Hankel norm approximation for infinite-dimensional systems

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Chapter 2

Classes of well-posed linear systems

2.1 Introduction

In this chapter we recall the notion of well-posed linear systems, which forms the general framework for studying infinite-dimensional systems in a state space context. In particular, we introduce two important classes of well-posed linear systems which are central in this thesis and list some of their properties, which will be used in the sequel. Figure 2.1 shows the hierarchy of the various classes of systems that we will encounter in this chapter.

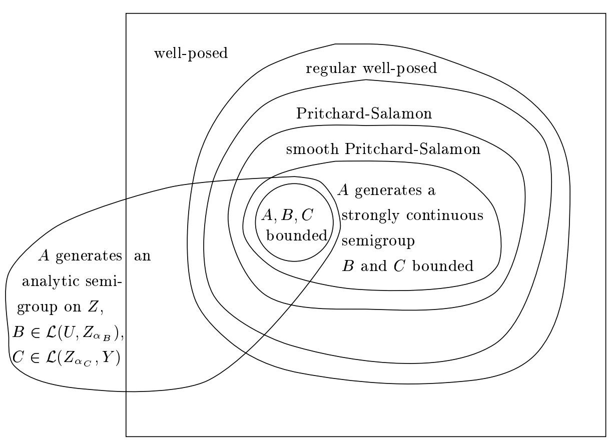


Figure 2.1: Various classes of systems.

Finally, in the last section of this chapter we will introduce several frequency domain spaces and prove a few elementary lemmas which will be used in what follows.

2.2 Well-posed linear systems

We assume that X is a Hilbert space and $A : D(A) \to X$ is the generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X. We define the Hilbert space X_1 as $D(A)$ with the norm $\|x\|_1 = \|(\beta I - A)x\|$, where $\beta \in \rho(A)$ is fixed[1]. The Hilbert space X_{-1} is the completion of X with respect to the norm $\|x\|_{-1} = \left\|(\beta I - A)^{-1}x\right\|$. We have

$$X_1 \subset X \subset X_{-1}, \tag{2.1}$$

densely and with continuous embeddings. The semigroup $\{T(t)\}_{t\geq 0}$ extends to a semigroup on X_{-1}, denoted by the same symbol. The generator of this extended semigroup is an extension of A, whose domain is X, so that $A : X\,(\subset X_{-1}) \to X_{-1}$, see Weiss [92].

We assume that U is a Hilbert space. The operator $B \in \mathcal{L}(U, X_{-1})$ is said to be an *admissible control operator* for $\{T(t)\}_{t\geq 0}$, if the input maps $\{\Phi_t\}_{t\geq 0}$ are bounded from $L_2([0,\infty), U)$ to X for all finite $t \geq 0$, where

$$\Phi_t u := \int_0^t T(t-\tau)Bu(\tau)d\tau, \quad \text{for all } u \in L_2([0,\infty), U). \tag{2.2}$$

The above integration is done in X_{-1}, but the result is in X. If x is the solution of $\dot{x}(t) = Ax(t) + Bu(t)$, $t \geq 0$, which is an equation in X_{-1}, with $x(0) = x_0 \in X$ and $u \in L_2([0,\infty), U)$, then $x(t) \in X$ for all $t \geq 0$. In this case, x is a continuous X-valued function of t. We have that for all $t \geq 0$,

$$x(t) = T(t)x_0 + \Phi_t u. \tag{2.3}$$

The operator B is called *bounded* if $B \in \mathcal{L}(U, X)$ (and *unbounded* otherwise).

We assume that Y is another Hilbert space. The operator $C \in \mathcal{L}(X_1, Y)$ is called an *admissible observation operator* for $\{T(t)\}_{t\geq 0}$, if for every $t > 0$ there exists a number $K_t \geq 0$ such that

$$\int_0^t \|CT(\tau)x_0\|^2 d\tau \leq K_t^2 \|x_0\|^2 \quad \text{for all } x_0 \in D(A). \tag{2.4}$$

The operator C is called *bounded* if it can be extended such that $C \in \mathcal{L}(X, Y)$. We regard $L_2^{\mathrm{loc}}([0,\infty), Y)$ as a Fréchet space with the seminorms being the

[1] This norm is equivalent to the graph norm.

L_2−norms on the intervals $[0,n]$, $n\in\mathbb{N}$. Then the admissibility of C means that there is a continuous operator $\Psi : X \to L_2^{\mathrm{loc}}([0,\infty),Y)$ such that

$$(\Psi x_0)(t) = CT(t)x_0 \text{ for all } x_0 \in D(A). \tag{2.5}$$

The operator Ψ is completely determined by (2.5), because $D(A)$ is dense in X.

If $B\in\mathcal{L}(U,X_{-1})$ is an admissible control operator for $\{T(t)\}_{t\geq 0}$ and $C\in\mathcal{L}(X_1,Y)$ is an admissible observation operator for $\{T(t)\}_{t\geq 0}$, then the *transfer functions* of the system Σ given by the triple (A,B,C) are solutions $G:\rho(A)\to\mathcal{L}(U,Y)$ of

$$G(s)-G(\beta) = -(s-\beta)C(sI-A)^{-1}(\beta I-A)^{-1}B \tag{2.6}$$

for s and β in $\rho(A)$. We remark that since B is an admissible control operator for $\{T(t)\}_{t\geq 0}$, $(\cdot I-A)^{-1}B$ is an $\mathcal{L}(U,X)$−valued analytic function and since C is an admissible observation operator $\{T(t)\}_{t\geq 0}$, $C(\cdot I-A)^{-1}$ is a $\mathcal{L}(X,Y)$−valued analytic function. Both $(\cdot I-A)^{-1}B$ and $C(\cdot I-A)^{-1}$ are analytic on some right half-plane $\mathbb{C}_\alpha^+ = \{s\in\mathbb{C}\mid \mathrm{Re}(s)>\alpha\}$. Thus any transfer function is a $\mathcal{L}(U,Y)$−valued function which is analytic in some $\mathbb{C}_\alpha^+$. Moreover any two transfer functions differ only by an additive constant, $D\in\mathcal{L}(U,Y)$. The point is that they need not necessarily be bounded on any $\mathbb{C}_\alpha^+$. We impose this as an extra assumption on the triple (A,B,C) and call this well-posedness. The system Σ given by the triple (A,B,C) is said to be *well-posed linear system* if B is an admissible control operator for $\{T(t)\}_{t\geq 0}$, C is an admissible observation operator for $\{T(t)\}_{t\geq 0}$ and its transfer functions are bounded on some half-plane $\mathbb{C}_\alpha^+$. For a well-posed linear system, the operator from the initial state and the input function to the final state and the output function is bounded. The input and output functions u and y are locally L_2 functions with values in U and in Y, respectively. The state trajectory x is an X-valued function. The boundedness property mentioned earlier means that for every $t>0$ there is a $c_t\geq 0$ such that

$$\|x(t)\|^2 + \int_0^t \|y(\tau)\|^2\,d\tau \leq c_t^2\left[\|x(0)\|^2 + \int_0^t \|u(\tau)\|^2\,d\tau\right] \tag{2.7}$$

(with c_t independent of $x(0)$ and of u). For the detailed definition, background and examples we refer to Salamon [78], [79], Staffans [82], [83], [81], Weiss [95], [94], Weiss and Rebarber [73] and Weiss[2] [96].

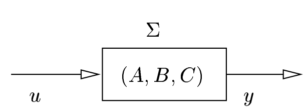


Figure 2.2: Well-posed linear systems.

$\{T(t)\}_{t\geq 0}$ is called the *semigroup* of Σ and A is called its *infinitesimal generator*. The well-posed linear system Σ is called *regular* if the limit

$$\lim_{\lambda\to+\infty} G(\lambda)v = Dv \tag{2.8}$$

exists for every $v \in U$, where λ is real (see [93], [95]). In this case, the operator $D \in \mathcal{L}(U,Y)$ is called the *feedthrough operator* of Σ. The operators A, B, C, D are called the *generating operators* of Σ, and we write $\Sigma = (A,B,C,D)$.

2.3 Duality theory

We will use the notion of "duality" in this book following the notation and results from Keulen [90]. For a short introduction about representations of dual spaces, duality pairings and pivot spaces we refer to Aubin [4].

The dual space of a Hilbert space Z is just the linear space of bounded linear functionals, that is, $\mathcal{L}(Z,\mathbb{C})$. We shall denote $\mathcal{L}(Z,\mathbb{C})$ by Z^d. The vector space Z^d is a Banach space with the dual norm $||\cdot||_d$ defined as follows: If $z' \in Z^d$, then

$$||z'||_d = \sup_{z\in Z, z\neq 0} \frac{|z'(z)|}{||z||}.$$

If Z is a Hilbert space, then there exists a surjective isometry J from Z to Z^d, defined by $(J(z_1))(z_2) = \langle z_2, z_1\rangle_Z$ for all $z_2 \in Z$. The dual space Z^d is a Hilbert space with the inner product defined as follows: if z_1' and $z_2' \in Z^d$, then

$$\langle z_1', z_2'\rangle_{Z^d} = \left\langle J^{-1}z_2', J^{-1}z_1'\right\rangle_Z = z_1'\left(J^{-1}(z_2')\right).$$

We call the isometry $J : Z \to Z^d$ the *duality map* [2] from Z to its dual Z^d. The inverse map $J^{-1} : Z^d \to Z$ is denoted by ι_Z and the inner product on Z^d satisfies

$$\langle z_1', z_2'\rangle_{Z^d} = \langle \iota_Z z_2', \iota_Z z_1'\rangle_Z \quad \text{for all } z_1' \text{ and } z_2' \text{ in } Z^d.$$

The duality pairing $\langle\cdot,\cdot\rangle_{\langle Z^d, Z\rangle}$ is defined by

$$\langle z', z\rangle_{\langle Z^d,Z\rangle} = z'(z) = \langle z, \iota_Z z'\rangle_Z, \quad \text{for all } z' \in Z^d \text{ and } z \in Z.$$

Let Z' be a Hilbert space and let j be a conjugate linear map which is an isometry from Z' onto Z^d, that is,

$$Z' \xrightarrow{j} Z^d \xrightarrow{\iota_Z} Z$$

[2] The duality map is *conjugate linear*, that is, it satisfies $J(\alpha_1 z_1 + \alpha_2 z_2) = \overline{\alpha_1}J(z_1) + \overline{\alpha_2}J(z_2)$ for all α_1 and α_2 in $\mathbb{C}$, and all z_1 and z_2 in Z.

and $\iota_Z j \in \mathcal{L}(Z', Z)$ is an isometry onto Z. The pair $\{Z', j\}$ is called a *representation of* Z^d. We define the *duality pairing* $\langle \cdot, \cdot \rangle_{\langle Z', Z \rangle}$ by

$$\langle z', z \rangle_{\langle Z', Z \rangle} = \langle jz', z \rangle_{\langle Z^d, Z \rangle} = (jz')(z) = \langle z, \iota_Z j z' \rangle_Z,$$

for all $z' \in Z'$ and $z \in Z$. In this case, we also call Z' the dual space of Z (this is standard abuse of terminology).

We say that Z is identified with its dual if the representation for Z^d is chosen to be $\{Z, \iota_Z^{-1}\}$. It follows that in this case, the duality pairing corresponds to the inner product on Z:

$$\langle z_1, z_2 \rangle_{\langle Z, Z \rangle} = \langle z_2, \iota_Z \iota_Z^{-1} z_1 \rangle_Z = \langle z_2, z_1 \rangle_Z$$

for all z_1 and z_2 in Z. If Z is identified with its dual, Z is called a *pivot space.*

Usually, it is assumed that whatever choice is made for the representation $\{Z', j\}$ of $\mathcal{L}(Z, \mathbb{C})$, the representation of the dual of this dual space (that is, $(Z')^d = \mathcal{L}(Z', \mathbb{C})$) is given by $\{Z, c_Z\}$, where c_Z is the isometry from Z to Z^d given by $c_Z = \iota_{Z'}^{-1} j^{-1} \iota_Z^{-1}$:

$$(Z')^d \xrightarrow{\iota_{Z'}} Z' \xrightarrow{j} Z^d \xrightarrow{\iota_Z} Z$$

(we could also say that $Z'' = Z$). This implies that

$$\begin{aligned}
\langle z'', z' \rangle_{\langle Z'', Z' \rangle} &= \langle z'', z' \rangle_{\langle Z, Z' \rangle} \\
&= \langle c_Z z'', z' \rangle_{\langle (Z')^d, Z' \rangle} \\
&= (c_Z z'')(z') \\
&= \left(\iota_{Z'}^{-1} j^{-1} \iota_Z^{-1} z'' \right)(z') \\
&= \left\langle z', j^{-1} \iota_Z^{-1} z'' \right\rangle_{Z'} \\
&= \left\langle \iota_Z^{-1} z'', jz' \right\rangle_{Z^d} \\
&= \langle \iota_Z j z', z'' \rangle_Z \\
&= \overline{(jz')(z'')} \\
&= \overline{\langle z', z'' \rangle_{\langle Z', Z \rangle}}.
\end{aligned}$$

We identify the bidual of Z with Z itself and the duality pairing $\langle \cdot, \cdot \rangle_{\langle Z, Z' \rangle}$ is given by

$$\langle z'', z' \rangle_{\langle Z, Z' \rangle} = \overline{\langle z', z'' \rangle_{\langle Z', Z \rangle}}.$$

If we have two Hilbert spaces W and Z with [3] $W \hookrightarrow Z$, then $Z^d \hookrightarrow W^d$. Here, we identify an element $z' \in Z^d = \mathcal{L}(Z, \mathbb{C})$ with $z'|_W \in W^d = \mathcal{L}(W, \mathbb{C})$. Similarly, if $w' \in W^d$ satisfies $|w'(z)| \leq$ (a constant) $||z||_Z$ for all $z \in W$,

[3] Here we use the notation $\hookrightarrow$ to mean $W \subset Z$, W is dense in Z, and the canonical injection $x \mapsto x : W \to Z$ is continuous.

then the unique continuous extension of w' to Z is also denoted by w'. Furthermore, for all $z' \in Z^d$ and $w \in W$ we have

$$\langle z', w \rangle_{\langle Z^d, Z \rangle} = z'(w) = \langle z', w \rangle_{\langle W^d, W \rangle}.$$

Now if Z is identified with its dual, we can define a space W' which is a representation of the dual of W such that $W \hookrightarrow Z = Z' \hookrightarrow W'$, as follows. Recall that ι_Z denotes the canonical map from Z^d to Z. Let W' be the completion of $Z' = Z$ with respect to the inner product

$$\langle w_1', w_2' \rangle_{W'} = \left\langle \iota_Z^{-1} w_2', \iota_Z^{-1} w_1' \right\rangle_{Z^d}$$

for w_1' and w_2' in $Z = Z'$. We have

$$\begin{array}{ccc} Z^d & \hookrightarrow & W^d \\ \iota_Z \downarrow & & \\ Z = Z' & \hookrightarrow & W'. \end{array}$$

For all $z' \in Z^d$,

$$\|\iota_Z z'\|_{W'} = \left\| \iota_Z^{-1} \iota_Z z' \right\|_{Z^d} = \|z'\|_{W^d}.$$

Thus it follows that ι_Z has a unique conjugate linear extension $\overline{\iota_Z}$ to W^d and $\overline{\iota_Z}$ is an isometry from W^d to W'. Furthermore, we can choose the pair $\{W', (\overline{\iota_Z})^{-1}\}$ as a representation of W^d so that the duality pairing $\langle \cdot, \cdot \rangle_{\langle W', W \rangle}$ is given by

$$\langle w', w \rangle_{\langle W', W \rangle} = \left(\overline{\iota_Z}^{-1} w' \right)(w)$$

and for all $(z', w) \in Z \times W$ we have

$$\langle z', w \rangle_{\langle W', W \rangle} = \left(\overline{\iota_Z}^{-1} z' \right)(w) = \left(\iota_Z^{-1} z' \right)(w) = \langle w, z' \rangle_Z.$$

Finally, $W \hookrightarrow Z = Z' \hookrightarrow W'$.

If in addition we have a third Hilbert space V with $Z \hookrightarrow V$, we have $V^d \hookrightarrow Z^d$ and we can define a representation V' of V^d such that $V' \hookrightarrow Z = Z' \hookrightarrow V$ as follows. Let $\iota_Z|_{V^d}$ denote the restriction of ι_Z to V^d and define V' as the image of V^d under this map. Furthermore, define the inner product on V' by

$$\langle v_1', v_2' \rangle_{V'} = \left\langle \left(\iota_Z|_{V^d} \right)^{-1} v_2', \left(\iota_Z|_{V^d} \right)^{-1} v_1' \right\rangle_{Z^d}.$$

Then the pair $\left\{ V', \left(\iota_Z|_{V^d} \right)^{-1} \right\}$ is a representation of V^d and the duality pairing is given by

$$\langle v', v \rangle_{\langle V', V \rangle} = \left(\left(\iota_Z|_{V^d} \right)^{-1} v' \right)(v).$$

Hence for all $(v', z) \in V' \times Z$ we have

$$\langle v', z\rangle_{\langle V',V\rangle} = \left((\iota_Z|_{V^d})^{-1} v' \right)(z) = \left(\iota_Z^{-1} v'\right)(z) = \langle z, v'\rangle_Z$$

and $V' \hookrightarrow Z' = Z \hookrightarrow V$. We depict this schematically in Figure 2.3. For all $(v', w) \in V' \times W$ we have

$$\langle v', w\rangle_{\langle W',W\rangle} = \langle w, v'\rangle_Z = \langle v', w\rangle_{\langle V',V\rangle}.$$

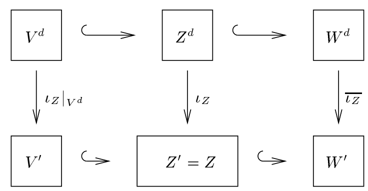


Figure 2.3: Interlacing of the dual spaces.

Suppose that X and Y are Hilbert spaces and that $\{X', j\}$ and $\{Y', k\}$ are representations of X^d and Y^d, with the pairings denoted by $\langle\cdot,\cdot\rangle_{\langle X',X\rangle}$ and $\langle\cdot,\cdot\rangle_{\langle Y',Y\rangle}$, respectively. Note that now $\iota_X j$ is an isometry from X' to X and that $\iota_Y k$ is an isometry from Y' to Y. The dual of a densely defined operator A from $D(A) \subset X$ to Y is an operator A' from $D(A') \subset Y'$ to X', defined as follows:

1. $D(A') = \left\{ y' \in Y' \mid \begin{array}{c} \exists\, x' \in X' \text{ such that for all } x \in D(A) \\ \langle y', Ax\rangle_{\langle Y',Y\rangle} = \langle x', x\rangle_{\langle X',X\rangle} \end{array} \right\}$, and

2. $\langle A'y', x\rangle_{\langle X',X\rangle} = \langle y', Ax\rangle_{\langle Y',Y\rangle}$ for all $y' \in D(A')$ and all $x \in D(A)$.

