Geometry of strings and branes

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Appendix A

Conventions

In this appendix, we will summarize our conventions. Furthermore, we will give some useful identities that have been used in the previous chapters.

A.1 Indices

The last two chapters have used a large amount of different indices. Below we will summarize the different ranges and meanings of these indices. First of all, the metric that we use is mostly plus: i.e. in five dimensions, we have $g_{\mu\nu} = (-++++)$. In chapter 5, we have used the following notations

$$\begin{array}{rcl} \mu,\nu & 0,1,\ldots,4 & \text{spacetime}\,, \\ a,b & 0,1,\ldots,4 & \text{tangent space}\,, \\ \alpha,\beta & 1,\ldots,4 & \text{spinor}\,, \\ i,j & 1,2 & \mathrm{SU}(2)\,, \end{array} \tag{A.1}$$

In chapter 6, we have furthermore used indices labelling the components of matter multiplet. In particular, we have used

$$\begin{array}{rcl} \widetilde{I},\widetilde{J} & 1,2,\ldots n_{\mathrm{V}}+n_{\mathrm{T}} & \text{vector-tensor multiplet}\,, \\ I,J & 1,2,\ldots,n_{\mathrm{V}} & \text{vector multiplet}\,, \\ M,N & 1,2,\ldots,n_{\mathrm{T}} & \text{tensor multiplet}\,, \\ X,Y & 1,2,\ldots,4n_{\mathrm{H}} & \text{hypermultiplet target space}\,, \\ A,B & 1,2,\ldots,2n_{\mathrm{H}} & \text{hypermultiplet tangent space}\,, \\ i,j & 1,2 & \mathrm{SU}(2)\,. \end{array} \tag{A.2}$$

In all cases, we denote symmetrizations with parentheses around the indices, and anti-symmetrizations with brackets around the indices. Furthermore, we (anti-)symmetrize with weight one

$$X_{(ab)} \equiv \tfrac{1}{2}\left(X_{ab} + X_{ba}\right) , \quad X_{[ab]} \equiv \tfrac{1}{2}\left(X_{ab} - X_{ba}\right) . \tag{A.3}$$

A.2 Tensors

Our conventions for the D-dimensional Levi–Civita tensor are

$$\varepsilon_{a_1 \dots a_D} = -\varepsilon^{a_1 \dots a_D} = 1 . \tag{A.4}$$

The Levi-Civita tensor with spacetime indices can be obtained from (A.4 by using vielbeins to convert the tangent space indices to spacetime indices, and multiplying the result with the vielbein determinant gives

$$\varepsilon_{\mu_1 \dots \mu_D} = e^{-1} e_{\mu_1}{}^{a_1} \cdots e_{\mu_d}{}^{a_D} \varepsilon_{a_1 \dots a_D} , \quad \varepsilon^{\mu_1 \dots \mu_D} = e\, e^{\mu_1}{}_{a_1} \cdots e^{\mu_D}{}_{a_D} \varepsilon^{a_1 \dots a_D} , \tag{A.5}$$

where we have used the Einstein summation convention in which repeated indices are summed over.

Note that raising and lowering the indices of the Levi-Civita tensor with spacetime indices is done with the metric, which for the Levi-Civita tensor with tangent space indices is done by using the definition (A.4). Contractions of the Levi-Civita tensor give products of delta-functions which are normalized as

$$\varepsilon_{a_1 \dots a_p b_1 \dots b_q} \varepsilon^{a_1 \dots a_p c_1 \dots c_q} = -p! q! \delta^{[c_1}_{[b_1} \dots \delta^{c_q]}_{b_q]} , \tag{A.6}$$

We have defined the dual of five-dimensional tensors as

$$\widetilde{A}^{a_1 \dots a_{5-n}} = \tfrac{1}{n!}\, \mathrm{i}\, \varepsilon_{a_1 \dots a_{5-n} b_1 \dots b_n} A^{b_n \dots b_1} . \tag{A.7}$$

Using (A.6), one finds the following identities

$$\widetilde{\widetilde{A}} = A , \qquad \frac{1}{n!} A^{a_1 \dots a_n} B_{a_1 \dots a_n} = \frac{1}{n!} A \cdot B = \frac{1}{(n-5)!} \widetilde{A} \cdot \widetilde{B} , \tag{A.8}$$

where we have introduced the generalized inner product notation $A \cdot B$ that we use throughout this thesis.

We use the same conventions for the Riemann tensor and its contractions as [92]. In particular, we define the Riemann tensor as

$$R^{\mu}{}_{\nu\lambda\rho} = \partial_\lambda \Gamma^{\mu}_{\rho\nu} - \partial_\rho \Gamma^{\mu}_{\lambda\nu} + \Gamma^{\mu}_{\sigma\lambda} \Gamma^{\sigma}_{\rho\nu} - \Gamma^{\mu}_{\sigma\rho} \Gamma^{\sigma}_{\lambda\nu} . \tag{A.9}$$

The Ricci tensor and Ricci scalar in this thesis are given by

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} , \quad R = g^{\mu\nu} R_{\mu\nu} . \tag{A.10}$$

With these conventions, the Einstein-Hilbert action has a positive sign.

