Hankel norm approximation for infinite-dimensional systems

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Chapter 1

Introduction

Systems and control theory is a subject in which beautiful and deep mathematical theorems have exactly matched the needs of a vital branch of technology. This conjunction of mathematics and engineering has been enriching for both fields. In this thesis we present a solution to the sub-optimal Hankel norm approximation problem which has been studied by both mathematicians and engineers. In this chapter, among other things, we explain the problem and the relevance of its solution to the important engineering problem of model reduction, in which the aim is to replace a high order (possibly infinite-dimensional) system by a low order model, without undue loss of accuracy. Finally we will give a brief outline of the chapters in this thesis.

In 1957, Zeev Nehari [59] studied the following natural problem. Suppose we are given a sequence $\{a_n\}_{n\geq 0}$ of complex numbers. Under what conditions does the infinite *Hankel matrix* [1]

$$H = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.1}$$

define a bounded linear operator on $\ell_2(\mathbb{N})$?

He found a striking answer: a necessary and sufficient condition is that there should exist an essentially bounded function φ on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ (parameterized as $\{e^{i\theta} \mid 0 \leq \theta \leq 2\pi\}$) such that

$$\text{the } n^{\mathtt{th}} \text{ Fourier coefficient } \hat{\varphi}(n) = a_n, \text{ for all } n \geq 0. \tag{1.2}$$

[1] That is, a matrix $\{c_{ij}\}_{i,j\geq 0}$, where c_{ij} depends only on $i+j$ and so $c_{ij} = a_{i+j}$ for some sequence $\{a_n\}_{n\geq 0}$.

Hermann Hankel (1839-1873).

Furthermore, when (1.2) is satisfied,

$$\|H\|_{\mathcal{L}(\ell_2(\mathbb{N}))} = \text{infimum of the } L_\infty - \text{norms of all such } \varphi. \tag{1.3}$$

Example 1.0.1 If

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

the *Hilbert matrix*, Schur proved that $\|H\|_{\mathcal{L}(\ell_2(\mathbb{N}))} = \pi$. By taking $\varphi\left(e^{i\theta}\right) = i(\pi-\theta)e^{i\theta}$, we have a function with L_∞−norm at most π, and

$$\begin{aligned} \hat{\varphi}(n) &= \frac{1}{2\pi}\int_0^{2\pi} i(\pi-\theta)e^{i\theta}e^{-in\theta}d\theta \\ &= \left[\frac{-i(\pi-\theta)e^{-i(n+1)\theta}}{2\pi(n+1)}\right]_0^{2\pi} - \frac{i}{2\pi(n+1)}\int_0^{2\pi} e^{-i(n+1)\theta}d\theta \\ &= \frac{1}{n+1}, \end{aligned}$$

for all $n \geq 0$. ◊

We can reformulate the above result as follows. Let

$$G(z) = \sum_{n=-\infty}^{\infty} g_n z^n \text{ and } g_n = a_n \text{ for all } n \geq 0.$$

In (1.3), the right hand side is the infimum of $\|\varphi\|_\infty$ over all $L_\infty-$functions φ on the circle such that

$$\hat{\varphi}(n) = a_n, \quad n \geq 0.$$

This class of φ is exactly the coset $G + H_\infty(\mathbb{D})^\perp$ in $L_\infty(\mathbb{T})$, where $H_\infty(\mathbb{D})^\perp$ is the subspace (of $L_\infty(\mathbb{T})$) of functions whose nonnegative Fourier coefficients vanish, which is the space of functions which are analytic and bounded in $\{z \in \mathbb{C} \mid |z| > 1\}$. Hence

$$\begin{aligned} \|H\|_{\mathcal{L}(\ell_2(\mathbb{N}))} &= \inf_{K \in H_\infty(\mathbb{D})^\perp} \|G + K\|_\infty \\ &= \operatorname{dist}_{L_\infty(\mathbb{T})}\left(G, H_\infty(\mathbb{D})^\perp\right). \end{aligned}$$

So Nehari's theorem gives us an expression for the distance of a bounded function on the circle from $H_\infty(\mathbb{D})^\perp$ as the norm of a Hankel matrix.