The adjoint of A (denoted by A^*) from $D(A^*) \subset Y$ to X is related to A' by

$$\begin{aligned} D(A^*) &= (\iota_Y k) D(A'), \\ A^* y &= (\iota_X j) A' (\iota_Y k)^{-1} y \text{ for all } y \in D(A^*), \end{aligned}$$

so that for all $y \in D(A^*)$ and $x \in D(A)$ we have

$$\begin{aligned} \langle x, A^* y\rangle_X &= \left\langle (\iota_X j)^{-1} (\iota_X j) A' (\iota_Y k)^{-1} y, x \right\rangle_{\langle X',X\rangle} \\ &= \left\langle (\iota_Y k)^{-1} y, Ax \right\rangle_{\langle Y',Y\rangle} \\ &= \langle Ax, y\rangle_Y . \end{aligned}$$

For a bounded linear operator $T \in \mathcal{L}(X,Y)$ we have $T' \in \mathcal{L}(Y', X')$, $T^* = (\iota_X j) T' (\iota_Y k)^{-1} \in \mathcal{L}(Y, X)$. Owing to the identification of the bidual of a Hilbert space with itself, we have that an operator $T \in \mathcal{L}(X,Y)$ satisfies $(T')' = T$.

The dual of a linear densely defined closed operator $A : D(A)\ (\subset X) \to Y$, denoted by $A' : D(A')\ (\subset Y') \to X'$ is closed and densely defined. Hence the dual of A', denoted by $(A')' : D((A')')\ (\subset X'' = X) \to Y'' = Y$ is well-defined and moreover, $D((A')') = D(A)$ and $(A')'x = Ax$ for all $x \in D(A)$.

Since we identify the bidual of a Hilbert space with itself, it follows that the dual of an operator $T \in \mathcal{L}(X, X')$ satisfies $T' \in \mathcal{L}(X, X')$. We call an operator $T \in \mathcal{L}(X, X')$ self-dual if $T = T'$. In this case the operator $S := (\iota_X j)T \in \mathcal{L}(X)$ is self-adjoint and satisfies $\langle y, Sx\rangle_X = \langle Tx, y\rangle_{\langle X', X\rangle}$.

2.4 The Pritchard-Salamon class

In this section we introduce a special class of well-posed linear systems for which there exists a rich literature (see Salamon [78], [79], Pritchard and Salamon [70], Curtain et al. [21]). Most systems containing delays and some systems described by partial differential equations fit into this framework. This class generalizes the class of state linear systems with bounded input and output operators studied in [34].

Let V and W be separable Hilbert spaces with continuous, dense injections and which satisfy

$$W \hookrightarrow Z \hookrightarrow V.$$

Suppose that A is the infinitesimal generator of strongly continuous semigroups $\{T^W(t)\}_{t\geq 0}$, $\{T^Z(t)\}_{t\geq 0}$ and $\{T^V(t)\}_{t\geq 0}$ on W, Z and V, respectively, such that

$$T^V(t)|_Z = T^Z(t) \text{ and } T^Z(t)|_W = T^W(t).$$

Since these semigroups are consistent, we shall simply use the notation $\{T(t)\}_{t\geq 0}$. Assume further that U and Y are separable Hilbert spaces (the input and output spaces), respectively.

1. $B \in \mathcal{L}(U, V)$ is a *Pritchard-Salamon admissible control operator* for $\{T(t)\}_{t\geq 0}$ if there exist a $t > 0$ and a constant $\beta > 0$ such that

$$\left\| \int_0^t T(t-\tau)Bu(\tau)d\tau \right\|_W \leq \beta \, \|u\|_{L_2([0,t),U)} \tag{2.9}$$

   for all $u \in L_2([0,t), U)$.

2. $C \in \mathcal{L}(W, Y)$ is a *Pritchard-Salamon admissible observation operator* for $\{T(t)\}_{t\geq 0}$ if there exist a $t > 0$ and a constant $\gamma > 0$ such that

$$\|CT(\cdot)z\|_{L_2([0,t),Y)} \leq \gamma \, \|z\|_V \tag{2.10}$$

   for all $z \in W$.

Remarks: In the above definition, Z is not essential and in the applications it is usual to take either $Z = V$ or $Z = W$. Furthermore, we remark that if (2.9) holds for some t, then it can be shown that it holds for all $t > 0$, where β will depend on t. Similarly, if (2.10) holds for some t, then it can be shown that it holds for all $t > 0$, where γ will depend on t in general. See for example, Remark 2.10, page 11, Curtain et al. [21].

Under the above assumptions, the state linear system $\Sigma(A,B,C,D)$ is called a *Pritchard-Salamon system* for any $D \in \mathcal{L}(U,Y)$. If, in addition, $D\left(A^V\right) \hookrightarrow W$, then $\Sigma(A,B,C,D)$ is called a *smooth Pritchard-Salamon system*.

When confusion may arise we use superscripts, for example A^X, $T^X(t)$, to denote the operators on $X = V$, Z or W. We denote the growth bound[4] on X by ω_0^X (the growth bounds ω_0^W and ω_0^V are different in general). We now enumerate a few properties of Pritchard-Salamon systems which have been proved in Curtain et al. [21] and in Curtain and Zwart [33] and will be used in the proofs.

1. For a smooth Pritchard-Salamon system, $\sigma_p\left(A^W\right) = \sigma_p\left(A^V\right)$ and $\sigma\left(A^W\right) = \sigma\left(A^V\right)$.

2. Let $L \in \mathcal{L}(Y,V)$ be a Pritchard-Salamon admissible input operator for $\{T(t)\}_{t\geq 0}$ and assume that $\Sigma(A,B,C,D)$ is a Pritchard-Salamon system. Then $\Sigma(A+LC,B,C,D)$ is also a Pritchard-Salamon system, where $A+LC$ is the infinitesimal generator of $\{T_{LC}(t)\}_{t\geq 0}$ which is the unique solution of

$$T_{LC}(t)z = T(t)z + \int_0^t T_{LC}(t-\tau)LCT(\tau)zd\tau.$$

   Furthermore, $\{T_{LC}(t)\}_{t\geq 0}$ is also the unique solution of

$$T_{LC}(t)z = T(t)z + \int_0^t T(t-\tau)LCT_{LC}(\tau)zd\tau.$$

   We note that $L \in \mathcal{L}(Y,W)$ is always a Pritchard-Salamon admissible control operator.

3. A smooth Pritchard-Salamon system $\Sigma(A,B,C,D)$ has a well-defined transfer operator $G(s)$ given by $\hat{y}(s) = G(s)\hat{u}(s)$, for all $s \in \mathbb{C}_\alpha^+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$, where $\alpha > \max\left\{\omega_0^V, \omega_0^W\right\}$, and

$$G(s) = D + C\left(sI - A^V\right)^{-1} B.$$

[4]The *growth bound* of a semigroup $\{T(t)\}_{t\geq 0}$, denoted by ω_0, is defined to be $\inf_{t>0} \frac{\log\,\|T(t)\|}{t}$

Furthermore, $C\left(\cdot I - A^V\right)^{-1} B \in H_\infty\left(\mathbb{C}_\alpha^+, \mathcal{L}(U,Y)\right)$, where

$$H_\infty\left(\mathbb{C}_\alpha^+, \mathcal{L}(U,Y)\right) = \left\{ G : \mathbb{C}_\alpha^+ \to \mathcal{L}(U,Y) \mid \begin{array}{l} G \text{ is analytic, and} \\ \sup_{s\in\mathbb{C}_\alpha^+} \|G(s)\|_{\mathcal{L}(U,Y)} < \infty \end{array} \right\}.$$

4. Let us assume that the Hilbert spaces Z, U, Y are identified with their duals. Then $B' \in \mathcal{L}(V', U)$, $C' \in \mathcal{L}(Y, W')$, $D' \in \mathcal{L}(Y, U)$ and $V' \hookrightarrow Z' \hookrightarrow W'$.

   If $\Sigma(A, B, C, D)$ is a smooth Pritchard-Salamon system, then $\Sigma(A', C', B', D')$ is also a smooth Pritchard-Salamon system, and $D\left((A')^{W'}\right) \hookrightarrow V'$.

If A generates an exponentially stable strongly continuous semigroup on V and W, then we say that the Pritchard-Salamon system $\Sigma(A,B,C,D)$ is *exponentially stable*. Its *controllability map* $\mathcal{B} \in \mathcal{L}(L_2([0,\infty),U),W)$ is defined by

$$\mathcal{B}u = \int_0^\infty T(s)Bu(s)ds$$

and its *observability map* $\mathcal{C} \in \mathcal{L}(V, L_2([0,\infty),Y))$ is defined by

$$\mathcal{C}z = CT(\cdot)z$$

for $z \in W$. The *controllability Gramian* $L_B \in \mathcal{L}(W', W)$ and *observability Gramian* $L_C \in \mathcal{L}(V, V')$ are defined by $L_B = \mathcal{B}\mathcal{B}'$, $L_C = \mathcal{C}'\mathcal{C}$. Its *Hankel operator* $\Gamma \in \mathcal{L}\left(L_2([0,\infty),U), L_2([0,\infty),Y)\right)$ is defined by $\Gamma = \mathcal{C}\mathcal{B}$. For an exponentially stable, smooth Pritchard-Salamon system, $C(sI-A)^{-1}B \in H_\infty^c(\mathcal{L}(U,Y))$, where $H_\infty^c(\mathcal{L}(U,Y))$ denotes the space of maps G defined in the closed right half-plane Re$(s)\geq 0$, taking values in $\mathcal{L}(U,Y)$, such that G is analytic in the open right half-plane Re$(s)> 0$, and G is bounded and continuous in the closed right half-plane Re$(s)\geq 0$. Moreover,

$$\lim_{\substack{|s|\to\infty \\ \mathrm{Re}(s)\geq 0}} \|G(s)\| = 0. \tag{2.11}$$

(That $e^{\epsilon t}CT(\cdot)B \in L_2([0,\infty), \mathbb{C}^{p\times m})$ for some $\epsilon > 0$ is shown in Curtain et al. [21], where $CT(\cdot)B$ has to be interpreted in the sense of Remark 2.10.(iv) [21]. Using the Cauchy-Schwarz inequality, it follows that $CT(\cdot)B \in L_1([0,\infty), \mathbb{C}^{p\times m})$. Finally, using Property A.6.2.g, page 636, Curtain and Zwart [34], we obtain the desired result.)

We now quote the following lemma from Curtain and Zwart [33] on Lyapunov equations:

Lemma 2.4.1 *Suppose that* $\Sigma(A,B,C,D)$ *is an exponentially stable, smooth Pritchard-Salamon system. Then*

1. *L_B and L_C are the unique solutions in $\mathcal{L}(W',W)$ and $\mathcal{L}(V,V')$ respectively of the following Lyapunov equations*

$$\langle z_1, (AL_B + L_B A' + BB')z_2 \rangle_{\langle W',W \rangle} = 0 \tag{2.12}$$

for all $z_1, z_2 \in D\left((A')^{W'}\right)$;

$$\langle z_1, (A'L_C + L_C A + C'C)z_2 \rangle_{\langle V,V' \rangle} = 0 \tag{2.13}$$

for all $z_1, z_2 \in D\left(A^V\right)$. *Moreover,*

$$L_B \in \mathcal{L}\left(D\left((A')^{W'}\right), D\left(A^V\right)\right)$$

and

$$L_C \in \mathcal{L}\left(D\left(A^V\right), D\left((A')^{W'}\right)\right).$$

2. $L_B L_C \in \mathcal{L}(V,W) \cap \mathcal{L}(V) \cap \mathcal{L}(W)$, $L_C L_B \in \mathcal{L}(W',V') \cap \mathcal{L}(V') \cap \mathcal{L}(W')$, *and* $\rho^V(L_B L_C) = \rho^W(L_B L_C)$ *and for* $-\frac{1}{\alpha} \in \rho(L_B L_C)$, *there holds*

$$\begin{aligned} (I + \alpha L_B L_C)^{-1} &\in \mathcal{L}(V) \cap \mathcal{L}(W), \\ (I + \alpha L_C L_B)^{-1} &\in \mathcal{L}(W') \cap \mathcal{L}(V') \end{aligned}$$

and

$$\begin{aligned} (I + \alpha L_B L_C)^{-1} &\in \mathcal{L}\left(D\left(A^V\right)\right), \\ (I + \alpha L_C L_B)^{-1} &\in \mathcal{L}\left(D\left((A')^{W'}\right)\right). \end{aligned}$$

From Curtain and Zwart [33], we have

Lemma 2.4.2 *Suppose that $\Sigma(A,B,C)$ is an exponentially stable, smooth Pritchard-Salamon system. Then $\sigma(\Gamma^*\Gamma)\backslash\{0\} = \sigma(L_B L_C)\backslash\{0\}$ and $r(L_C) = r(\Gamma^*\Gamma) = \|\Gamma\|^2$, where $r(\cdot)$ denotes the spectral radius.*

If $U = \mathbb{C}^m$ and $Y = \mathbb{C}^p$, then the Hankel operator of an exponentially stable Pritchard-Salamon system is compact (Lemma 3.5 in Curtain et al. [21] and Glover et al. [18]).

Lemma 2.4.3 *Suppose that $\Sigma(A,B,C)$ is an exponentially stable, smooth Pritchard-Salamon system with input space $\mathbb{C}^m$ and output space $\mathbb{C}^p$ and let $\sigma_{l+1} < \sigma < \sigma_l$. The operator $J := I - \frac{1}{\sigma^2}\Gamma^*\Gamma \in \mathcal{L}(L_2([0,\infty),\mathbb{C}^m))$ has a spectrum $\sigma(J)$ contained in $(-\infty,-\delta) \cup (\delta,\infty)$ for some $\delta > 0$, and $\sigma(J) \cap (-\infty,-\delta)$ consists of exactly l negative eigenvalues.*

Proof $\Gamma^*\Gamma$ is compact and has a pure point spectrum with 0 as the accumulation point. Considering the resolvent $\left[\left(I - \frac{1}{\sigma^2}\Gamma^*\Gamma\right) - \lambda I\right]^{-1}$, it is easy to see that J has a spectrum which is a shifted version of the spectrum of $\frac{1}{\sigma^2}\Gamma^*\Gamma$. Finally, since $\Gamma^*\Gamma$ has a pure point spectrum $\{\sigma_1^2, \sigma_2^2, \dots\}$ with 0 as the accumulation point, and since $\sigma_{l+1} < \sigma < \sigma_l$, J has exactly l negative eigenvalues. ■

Finally we give a simple example[5] of a delay system which can be modelled as a smooth Pritchard-Salamon system.

Example 2.4.4 Let $G(s) = \frac{e^{-s\tau}}{s+1}$, where $\tau > 0$. We will first give a realization of G. Let $Z = \mathbb{C} \times L_2(-\tau, 0)$ (with the obvious inner product). Consider the Sobolev space

$$W^{1,2}(-\tau,0) = \left\{ f \in L_2(-\tau,0) \mid \begin{array}{l} \text{the derivative of } f \text{ (in the sense of} \\ \text{distributions) is a regular distribution} \\ T_g, \text{ with } g \in L_2(-\tau,0) \end{array} \right\},$$

equipped with the inner product

$$\langle f, g\rangle_{W^{1,2}(-\tau,0)} = \int_{-\tau}^{0} \left[f(x)\overline{g(x)} + f'(x)\overline{g'(x)} \right] dx.$$

(With this inner product, $W^{1,2}(-\tau,0)$ is a Hilbert space: see for example, Proposition 5, page 55, Yosida [98]. Furthermore, from Corollary 7.3 [85], it follows that $W^{1,2}(-\tau,0) \subset \mathcal{C}(-\tau,0)$.) Define

$$A\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -z_2(0) \\ z_2' \end{bmatrix},$$

with

$$D(A) = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in Z \mid z_2 \in W^{1,2}(-\tau,0),\ z_1 = z_2(0) \right\}.$$

Then it can be shown that A is the infinitesimal generator of an exponentially stable strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on Z (see for instance, Example 2.8, van Keulen [90]). The space $W := D(A)$ with the inner product

$$\left\langle \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} \right\rangle_W = \langle z_2, \tilde{z}_2\rangle_{W^{1,2}(-\tau,0)}$$

is a Hilbert space and $\{T(t)\}_{t\geq 0}$ restricts to a strongly continuous semigroup on W. Let $B \in \mathcal{L}(\mathbb{C}, Z)$ be defined as follows:

$$Bu = \begin{bmatrix} u \\ 0 \end{bmatrix},$$

[5] $\left\{ \begin{array}{l} \dot{z}(t) = -z(t) + u(t), \\ y(t) = z(t-\tau). \end{array} \right.$

Finally, if $C : D(A) \to \mathbb{C}$ is given by

$$C \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_2(-\tau),$$

then $C \in \mathcal{L}(W, \mathbb{C})$. We now choose $V = Z$. In Pritchard and Salamon [70] it is explained that B and C are Pritchard-Salamon admissible control and observation operators, respectively. Finally we note that $D\left(A^V\right) \hookrightarrow W$ is trivially satisfied since $W = D(A)$, and so in fact we have a smooth Pritchard-Salamon system, with a transfer function $\frac{e^{-\tau s}}{s+1}$. ◊

2.5 The analytic class

In this section we introduce the class of exponentially stable analytic infinite-dimensional systems and establish some basic system theoretic properties.

First we recall the basic properties of exponentially stable analytic semigroups using the notation from Staffans [84] (see also Pazy [67]).

1. Let $-A$ be the infinitesimal generator of an exponentially stable analytic semigroup $\{T(t)\}_{t\geq 0}$ in Z.
2. For each $\theta \in \mathbb{R}$, we let $Z_\theta = A^{-\theta} Z$ be $D\left(A^\theta\right)$ with the norm $\|z\|_{Z_\theta} = \left\|A^\theta z\right\|_Z$ and inner product $\langle z_1, z_2\rangle_{Z_\theta} = \left\langle A^\theta z_1, A^\theta z_2\right\rangle_Z$.
3. The restrictions of $-A$ to Z_θ for $\theta > 0$ and the extensions of $-A$ to Z_θ for $\theta < 0$ (which we still denote by $-A$) generate analytic semigroups in Z_θ for all $\theta \in \mathbb{R}$. The generator of the semigroup $\{T(t)\}_{t\geq 0}$ on Z_θ is then $-A \in \mathcal{L}(Z_{\theta+1}, Z_\theta)$.

