topology, where

$$\mathtt{R}_n := \begin{bmatrix} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & & & \\ & \ddots & & & \\ & & \boxed{\begin{matrix} 0 & -\frac{1}{n} \\ \frac{1}{n} & 0 \end{matrix}} & & \\ & & & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} & \\ & & & & \ddots \end{bmatrix} \in \mathcal{L}(\ell_2(\mathbb{N})).$$

Since each $\mathtt{R}_n$ has finite rank $2n$, A^{-1} is the uniform limit of a sequence of compact operators and so A^{-1} is compact (see for example Theorem A.3.22.e, page 587, Curtain and Zwart [34]). From Theorem 6.29 (page 187, Kato [52]) if follows that A has compact resolvent.

Let $B \in \mathcal{L}(\mathbb{C}, \ell_2(\mathbb{N}))$ be defined by

$$B = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{3} \\ 0 \\ \vdots \end{bmatrix}.$$

It follows from Theorem 4.2.3 (page 164, Curtain and Zwart [34]) that $\Sigma(A, B, -)$ is approximately controllable. Dually, $\Sigma(A^*, -, B^*)$ is approximately observable. Consequently, from Lemma 2.2.6 (page 23, Oostveen [63]) it follows that $A_B := A - BB^*$ and $A_B^* = A^* - BB^*$ generate strongly stable semigroups $\{T_B(t)\}_{t \geq 0}$ and $\{T_B(t)^*\}_{t \geq 0}$, respectively, on $\ell_2(\mathbb{N})$.

Thus $A - BB^*$ generates a strongly stable semigroup, but the state linear system $\Sigma(A - BB^*, B, B^*)$ has a time-domain Hankel operator Γ_h that is bounded, but not compact. $\Diamond$

The following example shows that a system with the Class 2 structure can have a compact Hankel operator although $\{T_B(t)\}_{t \geq 0}$ is not strongly stable.

Example 3.2.2 We start by showing that the transfer function

$$G(s) = \frac{1}{1 + \sqrt{s^2 + 1}},$$

(with an appropriate choice of an analytic branch of the complex square root function) has a realization of the form $\Sigma(A - BB^*, B, B^*)$. We do this by interpreting it as the closed-loop system formed by applying the static output feedback to the system with transfer function

$$G_0(s) = \frac{1}{\sqrt{s^2 + 1}}.$$

Following Baras and Brockett [8], we show that $G_0(s)$ has a realization $\Sigma(A, B, B^*)$, where A generates a contraction semigroup. Let $A \in \mathcal{L}(\ell_2(\mathbb{Z}))$ be given by

$$A = \frac{1}{2} \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & 0 & -1 & & \\ & 1 & \boxed{0} & -1 & \\ & & 1 & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}.$$

It is easy to see that A is bounded and that $A + A^* = 0$. It can be shown that the spectrum of A (see Figure 3.2) is the set

$$[-i, i] = \{s \in \mathbb{C} \mid -1 \leq \operatorname{Im}(s) \leq 1\}.$$

(See Problem 241, page 135, and Problem 67, page 36, Halmos, [48].)

Let $B \in \mathcal{L}(\mathbb{C}, \ell_2(\mathbb{Z}))$ be defined as

$$B = \begin{bmatrix} \vdots \\ 0 \\ \boxed{1} \\ 0 \\ \vdots \end{bmatrix}.$$

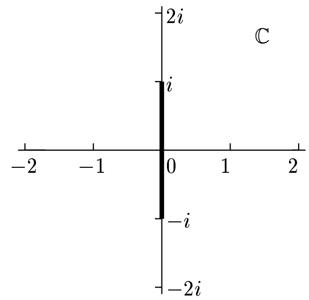


Figure 3.2: $\sigma(A) = \sigma_c(A) = [-i, i]$.

Claim: $\Sigma(A, B, B^*)$ realizes the transfer function $\frac{1}{\sqrt{s^2+1}}$.

We know that $s \mapsto B^*(sI - A)^{-1}B$ is analytic in $\mathbb{C} \setminus [-i, i]$. Moreover for $\omega > 1$, it can be checked easily that $B^*(i\omega - A)^{-1}B = \frac{-i}{\sqrt{\omega^2-1}}$, as follows. Let $L_2(\mathbb{T})$ be the the space of square integrable functions on the unit circle and let L_φ be the multiplication operator corresponding to the bounded function

$$\varphi(z) = -\frac{1}{2}z^{-1} + i\omega + \frac{1}{2}z.$$

From Exercise 241 (page 135, Halmos [48]), it follows that the bilaterally infinite matrix

$$i\omega I - A = \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & i\omega & \frac{1}{2} & & \\ & -\frac{1}{2} & i\omega & \frac{1}{2} & \\ & & -\frac{1}{2} & i\omega & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