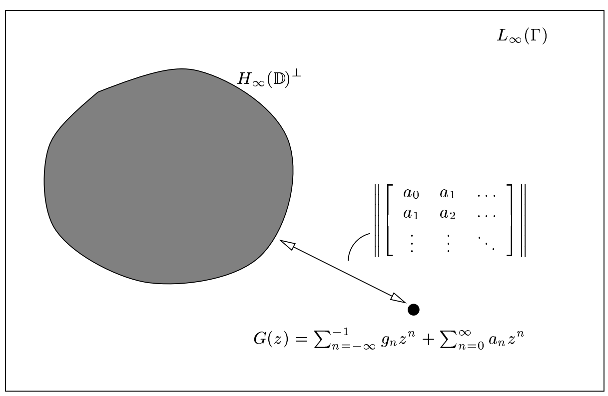


Figure 1.1: Nehari's theorem.

More detailed information relating to this type of $L_\infty-$approximation on the basis of Hankel operator theory was subsequently obtained by V.M. Adamjan, D.Z. Arov and M.G. Krein [2], D. Sarason [80] and S.R. Treil [87]. An excellent introduction to the subject is Young [100] and a survey of results can be found in Nikol'skii [60]. The fact that results on such $L_\infty-$approximation problems were relevant to the preoccupations of some engineers was realized by a number of mathematicians in the seventies. This stimulated further research and the birth of an intensive period of research and interaction by both engineers and mathematicians– the $H_\infty-$era of the 80's (see Francis [38]).

1.1 The sub-optimal Nehari problem

A reformulation of the question answered[2] by Nehari is the following: Given a bounded $m \times n$ matrix-valued function G on the imaginary axis, find the distance of G from $H_\infty(\mathbb{C}^{p\times m})^\perp$ in the L_∞−norm, where $H_\infty(\mathbb{C}^{p\times m})^\perp$ denotes the space of $p \times m$ matrix-valued functions which are bounded and analytic in the open left half-plane. That is, find

$$\begin{aligned}\operatorname{dist}_{L_\infty}\left(G, H_\infty\left(\mathbb{C}^{p\times m}\right)^\perp\right) &= \inf_{K(-\cdot)\in H_\infty(\mathbb{C}^{p\times m})} \|G+K\|_\infty \\ &= \inf_{K(-\cdot)\in H_\infty(\mathbb{C}^{p\times m})} \operatorname{esssup}_{\omega\in\mathbb{R}} \|G(i\omega)+K(i\omega)\|,\end{aligned}$$

where we use the notation $H_\infty(\mathbb{C}^{p\times m})$ for the space of $p \times m$ matrix-valued functions which are bounded and analytic in the open right half-plane. Another way of phrasing the question is to say that we are given a point G in the Banach space L_∞ and we seek the closest point or points in the closed subspace $H_\infty(\mathbb{C}^{p\times m})^\perp$. It is perhaps not surprising that in any Banach space other than a Hilbert space, best approximation problems are usually difficult and one does not expect to solve them exactly. Fortunately for control theory, the Nehari problem is one of the rare cases which does admit a precise solution.

If E is a Banach space, then let $H_2(E)$ denote the set of all analytic functions $f : \mathbb{C}_0^+ \to E$ such that

$$\|f\|_2 := \sup_{\zeta>0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\|f(\zeta+i\omega)\|^2 d\omega\right) < \infty.$$

For $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$ we define the *Hankel operator with symbol* G, denoted by H_G, acting from $H_2(\mathbb{C}^m)$ to $H_2(\mathbb{C}^p)$, as follows:

$$H_G f = \Pi(\Lambda_G f_-) \text{ for } f \in H_2(\mathbb{C}^m),$$

where Λ_G is the multiplication map on $L_2(i\mathbb{R}, \mathbb{C}^m)$ induced by G, Π is the orthogonal projection operator from $L_2(i\mathbb{R}, \mathbb{C}^p)$ onto $H_2(\mathbb{C}^p)$ and $f_-(s) := f(-s)$. The operator H_G **is** related to the infinite Hankel matrix H in (1.1): H_G has a matrix of that form with respect to a suitable orthonormal basis. Nehari's theorem says that for any $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$,

$$\operatorname{dist}_{L_\infty}\left(G, H_\infty(\mathbb{C}^{p\times m})^\perp\right) = \|H_G\|.$$

One is of course interested not only in the value of the infimum, but also in characterizing the set of K's for which the distance is attained. In the case of rational transfer functions there are explicit solutions for K (see Glover [40]). However, for control applications, the focus is on the sub-optimal Nehari problem. The *sub-optimal Nehari problem* is the following:

[2]The matrix case was settled by Adamjan, Arov and Krein in [1].