   These semigroups are all similar to each other, and they commute with A^β for all $\beta \in \mathbb{R}$. We therefore denote all of them by the same symbol $\{T(t)\}_{t\geq 0}$.
4. For each $t > 0$ and $\theta \in \mathbb{R}$, $T(t)$ maps Z_θ into $\cap_{\beta\in\mathbb{R}} Z_\beta$.
5. For each $\theta \geq 0$, there exist $K_1 > 0$ and $\epsilon > 0$ such that

   $$\left\|A^\theta T(t)\right\| \leq K_1 \frac{e^{-\epsilon t}}{t^\theta}, \quad t > 0$$

   where the norm represents the operator norm in any one of the spaces Z_θ.

   If $\theta < 0$, then there exist $K_2 > 0$ and $\epsilon > 0$ such that

   $$\left\|A^\theta T(t)\right\|_{\mathcal{L}(Z)} \leq K_2 e^{-\epsilon t}, \quad t > 0.$$

   (This follows from Lemma 6.3, page 71, Pazy [67].)

6. The map $t \mapsto T(t) : (0,\infty) \to \mathcal{L}(Z)$ is continuous in the norm topology. (See for example, Proposition 2.1.1.*(iv)*, page 35, Lunardi, [56].)

   This implies that for any $\theta \in \mathbb{R}$, the map $t \mapsto T(t) : (0,\infty) \to \mathcal{L}(Z_\theta)$ is continuous in the norm topology.

7. The same conclusion as above can be repeated with A replaced by A' to give another chain of Hilbert spaces $Z'_\theta = (A')^{-\theta} Z$ with similar properties.

   We identify Z'_θ with the dual of $Z_{-\theta}$ by using Z as the pivot space. Note that $Z_0 = Z'_0 = Z$. For $z_1 \in Z'_{-\theta}$ and $z_2 \in Z_\theta$, we have

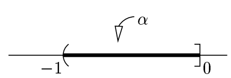
$$\langle z_1, z_2 \rangle_{\langle Z'_{-\theta}, Z_\theta \rangle} = \left\langle A^\theta z_2, (A')^{-\theta} z_1 \right\rangle_Z ,$$

   where $\langle \cdot, \cdot \rangle_{\langle Z'_{-\theta}, Z_\theta \rangle}$ denotes the duality pairing.

Next we analyze the system theoretic properties of the class of systems described by the triple $(-A, B, C)$ on the Hilbert space Z, under the assumptions

A1. $-A$ is the infinitesimal generator of an exponentially stable analytic semigroup $\{T(t)\}_{t \geq 0}$ on the Hilbert space Z.

A2. $B \in \mathcal{L}(U, Z_\alpha)$, where U is a Hilbert space and α is a fixed number satisfying $-1 < \alpha \leq 0$.



A3. $C \in \mathcal{L}(Z, Y)$, where Y is a Hilbert space.

We note that the assumption A2 implies that $A^\alpha B \in \mathcal{L}(U, Z)$ and $B \in \mathcal{L}(U, Z_\beta)$ for every β satisfying $-1 < \beta \leq \alpha$: $Z \hookrightarrow Z_\alpha \hookrightarrow Z_\beta \hookrightarrow Z_{-1}$.

Remark: We remark that what follows is similar to what was done in Curtain and Ichikawa [20], but there are differences in assumptions (see Table 2.1).

First we derive several results about B and its controllability Gramian. In Lemma 2.5.1.1 below we prove that if β is a real number satisfying $\beta < \frac{1}{2} + \alpha$, then the *controllability map* $\mathcal{B} : L_2([0,\infty), U) \to Z_\beta$ given by

$$\mathcal{B}u = \int_0^\infty T(t) B u(t) dt \tag{2.14}$$

is a well-defined bounded linear map, that is, B is an admissible input operator (see Weiss [92]) for the semigroup $\{T(t)\}_{t \geq 0}$ on the Hilbert space Z_β, where

	Curtain and Ichikawa [20]	Present approach
1.	They considered A to be a uniformly strongly elliptic operator of order two with smooth coefficients defined on a bounded open domain Ω of $\mathbb{R}^d$. Thus $-A$ is the infinitesimal generator of an exponentially stable, analytic, *compact* semigroup on Z.	Here we simply assume that $-A$ is the infinitesimal generator of an exponentially stable, analytic semigroup on Z.
2.	$B \in \mathcal{L}(U, Z_\alpha)$, where $-1 < \alpha < -\frac{3}{4}$.	$B \in \mathcal{L}(U, Z_\alpha)$, where $-1 < \alpha \leq 0$.
3.	$U = L_2(\partial\Omega)$. Thus U is *infinite*-dimensional, unless Ω is, for example, $(0,1)$, in which case U can be identified with $\mathbb{C}^2$.	$U = \mathbb{C}^m$

Table 2.1: Differences in assumptions.

$\beta < \frac{1}{2} + \alpha$. Moreover, in part 2. we show that if β and γ are real numbers satisfying $\beta - \gamma < 1 + 2\alpha$, $\beta < 1 + \alpha$, $\gamma > -(1+\alpha)$, then the *controllability Gramian* $L_B : Z'_\gamma \to Z_\beta$ given by

$$L_B z = \int_0^\infty T(t)BB'T(t)'z dt \tag{2.15}$$

is a well-defined bounded linear map from Z'_γ to Z_β.

Lemma 2.5.1 *$\mathcal{B}$ and L_B given by (2.14) and (2.15) are well-defined and have the following properties:*

1. *$\mathcal{B} \in \mathcal{L}\left(L_2([0,\infty), U), Z_\beta\right)$, and $\mathcal{B}' \in \mathcal{L}\left(Z'_{-\beta}, L_2([0,\infty), U)\right)$, for all β satisfying $\beta < \frac{1}{2} + \alpha$.*
2. *$L_B \in \mathcal{L}\left(Z'_\gamma, Z_\beta\right)$ for all γ and β satisfying $\beta - \gamma < 1 + 2\alpha$, $\beta < 1 + \alpha$, $\gamma > -(1+\alpha)$.*
3. *$L_B = \mathcal{B}\mathcal{B}' \in \mathcal{L}\left(Z'_\gamma, Z_\beta\right)$ for $\beta < \frac{1}{2} + \alpha$ and $\gamma > -\frac{1}{2} - \alpha$ and it is self-dual in $\mathcal{L}\left(Z'_{-\beta}, Z_\beta\right)$ for $\beta < \frac{1}{2} + \alpha$.*

4. $L_B \in \mathcal{L}\left(Z'_{-\beta}, Z_{\beta+1}\right) \cap \mathcal{L}\left(Z'_{-\beta-1}, Z_\beta\right)$ *is a solution of the Lyapunov equation*

$$AL_B z + L_B A' z = BB'z \quad \text{in } Z_\beta \tag{2.16}$$

for any $z \in Z'_{-\beta}$, *where* $-1 < \beta < \alpha$.

Proof

1. Let $u \in L_2([0,\infty), U)$. Since for $t > 0$, $T(t)Z_\alpha \subset \cap_{\beta\in\mathbb{R}} Z_\beta$, it follows that $T(t)Bu(t) \in Z_\beta$ for all $\beta \in \mathbb{R}$. Next we show that $A^\beta T(\cdot)Bu(\cdot) \in L_1([0,\infty), Z)$ if $\beta < \frac{1}{2} + \alpha$:

$$\begin{aligned} & \int_0^\tau \left\|A^\beta T(t)Bu(t)\right\|_Z dt \\ = & \int_0^\tau \left\|A^{\beta-\alpha}T(t)\right\|_{\mathcal{L}(Z)} \|A^\alpha B\|_{\mathcal{L}(U,Z)} \|u(t)\|_U dt \\ \le & \left[\int_0^\tau \left\|A^{\beta-\alpha}T(t)\right\|^2_{\mathcal{L}(Z)} dt\right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)}. \end{aligned}$$

<ins>`Case 1:`</ins> $\beta - \alpha \ge 0$. In this case,

$$\begin{aligned} & \int_0^\tau \left\|A^\beta T(t)Bu(t)\right\|_Z dt \\ \le & \left[\int_0^\tau \left\|A^{\beta-\alpha}T(t)\right\|^2_{\mathcal{L}(Z)} dt\right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)} \\ \le & \left[\int_0^\tau \frac{K_1^2 e^{-2\epsilon t}}{t^{2(\beta-\alpha)}} dt\right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)}. \end{aligned}$$

For β satisfying $2(\beta-\alpha) < 1$, we have $\int_0^\infty \frac{e^{-2\epsilon t}}{t^{2(\beta-\alpha)}} dt < \infty$, and so

$$\int_0^\tau \left\|A^\beta T(t)Bu(t)\right\|_Z dt \le \left[\int_0^\infty \frac{K_1^2 e^{-2\epsilon t}}{t^{2(\beta-\alpha)}} dt\right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)}. \tag{2.17}$$

<ins>`Case 2:`</ins> $\beta - \alpha < 0$. In this case,

$$\begin{aligned} & \int_0^\tau \left\|A^\beta T(t)Bu(t)\right\|_Z dt \\ \le & \left[\int_0^\tau \left\|A^{\beta-\alpha}T(t)\right\|^2_{\mathcal{L}(Z)} dt\right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)} \\ \le & \left[\int_0^\tau K_2^2 e^{-2\epsilon t} dt\right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)}. \end{aligned}$$

But $\int_0^\infty e^{-2\epsilon t}dt < \infty$, and so

$$\int_0^\tau \left\|A^\beta T(t)Bu(t)\right\|_Z dt \leq \left[\int_0^\infty K_2^2 e^{-2\epsilon t}dt\right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)}\|A^\alpha B\|_{\mathcal{L}(U,Z)}. \tag{2.18}$$

From the above two cases, it follows that $A^\beta T(\cdot)Bu(\cdot) \in L_1([0,\infty),Z)$. Consequently, (see for example Theorem A.5.23, page 628, Curtain and Zwart [34]: with f, A, $D(A)$, Z_1, Z_2 and Af in the theorem replaced by $T(\cdot)Bu(\cdot)$, A^β, Z_β, Z_β, Z and $A^\beta T(\cdot)Bu(\cdot)$, respectively)

$$A^\beta \int_0^\infty T(t)Bu(t)dt = \int_0^\infty A^\beta T(t)Bu(t)dt,$$

for every $\beta < \frac{1}{2} + \alpha$. Moreover, from (2.17) and (2.18) above, there exists a $M > 0$ such that

$$\left\|\int_0^\infty T(t)Bu(t)dt\right\|_{Z_\beta} = \left\|A^\beta \int_0^\infty T(t)Bu(t)dt\right\|_Z \leq M\|u\|_{L_2([0,\infty),U)}$$

and so $\mathcal{B} \in \mathcal{L}\left(L_2([0,\infty),U), Z_\beta\right)$. Consequently, we also obtain that $\mathcal{B}' \in \mathcal{L}\left(Z'_{-\beta}, L_2([0,\infty),U)\right)$ for $\beta < \frac{1}{2} + \alpha$.

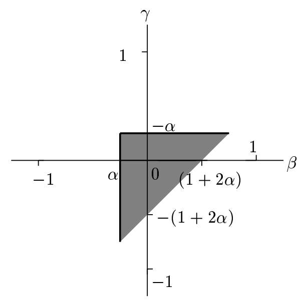
2. From property (4), for every $t > 0$ we have $T(t)BB'T(t)'Z'_\gamma \subset Z_\beta$, for any real β and γ. We first investigate the values of γ and β for which $A^\beta T(t)BB'T(t)'(A')^{-\gamma}z \in L_1([0,\infty),Z)$ for $z \in Z' = Z$. For $z \in Z$, we have

$$\begin{aligned}
&\int_0^\tau \left\|A^\beta T(t)BB'T(t)'(A')^{-\gamma}z\right\|_Z dt \\
= &\int_0^\tau \left\|A^{\beta-\alpha}T(t)A^\alpha BB'(A')^\alpha T(t)'(A')^{-\alpha-\gamma}z\right\|_Z dt \\
= &\int_0^\tau \left\|A^{\beta-\alpha}T(t)\right\|_{\mathcal{L}(Z)} \|A^\alpha B\|_{\mathcal{L}(U,Z)}\|B'(A')^\alpha\|_{\mathcal{L}(Z,U)} \\
&\left\|T(t)'(A')^{-\alpha-\gamma}\right\|_{\mathcal{L}(Z)} \|z\|_Z dt.
\end{aligned}$$

<u>`Case 1:`</u> Suppose that $\beta - \alpha \geq 0$, $-\alpha - \gamma \geq 0$.

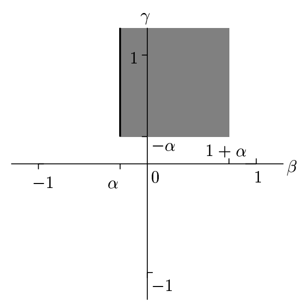
Then we have for $z \in Z$,

$$\begin{aligned}
&\int_0^\tau \left\|A^\beta T(t)BB'T(t)'(A')^{-\gamma}z\right\|_Z dt \\
\leq &\left[\int_0^\tau \frac{K_1^2 e^{-2\epsilon t}}{t^{\beta-2\alpha-\gamma}}dt\right] \|A^\alpha B\|_{\mathcal{L}(U,Z)}\|B'(A')^\alpha\|_{\mathcal{L}(Z,U)}\|z\|_Z \\
\leq &M_1\|z\|_Z,
\end{aligned}$$



where M_1 is a constant independent of τ. Thus $A^{\beta}T(t)BB'T(t)'(A')^{-\gamma}z \in L_1([0,\infty),Z)$ for $z \in Z$ and $\beta - 2\alpha - \gamma < 1$.

Case 2: Suppose that $\beta - \alpha \geq 0$, $-\alpha - \gamma < 0$.



Then we have for $z \in Z$,

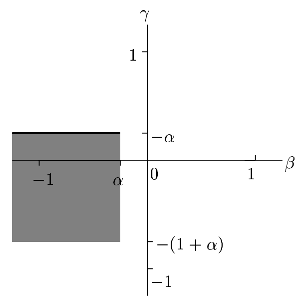
$$\begin{aligned} & \int_0^{\tau} \left\| A^{\beta}T(t)BB'T(t)'(A')^{-\gamma}z \right\|_Z dt \\ \leq & \left[\int_0^{\tau} \frac{K_1K_2e^{-2\epsilon t}}{t^{\beta-\alpha}} dt\right] \|A^{\alpha}B\|_{\mathcal{L}(U,Z)} \|B'(A')^{\alpha}\|_{\mathcal{L}(Z,U)} \|z\|_Z \\ \leq & M_2\|z\|_Z, \end{aligned}$$

where M_2 is a constant independent of τ. Thus $A^{\beta}T(t)BB'T(t)'(A')^{-\gamma}z \in L_1([0,\infty),Z)$ for $z \in Z$ and $\beta - \alpha < 1$.

Case 3: Suppose that $\beta - \alpha < 0$, $-\alpha - \gamma \geq 0$.

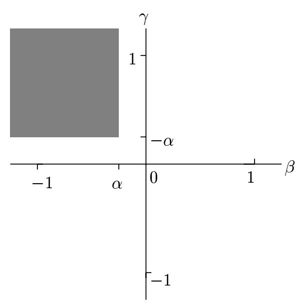
Then we have for $z \in Z$,

$$\begin{aligned} & \int_0^{\tau} \left\| A^{\beta}T(t)BB'T(t)'(A')^{-\gamma}z \right\|_Z dt \\ \leq & \left[\int_0^{\tau} \frac{K_1K_2e^{-2\epsilon t}}{t^{-\alpha-\gamma}} dt\right] \|A^{\alpha}B\|_{\mathcal{L}(U,Z)} \|B'(A')^{\alpha}\|_{\mathcal{L}(Z,U)} \|z\|_Z \\ \leq & M_3\|z\|_Z, \end{aligned}$$



where M_3 is a constant independent of τ. Thus $A^\beta T(t)BB'T(t)'(A')^{-\gamma}z \in L_1([0,\infty),Z)$ for $z \in Z$ and $-\alpha-\gamma < 1$.

Case 4: Suppose that $\beta - \alpha < 0$, $-\alpha-\gamma \geq 0$.



Then we have for $z \in Z$,

$$\begin{aligned} & \int_0^\tau \left\| A^\beta T(t)BB'T(t)'(A')^{-\gamma}z \right\|_Z dt \\ \leq & \left[\int_0^\tau K_2^2 e^{-2\epsilon t} dt \right] \|A^\alpha B\|_{\mathcal{L}(U,Z)} \|B'(A')^\alpha\|_{\mathcal{L}(Z,U)} \|z\|_Z \\ \leq & M_4 \|z\|_Z, \end{aligned}$$

where M_4 is a constant independent of τ. Thus $A^\beta T(t)BB'T(t)'(A')^{-\gamma}z \in L_1([0,\infty),Z)$ for $z \in Z$.

If $z \in Z'_\gamma$, then $(A')^\gamma z \in Z$ and applying the above results,

$$\begin{aligned} \int_0^\infty \left\| A^\beta T(t)BB'T(t)'z \right\|_Z dt & \leq \int_0^\infty \left\| A^\beta T(t)BB'T(t)'(A')^{-\gamma}(A')^\gamma z \right\|_Z dt \\ & \leq M \|(A')^\gamma z\|_Z = M\|z\|_{Z'_\gamma}, \end{aligned}$$

for some constant $M > 0$. So if $z \in Z'_\gamma$, then

$$\|L_B z\|_{Z_\beta} = \left\| A^\beta \int_0^\infty T(t)BB'T(t)'z dt \right\|_Z$$

$$\begin{aligned} &= \left\| \int_0^\infty A^\beta T(t) B B' T(t)' z dt \right\|_Z \\ &\quad \text{(using Theorem A.5.23, page 628, [34])} \\ &\leq \int_0^\infty \left\| A^\beta T(t) B B' T(t)' z \right\|_Z dt \\ &\leq M \|z\|_{Z'_\gamma}, \end{aligned}$$

for $\beta - 2\alpha - \gamma < 1$, $\beta < 1 + \alpha$, $\gamma > -(1 + \alpha)$. We depict this region (shaded) in Figure 2.4.

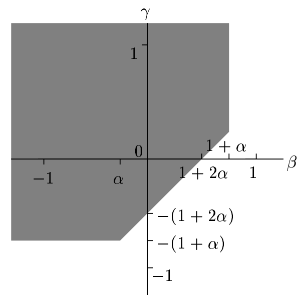


Figure 2.4: The feasible region of parameter values for L_B.

We note that the point $(1 + 2\alpha, 0)$ lies to the right of the point $(\alpha, 0)$ since $1 + \alpha > 0$. However, whether the point $(1 + 2\alpha, 0)$ lies to the right or to the left or coincides with the origin depends on the value of α:

$$\begin{aligned} -1 < \alpha < -\frac{1}{2} &\quad : \quad \text{left,} \\ \alpha = -\frac{1}{2} &\quad : \quad \text{coincides,} \\ -\frac{1}{2} < \alpha \leq 0 &\quad : \quad \text{right.} \end{aligned}$$

(In Figure 2.4, we have shown the case when α satisfies $-\frac{1}{2} < \alpha \leq 0$.)