A.3 Differential forms

In chapter 1, we have used differential form notation to simplify the supergravity actions. A p-form is related to a rank-p anti-symmetric tensor according to

$$F_{(p)} = \frac{1}{p!} \mathrm{d}x^{\mu_1} \ldots \mathrm{d}x^{\mu_p} F_{\mu_1 \ldots \mu_p} \,. \tag{A.11}$$

The analog of the dual of an anti-symmetric tensor (A.7), is given by the Hodge-dual: i.e a differential p-form A has a $D-p$-form $B = \star A$ as its dual with components

$$B_{\mu_1 \ldots \mu_q} = \frac{1}{p!} e \, \varepsilon_{\mu_1 \ldots \mu_q}{}^{\nu_1 \ldots \nu_p} A_{\nu_1 \ldots \nu_p} \,, \quad q = D - p \,. \tag{A.12}$$

Note in particular the different order in which the indices in (A.12) are contracted with respect to (A.7). With this definition, we have the usual identity

$$\star \star A_{(p)} = (-)^{pq+1} A_{(p)} \,, \quad q = D - p \,. \tag{A.13}$$

Furthermore, the D-dimensional invariant volume element can then be written as the star of the unit number

$$\star \mathbb{1} \equiv \mathrm{d}^D x \, \sqrt{|g|} \,. \tag{A.14}$$

A.4 Spinors

Our five-dimensional spinors are symplectic-Majorana spinors that transform in the $(4, 2)$ of $\overline{\mathrm{SO}(5)} \otimes \mathrm{SU}(2)$. The generators U_{ij} of the R-symmetry group $\mathrm{SU}(2)$ are defined to be anti-Hermitian and symmetric, i.e.

$$(U_i{}^j)^* = -U_j{}^i \,, \qquad U_{ij} = U_{ji} \,. \tag{A.15}$$

A symmetric traceless $U_i{}^j$ corresponds to a symmetric U^{ij} since we lower or raise $\mathrm{SU}(2)$ indices using the ε-symbol contracting the indices in a northwest-southeast (NW–SE) convention

$$X^i = \varepsilon^{ij} X_j \,, \qquad X_i = X^j \varepsilon_{ji} \,, \qquad \varepsilon_{12} = -\varepsilon_{21} = \varepsilon^{12} = 1 \,. \tag{A.16}$$

The actual value of ε is here given as an example. It is in fact arbitrary as long as it is antisymmetric, $\varepsilon^{ij} = (\varepsilon_{ij})^*$ and $\varepsilon_{jk} \varepsilon^{ik} = \delta_j{}^i$. When the $\mathrm{SU}(2)$ indices on spinors are omitted, NW-SE contraction is understood

$$\bar{\lambda} \psi = \bar{\lambda}^i \psi_i \,, \tag{A.17}$$

The charge conjugation matrix $\mathcal{C}$ and $\mathcal{C}\gamma_a$ are antisymmetric. The matrix $\mathcal{C}$ is unitary and γ_a is Hermitian apart from the timelike one, which is anti-Hermitian. The bar is the Majorana bar

$$\bar{\lambda}^i = (\lambda^i)^T \mathcal{C} \,. \tag{A.18}$$

We define the charge conjugation operation on spinors as

$$(\lambda^i)^C \equiv \alpha^{-1} B^{-1} \varepsilon^{ij} (\lambda^j)^* , \qquad \bar{\lambda}^{iC} \equiv \overline{(\lambda^i)^C} = \alpha^{-1} \left(\bar{\lambda}^k\right)^* B \varepsilon^{ki} , \tag{A.19}$$

where $B = \mathcal{C}\gamma_0$, and $\alpha = \pm 1$ when one uses the convention that complex conjugation does not interchange the order of spinors, or $\alpha = \pm\,\mathrm{i}$ when it does. Symplectic Majorana spinors satisfy $\lambda = \lambda^C$. Charge conjugation acts on gamma-matrices as $(\gamma_a)^C = -\gamma_a$, does not change the order of matrices, and works on matrices in $\mathrm{SU}(2)$ space as $M^C = \sigma_2 M^* \sigma_2$. Complex conjugation can then be replaced by charge conjugation, if for every bi-spinor one inserts a factor -1. Then, e.g., the expressions

$$\bar{\lambda}^i \gamma_\mu \lambda^j , \qquad \mathrm{i}\, \bar{\lambda}^i \lambda_i \tag{A.20}$$

are real for symplectic Majorana spinors. For more details, see [186].