If $\sigma > \|H_G\|$, then find all $K(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$ such that $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$.

The solutions K to the sub-optimal Nehari problem are the key to the design of controllers which maximize robustness with respect to uncertainty or minimize sensitivity to disturbances of sensors (see McFarlane and Glover [57] and Georgiou and Smith [39]). For the purpose of "model reduction", a notion to be explained later in this chapter, one considers a natural generalization of the Nehari problem, and this problem turns out to be much harder.

1.2 The sub-optimal Hankel norm approximation problem

Let $H_{\infty,l}(\mathbb{C}^{p\times m})$ denote the space of all $p \times m$ matrix-valued functions K of a complex variable defined in the right half-plane such that $K = G_* + F$, where F is an element in $H_\infty(\mathbb{C}^{p\times m})$ and G_* is the transfer function of a finite-dimensional system with MacMillan degree [3] at most l, with all its poles in the open right half-plane. $H_{\infty,l}(\mathbb{C}^{p\times m})$ is a subset of $L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$.

Now we recall the notion of singular values of a bounded linear operator from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$. For $k \in \{1, 2, \ldots\}$ the $k^{\mathtt{th}}$ *singular value* of an operator $\mathtt{H} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ (denoted by $\sigma_k(\mathtt{H})$) is defined to be the distance with respect to the norm in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ of $\mathtt{H}$ from the set of operators in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ of rank at most $k-1$. Thus $\sigma_1(\mathtt{H}) = \|\mathtt{H}\|$, and $\sigma_1(\mathtt{H}) \geq \sigma_2(\mathtt{H}) \geq \sigma_3(\mathtt{H}) \geq \ldots \geq 0$. If $\mathtt{H}$ is compact, then $\mathtt{H}^*\mathtt{H}$ is compact and nonnegative, and so the spectrum of $\mathtt{H}^*\mathtt{H}$ consists of a pure point spectrum with countably many nonnegative eigenvalues. The square roots of these eigenvalues are then the singular values of $\mathtt{H}$.

If the Hankel operator with symbol $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$ is compact, we sometimes denote the singular values of H_G, $\sigma_k(H_G)$, simply by $\sigma_k(G)$. The $\sigma_k(G)$'s are then referred to as the *Hankel singular values of* G.

The following theorem[4] of Adamjan, Arov and Krein [1] is a natural extension

[3] The *MacMillan degree* of a proper rational function G is defined to be the minimal state dimension of all possible (A, B, C, D)'s that realize G, that is

$$\text{MacMillan degree of } G = \min_{\substack{A \in \mathbb{C}^n, \\ B \in \mathbb{C}^{m\times n}, \\ C \in \mathbb{C}^{n\times p}, \\ D \in \mathbb{C}^{p\times m}}} \{n \mid G(s) = C(sI - A)^{-1}B + D\}.$$

[4] the matrix case was proved in Kung and Lin [53].

of Nehari's theorem:

Theorem 1.2.1 *For any* $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$,

$$\inf_{K(-\cdot)\in H_{\infty,l}(\mathbb{C}^{p\times m})} \|G(i\cdot) + K(i\cdot)\|_\infty = \sigma_{l+1}(G).$$

The Nehari theorem corresponds to the case $l = 0$. In Theorem 1.2.1, if $K(\cdot) = G_*(\cdot) + F(\cdot)$, where $G_*(-\cdot)$ is the rational transfer function of a finite-dimensional system with MacMillan degree at most l and with all its poles in the open right half-plane, and $F(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$, then $-G_*(\cdot)$ is said to be an *optimal Hankel norm approximant* of G. The *sub-optimal Hankel norm approximation problem* is the following:

If $\sigma_{l+1} < \sigma < \sigma_l$, then find all $K(-\cdot) \in H_{\infty,l}(\mathbb{C}^{p\times m})$ such that $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$.

K is then called a *solution* of the sub-optimal Hankel norm approximation problem.