3. That $\mathcal{B}\mathcal{B}' \in \mathcal{L}\left(Z'_\gamma, Z_\beta\right)$ for $\beta < \frac{1}{2} + \alpha$ and $\gamma > -\frac{1}{2} - \alpha$ and is self-dual in $\mathcal{L}\left(Z'_{-\beta}, Z_\beta\right)$ for $\beta < \frac{1}{2} + \alpha$ is a consequence of part 1 above.

So it remains to prove the equality $L_B = \mathcal{B}\mathcal{B}' \in \mathcal{L}\left(Z'_\gamma, Z_\beta\right)$ for $\beta < \frac{1}{2} + \alpha$ and $\gamma > -\frac{1}{2} - \alpha$:

It can be easily shown that if $z \in Z'_{-\beta}$ for $\beta < \frac{1}{2} + \alpha$, then

$$\left(\mathcal{B}' z\right)(t) = B' T(t)' z, \quad t \geq 0.$$

Thus if $z \in Z'_\gamma$ for $\gamma > -\frac{1}{2} - \alpha$, then $\mathcal{B}'z = B'T(\cdot)'z \in L_2([0,\infty), U)$, and since $\mathcal{B} \in \mathcal{L}\left(L_2([0,\infty), U), Z_\beta\right)$ for $\beta < \frac{1}{2} + \alpha$,

$$\mathcal{B}\mathcal{B}'z = \int_0^\infty T(t)B\mathcal{B}'zdt = \int_0^\infty T(t)BB'T(t)'zdt \quad \in Z_\beta.$$

But since $\beta - \gamma < \left(\frac{1}{2} + \alpha\right) + \left(\frac{1}{2} + \alpha\right) = 1 + 2\alpha$, $\beta < \frac{1}{2} + \alpha < 1 + \alpha$, $\gamma > -\frac{1}{2} - \alpha > -1 - \alpha$, it follows from part 2 that

$$\int_0^\infty T(t)BB'T(t)'zdt = L_B z \quad \in Z_\beta.$$

Thus $L_B = \mathcal{B}\mathcal{B}' \in \mathcal{L}\left(Z'_\gamma, Z_\beta\right)$ for $\beta < \frac{1}{2} + \alpha$ and $\gamma > -\frac{1}{2} - \alpha$.

4. From A2, $B'(A')^\beta$ belongs to $\mathcal{L}(Z,U)$ for β satisfying $-1 < \beta \leq \alpha$. Since $-A$ is the infinitesimal generator of an exponentially stable semigroup on the Hilbert space Z, it follows from the standard theory for Lyapunov equations (see for example Theorem 4.1.23, page 140, Curtain and Zwart [34]), applied to $\Sigma\left(-A, B'(A')^\beta, -\right)$ with state space Z, that there exists a unique self-adjoint solution $P \in \mathcal{L}(Z)$ of the operator Lyapunov equation

$$\langle A'z_1, Pz_2\rangle_Z + \langle Pz_1, A'z_2\rangle_Z = \left\langle B'(A')^\beta z_1, B'(A')^\beta z_2\right\rangle_U,$$

where z_1 and z_2 belong to $D(A')$ and P is given by

$$Pz = \int_0^\infty T(t)A^\beta BB'(A')^\beta T(t)'zdt \ \text{ for } \ z \in Z.$$

If w_1 and w_2 belong to $Z'_{1-\beta}$, then $(A')^{-\beta}w_1$ and $(A')^{-\beta}w_2$ belong to $\in D(A')$. Consequently, we have

$$\left\langle A'(A')^{-\beta}w_1, P(A')^{-\beta}w_2\right\rangle_Z + \left\langle P(A')^{-\beta}w_1, A'(A')^{-\beta}w_2\right\rangle_Z = \langle B'w_1, B'w_2\rangle_U. \tag{2.19}$$

We now proceed to relate P to L_B defined in part 2. If $\beta < \alpha$, we have, using part 2, that $L_B \in \mathcal{L}\left(Z'_{-\beta}, Z_{\beta+1}\right)$ and so $AL_B \in \mathcal{L}\left(Z'_{-\beta}, Z_\beta\right)$. If $w \in Z'_{1-\beta}$, then $w \in Z'_{-\beta}$, since $Z'_{1-\beta} \subset Z'_{-\beta}$. If $\langle\cdot,\cdot\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle}$ denotes the duality pairing between $Z'_{-\beta}$ and Z_β, then for w_1 and w_2 in $Z'_{1-\beta}$, with $-1 < \beta < \alpha$, we have

$$\begin{aligned}\langle w_1, AL_B w_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle} &= \left\langle w_1, A\int_0^\infty T(t)BB'T(t)'w_2dt\right\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle} \\ &= \left\langle A'w_1, \int_0^\infty T(t)BB'T(t)'w_2dt\right\rangle_{\langle Z'_{-\beta-1}, Z_{\beta+1}\rangle},\end{aligned}$$

where we have used $A \in \mathcal{L}\left(Z_{\beta+1}, Z_\beta\right)$ and $A' \in \mathcal{L}\left(Z'_{-\beta}, Z'_{-\beta-1}\right)$. Since $w_1 \in Z'_{1-\beta} = D\left((A')^{1-\beta}\right)$, $A'w_1$ belongs to $D\left((A')^{-\beta}\right) = Z'_{-\beta}$. From part 2,

$L_B \in \mathcal{L}\left(Z'_{-\beta}, Z_{\beta+1}\right)$ and since w_2 belongs to $Z'_{1-\beta} \subset Z'_{-\beta}$, it follows that $L_B w_2 \in Z_{\beta+1} \subset Z_\beta$ and so

$$\begin{aligned}\langle w_1, AL_B w_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle} &= \left\langle A'w_1, \int_0^\infty T(t)BB'T(t)'w_2 dt\right\rangle_{\langle Z'_{-\beta-1}, Z_{\beta+1}\rangle} \\ &= \left\langle A'w_1, \int_0^\infty T(t)BB'T(t)'w_2 dt\right\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle}.\end{aligned}$$

Hence

$$\langle w_1, AL_B w_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle} = \int_0^\infty \langle A'w_1, T(t)BB'T(t)'w_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle}\, dt,$$

since $A'w_1 \in D\left((A')^{-\beta}\right)$ and $T(t)BB'T(t)'w_2 \in Z_\beta$. Consequently,

$$\begin{aligned}&\langle w_1, AL_B w_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle} \\ &= \int_0^\infty \left\langle A'w_1, A^{-\beta}T(t)A^\beta BB'(A')^\beta T(t)'(A')^{-\beta}w_2\right\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle} dt \\ &= \int_0^\infty \left\langle A'(A')^{-\beta}w_1, T(t)A^\beta BB'(A')^\beta T(t)'(A')^{-\beta}w_2\right\rangle_Z dt,\end{aligned}$$

since $A^{-\beta} \in \mathcal{L}(Z, Z_\beta)$ and $A'(A')^{-\beta}w_1 \in Z$. As a result we obtain that

$$\begin{aligned}&\langle w_1, AL_B w_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle} \\ &= \left\langle A'(A')^{-\beta}w_1, \int_0^\infty T(t)A^\beta BB'(A')^\beta T(t)'(A')^{-\beta}w_2 dt\right\rangle_Z \\ &= \left\langle A'(A')^{-\beta}w_1, P(A')^{-\beta}w_2\right\rangle_Z. \qquad (2.20)\end{aligned}$$

Similarly, it can be shown that for w_1 and w_2 in $Z'_{1-\beta}$, we have

$$\langle w_1, L_B A'w_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle} = \left\langle P(A')^{-\beta}w_1, A'(A')^{-\beta}w_2\right\rangle_Z. \qquad (2.21)$$

(Here, using part 2, we consider $L_B \in \mathcal{L}\left(Z'_{-\beta-1}, Z_\beta\right)$ for $\beta < \alpha$; we have $L_B A' \in \mathcal{L}\left(Z'_{-\beta}, Z_\beta\right)$.)

So by substituting (2.20) and (2.21) in (2.19), we see that L_B satisfies

$$\langle z_1, AL_B z_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle} + \langle z_1, L_B A'z_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle} = \langle z_1, BB'z_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle}$$

for z_1 and z_2 in $Z'_{-\beta+1}$ $(\subset Z'_{-\beta})$. But since $Z'_{-\beta+1}$ is dense in $Z'_{-\beta}$, and AL_B, $L_B A'$ and $BB' \in \mathcal{L}\left(Z'_{-\beta}, Z_\beta\right)$, it follows that

$$\langle z_1, AL_B z_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle} + \langle z_1, L_B A'z_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle} = \langle z_1, BB'z_2\rangle_{\langle Z'_{-\beta}, Z_\beta\rangle}$$

holds for z_1 and z_2 in $Z'_{-\beta}$. ∎

Next we derive some properties of the *observability map* $\mathcal{C} : Z \to L_2([0,\infty), Y)$ given by

$$\mathcal{C}z = CT(\cdot)z, \tag{2.22}$$

and the *observability Gramian* $L_C : Z \to Z$ given by

$$L_C z = \int_0^\infty T(t)'C'CT(t)z dt. \tag{2.23}$$

Owing to the smoothing property of $\{T(t)\}_{t\geq 0}$, both $\mathcal{C}$ and L_C inherit some smoothness too.

Lemma 2.5.2 *$\mathcal{C}$ and L_C defined by (2.22) and (2.23) have the following properties:*

1. *$\mathcal{C} \in \mathcal{L}(Z_\beta, L_2([0,\infty), Y))$ and $\mathcal{C}' \in \mathcal{L}\left(L_2([0,\infty), Y), Z'_{-\beta}\right)$, for all β satisfying $\beta > -\frac{1}{2}$.*
2. *$L_C \in \mathcal{L}\left(Z_\beta, Z'_\gamma\right)$ for all γ and β satisfying $\gamma - \beta < 1$, $\beta > -1$, $\gamma < 1$.*
3. *$L_C = \mathcal{C}'\mathcal{C} \in \mathcal{L}\left(Z_\beta, Z'_\gamma\right)$ for $\beta > -\frac{1}{2}$ and $\gamma < \frac{1}{2}$ and it is self-dual in $\mathcal{L}\left(Z_\beta, Z'_{-\beta}\right)$ for $\beta > -\frac{1}{2}$.*
4. *If $\epsilon > 0$, then $L_C \in \mathcal{L}\left(Z_\epsilon, Z'_{-\epsilon+1}\right) \cap \mathcal{L}\left(Z_{\epsilon-1}, Z'_{-\epsilon}\right)$ is a solution of the Lyapunov equation*

$$A'L_C z + L_C A z = C'Cz \quad \text{in } Z'_{-\epsilon} \tag{2.24}$$

for any $z \in Z_\epsilon$.

Proof

1. If $z \in Z_\beta$, then

$$\begin{aligned}
\int_0^\tau \|CT(t)z\|_Y^2 dt &\leq \int_0^\tau \|C\|^2_{\mathcal{L}(Z,Y)} \|T(t)z\|_Z^2 dt \\
&= \|C\|^2_{\mathcal{L}(Z,Y)} \int_0^\tau \left\|A^{-\beta}T(t)A^\beta z\right\|_Z^2 dt \\
&\leq \|C\|^2_{\mathcal{L}(Z,Y)} \int_0^\tau \left\|A^{-\beta}T(t)\right\|^2_{\mathcal{L}(Z)} \|A^\beta z\|_Z^2 dt \\
&= \|C\|^2_{\mathcal{L}(Z,Y)} \|z\|^2_{Z_\beta} \int_0^\tau \left\|A^{-\beta}T(t)\right\|^2_{\mathcal{L}(Z)} dt.
\end{aligned}$$

<u>`Case 1:`</u> $-\frac{1}{2} < \beta \leq 0$. If $z \in Z_\beta$, then

$$\begin{aligned}\int_0^\tau \|CT(t)z\|_Y^2 dt &\leq \|C\|^2_{\mathcal{L}(Z,Y)}\|z\|^2_{Z_\beta}\int_0^\tau \frac{K_1^2 e^{-2\epsilon t}}{t^{-2\beta}}dt \\ &\leq \|C\|^2_{\mathcal{L}(Z,Y)}\|z\|^2_{Z_\beta}\int_0^\infty \frac{K_1^2 e^{-2\epsilon t}}{t^{-2\beta}}dt \\ &\leq M_1^2\|z\|^2_{Z_\beta},\end{aligned}$$

for some constant $M_1 > 0$ (independent of τ), since $-2\beta < 1$.

<u>`Case 2:`</u> $0 < \beta$. If $z \in Z_\beta$, then

$$\begin{aligned}\int_0^\tau \|CT(t)z\|_Y^2 dt &\leq \|C\|^2_{\mathcal{L}(Z,Y)}\|z\|^2_{Z_\beta}\int_0^\tau K_2^2 e^{-2\epsilon t}dt \\ &\leq \|C\|^2_{\mathcal{L}(Z,Y)}\|z\|^2_{Z_\beta}\int_0^\infty K_2^2 e^{-2\epsilon t}dt \\ &\leq M_2^2\|z\|^2_{Z_\beta},\end{aligned}$$

for some constant $M_2 > 0$ (independent of τ).

Thus we have $CT(\cdot)z \in L_2([0,\infty),Y)$ and moreover from the above we obtain $\|\mathcal{C}z\|_{L_2([0,\infty),Y)} \leq M\|z\|_{Z_\beta}$, for some $M > 0$. So it follows that $\mathcal{C} \in \mathcal{L}\left(Z_\beta, L_2([0,\infty),Y)\right)$ for $\beta > -\frac{1}{2}$ and $\mathcal{C}' \in \mathcal{L}\left(L_2([0,\infty),Y), Z'_{-\beta}\right)$ for $\beta > -\frac{1}{2}$.

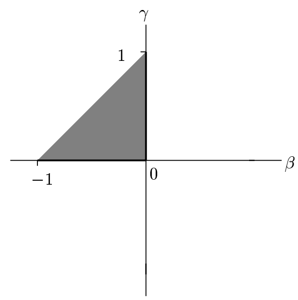
2. From property (4), foe every $t > 0$ we have $T(t)'C'CT(t)Z_\beta \subset Z'_{-\gamma}$, for any real β and γ. We now investigate the values of γ and β for which $(A')^\gamma T(t)'C'CT(t)A^{-\beta}z \in L_1\left([0,\infty),Z'\right)$ for $z \in Z$. If $z \in Z$,

$$\begin{aligned}&\int_0^\tau \left\|(A')^\gamma T(t)'C'CT(t)A^{-\beta}z\right\|_{Z'} dt \\ =\ &\int_0^\tau \|(A')^\gamma T(t)'\|_{\mathcal{L}(Z')}\|C'\|_{\mathcal{L}(Y,Z')}\|C\|_{\mathcal{L}(Z,Y)}\left\|T(t)A^{-\beta}\right\|_{\mathcal{L}(Z)}\|z\|_Z dt \\ \leq\ &\|C\|^2_{\mathcal{L}(Z,Y)}\|z\|_Z\int_0^\tau \|(A')^\gamma T(t)'\|_{\mathcal{L}(Z')}\left\|T(t)A^{-\beta}\right\|_{\mathcal{L}(Z)}dt.\end{aligned}$$

<u>`Case 1:`</u> $\gamma \geq 0,\ \beta \leq 0$.

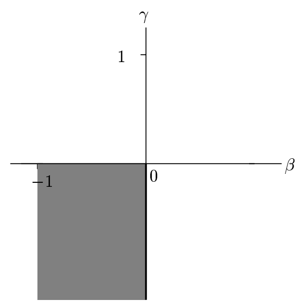
If $z \in Z$,

$$\begin{aligned}\int_0^\tau \left\|(A')^\gamma T(t)'C'CT(t)A^{-\beta}z\right\|_{Z'} dt &\leq \|C\|^2_{\mathcal{L}(Z,Y)}\|z\|_Z\int_0^\tau \frac{K_1^2 e^{-2\epsilon t}}{t^{\gamma-\beta}}dt \\ &\leq \|C\|^2_{\mathcal{L}(Z,Y)}\|z\|_Z\int_0^\infty \frac{K_1^2 e^{-2\epsilon t}}{t^{\gamma-\beta}}dt \\ &\leq M_1\|z\|,\end{aligned}$$



for some constant $M_1 > 0$ (independent of τ) if $\gamma - \beta < 1$.

Case 2: $\gamma < 0, \ \beta \leq 0$.



If $z \in Z$,

$$\begin{aligned}\int_0^\tau \left\|(A')^\gamma T(t)'C'CT(t)A^{-\beta}z\right\|_Z dt &\leq \|C\|^2_{\mathcal{L}(Z,Y)}\|z\|_Z \int_0^\tau \frac{K_1K_2e^{-2\epsilon t}}{t^{-\beta}}dt \\ &\leq \|C\|^2_{\mathcal{L}(Z,Y)}\|z\|_Z \int_0^\infty \frac{K_1K_2e^{-2\epsilon t}}{t^{-\beta}}dt \\ &\leq M_2\|z\|,\end{aligned}$$

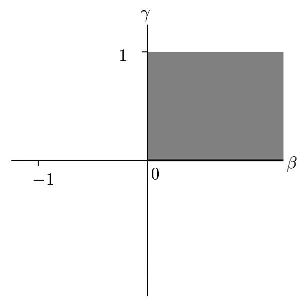
for some constant $M_2 > 0$ (independent of τ) if $-\beta < 1$, that is, if $\beta > -1$.

Case 3: $\gamma \geq 0, \ \beta > 0$.

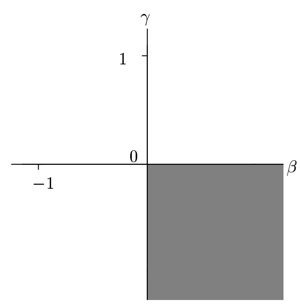
If $z \in Z$,

$$\begin{aligned}\int_0^\tau \left\|(A')^\gamma T(t)'C'CT(t)A^{-\beta}z\right\|_Z dt &\leq \|C\|^2_{\mathcal{L}(Z,Y)}\|z\|_Z \int_0^\tau \frac{K_1K_2e^{-2\epsilon t}}{t^{\gamma}}dt \\ &\leq \|C\|^2_{\mathcal{L}(Z,Y)}\|z\|_Z \int_0^\infty \frac{K_1K_2e^{-2\epsilon t}}{t^{\gamma}}dt \\ &\leq M_3\|z\|,\end{aligned}$$

for some constant $M_3 > 0$ (independent of τ) if $\gamma < 1$.