is the Laurent matrix corresponding to the multiplication operator L_φ on $L_2(\mathbb{T})$ with respect to the familiar standard orthonormal basis in $L_2(\mathbb{T})$, namely $\{e_n(z) = z^n\}_{n\in\mathbb{Z}}$. Thus $(i\omega I - A)^{-1}$ corresponds to multiplication by

$$\begin{aligned} \frac{1}{\varphi(z)} &= \frac{1}{-\frac{1}{2}z^{-1} + i\omega + \frac{1}{2}z} \\ &= \frac{1}{i\omega\left[1 - \frac{1}{2i\omega}\left(-z + \frac{1}{z}\right)\right]} \\ &= \frac{1}{i\omega}\left[1 + \frac{1}{2i\omega}\left(-z + \frac{1}{z}\right) + \left[\frac{1}{2i\omega}\left(-z + \frac{1}{z}\right)\right]^2 + \dots\right]. \end{aligned} \tag{3.8}$$

Since we want to find out $B^*(i\omega I - A)^{-1}B$, we are actually finding the $(0, 0)^{\text{th}}$

entry of $(i\omega I - A)^{-1}$, which is the coefficient of z^0 in (3.8): namely,

$$\frac{1}{i\omega}\sum_{n=0}^{\infty}\left(\frac{1}{2i\omega}\right)^{2n}(-1)^n\binom{2n}{n}.$$

But for $\omega > 1$, we have

$$\begin{aligned}\frac{-i}{\sqrt{\omega^2-1}} &= \frac{1}{i\omega}\left(1-\frac{1}{\omega^2}\right)^{-\frac{1}{2}}\\ &= \frac{1}{i\omega}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\left(\frac{1}{\omega^2}\right)^n\\ &= \frac{1}{i\omega}\sum_{n=0}^{\infty}\left(\frac{1}{2i\omega}\right)^{2n}(-1)^n 2^{2n}\binom{-\frac{1}{2}}{n}(-1)^n.\end{aligned}$$

It is easy to see that

$$2^{2n}\binom{-\frac{1}{2}}{n}(-1)^n = \binom{2n}{n},$$

and so it follows that for $\omega > 1$, $B^*(i\omega - A)^{-1}B = \frac{-i}{\sqrt{\omega^2-1}}$.

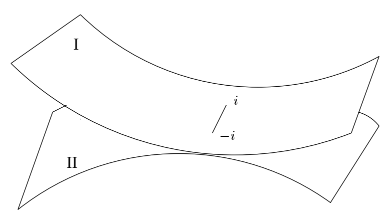


Figure 3.3: The Riemann surface of the map $s \mapsto \frac{1}{\sqrt{s^2+1}}$.

Consider the map $s \mapsto \frac{1}{\sqrt{s^2+1}}$. We shall take two copies of the s−plane, with cuts along the segment $[-i, i]$. The formation of an analytic branch of the map $s \mapsto \frac{1}{\sqrt{s^2+1}}$ takes place in each of these planes. Now we shall join the edges of our cuts cross-wise with the help of two segments $[-i, i]$, the inner points of which will be considered to be different, although geometrically they coincide. This is the Riemann surface of the function $s \mapsto \frac{1}{\sqrt{s^2+1}}$ (see Figure 3.3). It is double-sheeted, with two branch points on i and $-i$. The function $s \mapsto \frac{1}{\sqrt{s^2+1}}$ is single-valued and continuous on this surface. We consider the analytic branch of $s \mapsto \frac{1}{\sqrt{s^2+1}}$ corresponding to the plane I; $i\omega \mapsto \frac{-i}{\sqrt{\omega^2-1}}$ for $\omega > 1$. Since $G_0(s)$ and this analytic branch match on $\{i\omega |\ \omega > 1\}$, the claim follows.

Thus $\Sigma(A - BB^*, B, B^*)$ realizes the transfer function $G(s) = \frac{1}{1+\sqrt{s^2+1}}$. Since $G(s) = \frac{1}{1+\sqrt{s^2+1}}$, it can be easily checked that $G(s) \in H_\infty(\mathbb{C})$. Moreover,

$\omega \mapsto G(i\omega)$ is continuous and has the limit 0 as $\omega \to \pm\infty$. So it follows from Theorem 3.1.1, that G determines a compact Hankel operator. However, our earlier discussion shows that in this case $A - BB^*$ cannot generate a strongly stable semigroup. $\Diamond$

3.3 Nuclearity

In this section we study the nuclearity property of the Hankel operator of a system in terms of a given realization (A, B, C).

First we show that an exponentially stable state linear system with bounded inputs and outputs and finite-dimensional input and output spaces is nuclear.