1.3 The model reduction problem

An elaborate model of a physical system can sometimes be replaced by a simpler one, which is easier to analyze and use, without a significant loss of accuracy. In this section we shall see how a solution to the sub-optimal Hankel norm approximation problem gives rise to a finite-dimensional reduced model to the original infinite-dimensional system.

1.3.1 The Hankel norm of systems

In practice, many physical systems and engineering devices are modelled by mathematical systems of the type

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \qquad t \geq 0,$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on the Hilbert space X, $B \in \mathcal{L}(\mathbb{C}^m, X)$ and $C \in \mathcal{L}(X, \mathbb{C}^p)$. [5]

[5] More generally, the input space $\mathbb{C}^m$ and the output space $\mathbb{C}^p$ could be Hilbert spaces U and Y, respectively, and B and C could be *un*bounded.

If A is *exponentially stable* (that is, there exists an $\epsilon < 0$ and a $M > 0$ such that $\|T(t)\| \leq Me^{-\epsilon t}$ for all $t \geq 0$), then the *impulse response* $h(\cdot) = CT(\cdot)B$ is an element belonging to $L_1([0,\infty),\mathbb{C}^{p\times m})$, and h induces a compact *time-domain Hankel operator* $\Gamma_h \in \mathcal{L}\left(L_2([0,\infty),\mathbb{C}^m), L_2([0,\infty),\mathbb{C}^p)\right)$, defined as follows: given $u \in L_2([0,\infty),\mathbb{C}^m)$,

$$(\Gamma_h u)(t) = \int_0^\infty h(t+\tau)u(\tau)d\tau \text{ for all } t \geq 0.$$

The impulse response h has the Laplace transform $G(s) = C(sI-A)^{-1}B$ with $G(\cdot) : i\mathbb{R} \to \mathbb{C}^{p\times m} \in L_\infty(i\mathbb{R},\mathbb{C}^{p\times m})$. This induces a compact Hankel operator H_G with symbol G, which is related to Γ_h via Laplace transformation:

$$H_G\hat{f}(s) = \left(\widehat{\Gamma_h f}\right)(s) \text{ for all } s \text{ in the open right half-plane,}$$

and $\|\Gamma_h\| = \|H_G\| \leq \|G\|_\infty \leq \|h\|_{L_1}$. If $u_* \in L_2((-\infty,0],\mathbb{C}^m)$, then $u(\cdot) := u_*(-\cdot) \in L_2([0,\infty),\mathbb{C}^m)$, and we have for $t \geq 0$

$$\left(\Gamma_h u\right)(t) = \int_0^\infty CT(t+\tau)u(\tau)d\tau = \int_{-\infty}^0 CT(t-\tau)u_*(\tau)d\tau.$$

Roughly speaking, if we run the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu_*(t), \\ y(t) &= Cx(t), \end{aligned}$$

from time $-\infty$ to 0 with input function u_* and "initial state $x(-\infty) = 0$", and then cease to input anything and observe the output $y(\cdot)$ from time 0 onwards, then $\left(\Gamma_h u\right)(t) = y(t)$ for all $t \geq 0$ (see Figure 1.2).

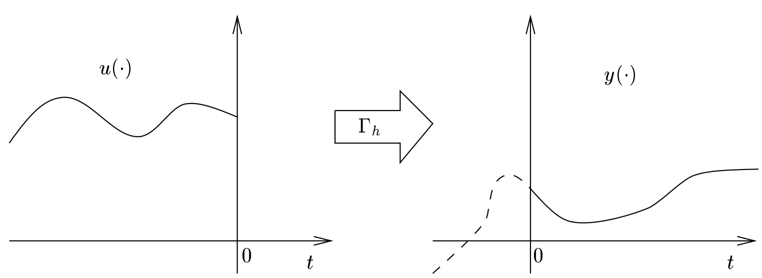


Figure 1.2: The time-domain Hankel operator.

In other words, Γ_h is the mapping from past inputs to future outputs. The *Hankel norm of a system with transfer function* G (sometimes denoted by $\|G\|_H$) is defined to be $\|H_G\|$. It is now evident that this is a physically meaningful notion of closeness of systems. To say that G_* is close to G in the Hankel norm means that the corresponding systems acting on a past input signal give future outputs which are close to each other.