Case 4: $\gamma < 0,\ \beta > 0.$



If $z \in Z$,

$$\begin{aligned}\int_0^\tau \left\|(A')^\gamma T(t)'C'CT(t)A^{-\beta}z\right\|_Z dt &\le \|C\|^2_{\mathcal{L}(Z,Y)}\|z\|_Z \int_0^\tau K_2^2 e^{-2\epsilon t}dt \\ &\le \|C\|^2_{\mathcal{L}(Z,Y)}\|z\|_Z \int_0^\infty K_2^2 e^{-2\epsilon t}dt \\ &\le M_4\|z\|,\end{aligned}$$

for some constant $M_4 > 0$ (independent of τ).

So $(A')^\gamma T(t)'C'CT(t)A^{-\beta}z \in L_1([0,\infty), Z)$ for $z \in Z$. If $z \in Z_\beta$, then $A^\beta z \in Z_\beta$ and using the above, we obtain for $\gamma - \beta < 1,\ \gamma < 1, \beta > -1$,

$$\begin{aligned}\|L_C z\|_{Z'_\gamma} &= \left\|(A')^\gamma \int_0^\infty T(t)'C'CT(t)zdt\right\|_Z \\ &= \left\|\int_0^\infty (A')^\gamma T(t)'C'CT(t)zdt\right\|_Z \\ &= \left\|\int_0^\infty (A')^\gamma T(t)'C'CT(t)A^{-\beta}A^\beta zdt\right\|_Z \\ &\le \int_0^\infty \left\|(A')^\gamma T(t)'C'CT(t)A^{-\beta}A^\beta z\right\| dt \\ &\le M\|A^\beta z\|_Z = M\|z\|_{Z_\beta},\end{aligned}$$

for some constant $M > 0$. We depict this region (shaded) in Figure 2.5.

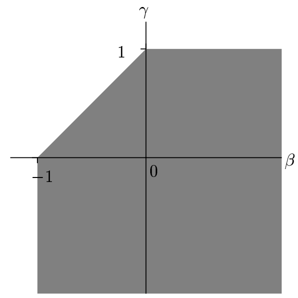


Figure 2.5: The feasible region of parameter values for L_C.

3. That $\mathcal{C}'\mathcal{C} \in \mathcal{L}\left(Z_\beta, Z'_\gamma\right)$ for $\beta > -\frac{1}{2}$ and $\gamma < \frac{1}{2}$ and is self-dual in $\mathcal{L}\left(Z_\beta, Z'_{-\beta}\right)$ for $\beta > -\frac{1}{2}$ is a consequence of part 1 above.

We only need to prove that the equality $L_C = \mathcal{C}'\mathcal{C} \in \mathcal{L}\left(Z_\beta, Z'_\gamma\right)$ holds for $\beta > -\frac{1}{2}$ and $\gamma < \frac{1}{2}$:

From 1, $\mathcal{C}' \in \mathcal{L}\left(L_2([0,\infty), Y), Z'_\gamma\right)$ for $\gamma < \frac{1}{2}$ and it can be easily shown that for any $y \in L_2([0,\infty), Y)$

$$\mathcal{C}'y = \int_0^\infty T(t)'C'y(t)dt \quad \in Z'_\gamma.$$

If z with Z_β for $\beta > -\frac{1}{2}$, then since $\mathcal{C} \in \mathcal{L}\left(Z_\beta, L_2([0,\infty), Y)\right)$,

$$(\mathcal{C}z)(\cdot) = CT(\cdot)z \quad \in L_2([0,\infty), Y),$$

and so

$$\mathcal{C}'\mathcal{C}z = \mathcal{C}'(CT(\cdot)z) = \int_0^\infty T(t)'C'CT(t)zdt \quad \in Z'_\gamma.$$

Now since $\gamma - \beta < \left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right) = 1$, $\gamma < \frac{1}{2} < 1$, $\beta > -\frac{1}{2} > -1$, it follows from part 2 that

$$\int_0^\infty T(t)'C'CT(t)zdt = L_C z \quad \in Z'_\gamma.$$

So $L_C = \mathcal{C}'\mathcal{C} \in \mathcal{L}\left(Z_\beta, Z'_\gamma\right)$ for $\beta > -\frac{1}{2}$ and $\gamma < \frac{1}{2}$.

4. $C \in \mathcal{L}(Z, Y)$ and so $CA^{-\epsilon} \in \mathcal{L}(Z, Y)$ for $\epsilon > 0$ (using Lemma 6.3, page 71, Pazy [67]). Since $-A$ is the infinitesimal generator of an exponentially stable strongly continuous semigroup on the Hilbert space Z, it follows from the standard theory for Lyapunov equations (see for example Theorem 4.1.23, page 140, Curtain and Zwart [34]) applied to $\Sigma\left(-A, -, CA^{-\epsilon}\right)$ with state space

Z, that there exists a unique self-adjoint solution Q of the operator Lyapunov equation

$$\langle Az_1, Qz_2\rangle_Z + \langle Qz_1, Az_2\rangle_Z = \left\langle CA^{-\epsilon}z_1, CA^{-\epsilon}z_2\right\rangle_Y,$$

for all z_1 and z_2 in $D(A)$ and Q is given by

$$Qz = \int_0^\infty T(t)'(A')^{-\epsilon}C'CA^{-\epsilon}T(t)z dt,$$

for all $z \in Z$. If w_1 and w_2 belong to $Z_{1+\epsilon}$, then $A^\epsilon w_1$ and $A^\epsilon w_2$ belong to $D(A)$. Consequently, we have

$$\langle AA^\epsilon w_1, QA^\epsilon w_2\rangle_Z + \langle QA^\epsilon w_1, AA^\epsilon w_2\rangle_Z = \langle Cw_1, Cw_2\rangle_Y. \tag{2.25}$$

If $\epsilon > 0$, we have, using part 2, that $L_C \in \mathcal{L}\left(Z_\epsilon, Z'_{-\epsilon+1}\right)$ and so $A'L_C \in \mathcal{L}\left(Z_\epsilon, Z'_{-\epsilon}\right)$. If $w \in Z_{1+\epsilon}$, then $w \in Z_\epsilon$, since $Z_{1+\epsilon} \subset Z_\epsilon$. If $\langle\cdot,\cdot\rangle_{\left\langle Z_\epsilon, Z'_{-\epsilon}\right\rangle}$ denotes the duality pairing between Z_ϵ and $Z'_{-\epsilon}$, with $\epsilon > 0$, then for $w_1, w_2 \in Z_{1+\epsilon}$, we have

$$\begin{aligned}\langle w_1, A'L_C w_2\rangle_{\left\langle Z_\epsilon, Z'_{-\epsilon}\right\rangle} &= \left\langle w_1, A'\int_0^\infty T(t)'C'CT(t)w_2 dt\right\rangle_{\left\langle Z_\epsilon, Z'_{-\epsilon}\right\rangle} \\ &= \left\langle Aw_1, \int_0^\infty T(t)'C'CT(t)w_2 dt\right\rangle_{\left\langle Z_{\epsilon-1}, Z'_{1-\epsilon}\right\rangle},\end{aligned}$$

since $A' \in \mathcal{L}\left(Z'_{-\epsilon+1}, Z'_{-\epsilon}\right)$ and $A \in \mathcal{L}\left(Z_\epsilon, Z'_{\epsilon-1}\right)$. Since $w_1 \in Z_{1+\epsilon} = D\left(A^{1+\epsilon}\right)$, Aw_1 belongs to $D\left(A^\epsilon\right)$. From part 2, $L_C \in \mathcal{L}\left(Z_\epsilon, Z'_{-\epsilon+1}\right)$ and since $w_2 \in Z_{1+\epsilon} \subset Z_\epsilon$, $L_C w_2 \in Z'_{-\epsilon+1} \subset Z'_{-\epsilon}$ and so

$$\begin{aligned}\langle w_1, A'L_C w_2\rangle_{\left\langle Z_\epsilon, Z'_{-\epsilon}\right\rangle} &= \left\langle Aw_1, \int_0^\infty T(t)'C'CT(t)w_2 dt\right\rangle_{\left\langle Z_{\epsilon-1}, Z'_{1-\epsilon}\right\rangle} \\ &= \left\langle Aw_1, \int_0^\infty T(t)'C'CT(t)w_2 dt\right\rangle_{\left\langle Z_\epsilon, Z'_{-\epsilon}\right\rangle}.\end{aligned}$$

Since $Aw_1 \in D\left(A^\epsilon\right)$ and $T(t)'C'CT(t)w_2 \in Z'_{-\epsilon}$,

$$\langle w_1, A'L_C w_2\rangle_{\left\langle Z_\epsilon, Z'_{-\epsilon}\right\rangle} = \int_0^\infty \langle Aw_1, T(t)'C'CT(t)w_2\rangle_{\left\langle Z_\epsilon, Z'_{-\epsilon}\right\rangle} dt.$$

Consequently,

$$\begin{aligned}&\langle w_1, A'L_C w_2\rangle_{\left\langle Z_\epsilon, Z'_{-\epsilon}\right\rangle} \\ =\ &\int_0^\infty \left\langle Aw_1, (A')^\epsilon T(t)'(A')^{-\epsilon}C'CA^{-\epsilon}T(t)A^\epsilon w_2\right\rangle_{\left\langle Z_\epsilon, Z'_{-\epsilon}\right\rangle} dt \\ =\ &\int_0^\infty \left\langle AA^\epsilon w_1, T(t)'(A')^{-\epsilon}C'CA^{-\epsilon}T(t)A^\epsilon w_2\right\rangle_Z dt\end{aligned}$$

since $(A')^\epsilon \in \mathcal{L}\left(Z, Z'_{-\epsilon}\right)$ and $AA^\epsilon w_1 \in Z$. As a result we obtain that

$$\begin{aligned} & \langle w_1, A'L_C w_2\rangle_{\langle Z_\epsilon, Z'_{-\epsilon}\rangle} \\ = \; & \left\langle AA^\epsilon w_1, \int_0^\infty T(t)'(A')^{-\epsilon}C'CA^{-\epsilon}T(t)A^\epsilon w_2 dt \right\rangle_Z \\ = \; & \langle AA^\epsilon w_1, QA^\epsilon w_2\rangle_Z . \end{aligned} \tag{2.26}$$

Similarly, it can be shown that for w_1 and w_2 in $Z_{1+\epsilon}$, we have

$$\langle w_1, L_C A w_2\rangle_{\langle Z_\epsilon, Z'_{-\epsilon}\rangle} = \langle QA^\epsilon w_1, AA^\epsilon w_2\rangle_Z . \tag{2.27}$$

(Here, using part 2, we consider $L_C \in \mathcal{L}\left(Z_{\epsilon-1}, Z'_{-\epsilon}\right)$ for $\epsilon > 0$; we have $L_C A \in \mathcal{L}\left(Z_\epsilon, Z'_{-\epsilon}\right)$.)

So by substituting (2.26) and (2.27) in (2.25), we see that L_C satisfies

$$\langle z_1, A'L_C z_2\rangle_{\langle Z_\epsilon, Z'_{-\epsilon}\rangle} + \langle z_1, L_C A z_2\rangle_{\langle Z_\epsilon, Z'_{-\epsilon}\rangle} = \langle z_1, C'C z_2\rangle_{\langle Z_\epsilon, Z'_{-\epsilon}\rangle}$$

for z_1 and z_2 in $Z_{1+\epsilon}$ $(\subset Z_\epsilon)$. But since $Z_{1+\epsilon}$ is dense in Z_ϵ, and $A'L_C$, $L_C A$ and $C'C \in \mathcal{L}\left(Z_\epsilon, Z'_{-\epsilon}\right)$, it follows that

$$\langle z_1, A'L_C z_2\rangle_{\langle Z_\epsilon, Z'_{-\epsilon}\rangle} + \langle z_1, L_C A z_2\rangle_{\langle Z_\epsilon, Z'_{-\epsilon}\rangle} = \langle z_1, C'C z_2\rangle_{\langle Z_\epsilon, Z'_{-\epsilon}\rangle}$$

holds for z_1 and z_2 in Z_ϵ. ■

Remark: We note that Lemma 2.5.2.1 shows that C is an admissible observation operator for the semigroup $\{T(t)\}_{t\geq 0}$ on the Hilbert space Z_β, for $\beta > -\frac{1}{2}$.

We now show that the system Σ given by the triple $(-A, B, C)$ is a well-posed linear system on Z_β for any $\beta \in \mathbb{R}$ satisfying $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$.

Theorem 2.5.3 *Let $\beta \in \mathbb{R}$ satisfy $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$. If A, B and C satisfy the assumptions A1, A2 and A3 listed at the beginning of this section, then the system given by the triple $(-A, B, C)$ is a well-posed linear system with input space U, state space Z_β and output space Y, and has a transfer function given by $G(s) = C(sI + A)^{-1}B$.*

Proof From Theorems 2.5.1.1 and 2.5.2.1, it follows that B and C are admissible input and output operators for the semigroup $\{T(t)\}_{t\geq 0}$ with state space Z_β. We show that $\Theta(s) = C(sI + A)^{-1}B$ is well-defined for $s \in \rho(-A)$.

1. Let $s \in \rho(-A)$. Firstly, if $u \in U$, then $Bu \in Z_\alpha$ and since $(sI + A)^{-1} \in \mathcal{L}(Z_\alpha, Z_{\alpha+1})$, it follows that $(sI + A)^{-1}Bu \in Z_{\alpha+1}$. Furthermore, since $\alpha > -1$, it follows that $(sI + A)^{-1}Bu \in Z$. Hence $C(sI + A)^{-1}Bu \in Y$.

2. Furthermore, we have for $u \in U$,

$$\begin{aligned}
\|\Theta u\|_Y &\leq \left\|C(sI+A)^{-1}Bu\right\|_Y \\
&\leq \|C\|_{\mathcal{L}(Z_{\alpha+1},Y)} \left\|(sI+A)^{-1}Bu\right\|_{Z_{\alpha+1}} \\
&\leq \|C\|_{\mathcal{L}(Z_{\alpha+1},Y)} \left\|(sI+A)^{-1}\right\|_{\mathcal{L}(Z_\alpha,Z_{\alpha+1})} \|Bu\|_{Z_\alpha} \\
&\leq \|C\|_{\mathcal{L}(Z_{\alpha+1},Y)} \left\|(sI+A)^{-1}\right\|_{\mathcal{L}(Z_\alpha,Z_{\alpha+1})} \|B\|_{\mathcal{L}(U,Z_\alpha)} \|u\|_U.
\end{aligned}$$

Thus for each $s \in \rho(-A)$, $\Theta(s) \in \mathcal{L}(U,Y)$. In fact we have shown that

$$\left\|C(sI+A)^{-1}B\right\|_{\mathcal{L}(U,Y)} \leq \|C\|_{\mathcal{L}(Z_{\alpha+1},Y)} \left\|(sI+A)^{-1}\right\|_{\mathcal{L}(Z_\alpha,Z_{\alpha+1})} \|B\|_{\mathcal{L}(U,Z_\alpha)}. \tag{2.28}$$

3. Using the resolvent identity, it is easy to check that for s and β in $\rho(-A)$, we have

$$\Theta(s) - \Theta(\beta) = -(s-\beta)C(sI+A)^{-1}(\beta+A)^{-1}B.$$

4. Finally we show that the system Σ given by the triple $(-A,B,C)$ is well-posed by showing that the map $\Theta(\cdot)$ is bounded in the half-plane $\mathbb{C}_0^+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$. In order to do this, we use the exponential stability of the generator of the semigroup and (2.28): We have for all $s \in \mathbb{C}_0^+$

$$\begin{aligned}
\left\|(sI+A)^{-1}\right\|_{\mathcal{L}(Z_\alpha,Z_{\alpha+1})} &\leq \left\|A(sI+A)^{-1}\right\|_{\mathcal{L}(Z_\alpha)} \\
&= \left\|I - s(sI+A)^{-1}\right\|_{\mathcal{L}(Z_\alpha)} \\
&\leq \|I\|_{\mathcal{L}(Z_\alpha)} + |s| \left\|(sI+A)^{-1}\right\|_{\mathcal{L}(Z_\alpha)} \\
&\leq 1 + |s| \frac{M}{|s|} \quad \text{(using Theorem 5.2.(c), Pazy [67])} \\
&= 1 + M,
\end{aligned}$$

for some $M > 0$. This completes the proof. ■

Remark: A more general result with unbounded C can be proven in a similar manner. We quote the following from Staffans [84]:

Theorem 2.5.4 *Let* $-A$ *generate an exponentially stable analytic strongly continuous semigroup* $\{T(t)\}_{t\geq 0}$ *in* Z, $B \in \mathcal{L}(U, Z_{\alpha_B})$, $C \in \mathcal{L}(Z_{\alpha_C}, Y)$ *and* $D \in \mathcal{L}(U,Y)$, *where* $\alpha_B \leq \alpha_C < \alpha_B + 1$. *Fix any* γ *satisfying* $\alpha_C - \frac{1}{2} < \gamma < \alpha_B + \frac{1}{2}$. *Then* $-A$, B, C *generate a regular well-posed system on* (U, Z_γ, Y), *where* $X = Z_\gamma$. *The transfer function is given by* $G(s) = C(sI+A)^{-1}B + D$, *and it satisfies*

$$\lim_{\substack{|s|\to\infty \\ s \in \mathbb{C}_0^+}} G(s) = D.$$

In the sequel, the operators $L_B L_C$ and $L_C L_B$ play an important role, so we collect some properties here which readily follow from Lemmas 2.5.1.2 and 2.5.2.2.

Corollary 2.5.5 *L_B and L_C defined by (2.15) and (2.23), respectively, satisfy:*

1. *$L_B L_C \in \mathcal{L}(Z_\beta)$ for any β satisfying $-1 < \beta < 1 + \alpha$.*
2. *$L_C L_B \in \mathcal{L}(Z'_\gamma)$ for any γ satisfying $-(1+\alpha) < \gamma < 1$.*

Proof From Lemma 2.5.1.2, we know that $L_B \in \mathcal{L}(Z'_\gamma, Z_\beta)$ for γ, β satisfying $\beta - \gamma < 1 + 2\alpha$, $\beta < 1 + \alpha$, $\gamma > -(1+\alpha)$. Furthermore, Lemma 2.5.2.2 states that $L_C \in \mathcal{L}(Z_\beta, Z'_\gamma)$ for γ, β satisfying $\gamma - \beta < 1$, $\beta > -1$, $\gamma < 1$.

Since $-1 < \alpha$, the intersection of the feasible regions depicted in Figures 2.4 and 2.5 is nonempty and is sketched in Figure 2.6.

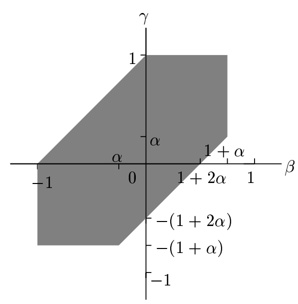


Figure 2.6: The intersection of feasible regions.

Hence the claims in parts 1 and 2 follow. ■

We now examine the Hankel operator associated with $h(\cdot) = CT(\cdot)B$. Throughout the remainder of this section, we assume that the input and output spaces are finite-dimensional: say, $U = \mathbb{C}^m$ and $Y = \mathbb{C}^p$.