Theorem 3.3.1 *Let A be the infinitesimal generator of an exponentially stable strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on the separable Hilbert space X, with $B \in \mathcal{L}(\mathbb{C}^m, X)$, $C \in \mathcal{L}(X, \mathbb{C}^p)$. Then*

1. *The observability operator $\mathcal{C} : X \to L_2([0,\infty), \mathbb{C}^p)$ defined by $(\mathcal{C}x)(\cdot) = CT(\cdot)x$ is Hilbert-Schmidt.*
2. *The controllability operator $\mathcal{B} : L_2([0,\infty), \mathbb{C}^m) \to X$ defined by $\mathcal{B}u = \int_0^\infty T(t)Bu(t)dt$ is Hilbert-Schmidt.*
3. *$L_C = \mathcal{C}^*\mathcal{C}$, $L_B = \mathcal{B}\mathcal{B}^*$ and $\Gamma_h = \mathcal{C}\mathcal{B}$ are all nuclear.*

Proof *1.* Define $\mathcal{C}_i : X \to L_2(0,\infty)$, $i \in \{1, \ldots, p\}$ by

$$(\mathcal{C}_i x)(t) = \langle CT(t)x, e_i \rangle = \langle x, T(t)^* C^* e_i \rangle,$$

where $\{e_1, \ldots, e_p\}$ is the standard basis for $\mathbb{C}^p$. We have

$$\begin{aligned} |(\mathcal{C}_i x)(t)| &= |\langle x, T(t)^* C^* e_i \rangle| \\ &\leq \|x\| \, \|T(t)^* C^* e_i\| \leq \|x\| \, \|T(t)^*\| \, \|C^*\| \, \|e_i\| \\ &\leq \|x\| \, M \, e^{-\alpha t} \, \|C^*\|, \end{aligned}$$

and $\int_0^\infty |M \, e^{-\alpha t} \, \|C^*\||^2 \, dt < \infty$. We will now use the following result which is an adaptation of Theorem 6.12 (page 140, Weidmann [91]):

Theorem 3.3.2 *Let $\mathtt{K}$ be a bounded linear operator from a Hilbert space $\mathcal{H}$ into $L_2(0,\infty)$. If there exists a function $\kappa \in L_2(0,\infty)$ such that $|(\mathtt{K}v)(t)| \leq \kappa(t)\|v\|$ for almost all $t \in (0,\infty)$, and all $v \in \mathcal{H}$, then $\mathtt{K}$ is Hilbert-Schmidt.*

Applying this result we obtain that $\mathcal{C}_i$ is Hilbert-Schmidt. Consequently, for an arbitrary orthonormal basis $\{x_i\}$ of X,

$$\sum_{j=1}^{\infty} \|\mathcal{C}_i x_j\|^2_{L_2(0,\infty)} < \infty \text{ for all } i \in \{1, \ldots, p\},$$

and

$$\sum_{i=1}^{p} \sum_{j=1}^{\infty} \|\mathcal{C}_i x_j\|^2_{L_2(0,\infty)} < \infty.$$

Thus

$$\sum_{j=1}^{\infty} \|\mathcal{C} x_j\|^2_{L_2([0,\infty),\mathbb{C}^p)} < \infty,$$

and so $\mathcal{C}$ is Hilbert-Schmidt.

2. From Theorem 6.9 (Weidmann [91]), it follows that $\mathcal{B}$ is Hilbert-Schmidt iff $\mathcal{B}^*$ is Hilbert-Schmidt. But $\mathcal{B}^*$ is Hilbert-Schmidt by applying the first part of the lemma to the dual system $\Sigma(A^*, C^*, B^*)$.

3. Using Theorem 7.10(b) (Weidmann [91]), we obtain that $L_C = \mathcal{C}^*\mathcal{C}$, $L_B = \mathcal{B}\mathcal{B}^*$ and $\Gamma_h = \mathcal{C}\mathcal{B}$ are all nuclear. ■

We remark that the Hilbert-Schmidt property of $\mathcal{C}$ was already shown in Dumortier [36] (page 24, Proposition 1.0.2).

We now show that a class of regular linear systems with an exponentially stable analytic semigroup and unbounded B and C has a nuclear Hankel operator. We use the notation introduced in Section 2.5.

Theorem 3.3.3 *Let*

1. $-A$ *generate an exponentially stable, analytic, strongly continuous semigroup* $\{T(t)\}_{t\geq 0}$ *on* Z,
2. $B \in \mathcal{L}(\mathbb{C}^m, Z_{\alpha_B})$, *and*
3. $C \in \mathcal{L}(Z_{\alpha_C}, \mathbb{C}^p)$,

where $\alpha_B \leq \alpha_C < \alpha_B + 1$. *If* γ *satisfies* $\alpha_C - \frac{1}{2} < \gamma < \alpha_B + \frac{1}{2}$, *then*