1.3.2 The model reduction problem

In some areas of engineering, linear infinite-dimensional state-space models of dynamic systems are derived and it is desirable if they can be replaced by finite-dimensional systems without incurring too much error. This would make the subsequent control system design computationally less demanding and possibly numerically more reliable. A wide variety of methods for model reduction have been proposed over the years; here we list a few:

1. Balanced model reduction: See for example Moore [58] and Pernebo and Silverman [68] for the finite-dimensional case and Curtain and Glover [17], Ober [61], Young [99] for the infinite-dimensional case.

2. Approximation of systems with $h \in L_1 \cap L_2$ with a nuclear Hankel operator: See Curtain, Glover and Partington [18].

3. Other miscellaneous techniques: See Curtain [15], Glover, Lam and Partington [41], [42], Gu, Khargonekar and Lee [47].

However, in this thesis, we consider the *model reduction problem with respect to the Hankel norm*:

Given a stable G and a nonnegative integer l, find a stable rational G_* of MacMillan degree at most l such that $\|G - G_*\|_H$ is small.

We explain the relation of this question to the problem of sub-optimal Hankel norm approximation in the next section.

1.3.3 Hankel norm approximation and model reduction

Suppose that $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p\times m})$ has a compact Hankel operator and that if $\sigma_{l+1} < \sigma < \sigma_l$, $K(-\cdot) \in H_{\infty,l}(\mathbb{C}^{p\times m})$ is a solution to the sub-optimal Hankel norm approximation problem, that is,

$$\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma.$$

Let $K(\cdot) = G_*(\cdot) + F(\cdot)$, where $G_*(-\cdot)$ is the rational transfer function of a finite-dimensional system with MacMillan degree at most l and with all its poles in the open right half-plane, and $F(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$. Then $-G_*(\cdot)$ (called a *sub-optimal Hankel norm approximant*) is a reduced order model of the original system G in the Hankel norm, with error at most equal to σ. Indeed, we have

$$\|G + G_*\|_H = \|G + G_* + F\|_H = \|G + K\|_H \leq \|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma. \quad (1.4)$$

It is not clear that closeness in the Hankel norm is an appropriate criterion for a reduced order model. Most robust control designs are robust with respect to the stronger L_∞−norm: $\|G\|_H \leq \|G\|_\infty$. However, in practice the Hankel norm approximants performed well. The explanation of this fortunate fact was provided by Glover in his seminal paper [40], which was generalized in [18].

In Glover et al. [18] it was shown that if $h \in L_1([0,\infty), \mathbb{C}^{p\times m}) \cap L_2([0,\infty), \mathbb{C}^{p\times m})$, and if $\sum_{k=1}^{\infty} \sigma_k < \infty$, then for a given integer l, there exists an **optimal** Hankel-norm approximation G_0 of order l, such that

$$\|G - G_0\|_H \leq \|G - G_0\|_\infty \leq \sum_{k=l+1}^{\infty} \sigma_k(G).$$

It is very commonly the case that the Hankel operator has a few sizeable singular values and the remaining tail away very quickly to zero. In such a case, the right hand side can be made very small, and one is assured that an optimal Hankel norm approximant is also good with respect to the L_∞−norm. This norm is extremely desirable because there exist many robustness theories with respect to this norm.

In this thesis, we only consider the sub-optimal Hankel-norm approximation problem. In (1.4) above we showed how this yields a sub-optimal Hankel norm approximant $-G_*(\cdot)$ which is close in the Hankel norm. However, for control applications, we prefer closeness in the L_∞−norm. Since we were unable to find any results on this, even in the finite-dimensional literature, we proved the following new result in section 3.4: If $h \in L_1([0,\infty), \mathbb{C}^{p\times m})$ and $\sum_{k=1}^{\infty} \sigma_k < \infty$, then any **sub-optimal** Hankel norm approximant G_* as in (1.4) of MacMillan degree l satisfies

$$\|G(i\cdot) + G_*(i\cdot)\|_\infty \leq 4l\, \sigma_l(G) + 2 \sum_{k=l+1}^{\infty} \sigma_k(G).$$

Moreover we will show that the error bound

$$E_l := 4l\, \sigma_l(G) + 2 \sum_{k=l+1}^{\infty} \sigma_k(G)$$

tends to zero as $l \to \infty$. So the sub-optimal Hankel norm approximant is also a good approximation of G in the L_∞−norm.