Lemma 2.5.6 *If A, B, and C satisfy the assumptions A1, A2 and A3 listed at the beginning of this section with $U = \mathbb{C}^m$ and $Y = \mathbb{C}^p$, then:*

1. *The impulse response $h(\cdot) = CT(\cdot)B \in L_1([0,\infty), \mathbb{C}^{p\times m})$ and so it has a well-defined compact Hankel operator $\Gamma : L_2([0,\infty), \mathbb{C}^m) \to$*

$L_2([0,\infty),\mathbb{C}^p)$ *given by*

$$(\Gamma u)(t) = \int_0^\infty h(t+\tau)u(\tau)d\tau \quad t \geq 0, \quad \text{for } u \in L_2([0,\infty),\mathbb{C}^m). \tag{2.29}$$

2. If $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$, *then* $\Gamma = \mathcal{C}\mathcal{B}$, *where* $\mathcal{B} \in \mathcal{L}(L_2([0,\infty),\mathbb{C}^m), Z_\beta)$ *and* $\mathcal{C} \in \mathcal{L}(Z_\beta, L_2([0,\infty),\mathbb{C}^p))$.

Proof *1.* We have

$$\begin{aligned}
\|CT(t)Bu\|_{\mathbb{C}^p} &\leq \|C\|_{\mathcal{L}(Z,\mathbb{C}^p)}\|T(t)Bu\|_Z \\
&\leq \|C\|_{\mathcal{L}(Z,\mathbb{C}^p)}\|A^{-\alpha}T(t)A^\alpha Bu\|_Z \\
&\leq \|C\|_{\mathcal{L}(Z,\mathbb{C}^p)}\|A^{-\alpha}T(t)\|_{\mathcal{L}(Z)}\|A^\alpha B\|_{\mathcal{L}(\mathbb{C}^m,Z)}\|u\|_{\mathbb{C}^m} \\
&\leq \|C\|_{\mathcal{L}(Z,\mathbb{C}^p)}\frac{Ke^{-\epsilon t}}{t^{-\alpha}}\|A^\alpha B\|_{\mathcal{L}(\mathbb{C}^m,Z)}\|u\|_{\mathbb{C}^m},
\end{aligned}$$

and since $-\alpha < 1$, it follows that $h(\cdot) \in L_1([0,\infty),\mathbb{C}^{p\times m})$. Compactness follows from Lemma 8.2.4 (page 399, Curtain and Zwart [34]).

2. From Lemma 2.5.1.1 it follows that $\mathcal{B} \in \mathcal{L}(L_2([0,\infty),\mathbb{C}^m), Z_\beta)$ and from Lemma 2.5.2.1, $\mathcal{C} \in \mathcal{L}(Z_\beta, L_2([0,\infty),\mathbb{C}^p))$. Consequently, if $u \in L_2([0,\infty),\mathbb{C}^m)$, $\mathcal{B}u \in Z_\beta$, and

$$(\mathcal{C}(\mathcal{B}u))(t) = CT(t)\mathcal{B}u = CT(t)\int_0^\infty T(\tau)Bu(\tau)d\tau.$$

But $CT(t) \in \mathcal{L}(Z_\beta,\mathbb{C}^p)$ and so we have (see for example Theorem A.5.23, page 628, Curtain and Zwart [34])

$$\begin{aligned}
(\mathcal{C}(\mathcal{B}u))(t) &= \int_0^\infty CT(t)T(\tau)Bu(\tau)d\tau = \int_0^\infty CT(t+\tau)Bu(\tau)d\tau \\
&= \int_0^\infty h(t+\tau)u(\tau)d\tau = (\Gamma u)(t).
\end{aligned}$$

■

Remark: In fact, Γ is nuclear, that is $\sum_{i=1}^\infty \sigma_i < \infty$ (see Theorem 3.3.3 in Chapter 3).

We will first show that $L_B L_C \in \mathcal{L}(Z_\beta)$ is compact for all β satisfying $-1 < \beta < 1+\alpha$. This was shown in Curtain and Ichikawa [20] under the extra assumption that $T(t)$ was compact for each $t > 0$. Here we do not need this assumption, since we assume that B has finite rank.

Lemma 2.5.7 *Under the assumptions A1, A2 and A3, with* $U = \mathbb{C}^m$ *and* $Y = \mathbb{C}^p$, $L_B L_C \in \mathcal{L}(Z_\beta)$ *is compact for all* β *satisfying* $-1 < \beta < 1+\alpha$.

Proof The map $t \mapsto T(t)BB'T(t)' : (0,\infty) \to \mathcal{L}\left(Z'_\gamma, Z_\beta\right)$ is continuous in the norm topology, and since B is compact, for each t greater than zero $T(t)BB'T(t)'$ is compact. Let $\mathtt{C}\left(Z'_\gamma, Z_\beta\right)$ denote the Banach space of compact operators from Z'_γ to Z_β. Thus we have that the map $t \mapsto T(t)BB'T(t)' : (0,\infty) \to \mathtt{C}\left(Z'_\gamma, Z_\beta\right)$ is continuous in the norm topology. Moreover, as in the proof of Lemma 2.5.1.3, it can be shown that for γ and β satisfying $\beta - \gamma < 1 + 2\alpha$, $\beta < 1 + \alpha$ and $\gamma > -(1+\alpha)$, we have

$$\int_0^\infty \|T(t)BB'T(t)'\|_{\mathtt{C}\left(Z'_\gamma, Z_\beta\right)} dt < \infty.$$

Consequently, the Bochner integral $\int_0^\infty T(t)BB'T(t)'dt \in \mathtt{C}\left(Z'_\gamma, Z_\beta\right)$ (see for example, Thomas [86] or Hille and Phillips [50]). Proceeding as in the proof of Corollary 2.5.5.1, it follows that $L_B L_C \in \mathcal{L}(Z_\beta)$ and is compact for all β satisfying $-1 < \beta < 1 + \alpha$. ■

Next we show that the spectra of $\Gamma^*\Gamma \in \mathcal{L}(L_2([0,\infty), \mathbb{C}^m))$ and $L_B L_C \in \mathcal{L}(Z_\beta)$ are identical for all β satisfying $-1 < \beta < 1 + \alpha$.

Lemma 2.5.8 *Under the assumptions A1, A2 and A3 with $U = \mathbb{C}^m$ and $Y = \mathbb{C}^p$, the nonzero Hankel singular values are equal to the square roots of the nonzero eigenvalues of $L_B L_C \in \mathcal{L}(Z_\beta)$ for β satisfying $-1 < \beta < 1 + \alpha$.*

Proof First we prove the result for $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$. From Lemma 2.5.1.3, we know that $L_B = \mathcal{B}\mathcal{B}' \in \mathcal{L}\left(Z'_\gamma, Z_\beta\right)$ for $\beta < \frac{1}{2} + \alpha$ and $\gamma > -\frac{1}{2} - \alpha$. Furthermore, from Lemma 2.5.2.3, $L_C = \mathcal{C}'\mathcal{C} \in \mathcal{L}\left(Z_\beta, Z'_\gamma\right)$ for $\beta > -\frac{1}{2}$ and $\gamma < \frac{1}{2}$.

Let us choose β and γ satisfying

$$-\frac{1}{2} - \alpha < \gamma < \frac{1}{2}, \quad -\frac{1}{2} < \beta < \frac{1}{2} + \alpha.$$

(We remark that such a choice is possible, since $\alpha > -1$.) Consequently,

$$L_B L_C = \mathcal{B}\mathcal{B}'\mathcal{C}'\mathcal{C} = \mathcal{B}\Gamma^*\mathcal{C} \in \mathcal{L}(Z_\beta),$$

and owing to the compactness of Γ (and hence of Γ^*), $L_B L_C \in \mathcal{L}(Z_\beta)$ is compact for $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$.

Let $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$ and suppose that $\lambda \neq 0$ is an eigenvalue of $L_B L_C \in \mathcal{L}(Z_\beta)$ and v a corresponding eigenvector; $L_B L_C v = \lambda v$. Then

$$\Gamma\Gamma^*\mathcal{C}v = \mathcal{C}\mathcal{B}\mathcal{B}'\mathcal{C}'\mathcal{C}v = \mathcal{C}L_B L_C v = \mathcal{C}(\lambda v) = \lambda\mathcal{C}v,$$

and so λ is an eigenvalue of $\Gamma\Gamma^*$, since $\mathcal{C}v \neq 0$ (for otherwise $L_B L_C v = L_B \mathcal{C}'\mathcal{C}v = 0$, a contradiction).

Conversely, suppose that μ is an eigenvalue of $\Gamma\Gamma^*$, with corresponding eigenvector y. Then

$$L_B L_C L_B \mathcal{C}' y = L_B L_C \mathcal{B}\mathcal{B}'\mathcal{C}' y = L_B L_C \mathcal{B}\Gamma^* y,$$

and

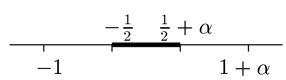
$$L_B L_C L_B \mathcal{C}' y = \mathcal{B}\mathcal{B}'\mathcal{C}'\mathcal{C}\mathcal{B}\mathcal{B}'\mathcal{C}' y = \mathcal{B}\Gamma^*\Gamma\Gamma^* y = \mathcal{B}\Gamma^*(\mu y) = \mu\mathcal{B}\Gamma^* y.$$

So μ is an eigenvalue of $L_B L_C$, since $\mathcal{B}\Gamma^* y \neq 0$ (for otherwise $\Gamma\Gamma^* y = \mathcal{C}\mathcal{B}\Gamma^* y = 0$, a contradiction).

Finally, we extend the result to all β satisfying $-1 < \beta < 1 + \alpha$. Since $L_B L_C \in \mathcal{L}(Z_\beta)$ is compact for $-1 < \beta < 1 + \alpha$, it has a pure point spectrum. Moreover, by Lemmas 2.5.1 and 2.5.2,

$$L_B L_C \in \mathcal{L}(Z_\beta, Z_{\beta+\theta}) \quad \text{for} \quad -1 < \beta < 1 + \alpha - \theta,$$

for any θ satisfying $0 < \theta < \min\left\{1 + \alpha, \frac{1}{2}\right\}$. From this it follows that the nonzero point spectra of $L_B L_C \in \mathcal{L}(Z_\beta)$ and $L_B L_C \in \mathcal{L}(Z_{\beta+\theta})$ are the same for all $\beta \in (-1, 1 + \alpha - \theta)$.



Hence the nonzero point spectrum of $L_B L_C \in \mathcal{L}(Z_\beta)$ is independent of β and so $\sigma(L_B L_C) = \sigma(\Gamma^*\Gamma)$ for all β satisfying $-1 < \beta < 1 + \alpha$. ■

Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ denote the Hankel singular values of the system. Furthermore, let σ be such that $\sigma_{l+1} < \sigma < \sigma_l$. Then we have the following useful lemma which will be used in the next section.

Lemma 2.5.9 *Under the assumptions A1, A2 and A3 with $U = \mathbb{C}^m$ and $Y = \mathbb{C}^p$, if $\sigma_{l+1} < \sigma < \sigma_l$, then $J := I - \frac{1}{\sigma^2}\Gamma^*\Gamma \in \mathcal{L}(L_2([0,\infty), \mathbb{C}^m))$ has a spectrum $\sigma(J)$ contained in $(-\infty, -\delta) \cup (\delta, \infty)$ for some $\delta > 0$, and $\sigma(J) \cap (-\infty, -\delta)$ consists of exactly l negative eigenvalues.*

Proof $\Gamma^*\Gamma$ is compact and has a pure point spectrum with 0 as the accumulation point. Considering the resolvent $\left[\left(I - \frac{1}{\sigma^2}\Gamma^*\Gamma\right) - \lambda I\right]^{-1}$, it is easy to see that J has a spectrum which is a shifted version of the spectrum of $\frac{1}{\sigma^2}\Gamma^*\Gamma$. Finally, since $\Gamma^*\Gamma$ has a pure point spectrum $\left\{\sigma_1^2, \sigma_2^2, \ldots\right\}$ with 0 as the accumulation point, and since $\sigma_{l+1} < \sigma < \sigma_l$, J has exactly l negative eigenvalues. ■

Analytic semigroups are generated by parabolic and some hyperbolic partial differential equations. For a change, we give a different example of an analytic system from the class of fractional transfer functions.

Example 2.5.10 *(The fractional transfer function $\frac{1}{(1+s)^m}$, where $0 < m < 1$.)*

Let $\alpha \in \mathbb{R}$ satisfy $-1 < \alpha < m-1$ (< 0). We denote by Z the vector space

$$Z = \left\{ f \in L_2(0,\infty) \mid \int_0^\infty (1+x)^{-\alpha} |f(x)|^2 dx < \infty \right\},$$

with the inner product

$$\langle f_1, f_2 \rangle_Z = \int_0^\infty (1+x)^{-\alpha} f_1(x) \overline{f_2(x)} dx.$$

It is easy to see that Z is a Hilbert space. Let $R_A : Z \to Z$ be the multiplication operator by $x \mapsto \frac{1}{1+x}$:

$$(R_A f)(x) = \frac{1}{1+x} f(x) \text{ for all } f \in Z.$$

The operator $R_A \in \mathcal{L}(Z)$, since

$$\|R_A f\|_Z^2 = \int_0^\infty (1+x)^{-\alpha} \frac{1}{(1+x)^2} |f(x)|^2 dx \le \int_0^\infty (1+x)^{-\alpha} |f(x)|^2 dx = \|f\|_Z^2.$$

Furthermore, R_A is injective, since if $0 = (R_A f)(x) = \frac{1}{1+x} f(x)$ for almost all $x \in (0,\infty)$, then $f(x) = 0$ for almost all $x \in (0,\infty)$. Thus one can define an algebraic inverse of R_A, which we denote by A, as follows:

$$D(A) = \operatorname{ran}(R_A) \text{ and } AR_A f = f.$$

It is immediate that A is well-defined. The operator A is closed, since the graphs of A and R_A are related by the following isomorphism:

$$\mathbb{I} : Z \times Z \to Z \times Z, \ \mathbb{I}(f_1, f_2) = (f_2, f_1) \text{ for all } (f_1, f_2) \in Z \times Z,$$

We have

$$D(A) = \operatorname{ran}(R_A) = \{R_A f \in Z \mid f \in Z\} = \{g \in Z \mid (1+\cdot) g(\cdot) \in Z\}.$$

Moreover, if $g \in D(A)$, $(Ag)(x) = (1+x)g(x)$. The set $D(A)$ contains continuous functions compactly supported in $(0,\infty)$, that is, $\mathcal{C}_{00}(0,\infty) \subset D(A)$. If $f \in Z$, then $(1+\cdot)^{-\frac{\alpha}{2}} f(\cdot) \in L_2(0,\infty)$. Since $\mathcal{C}_{00}(0,\infty)$ is dense in $L_2(0,\infty)$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{C}_{00}(0,\infty)$ such that $f_n \to (1+\cdot)^{-\frac{\alpha}{2}} f(\cdot)$ in $L_2(0,\infty)$, that is

$$(1+\cdot)^{-\frac{\alpha}{2}} \frac{f_n(\cdot)}{(1+\cdot)^{-\frac{\alpha}{2}}} \stackrel{L_2(0,\infty)}{\longrightarrow} (1+\cdot)^{-\frac{\alpha}{2}} f(\cdot).$$

Hence we have $\frac{f_n(\cdot)}{(1+\cdot)^{-\frac{\alpha}{2}}} \stackrel{Z}{\longrightarrow} f$, where $\frac{f_n(\cdot)}{(1+\cdot)^{-\frac{\alpha}{2}}} \in \mathcal{C}_{00}(0,\infty)$. So $D(A)$ is dense in Z.

It is clear that $R_A = R_A^*$: indeed, for any f and g in Z, we have

$$\begin{aligned}\langle R_A f, g\rangle_Z &= \int_0^\infty (1+x)^{-\alpha}\frac{1}{1+x}f(x)\overline{g(x)}dx \\ &= \int_0^\infty (1+x)^{-\alpha} f(x)\overline{\left[\frac{g(x)}{1+x}\right]}dx \\ &= \langle f, R_A g\rangle_Z .\end{aligned}$$

Thus from Lemma A.3.65.(c) (page 603, Curtain and Zwart [34]), we have

$$(A^*)^{-1} = \left(A^{-1}\right)^* = R_A^* = R_A.$$

So $R_A A^* f = f$ for all $f \in D(A^*)$, $A^* R_A f = f$ for all $f \in Z$ and $D(A^*) = \operatorname{ran}(R_A) = D(A)$. Furthermore, if $f \in D(A) = D(A^*)$, then

$$Af = A\left(R_A A^* f\right) = \left(AR_A\right)\left(A^* f\right) = A^* f$$

and so A is self-adjoint. If $f \in D(A)$, then we have

$$\begin{aligned}\langle Af, f\rangle_Z &= \int_0^\infty (1+x)^{-\alpha}(1+x)f(x)\overline{f(x)}dx \\ &= \int_0^\infty (1+x)^{-\alpha}(1+x)|f(x)|^2 dx \\ &\geq \int_0^\infty (1+x)^{-\alpha}|f(x)|^2 dx \\ &= \|f\|_Z^2.\end{aligned}$$

A is a nonnegative self-adjoint operator and so using Example 1.25 (page 493, Kato [52]) it follows that $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$.

<u>The resolvent of $-A$.</u> It can be shown that $\rho(-A) = \mathbb{C}(-\infty, -1]$ and for $\lambda \in \rho(-A)$,

$$(\lambda I + A)^{-1} f(x) = \frac{1}{\lambda + 1 + x} f(x) \quad \text{for all } f \in Z.$$

<u>The point spectrum of $-A$ is empty.</u> $\sigma(-A) = \mathbb{C}(\mathbb{C}(-\infty, -1]) = (-\infty, -1]$. Let $-Af = \lambda f$, $f \in D(A)$, and $f \neq 0$. So $(-Af)(x) = \lambda f(x)$, that is, $-(1+x)f(x) = \lambda f(x)$. But since $x \neq \lambda + 1$ almost everywhere, $f(x) = 0$ almost everywhere. Thus $f = 0$ (a contradiction!). So the point spectrum of A is empty, and its (continuous) spectrum is $(-\infty, -1]$, see Figure 2.7.

<u>The semigroup $\{T(t)\}_{t\geq 0}$.</u> Given $t \geq 0$, define $T_*(t) \in \mathcal{L}(Z)$ by $\left(T_*(t) f\right)(x) = e^{-(1+x)t} f(x)$. The operator $T_*(t)$ is well-defined since for all

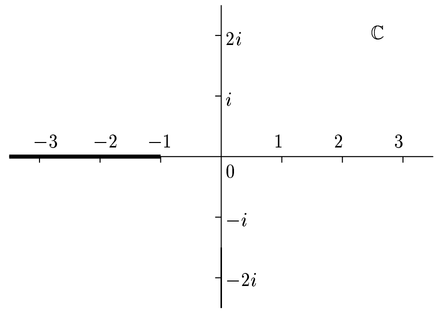


Figure 2.7: $\sigma(A) = \sigma_c(A) = (-\infty, -1]$.