1. $-A$, B, C *generate a regular linear system with state space* Z_γ, *input space* $\mathbb{C}^m$ *and output space* $\mathbb{C}^p$. *The transfer function is given by* $G(s) = C(sI + A)^{-1}B$, *and it satisfies*

$$\lim_{\substack{|s|\to\infty \\ s\in\overline{\mathbb{C}_0^+}}} \|G(s)\| = 0.$$

2. *The controllability map* $\mathcal{B} \in \mathcal{L}(L_2([0,\infty),\mathbb{C}^m), Z_\gamma)$ *and the observability map* $\mathcal{C} \in \mathcal{L}(Z_\gamma, L_2([0,\infty),\mathbb{C}^p))$ *are Hilbert-Schmidt operators.*

3. $h(\cdot) := CT(\cdot)B \in L_1([0,\infty),\mathbb{C}^{p\times m})$ *and the Hankel operator* Γ_h *satisfies* $\Gamma_h = \mathcal{C}\mathcal{B}$, *where* $\mathcal{B} \in \mathcal{L}(L_2([0,\infty),\mathbb{C}^m), Z_\gamma)$ *and* $\mathcal{C} \in \mathcal{L}(Z_\gamma, L_2([0,\infty),\mathbb{C}^p))$. *Furthermore,* Γ_h *is nuclear.*

Proof *1.* This part follows from Staffans [81] (page 251) or Staffans [84].

2. Just as in Theorem 3.3.1, it is sufficient to prove this for the case $p = 1$. From part 1 above, we know that C is an admissible observation operator for $\{T(t)\}_{t\geq 0}$ with state space Z_γ and so $\mathcal{C} \in \mathcal{L}(Z_\gamma, L_2([0,\infty),\mathbb{C}))$. Since $C \in \mathcal{L}(Z_{\alpha_C},\mathbb{C})$, for every $z \in Z_{\alpha_C}$, we have

$$\begin{aligned}
|(\mathcal{C}z)(t)| &= |CT(t)z| \\
&\leq \|C\|_{\mathcal{L}(Z_{\alpha_C},\mathbb{C})}\|T(t)z\|_{Z_{\alpha_C}} \\
&= \|C\|_{\mathcal{L}(Z_{\alpha_C},\mathbb{C})}\left\|A^{\alpha_C-\gamma}T(t)A^{\gamma}z\right\|_Z \\
&\leq \|C\|_{\mathcal{L}(Z_{\alpha_C},\mathbb{C})}\left\|A^{\alpha_C-\gamma}T(t)\right\|_{\mathcal{L}(Z)}\|A^{\gamma}z\|_Z \\
&= \|C\|_{\mathcal{L}(Z_{\alpha_C},\mathbb{C})}\left\|A^{\alpha_C-\gamma}T(t)\right\|_{\mathcal{L}(Z)}\|z\|_{Z_\gamma}.
\end{aligned}$$

Case 1: If $\alpha_C - \gamma \geq 0$, then we have

$$\begin{aligned}
|(\mathcal{C}z)(t)| &= \|C\|_{\mathcal{L}(Z_{\alpha_C},\mathbb{C})}\left\|A^{\alpha_C-\gamma}T(t)\right\|_{\mathcal{L}(Z)}\|z\|_{Z_\gamma} \\
&\leq \|C\|_{\mathcal{L}(Z_{\alpha_C},\mathbb{C})}K_1\frac{e^{-\epsilon t}}{t^{\alpha_C-\gamma}}\|z\|_{Z_\gamma},
\end{aligned}$$

and $t \mapsto \frac{e^{-\epsilon t}}{t^{\alpha_C-\gamma}} \in L_2([0,\infty),\mathbb{C})$, for $\gamma > \alpha_C - \frac{1}{2}$. So by Theorem 6.12 (page 140, Weidmann [91]), it follows that that $\mathcal{C}$ is Hilbert-Schmidt for all γ satisfying $\alpha_C \geq \gamma > \alpha_C - \frac{1}{2}$.

Case 2: If $\alpha_C - \gamma < 0$, then we have

$$\begin{aligned}
|(\mathcal{C}z)(t)| &= \|C\|_{\mathcal{L}(Z_{\alpha_C},\mathbb{C})}\left\|A^{\alpha_C-\gamma}T(t)\right\|_{\mathcal{L}(Z)}\|z\|_{Z_\gamma} \\
&\leq \|C\|_{\mathcal{L}(Z_{\alpha_C},\mathbb{C})}K_2e^{-\epsilon t}\|z\|_{Z_\gamma},
\end{aligned}$$

and $t \mapsto e^{-\epsilon t} \in L_2([0,\infty),\mathbb{C})$. So by Theorem 3.3.2, it follows that that $\mathcal{C}$ is Hilbert-Schmidt for all γ satisfying $\alpha_C < \gamma$.