While the natural formulation of the sub-optimal Hankel norm approximation problem is in terms of the transfer function, most systems in control applications are modelled in state-space form in terms of parameters A, B, C. This has been done for the rational case in Glover [40], Ball and Ran [7] and for the special class of Pritchard-Salamon infinite-dimensional systems in Curtain and Ran [23]. Our aim in this thesis has been two-fold:

1. To obtain sufficient conditions for the existence of a solution to the sub-optimal Hankel norm approximation problem in terms of a solution to a key $J-$spectral factorization problem. This allows a complete parameterization of all solutions in terms of the spectral factor (see Chapter 4).

2. For certain classes of well-posed linear infinite-dimensional systems, we use the frequency domain results established in Chapter 4 to obtain explicit solutions to the sub-optimal Hankel norm approximation problem in terms of the state-space parameters A, B, C. The classes considered are exponentially stable, approximately controllable smooth Pritchard-Salamon systems (Chapter 6), exponentially stable, approximately controllable analytic systems (Chapter 7) and non-exponentially stable systems (Chapter 8).

The above strategy is reminiscent of the strategy in Ball and Ran [7], Ran [71] and Curtain and Ran [23] with one important difference. In these three references, the starting point was to quote the following result from Ball and Helton [6].

Theorem 1.3.1 *Let* $h \in L_1([0,\infty),\mathbb{C}^{p\times m}) \cap L_2([0,\infty),\mathbb{C}^{p\times m})$ *define the compact time-domain Hankel operator* $\Gamma_h \in \mathcal{L}(L_2([0,\infty),\mathbb{C}^m), L_2([0,\infty),\mathbb{C}^p))$, *with Hankel singular values* $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$. *Let* G *denote the Laplace transform of* h.

If σ *satisfies* $\sigma_{l+1} < \sigma < \sigma_l$, *then there exists a* $(p+m)\times(p+m)$ *matrix function*

$$\theta(s) = \begin{bmatrix} \theta_{11}(s) & \theta_{12}(s) \\ \theta_{21}(s) & \theta_{22}(s) \end{bmatrix}$$

such that if

$$\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma,$$

where $K(-s) \in H_{\infty,l}(\mathbb{C}^{p\times m})$, *then*

$$K(s) + G(s) = (\theta_{11}(s)Q(s) + \theta_{12}(s))\,(\theta_{21}(s)Q(s) + \theta_{22}(s))^{-1} \tag{1.5}$$

for a unique $Q(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$ *with* $\|Q(i\cdot)\|_\infty \leq 1$. *Conversely, if* $Q(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$ *with* $\|Q(i\cdot)\|_\infty \leq 1$, *then the right hand side of (1.5) defines an element which is of the form* $G+K$, *where* $K(-\cdot) \in H_{\infty,l}(\mathbb{C}^{p\times m})$, *and* $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$.

The matrix function θ is any matrix function satisfying the two conditions

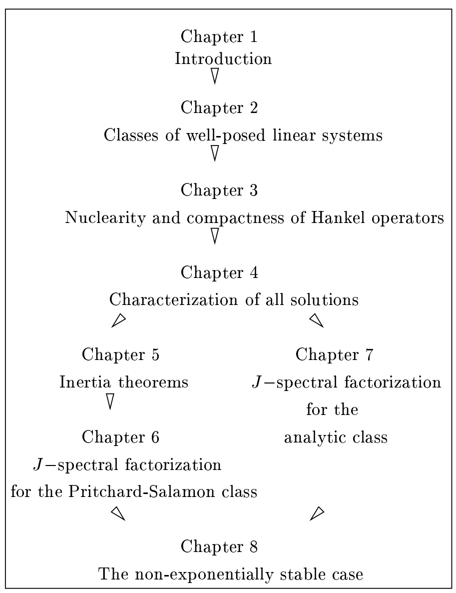
$$\theta(i\omega)^* \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \theta(i\omega) = \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix}, \text{ for all } \omega \in \mathbb{R}; \tag{1.6}$$

$$\theta H_2\left(\mathbb{C}^{p+m}\right) = \begin{bmatrix} I_p & G \\ 0 & -I_m \end{bmatrix} H_2\left(\mathbb{C}^{p+m}\right). \tag{1.7}$$

Since it is not directly clear to non experts and certainly not to the engineering community how this result follows from [6], we found it worthwhile to derive a simpler frequency domain result from first principles using elementary mathematics. We remark that an analogous strategy has been followed earlier to solve the special case of the sub-optimal Nehari problem in Curtain and Zwart [34], [33], Curtain and Oostveen [22] and Curtain and Ichikawa [20].