$f \in Z$, we have

$$\begin{aligned} \|T_*(t)f\|_Z^2 &= \int_0^\infty (1+x)^{-\alpha} \left|(T_*(t)f)(x)\right|^2 dx \\ &= \int_0^\infty (1+x)^{-\alpha} e^{-2(1+x)t} |f(x)|^2 dx \\ &\le e^{-2t} \int_0^\infty (1+x)^{-\alpha} |f(x)|^2 dx \\ &= e^{-2t} \|f\|_Z^2. \end{aligned} \tag{2.30}$$

Next we prove that $\{T_*(t)\}_{t \ge 0}$ is a strongly continuous semigroup on Z:

1. $(T_*(0)f)(x) = f(x)$ for all $f \in Z$, and so $T_*(0) = I$.
2. For any $f \in Z$, we have

$$\begin{aligned} (T_*(\tau+t)f)(x) &= e^{-(1+x)(\tau+t)} f(x) \\ &= e^{-(1+x)\tau} \left[e^{-(1+x)t} f(x) \right] \\ &= e^{-(1+x)\tau} (T_*(t)f)(x) \\ &= (T_*(\tau)(T_*(t)f))(x). \end{aligned}$$

   Thus $T_*(\tau+t) = T_*(\tau)T(t)$ for all nonnegative t and τ.
3. Finally we prove the strong continuity at $t = 0$. Let $0 \ne f \in Z$. We have

$$\|T_*(t)f - f\|_Z^2 = \int_0^\infty (1+x)^{-\alpha} \left| e^{-(1+x)t} - 1 \right|^2 |f(x)|^2 dx.$$

   Choose a $M > 0$ large enough such that $0 < \int_M^\infty |f(x)|^2 dx < \frac{\epsilon}{2}$. Next, choose a $\delta > 0$ small enough so that $0 \le t < \delta$ implies that

$$\sup_{x \in [0,M]} \left| e^{-(1+x)t} - 1 \right| < \frac{\epsilon}{2 \int_0^M (1+x)^{-\alpha} |f(x)|^2 dx}.$$

So $\lim_{t\searrow 0}\|T_*(t)f-f\|_2=0$, for all $f\in Z$. This proves the strong continuity.

Hence $\{T_*(t)\}_{t\geq 0}$ is a strongly continuous semigroup on $L_2(0,\infty)$.

Next we prove that the generator of $\{T_*(t)\}_{t\geq 0}$ is $(-A,D(A))$. Let us denote the infinitesimal generator of $\{T_*(t)\}_{t\geq 0}$ by $(A_*,D(A_*))$. From Lemma 2.1.11 (page 24, Curtain and Zwart [34]), for all $f\in Z$ we have

$$(\lambda I-A_*)^{-1}f=\int_0^\infty e^{-\lambda t}T_*(t)f dt$$

for Re(λ)> ω_{0*}, the growth bound of the semigroup $\{T_*(t)\}_{t\geq 0}$. Consequently, for $\lambda\in\mathbb{C}$ such that Re(λ)>max$\{-1,\omega_{0*}\}$ we obtain

$$\begin{aligned}\left((\lambda I-A_*)^{-1}f\right)(x) &= \int_0^\infty e^{-\lambda t}\left(T_*(t)f\right)(x)dt\\ &= \frac{1}{\lambda+1+x}f(x)\\ &= \left((\lambda I+A)^{-1}f\right)(x).\end{aligned}$$

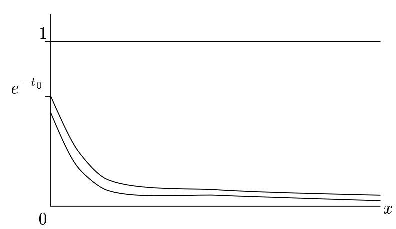
From this it follows easily that $D(A)=D(A_*)$ and $A_*=-A$.

Since the semigroup is determined by its infinitesimal generator (see for example, Theorem 5.5, page 21, Pazy [67]), it follows that the semigroup $\{T(t)\}_{t\geq 0}$ is in fact the semigroup $\{T_*(t)\}_{t\geq 0}$.

From (2.30) it follows that $\|T(t)\|\leq e^{-t}$ and so the growth bound of $\{T(t)\}_{t\geq 0}$ is less than -1. Hence the semigroup $\{T(t)\}_{t\geq 0}$ is exponentially stable.

Remarks:

1. Since the semigroup $\{T(t)\}_{t\geq 0}$ is analytic, the map $T(\cdot):(0,\infty)\to\mathcal{L}(Z)$ is continuous in the $\mathcal{L}(Z)-$topology (see for example, Proposition 2.1.1.*(iv)*, page 35, Lunardi [56]). This can also be seen directly:



First we show that for α, β both greater than zero,

$$\sup_{x\in(0,\infty)}\left|e^{-\alpha(1+x)}-e^{-\beta(1+x)}\right|<\frac{|\alpha-\beta|}{\min\{\alpha,\beta\}}.$$

From the mean value theorem applied to the function $\theta \mapsto e^{-\theta(1+x)}$: $(0,\infty) \to \mathbb{R}$, we have

$$\frac{e^{-\alpha(1+x)} - e^{-\beta(1+x)}}{\alpha - \beta} = (x+1)e^{-\gamma(1+x)},$$

for some $\gamma \in (\min\{\alpha,\beta\}, \max\{\alpha,\beta\})$. It is easy to see that the function $x \mapsto (x+1)e^{-\gamma(1+x)} : (0,\infty) \to \mathbb{R}$ is bounded by $\frac{1}{\gamma}$: we have

$$e^{\gamma(1+x)} = 1 + \gamma(1+x) + \frac{[\gamma(1+x)]^2}{2!} + \ldots > \gamma(1+x),$$

and so $(1+x)e^{-\gamma(1+x)} < \frac{1}{\gamma}$. Finally, using the fact that $\gamma > \min\{\alpha,\beta\}$, we have the desired result.

Thus for every $t > 0$, $\lim_{t\to 0} \|T(t+t_0) - T(t_0)\| = 0$.

2. However, the semigroup $\{T(t)\}_{t\geq 0}$ is not continuous at $t = 0$ in the norm topology. Indeed, if it were, then its infinitesimal generator A would be bounded (see for example Theorem 1.2, page 2, Pazy [67]); but we know that A is unbounded, since $D(A)$ is not closed.

We note that $A^\theta h(x) = (1+x)^\theta h(x)$, $h \in D\left(A^\theta\right)$, and for each $\theta \in \mathbb{R}$, we define Z_θ as described earlier.

Define $B : \mathbb{C} \to Z_\alpha$ by $(Bu)(x) = x^{-\frac{m}{2}}u$. Using

$$\int_0^\infty x^p(1+x)^q dx = \frac{\Gamma(p)\Gamma(1-p-q)}{\Gamma(-q)} \quad \text{if } p > -1 \text{ and } p+q < -1, \tag{2.31}$$

it is easy to see that $B \in \mathcal{L}(\mathbb{C}, Z_\alpha)$.

Let $C : Z \to \mathbb{C}$ be given by

$$Cf = \frac{\sin(m\pi)}{\pi} \int_0^\infty x^{-\frac{m}{2}} f(x)dx \text{ for all } f \in Z.$$

From (2.31) and the Cauchy-Schwarz inequality it follows that $C \in \mathcal{L}(Z, \mathbb{C})$.

Thus all the conditions in Theorem 2.5.3 are satisfied and so the triple $(-A, B, C)$ defines a regular well-posed linear system on Z_β, where $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$, and it realizes the transfer function $G(s) = \frac{1}{(1+s)^m}$: Indeed we have

$$\begin{aligned} G(s) &= C(sI+A)^{-1}B \\ &= \frac{\sin(m\pi)}{\pi} \int_0^\infty x^{-\frac{m}{2}}(s+1+x)^{-1}x^{-\frac{m}{2}}dx \\ &= \frac{\sin(m\pi)}{\pi} \int_0^\infty \frac{x^{-m}}{s+1+x}dx \\ &= \frac{1}{(1+s)^m}, \end{aligned}$$

where we choose the following branch: for $s \in \mathbb{C} \setminus (-\infty, -1]$,

$$\frac{1}{(1+s)^m} = e^{-m[\log|1+s| + i\,\mathrm{Arg}\,(1+s)]}, \quad -\pi < \mathrm{Arg}\,(1+s) < \pi$$

(see for instance pages 187-188, Lang [54]). ◊

2.6 Transfer function algebras

In this section, we introduce some function spaces which will be used in the sequel. Throughout this thesis, we use the following notation. For $r \in \mathbb{R}$ (see Figure 2.8)

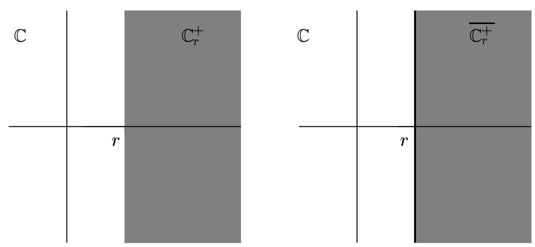


Figure 2.8: The half-planes $\mathbb{C}_r^+$ and $\overline{\mathbb{C}_r^+}$.

$$\mathbb{C}_r^+ = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > r\}, \qquad \overline{\mathbb{C}_r^+} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq r\},$$
$$\mathbb{C}_r^- = \{s \in \mathbb{C} \mid \mathrm{Re}(s) < r\}, \qquad \overline{\mathbb{C}_r^-} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \leq r\}.$$

1. $H_\infty^c(\mathbb{C}^{p\times m})$ denotes the set of complex $p \times m$ matrix-valued functions defined in the closed right half-plane, which are bounded and analytic in $\mathbb{C}_0^+$, and continuous in $\overline{\mathbb{C}_0^+}$. The set $H_\infty^c(\mathbb{C})$, with point-wise addition and multiplication, is a commutative ring with identity.

2. $H_{\infty,l}^c(\mathbb{C}^{p\times m})$ denotes the set of complex $p \times m$ matrix-valued functions $X(\cdot)$ of a complex variable defined in the closed right half-plane with a decomposition $X = G_* + F$, where G_* is the matrix transfer function of a system of MacMillan degree *at most equal* to l, with all its poles in the open right half-plane, and $F \in H_\infty^c(\mathbb{C}^{p\times m})$.

3. $H_{\infty,[l]}^c(\mathbb{C}^{p\times m})$ denotes the set of complex $p \times m$ matrix-valued functions $X(\cdot)$ of a complex variable defined in the closed right half-plane with a decomposition $X = G_* + F$, where G_* is the matrix transfer function of a system of MacMillan degree *equal* to l, with all its l poles in the open right half-plane, and $F \in H_\infty^c(\mathbb{C}^{p\times m})$.

4. $\mathcal{S}$ denotes the set of complex-valued functions $g \in H_\infty^c(\mathbb{C})$ that have a nonzero limit at infinity in $\overline{\mathbb{C}_0^+}$, finitely many zeros in $\overline{\mathbb{C}_0^+}$, and the zeros are all contained in the open right half-plane.

5. $\mathcal{T}$ denotes the set of complex-valued functions $g \in H_\infty(\mathbb{C})$ that have a nonzero limit at infinity in $\mathbb{C}_0^+$, finitely many zeros in $\mathbb{C}_0^+$, and there exists a $\delta > 0$ such that $|g^{-1}|$ is bounded for $0 < \mathrm{Re}(s) < \delta$.

6. $H_{\infty,l}(\mathbb{C}^{p\times m})$ denotes the set of $p \times m$ matrix-valued transfer functions $X(\cdot)$ of a complex variable defined in the open right half-plane with a decomposition $X = G_* + F$, where G_* is the matrix transfer function of a system of MacMillan degree at most equal to l, with all its poles in the open right half-plane, and $F \in H_\infty(\mathbb{C}^{p\times m})$, the Hardy space of functions analytic and bounded in the open right half-plane.

7. $H_{\infty,[l]}(\mathbb{C}^{p\times m})$ denotes the set of $p \times m$ matrix-valued transfer functions $X(\cdot)$ of a complex variable defined in the open right half-plane with a decomposition $X = G_* + F$, where G_* is the matrix transfer function of a system of MacMillan degree l, with all its poles in the open right half-plane, and $F \in H_\infty(\mathbb{C}^{p\times m})$, the Hardy space of functions analytic and bounded in the open right half-plane.

8. $\mathcal{R}$ denotes the class of proper, rational functions g with complex coefficients such that g has no poles in $\overline{\mathbb{C}_0^+}$.

9. $\mathcal{R}_\infty$ denotes the class of proper, rational functions g with complex coefficients such that g has no poles in $\overline{\mathbb{C}_0^+}$, and has a nonzero limit at infinity.

10. $\mathcal{M}H_\infty^c$ denotes the set of matrices (of any size) with each entry in $H_\infty^c(\mathbb{C})$. Similarly, we use $\mathcal{M}H_\infty$, $\mathcal{M}\mathcal{R}$, $\mathcal{M}H_{\infty,\bullet}^c$, and so on.

Suppose M and N belong to $\mathcal{M}H_\infty^c$ ($\mathcal{M}H_\infty$) and have the same number of columns. Then the pair (M, N) is *right coprime* over $\mathcal{M}H_\infty^c$ ($\mathcal{M}H_\infty$) if there exist X and Y in $\mathcal{M}H_\infty^c$ ($\mathcal{M}H_\infty$) such that the following *Bezout identity* holds:

$$XM - YN = I \text{ for all } s \in \overline{\mathbb{C}_0^+} \quad \left(\mathbb{C}_0^+\right).$$

Suppose that G belongs to $H_{\infty,[l]}^c(\mathbb{C}^{p\times m})$ ($H_{\infty,[l]}(\mathbb{C}^{p\times m})$) and that the pair (M, N) is right coprime over $\mathcal{M}H_\infty^c$ ($\mathcal{M}H_\infty$). If M is such that $\det(M) \in \mathcal{S}$ ($\mathcal{T}$) and $G = NM^{-1}$, then we call this a *right coprime factorization* of G over $\mathcal{M}H_\infty^c$ ($\mathcal{M}H_\infty$, respectively).

Next we prove a few elementary facts concerning elements from the above classes of transfer functions, which are used in the proofs in the remainder of this thesis. Note that the following lemmas have two versions either with or without continuity on the imaginary axis. We only prove the results for $\mathcal{M}H_\infty^c$ unless the proofs for $\mathcal{M}H_\infty$ differ significantly.

Lemma 2.6.1 *If* $f \in H^c_\infty(\mathbb{C})$ *(*$H_\infty(\mathbb{C})$*) and* $g \in \mathcal{S}$ *(*$\mathcal{T}$*) has at most* l *zeros, then* $\frac{f}{g} \in H^c_{\infty,l}(\mathbb{C})$ *(*$H_{\infty,l}(\mathbb{C})$*).*

Proof $\frac{f}{g}$ has a Laurent expansion around each zero p_i of g. So

$$\frac{f(s)}{g(s)} = \sum_{i=1}^{r}\sum_{j=1}^{m_i} \frac{a_{i,j}}{(s-p_i)^j} + \alpha(s),$$

where $p_1, \ldots, p_r$ are the zeros of g with multiplicities $m_1, \ldots, m_r$ respectively ($\sum_{i=1}^r m_i \le l$), and α is analytic in $\mathbb{C}_0^+$.

1. Since g has a nonzero limit at infinity in $\overline{\mathbb{C}_0^+}$, given $\epsilon > 0$, there exists an M such that $|s| > M$ implies that

$$\left| g(s) - \lim_{\substack{s\to\infty \\ s\in\overline{\mathbb{C}_0^+}}} g(s) \right| < \epsilon.$$

   Thus $\frac{1}{|g(\cdot)|}$ is bounded for $|s| > M$. Since $f \in H^c_\infty$, $|f(\cdot)|$ is bounded for $|s| > M$. Moreover, choosing M large enough to include all the zeros p_i's in the open disk $\{|s| < M\}$, we have that $\left|\sum_{i=1}^r \sum_{j=1}^{m_i} \frac{a_{i,j}}{(\cdot - p_i)^j}\right|$ is bounded for $|s| > M$. Thus α is bounded for $\Xi_0 := \left\{s \in \mathbb{C}_0^+ \mid |s| > M\right\}$.

2. Let $\epsilon := \min\{\mathrm{Re}(p_1), \ldots, \mathrm{Re}(p_r)\}$. Consider the compact set $\Xi_1 := \left\{s \in \mathbb{C}_0^+ \mid |s| \le M,\ \mathrm{Re}(s) \ge \frac{\epsilon}{2}\right\}$. Since α is analytic in $\mathbb{C}_0^+$, in particular it is continuous in Ξ_1 and $|\alpha(\cdot)|$ is bounded for $s \in \Xi_1$.

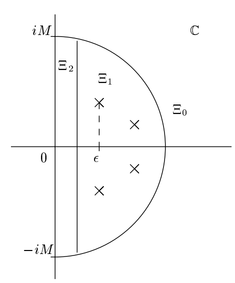


Figure 2.9: The sets Ξ_0, Ξ_1 and Ξ_2.

3. Finally consider the set $\Xi_2 := \left\{s \in \mathbb{C}_0^+ \mid |s| < M,\ \mathrm{Re}(s) < \frac{\epsilon}{2}\right\}$ (see Figure 2.9). We prove that $\inf_{s\in\Xi_2} |g(s)| > 0$. If not, there exists a sequence $\{s_n\}$ in Ξ_2 such that $g(s_n) \to 0$. Consider the compact set $\overline{\Xi_2}$, the closure of Ξ_2. There exists a convergent subsequence $\{s_{n_k}\}$ of $\{s_n\}$

such that $s_{n_k} \to s_0 \in \overline{\Xi_2}$ and $g(s_{n_k}) \to 0$. But since $g \in H^c_\infty(\mathbb{C})$, it follows that $g(s_0) = 0$, which is a contradiction, since the zeros of g are contained in the complement of $\overline{\Xi_2}$. Thus $\inf_{s\in\Xi_2} |g(s)| > 0$, which implies that $\sup_{s\in\Xi_2} \frac{1}{|g(s)|} < \infty$. Hence $\left|\frac{f(s)}{g(s)}\right|$ is bounded in Ξ_2, and since $\{p_1, \dots, p_r\} \subset \complement\overline{\Xi_2}$, $\left|\sum_{i=1}^r \sum_{j=1}^{m_i} \frac{a_{i,j}}{(\cdot - p_i)^j}\right|$ is bounded in Ξ_2. Thus $|\alpha(\cdot)|$ is bounded in Ξ_2.