From the above cases, it follows that $\mathcal{C}$ is Hilbert-Schmidt for all γ satisfying $\gamma > \alpha_C - \frac{1}{2}$.

Since $\mathcal{B}$ is an admissible control operator for $\{T(t)\}_{t\geq 0}$ with state space Z_γ, $\mathcal{B}$ is an element in $\mathcal{L}(L_2([0,\infty),\mathbb{C}^m), Z_\gamma)$ and for $u \in L_2([0,\infty),\mathbb{C}^m)$,

$$\mathcal{B}u = \int_0^\infty T(t)Bu(t)dt.$$

Thus the dual operator $\mathcal{B}' \in \mathcal{L}\left(Z'_{-\gamma}, L_2([0,\infty),\mathbb{C}^m)\right)$ and for $z \in Z'_{-\gamma}$,

$$(\mathcal{B}'z)(\cdot) = B'T(\cdot)'z.$$

Proceeding as above, it can be shown that $\mathcal{B}' \in \mathcal{L}\left(Z'_{-\gamma}, L_2([0,\infty),\mathbb{C}^m)\right)$ is Hilbert-Schmidt. Thus, using Proposition 2 (page 261, Aubin [4]) it follows that $\mathcal{B}$ is also Hilbert-Schmidt.

3. If $u \in \mathbb{C}^m$, then $Bu \in Z_{\alpha_B}$ and for $t > 0$, $T(t)Bu \in \cap_{\beta\in\mathbb{R}} Z_\beta$. Consequently $T(t)Bu \in Z_{\alpha_C}$ and $CT(t)Bu \in \mathbb{C}^p$ for every $t > 0$. Moreover,

$$\|CT(t)Bu\|_{\mathbb{C}^p} \leq \|C\|_{\mathcal{L}\left(Z_{\alpha_C},\mathbb{C}^p\right)} \left\|A^{\alpha_C-\alpha_B}T(t)\right\|_{\mathcal{L}(Z)} \|B\|_{\mathcal{L}\left(\mathbb{C}^m,Z_{\alpha_C}\right)} \|u\|_{\mathbb{C}^m}.$$

But since $\alpha_C - \alpha_B \geq 0$, it follows that

$$\|CT(t)B\|_{\mathbb{C}^{p\times m}} \leq \|C\|_{\mathcal{L}\left(Z_{\alpha_C},\mathbb{C}^p\right)} K_1 \frac{e^{-\epsilon t}}{t^{\alpha_C-\alpha_B}} \|B\|_{\mathcal{L}\left(\mathbb{C}^m,Z_{\alpha_C}\right)}.$$

Finally, since $\alpha_C - \alpha_B < 1$, we obtain that $h(\cdot) = CT(\cdot)B \in L_1([0,\infty),\mathbb{C}^{p\times m})$.

We know that $\mathcal{B} \in \mathcal{L}\left(L_2([0,\infty),\mathbb{C}^m), Z_\gamma\right)$ and $\mathcal{C} \in \mathcal{L}\left(Z_\gamma, L_2([0,\infty),\mathbb{C}^p)\right)$. Consequently, if $u \in L_2([0,\infty),\mathbb{C}^m)$, $\mathcal{B}u \in Z_\gamma$, and

$$(\mathcal{C}(\mathcal{B}u))(t) = CT(t)\mathcal{B}u = CT(t)\int_0^\infty T(\tau)Bu(\tau)d\tau.$$

But $CT(t) \in \mathcal{L}\left(Z_\gamma, \mathbb{C}^p\right)$ and so we have (see for example Theorem A.5.23, page 628, Curtain and Zwart [34])

$$\begin{aligned}
(\mathcal{C}(\mathcal{B}u))(t) &= \int_0^\infty CT(t)T(\tau)Bu(\tau)d\tau \\
&= \int_0^\infty CT(t+\tau)Bu(\tau)d\tau \\
&= \int_0^\infty h(t+\tau)u(\tau)d\tau \\
&= (\Gamma_h u)(t).
\end{aligned}$$

Furthermore, $\mathcal{B}$ and $\mathcal{C}$ are Hilbert-Schmidt operators, and so it follows from Theorem 7.10 (b) (page 175, Weidmann [91]) that $\Gamma_h = \mathcal{C}\mathcal{B}$ is nuclear. ■

As explained in the introduction, analytic semigroups are generated by parabolic and some hyperbolic partial differential equations and so the above theorem is relevant to many distributed parameter systems with unbounded sensing and control. For Example 2.5.10 considered earlier in Section 2.5 of Chapter 2, the transfer function $G(s) = \frac{1}{(1+s)^m}$ has a nuclear Hankel operator.

The analyticity assumption in Theorem 3.3.3 is crucial. In general, if A generates an exponentially stable semigroup and if B or C is unbounded, the Hankel operator will not be nuclear: to show this, we revisit Example 2.4.4 given in Chapter 2.