1.4 Organization of the thesis

In this section, we give an overview of the results in this thesis.



Chapter 2: We first define well-posed linear systems, which forms the general framework for studying infinite-dimensional systems with a state space representation. We recall a number of definitions concerning this class of systems which are relevant to the remainder of this thesis. Next we discuss two important classes of well-posed linear systems:

1. the Pritchard-Salamon class, and
2. the analytic class,

which are central in this thesis. Some of their properties, which will be used in Chapters 6 and 7 are listed. Finally, in the last section of this chapter we will introduce several frequency domain spaces and prove a few elementary lemmas which will be used later.

Chapter 3: In the theory of approximation of infinite-dimensional systems, the nuclearity and compactness properties of the Hankel operator play an essential role. In this chapter we investigate the relationships between the exponential (or strong) stability of certain classes of regular linear systems and the compactness and nuclearity properties of the Hankel operator by means of considering illustrative examples. New sufficient conditions for nuclearity are given for exponentially stable regular linear systems with an analytic semigroup. In the last section of this chapter we will derive a bound on the $L_\infty-$error of a sub-optimal Hankel norm approximant of a system with a nuclear Hankel operator.

Chapter 4: In this chapter, first we prove the existence of a solution to the sub-optimal Hankel norm approximation problem in terms of a solution (Λ) to a key $J-$spectral factorization problem. Subsequently, under the same (frequency domain) assumptions (on the $J-$spectral factor Λ), we give a characterization of all solutions to the sub-optimal Hankel norm approximation problem. These derivations are new.

Later in Chapters 6 and 7, we will show that for the Pritchard-Salamon class and the analytic class, there do exist $J-$spectral factors for each class satisfying the assumptions in this chapter, hence solving the sub-optimal Hankel norm approximation problem for these classes of infinite-dimensional systems. We will give an explicit formula for a $J-$spectral factor Λ in terms of the state-space parameters (A, B, C) of the original system.

Chapter 5: We prove new inertia theorems that relate the number of unstable eigenvalues of the infinitesimal generator to the number of negative eigenvalues of a solution to the operator Lyapunov inequality. The main result in this chapter will be used for the specific purpose of showing the existence of a $J-$spectral factor for the Pritchard-Salamon class, that satisfies the last assumption (S6) of Chapter 4.

Chapter 6: The sub-optimal Hankel norm approximation problem is solved for the smooth Pritchard-Salamon class of exponentially stable, approximately controllable infinite-dimensional systems with finite-dimensional input and output spaces by using the results from Section 2.4 of Chapter 2, Chapter 5 and Chapter 4: we give a formula for a $J-$spectral factor Λ in terms of the state-space parameters, and check that it has all the properties required in Chapter 4.

These results were obtained earlier in Curtain and Ran [23] by appealing to the Ball and Helton result [6]. In fact, they did not need the extra assumption of approximate controllability! However, this is because they did not need to verify that $\Lambda_{11}(-\cdot)^{-1}$ (assumption S6 of Chapter 4) has l unstable poles.

Chapter 7: In this chapter, the sub-optimal Hankel norm approximation problem is solved for the analytic class of exponentially stable, approximately controllable infinite-dimensional systems with finite-dimensional input and output spaces by using the results from Section 2.5 of Chapter 2 and Chapter 4. Again, we give a formula for a $J-$spectral factor Λ in terms of the state-space parameters, and check that it has all the properties required in Chapter 4. The results of this chapter are new and include the results on the Nehari problem in Curtain and Ichikawa [20] as a special case.

Chapter 8: In the previous chapters the sub-optimal Hankel norm approximation problem was solved for various classes of infinite-dimensional systems under the assumption that A generates an *exponentially* stable strongly continuous semigroup. In this chapter, we will solve the sub-optimal Hankel norm approximation problem for non-exponentially stable infinite-dimensional systems in terms of a solution to the sub-optimal Hankel norm approximation problem for an exponentially stable system. The exponentially stable system is obtained by shifting the generator of the semigroup of the original system. These results are new.