Since $\mathbb{C}_0^+ = \Xi_0 \cup \Xi_1 \cup \Xi_2$, it follows from the above that $\alpha(\cdot)$ is bounded in $\mathbb{C}_0^+$.

Finally, since f and g are in H^c_∞, and the zeros of g, namely $p_1, \dots, p_r$ are contained in $\mathbb{C}_0^+$, it follows that

$$s \mapsto \alpha(s) = \frac{f(s)}{g(s)} - \sum_{i=1}^{r} \sum_{j=1}^{m_i} \frac{a_{i,j}}{(s-p_i)^j}$$

is continuous in $\overline{\mathbb{C}_0^+} \cap \left\{ s \in \overline{\mathbb{C}_0^+} \mid \mathrm{Re}(s) \leq \frac{\epsilon}{2} \right\}$. Moreover, α is analytic in $\mathbb{C}_0^+$. Thus α is continuous in $\overline{\mathbb{C}_0^+}$ and so $\alpha \in H^c_\infty(\mathbb{C})$. This completes the proof of this lemma. ∎

Lemma 2.6.2 *If $K \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$ $(H_{\infty,[l]}(\mathbb{C}^{p\times m}))$, then there exists a right coprime factorization $K = NM^{-1}$, where M is rational,* $\det(M) \in \mathcal{R}_\infty$ *has exactly l zeros in $\overline{\mathbb{C}_0^+}$ and they are all contained in $\mathbb{C}_0^+$.*

Proof Let $K = G_* + F$, where G_* is the matrix transfer function of a system of MacMillan degree equal to l, with all its l poles in the open right half-plane, and $F \in H^c_\infty(\mathbb{C}^{p\times m})$. Let $G_* = N_1 M^{-1}$ be a right coprime factorization of G_* over $\mathcal{MR}$ (which exists by Lemma A.7.37, Curtain and Zwart [34]). We claim that $K = (FM + N_1)M^{-1}$ is a right coprime factorization over $\mathcal{M}H^c_\infty$ for K. M is square with $\det(M) \in \mathcal{R}_\infty \subset \mathcal{S}$ having exactly l zeros in $\overline{\mathbb{C}_0^+}$ (this follows for example, from page 287, Rugh [76] and the fact that G_* has MacMillan degree l), and they are all contained in $\mathbb{C}_0^+$ and $N := FM + N_1 \in \mathcal{M}H^c_\infty$ by the ring properties of $H^c_\infty(\mathbb{C})$. Since (N_1, M) is a rational coprime pair over $\mathcal{MR}$, there exist X_*, $Y_* \in \mathcal{MR}$ such that $X_*M - Y_*N_1 = I$. Defining X and Y in $\mathcal{M}H^c_\infty$ by $X = X_* + Y_*F$, $Y = Y_*$, we see that

$$XM - YN = [X_* + Y_*F]\,M - Y_*[FM + N_1] = I$$

and so (N, M) is a right coprime pair over $\mathcal{M}H^c_\infty$. ∎

Lemma 2.6.3 *If*

1. M *and* N *belong to* $\mathcal{M}H^c_\infty$ *(*$\mathcal{M}H_\infty$*),*
2. M *and* N *have the same number of columns,*
3. M *is a square matrix with* $\det(M) \in \mathcal{S}$ *(*$\mathcal{T}$*) and*
4. $\det(M)$ *has* l *zeros in the open right half-plane,*

then $NM^{-1} \in H^c_{\infty,l}(\mathbb{C}^{p\times m})$ *(*$H_{\infty,l}(\mathbb{C}^{p\times m})$*).*

Proof *1.* The inverse of the square matrix M is given by

$$M^{-1} = \frac{1}{\det(M)} \operatorname{adj}(M),$$

where $\operatorname{adj}(M) \in \mathcal{M}H^c_\infty$, since its components are sums and products of the components of M, which are all in $H^c_\infty(\mathbb{C})$ by assumption. Thus from Lemma 2.6.1, we have that all the entries in NM^{-1} belong to $H^c_{\infty,l}(\mathbb{C})$ and so NM^{-1} can be written as the sum of a proper rational matrix G_* and a matrix $F \in \mathcal{M}H^c_\infty$. Thus $NM^{-1} \in H^c_{\infty,k}(\mathbb{C}^{p\times m})$, for some nonnegative integer k. In the remainder of the proof, we will show that $k \leq l$, hence proving the claim.

2. Since $NM^{-1} \in H^c_{\infty,k}(\mathbb{C}^{p\times m})$, it follows from Lemma 2.6.2 that it has a right coprime factorization $N_* M_*^{-1}$, where M_* is rational, $\det(M_*) \in \mathcal{R}_\infty$ has exactly k zeros in $\overline{\mathbb{C}_0^+}$ and they are all contained in $\mathbb{C}_0^+$. Since (M_*, N_*) is right coprime, there exist X and Y in $\mathcal{M}H^c_\infty$ such that

$$XM_* - YN_* = I.$$

Upon post multiplication by $M_*^{-1}M$, we obtain

$$XM - YN_*M_*^{-1}M = M_*^{-1}M.$$

Using $N_*M_*^{-1} = NM^{-1}$, we have $M_*^{-1}M = XM - YN \in \mathcal{M}H^c_\infty$. Consequently $\frac{\det(M)}{\det(M_*)} \in H^c_\infty(\mathbb{C})$. It is now easy to see that $k \leq l$. ■

Lemma 2.6.4 *If*

$$\begin{array}{rcll} K & \in & H^c_{\infty,l}(\mathbb{C}^{p\times m}) & (H_{\infty,l}(\mathbb{C}^{p\times m}))\,, \\ K_1 & \in & H^c_\infty(\mathbb{C}^{p_*\times p}) & (H_\infty(\mathbb{C}^{p_*\times p}))\,, \quad \textit{and} \\ K_2 & \in & H^c_\infty(\mathbb{C}^{m\times m_*}) & (H_\infty(\mathbb{C}^{m\times m_*}))\,, \end{array}$$

then $K_1KK_2 \in H^c_{\infty,l}(\mathbb{C}^{p_*\times m_*})$ *(*$H_{\infty,l}(\mathbb{C}^{p_*\times m_*})$*).*

Proof From Lemma 2.6.2, $K = NM^{-1}$, where $N, \; M \in \mathcal{M}H_\infty^c$ and $\det(M) \in \mathcal{S}$ has, say $k \; (\leq l)$ zeros in the open right half-plane. Thus

$$K_1 K K_2 = \frac{1}{\det(M)} \, K_1 \, N \, \mathrm{adj}(M) \, K_2$$

and proceeding as in the proof of Lemma 2.6.3, we get that $K_1 K K_2 \in H^c_{\infty,k}(\mathbb{C}^{p_* \times m_*}) \subset H^c_{\infty,l}(\mathbb{C}^{p_* \times m_*})$. ■

Lemma 2.6.5 *If (N, M) is a right coprime factorization of $K \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$ $(H_{\infty,[l]}(\mathbb{C}^{p\times m}))$, and $V \in H^c_\infty(\mathbb{C}^{m\times m})$ $(H_\infty(\mathbb{C}^{m\times m}))$ is invertible as an element of $\mathcal{M}H^c_\infty$ $(\mathcal{M}H_\infty)$, then (NV, MV) is also a right coprime factorization of K.*

Moreover, any two right coprime factorizations of $K \in H^c_{\infty,[l]}(\mathbb{C}^{p\times m})$ $(H_{\infty,[l]}(\mathbb{C}^{p\times m}))$ are unique up to a common right multiplication by an invertible element in $\mathcal{M}H^c_\infty$ $(\mathcal{M}H_\infty)$.

Proof These proofs are analogous to parts b and c of the proof of Theorem 7.2.8, pages 353-354, Curtain and Zwart [34]. ■

Lemma 2.6.6 *If*

$$\begin{array}{rcll} K = NM^{-1} & \in & H^c_{\infty,[l]}(\mathbb{C}^{p\times m}) & \left(H_{\infty,[l]}(\mathbb{C}^{p\times m})\right), \\ N & \in & H^c_\infty(\mathbb{C}^{p\times m}) & \left(H_\infty(\mathbb{C}^{p\times m})\right), \; \textit{and} \\ M & \in & H^c_\infty(\mathbb{C}^{m\times m}) & \left(H_\infty(\mathbb{C}^{m\times m})\right), \end{array}$$

with (M, N) right coprime, then $\det(M)$ has exactly l zeros in $\overline{\mathbb{C}_0^+}$ $(\mathbb{C}_0^+)$, and the zeros are all contained in the open right half-plane.

Proof From Lemma 2.6.2, there exists another coprime factorization of $K = N_* M_*^{-1}$, where $N_* \in H^c_\infty(\mathbb{C}^{p\times m})$, $M_* \in H^c_\infty(\mathbb{C}^{m\times m})$, with (M_*, N_*) right coprime, M_* is rational and $\det(M_*) \in \mathcal{R}_\infty$ has exactly l zeros in $\overline{\mathbb{C}_0^+}$, and the zeros are all contained in $\mathbb{C}_0^+$. From Lemma 2.6.5, $M = M_* U$, where U is an invertible element in $\mathcal{M}H^c_\infty$. From $I = UU^{-1}$ we have that $[\det(U)] \left[\det\left(U^{-1}\right)\right] = 1$, and hence $\det(U(s)) \neq 0$ for $s \in \overline{\mathbb{C}_0^+}$. Since $\det(M) = \det(M_*) \, \det(U)$, it follows that $\det(M)$ has exactly l zeros in $\overline{\mathbb{C}_0^+}$, and the zeros are all contained in the open right half-plane. ■

The following technical lemma will be used in the characterization of solutions in Chapter 4.

Lemma 2.6.7 *If* $K(\cdot) \in \mathcal{M}H^c_{\infty,\bullet}$, *then given any* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that whenever* $0 \leq \zeta \leq \delta$, *we have*

$$\|K(\zeta + i\cdot)\|_\infty \leq \|K(i\cdot)\|_\infty + \epsilon,$$

where $\|\cdot\|_\infty$ *denotes the* $L_\infty-$*norm* [6].

In order to prove the above Lemma 2.6.7, we will use the following generalization of the so-called "3−lines theorem" to the matrix case. The statement and proof of the 3−lines theorem in the scalar case can be found, for instance, in Theorem 12.8, pages 257-258 of W. Rudin [75].

Lemma 2.6.8 *Suppose*

1. $\Omega = \{\zeta + i\omega \mid a < \zeta < b\}$, $\overline{\Omega} = \{\zeta + i\omega \mid a \leq \zeta \leq b\}$,
2. $K : \overline{\Omega} \to \mathbb{C}^{p\times m}$ *is continuous on* $\overline{\Omega}$,
3. K *is analytic in* Ω,
4. $\|K(s)\| \leq B$ *for all* $s \in \Omega$ *and for some fixed* $B < \infty$.

If $M(\zeta) = \sup\{\|K(\zeta + i\omega)\| \mid \omega \in \mathbb{R}\}$ *for* $a \leq \zeta \leq b$, *then we have*

$$M(\zeta)^{b-a} \leq M(a)^{b-\zeta} M(b)^{\zeta-a} \text{ for all } a < \zeta < b.$$

Proof *1.* We assume first that $M(a) = M(b) = 1$. In this case, we have to prove that $\|K(s)\| \leq 1$ for all $s \in \Omega$. For each $\epsilon > 0$, we define an auxiliary function

$$h_\epsilon(s) = \frac{1}{1 + \epsilon(s-a)} \text{ for all } s \in \overline{\Omega}. \tag{2.32}$$

Since $\mathrm{Re}(1 + \epsilon(\zeta + i\omega - a)) = 1 + \epsilon(\zeta - a) \geq 1$ in $\overline{\Omega}$, we have

$$\|h_\epsilon(s)K(s)\| = |h_\epsilon(s)|\,\|K(s)\| \leq 1 \text{ for all } s \in \partial\Omega, \tag{2.33}$$

the boundary of Ω. Also, $|1 + \epsilon(\zeta + i\omega - a)| \geq \epsilon|\omega|$, so that

$$\|h_\epsilon(s)K(s)\| = |h_\epsilon(s)|\,\|K(s)\| \leq \frac{B}{\epsilon|\omega|} \text{ for all } s = \zeta + i\omega \in \overline{\Omega} \setminus \mathbb{R}. \tag{2.34}$$

Let R be the rectangle cutoff from $\overline{\Omega}$ by the lines $y = \pm\frac{B}{\epsilon}$ (see Figure 2.10).

By (2.33) and (2.34), $\|h_\epsilon K(s)\| \leq 1$ on ∂R, the boundary of the rectangle R. Hence $\|h_\epsilon K(s)\| \leq 1$ on R, by the maximum modulus theorem (see for

[6] For a fixed $\xi \in \mathbb{R}$, $\|K(\xi + i\cdot)\|_\infty = \sup_{\omega\in\mathbb{R}} \|K(\xi + i\omega)\|$.

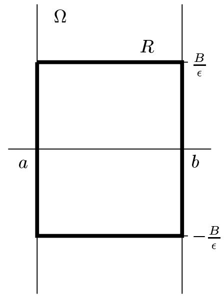


Figure 2.10: The rectangle R.

example, Theorem 3.18.4, page 115, Hille and Phillips [50]). But (2.34) shows that $\|h_\epsilon K(s)\| \leq 1$ on the rest of $\overline{\Omega}$. Thus $\|h_\epsilon K(s)\| \leq 1$ for all $s \in \Omega$ and all $\epsilon > 0$. If we fix $s \in \Omega$ and then let ϵ tend to zero, we obtain the desired result.

2. We now turn to the general case with $M(a) \neq 0$. Let

$$f(s) = M(a)^{\frac{b-s}{b-a}} M(b)^{\frac{s-a}{b-a}},$$

where for $M > 0$ and s complex, M^s is defined by

$$M^s = e^{s \log M},$$

and $\log M$ is real. Then

1. f is analytic in $\mathbb{C}$,
2. f has no zero,
3. $\frac{1}{f}$ is bounded in $\overline{\Omega}$,
4. $|f(a+i\omega)| = M(a)$,
5. $|f(b+i\omega)| = M(b)$,

and so $\frac{1}{f} K$ satisfies our previous assumptions. Thus $\left\| \frac{1}{f(s)} K(s) \right\| \leq 1$ in Ω, and hence

$$\|K(\zeta + i\omega)\| \leq |f(\zeta + i\omega)| = M(a)^{\frac{b-\zeta}{b-a}} M(b)^{\frac{\zeta-a}{b-a}}.$$

Hence $M(\zeta)^{b-a} \leq M(a)^{b-\zeta} M(b)^{\zeta-a}$.

3. Finally, we show that if $M(a) = 0$, then K is identically zero in $\overline{\Omega}$. Let

$$E_{11} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{p \times m}.$$

We have, for positive ϵ,

$$\sup_{\omega\in\mathbb{R}}\{\|\epsilon E_{11} + K(a+i\omega)\|\} = \sup_{\omega\in\mathbb{R}}\{\|\epsilon E_{11} + 0\|\} = \epsilon\|E_{11}\| \ (>0),$$

and

$$\sup_{\omega\in\mathbb{R}}\{\|\epsilon E_{11} + K(b+i\omega)\|\} \leq \sup_{\omega\in\mathbb{R}}\{\epsilon\|E_{11}\| + \|K(b+i\omega)\|\} = \epsilon\|E_{11}\| + M(b).$$

Thus from the above, we obtain

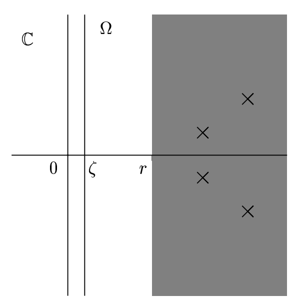
$$\|\epsilon E_{11} + K(\zeta+i\omega)\| \leq \epsilon^{\frac{b-\zeta}{b-a}}\|E_{11}\|^{\frac{b-\zeta}{b-a}}\left(\epsilon\|E_{11}\| + M(b)\right)^{\frac{\zeta-a}{b-a}}.$$

This is true for all $\epsilon > 0$. Passing the limit as ϵ tends to zero, we have

$$\|K(\zeta+i\omega)\| \leq 0\ \|E_{11}\|^{\frac{b-\zeta}{b-a}} M(b)^{\frac{\zeta-a}{b-a}} = 0.$$

Consequently $K(\zeta+i\omega)=0$ for all $\omega\in\mathbb{R}$ and $a\leq\zeta<b$. Finally the continuity of K yields that also $K(b+i\omega)=0$ for all $\omega\in\mathbb{R}$. Thus K is zero in $\overline{\Omega}$. ∎

Proof (of Lemma 2.6.7.) Let $K(s)=G(s)+F(s)$ where G is the matrix rational transfer operator of a system of MacMillan degree, say l, with all its poles in the open right half-plane and $F\in\mathcal{M}H^c_\infty$. Let the poles of G be contained in the half-plane $\mathbb{C}_r^+$ for some $r>0$. Consider the function $s\mapsto K(s)$ defined for s belonging to the infinite strip $\overline{\Omega} := \{s\in\mathbb{C} \mid 0\leq \mathrm{Re}(s)\leq r\}$. Clearly, $K(\cdot)$ is continuous in $\overline{\Omega}$ and analytic in the interior of $\overline{\Omega}$.



Using the triangle inequality, it is easy to see that $K(\cdot)$ is bounded in Ω:

1. $s\mapsto G(s)$ is bounded in Ω (since all its poles are in $\mathbb{C}_r^+$) and

2. $s\mapsto F(s)$ is bounded in Ω (in fact, in $\overline{\mathbb{C}_0^+}$).

For any $\zeta > 0$, define $M(\zeta) = \sup_{\omega\in\mathbb{R}}\{\|K(\zeta + i\omega)\|\}$. Using Lemma 2.6.8, we obtain

$$M(\zeta) \leq [\|K(i\cdot)\|_\infty]^{1-\frac{\zeta}{r}}\, M(r)^{\frac{\zeta}{r}}.$$

If $\|K(i\cdot)\|_\infty = 0$, then we have $M(\zeta) = 0$, and the result follows trivially. We now consider the case when $\|K(i\cdot)\|_\infty \neq 0$: we have

$$M(\zeta) \leq \|K(i\cdot)\|_\infty \left[\frac{M(r)}{\|K(i\cdot)\|_\infty}\right]^{\frac{\zeta}{r}}.$$

Since $\lim_{\zeta\to 0}\left[\frac{M(r)}{\|K(i\cdot)\|_\infty}\right]^{\frac{\zeta}{r}} = 1$, there exists a δ such that $0 < \delta < r$, and for any ζ satisfying $0 \leq \zeta \leq \delta$, we have $\|K(\zeta + i\cdot)\|_\infty \leq \|K(i\cdot)\|_\infty + \epsilon$. ■