Example 2.4.4 (continued) We note that the semigroup is not analytic, but it is exponentially stable, and B is bounded, but C is unbounded.

The Hankel singular values are $\sigma_k = \frac{1}{\sqrt{\mu_k^2+1}}$, where the μ_k's are the roots of the transcendental equation

$$\tan(\mu\tau) = \frac{-\mu(3-\mu^2)}{1-3\mu^2} \tag{3.9}$$

(see for example, Theorem 8.2.10, pages 402-403, Curtain and Zwart [34]).

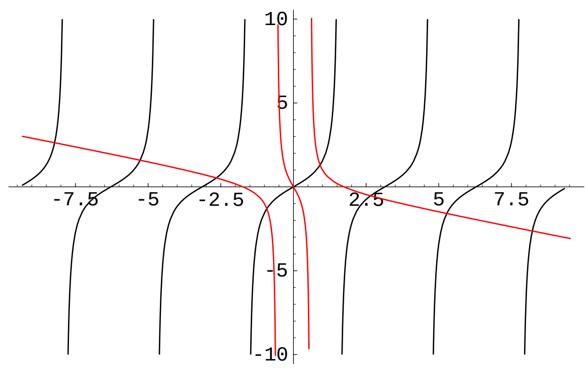


Figure 3.4: Roots of the transcendental equation.

For the sake of simplicity, we assume $\tau = 1$. Because of the periodicity of $\tan(\cdot)$, its monotonicity in each periodic interval of the type

$$\left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right), \quad k \in \mathbb{Z},$$

and the monotonicity of the function $f(x) = \frac{-x(3-x^2)}{1-3x^2}$, for $|x| > \sqrt{3}$, the positive roots of the transcendental equation above satisfy $1 < \mu_k < k\pi$ for $k > 2$ (see Figure 3.4). Thus we obtain

$$\sigma_k = \frac{1}{\sqrt{\mu_k^2+1}} > \frac{1}{\sqrt{\mu_k^2+\mu_k^2}} = \frac{1}{\sqrt{2}\mu_k} > \frac{1}{\sqrt{2}\pi k} \quad \text{for } k > 2.$$

Hence $\sum \sigma_k$ diverges, and so Γ_h is not nuclear. However, it is Hilbert-Schmidt. This follows from Exercise 8.9 (Curtain [34]). ◊

Finally, we make the point that exponential stability is not a necessary condition for nuclearity: in the following example, the Hankel operator is nuclear, although the semigroup is not exponentially stable.

Example 3.3.4 We construct the example below following Ober [61]. Let $\{\lambda_k\}_{k\geq 1}$ be a decreasing bounded sequence of distinct positive numbers converging to 0. Consider the system $\Sigma(A,B,B^*)$, with

$$A = \left[\frac{-\lambda_k\lambda_l}{kl\,(\lambda_k+\lambda_l)}\right]_{1\leq k,l<\infty}$$

and

$$B = \begin{bmatrix} \frac{\lambda_1}{1} \\ \frac{\lambda_2}{2} \\ \frac{\lambda_3}{3} \\ \vdots \end{bmatrix} \in \mathcal{L}\left(\mathbb{C}, \ell_2(\mathbb{N})\right).$$

Then

1. $A \in \mathcal{L}(\ell_2(\mathbb{N}))$ is Hilbert-Schmidt. This follows from the proof of Proposition 3.(i) (page 304, Ober [61] and Theorem 6.22, Weidmann [91]). Thus $0 \in \sigma(A)$ and this rules out exponential stability. However, from the proof of Proposition 3.(iii) (page 304, [61]), it follows that the semigroup $\left\{e^{At}\right\}_{t\geq 0}$ is strongly stable.

2. The Lyapunov equation

$$A\Lambda + \Lambda A = -BB^*. \tag{3.10}$$

has a solution

$$\Lambda_0 := \begin{bmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This is proved in Proposition 3 (page 304, [61]), and so it follows from Theorem 3.1 (page 10, Hansen and Weiss [49]) that B is an infinite-time admissible control operator. Since $A = A^*$ generates a strongly stable semigroup, it follows from [49] (page 10) that (3.10) has the unique solution

$$\mathcal{B}\mathcal{B}^* = \Lambda_0 = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which is clearly compact. Hence it follows from Theorem 6.4.(c) (page 131, Weidmann [91]) that $\mathcal{B}$ is compact.

3. The time domain Hankel operator[1] $\Gamma_h = \mathcal{B}^*\mathcal{B} \in \mathcal{L}\left(L_2(0,\infty)\right)$ is compact, with $\sigma_k\left(\Gamma_h\right) = \lambda_k$ for all $k \in \mathbb{N}$.

[1] The equality $\Gamma_h = \mathcal{B}^*\mathcal{B}$ follows from the remark after the proof of Corollary 4.4, page 276, Ober [62]. That $\sigma_k\left(\Gamma_h\right) = \lambda_k$ is established in Corollary 2 (page 304, [61]).

Finally, upon choosing the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}}\lambda_k < \infty$, we can obtain nuclearity: for example, we can take $\lambda_k = \frac{1}{k^2}$. ◊

Conclusions: There already exist necessary and sufficient conditions for the compactness and nuclearity of the Hankel operator in terms of the transfer function. However, in most applications, the model is given in terms of a triple (A, B, C) of operators. We have given new sufficient conditions for nuclearity

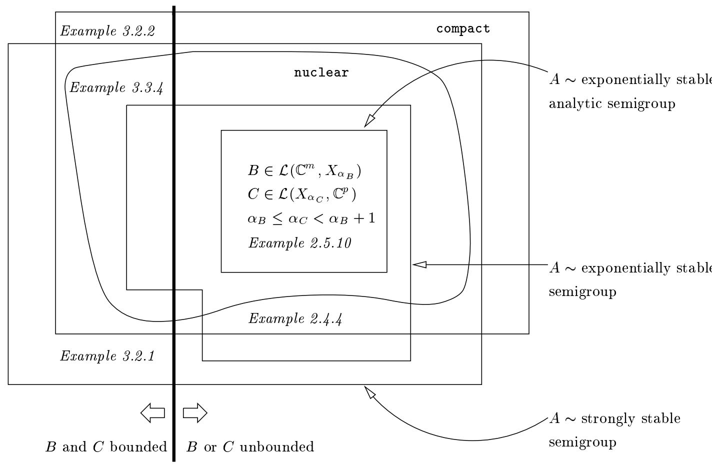


Figure 3.5: Overview of the results.

for two classes of realizations: an exponentially stable realization with bounded B and C (Theorem 3.3.1) and an exponentially stable analytic realization (Theorem 3.3.3). On the other hand, by means of examples, we have shown that the state space properties of exponential and strong stability of the semigroup have little to do with the compactness and nuclearity of the Hankel operator. Figure 3.5 gives a concise overview of the results shown in the previous sections of this chapter.

3.4 L_∞−error of sub-optimal Hankel norm approximants, assuming nuclearity

In this section, we prove that a sub-optimal Hankel norm approximant is also a good approximation with respect to the L_∞−norm under the assumption that the Hankel operator is nuclear. To prove Theorem 3.4.1 below, we will use the following results:

P1. If $h \in L_1([0,\infty), \mathbb{C}^{p\times m})$ and Γ_h is nuclear, then $\|h\|_{L_1([0,\infty),\mathbb{C}^{p\times m})} \le$

$2\sum_{k=1}^{\infty}\sigma_k$ (see for example, Theorem 2.1, page 866, Glover et al. [18]).

P2. If $\mathtt{K}_1, \mathtt{K}_2$ are compact operators from the Hilbert space $\mathcal{H}_1$ to the Hilbert space $\mathcal{H}_2$, then $\sigma_{j+k-1}(\mathtt{K}_1+\mathtt{K}_2) \leq \sigma_j(\mathtt{K}_1)+\sigma_k(\mathtt{K}_2)$ for all j and k in $\mathbb{N}$ (see for example, Theorem 7.7, page 171, Weidmann [91]).

Theorem 3.4.1 *Suppose that*

1. $h \in L_1([0,\infty), \mathbb{C}^{p\times m})$, *and let* G *denote the Laplace transform of* h ,
2. Γ_h *is a nuclear Hankel operator,*
3. $\sigma_{l+1}(G) < \sigma < \sigma_l(G)$, *and*
4. $K(-\cdot) \in H_{\infty,l}(\mathbb{C}^{p\times m})$ *is a solution to the sub-optimal Hankel norm approximation problem, that is,* $\|G(i\cdot)+K(i\cdot)\|_\infty \leq \sigma$.

If

$K(\cdot) = G_*(\cdot) + F(\cdot)$, *where*
$G_*(-\cdot)$ *is the rational transfer function of a finite-dimensional system with MacMillan degree at most* l, *and*
$F(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$,

then

$$\|G(i\cdot)+G_*(i\cdot)\|_\infty \leq 4l\ \sigma_l(G) + 2\sum_{k=l+1}^{\infty}\sigma_k(G).$$

Proof We know that

$$\sigma_k(H_{G+G_*}) \leq \|H_{G+G_*}\| = \|H_{G+K}\| \leq \|G+K\|_\infty \leq \sigma < \sigma_l, \tag{3.11}$$

for all $k \in \mathbb{N}$. Moreover from P2. above, we obtain

$$\sigma_{k+l}(H_{G+G_*}) \leq \sigma_k(H_G) + \sigma_{l+1}(H_{G_*}) = \sigma_k(H_G), \tag{3.12}$$

since G_* has MacMillan degree at most equal to l. Thus we have

$$\left.\begin{array}{rcl}\sigma_1(H_{G+G_*}) & \leq & \sigma_l(G)\\ & \vdots & \\ \sigma_l(H_{G+G_*}) & \leq & \sigma_l(G)\end{array}\right\}\ \text{using (3.11)}$$

$$\left.\begin{array}{rcl}\sigma_{l+1}(H_{G+G_*}) & \leq & \sigma_l(G)\\ & \vdots & \\ \sigma_{l+l}(H_{G+G_*}) & \leq & \sigma_l(G)\end{array}\right\}\ \begin{array}{l}\text{(3.11) and (3.12) both apply,}\\ \text{but (3.11) is sharper.}\end{array}$$

$$\left.\begin{array}{rcl}\sigma_{l+l+1}(H_{G+G_*}) & \leq & \sigma_{l+1}(G)\\ & \vdots & \end{array}\right\}\ \begin{array}{l}\text{(3.11) and (3.12) both apply,}\\ \text{but (3.12) is sharper.}\end{array}$$

Finally, employing the estimate from P1. above, we have

$$\|G(i\cdot) + G_*(i\cdot)\|_\infty \leq \|h + h_*\|_{L_1([0,\infty),\mathbb{C}^{p\times m})} \leq 2 \sum_{k=1}^{\infty} \sigma_k \left(H_{G+G_*}\right),$$

where h_* denotes the inverse Laplace transform of G_*. Thus we have

$$\|G(i\cdot) + G_*(i\cdot)\|_\infty \leq 4l\ \sigma_l(G) + 2 \sum_{k=l+1}^{\infty} \sigma_k(G).$$

■

The error bound in Theorem 3.4.1 is like the one in Theorem 6.4 (page 889, Glover et al. [18]).

Finally we investigate the nature of the error bound

$$E_l := 4l\ \sigma_l(G) + 2 \sum_{k=l+1}^{\infty} \sigma_k(G).$$

In Theorem 3.4.3 below, we prove that $E_l \to 0$ as $l \to \infty$. This shows that the sub-optimal Hankel norm approximants are also good approximations with respect to the L_∞−norm. In order to prove Theorem 3.4.3 below, we need the following lemma.

Lemma 3.4.2 *If* $a_1 \geq a_2 \geq \ldots \geq 0$ *and* $\sum_{n=1}^{\infty} a_n < \infty$*, then* $\lim_{n\to\infty} n\ a_n = 0$.

Proof We have

$$\sum_{k=n+1}^{\infty} a_k \geq a_{n+1} + \ldots + a_{2n} \geq n\ a_{2n} \geq 0.$$

Thus $\lim_{n\to\infty} n\ a_{2n} = 0$, and so

$$\lim_{n\to\infty} 2n\ a_{2n} = 0. \tag{3.13}$$

Furthermore,

$$\sum_{k=n+1}^{\infty} a_k \geq a_{n+2} + \ldots + a_{2n+1} \geq n\ a_{2n+1} \geq 0.$$

Thus $\lim_{n\to\infty} n\ a_{2n+1} = 0$, and so $\lim_{n\to\infty} 2n\ a_{2n+1} = 0$. But since $\lim_{n\to\infty} a_{2n+1} = 0$, we obtain

$$\lim_{n\to\infty} (2n+1)\ a_{2n+1} = 0. \tag{3.14}$$

From (3.13) and (3.14) above, the result follows. ∎

Remark: We remark that the assumption $a_1 \geq a_2 \geq \ldots \geq 0$ is important. Consider for example the lacunary series in which the $\left(n^2\right)^{\mathrm{th}}$ term is $\frac{1}{n^2}$, and all other terms are zero: then

$$\frac{1}{1^2} + 0 + 0 + \frac{1}{2^2} + 0 + 0 + 0 + 0 + \frac{1}{3^2} + 0 + \ldots < \infty,$$

but $\{n\, a_n\}$ has the constant subsequence $\{n^2\, \frac{1}{n^2}\}$.

Theorem 3.4.3 *If* H_G *is nuclear, then given any* $\epsilon > 0$ *, there exists a* $N \in \mathbb{N}$ *such that*

$$E_l = 4l\; \sigma_l(G) + 2 \sum_{k=l+1}^{\infty} \sigma_k(G) < \epsilon \;\; \textit{for all} \;\; l \geq N.$$

That is, $E_l \to 0$ *as* $l \to \infty$.

Proof Since H_G is nuclear, $\sum \sigma_k(G)$ converges, and so $\sum_{k=l+1}^{\infty} \sigma_k(G) \to 0$. From Lemma 3.4.2 above, it follows that $\lim_{l\to\infty} l\; \sigma_l(G) = 0$. Hence the result follows. ∎