A.5 Gamma-matrices

The gamma-matrices γ_a are defined as matrices that satisfy the Clifford-algebra

$$\{\gamma_a, \gamma_b\} \equiv \gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab} \tag{A.21}$$

Completely anti-symmetrized products of gamma-matrices are denoted in three different ways

$$\gamma_{(n)} = \gamma_{a_1 \cdots a_n} = \gamma_{[a_1} \cdots \gamma_{a_n]} . \tag{A.22}$$

The product of all gamma-matrices is proportional to the unit matrix in odd dimensions. We use

$$\gamma^{abcde} = \mathrm{i}\, \varepsilon^{abcde} . \tag{A.23}$$

This implies that the dual of a $(5-n)$-antisymmetric gamma-matrix is the n-antisymmetric gamma-matrix given by

$$\gamma_{a_1 \ldots a_n} = \tfrac{1}{(5-n)!}\, \mathrm{i}\, \varepsilon_{a_1 \ldots a_n b_1 \ldots b_{5-n}} \gamma^{b_{5-n} \ldots b_1} . \tag{A.24}$$

For convenience, we will give the values of gamma-contractions like

$$\gamma^{(m)} \gamma_{(n)} \gamma_{(m)} = c_{n,m} \gamma_{(n)} , \tag{A.25}$$

where the constants $c_{n,m}$ are given in table A.1. The constants for $n, m > 2$ can easily be obtained from (A.24) and table A.1.

Changing the order of spinors in a bilinear leads to the following signs

$$\bar{\psi}^{(1)} \gamma_{(n)} \chi^{(2)} = t_n\, \bar{\chi}^{(2)} \gamma_{(n)} \psi^{(1)} \qquad \begin{cases} t_n = +1 \text{ for } n = 0,1 \\ t_n = -1 \text{ for } n = 2,3 \end{cases} \tag{A.26}$$

where the labels (1) and (2) denote any $\mathrm{SU}(2)$ representation.

$c_{n,m}$	$m=1$	$m=2$
$n=0$	5	-20
$n=1$	-3	-4
$n=2$	1	4

Table A.1: Coefficients used in contractions of gamma-matrices.

A.6 Fierz-identities

The sixteen different gamma-matrices $\gamma_{(n)}$ for $n = 0, 1, 2$ form a complete basis for four-dimensional matrices. Similarly, the identity matrix $\mathbb{1}_2$ and the three Pauli-matrices σ^i for $i = 1, 2, 3$ form a basis for two-dimensional matrices. A change of basis in a product of two pseudo-Majorana spinors will give rise to so-called Fierz-rearrangement formulae, which in their simplest form are given by

$$\psi_j\bar{\lambda}^i = -\tfrac{1}{4}\bar{\lambda}^i\psi_j - \tfrac{1}{4}\bar{\lambda}^i\gamma^a\psi_j\gamma_a + \tfrac{1}{8}\bar{\lambda}^i\gamma^{ab}\psi_j\gamma_{ab}\,, \qquad \bar{\psi}^{[i}\lambda^{j]} = -\tfrac{1}{2}\bar{\psi}\lambda\varepsilon^{ij}\,. \tag{A.27}$$

Using such Fierz-rearrangements, other useful identities can be deduced for working with cubic fermion terms

$$\begin{aligned} &\lambda_j\bar{\lambda}^j\lambda^i = \gamma^a\lambda_j\bar{\lambda}^j\gamma_a\lambda^i = \tfrac{1}{8}\gamma^{ab}\lambda^i\bar{\lambda}\gamma_{ab}\lambda\,,\\ &\gamma^{cd}\gamma_{ab}\lambda^i\bar{\lambda}\gamma^{cd}\lambda = 4\lambda^i\bar{\lambda}\gamma^{ab}\lambda\,,\\ &\gamma_a\lambda\bar{\lambda}\gamma^{ab}\lambda = 0\,. \end{aligned} \tag{A.28}$$

When one multiplies three spinor doublets, one should be able to write the result in terms of $\binom{8}{3} = 56$ independent structures. From analyzing the representations, one can obtain that these are in the $(4, 2) + (4, 4) + (16, 2)$ representations of $\overline{\mathrm{SO}(5)} \times \mathrm{SU}(2)$. They are

$$\begin{aligned} &\lambda_j\bar{\lambda}^j\lambda^i = \gamma^a\lambda_j\bar{\lambda}^j\gamma_a\lambda^i = \tfrac{1}{8}\gamma^{ab}\lambda^i\bar{\lambda}\gamma_{ab}\lambda\,,\\ &\lambda^{(k}\bar{\lambda}^i\lambda^{j)}\,,\\ &\lambda_j\bar{\lambda}^j\gamma_a\lambda^i\,. \end{aligned} \tag{A.29}$$

As a final Fierz-identity, we give a three-spinor identity which is needed to prove the invariance under supersymmetry of the action for a vector multiplet

$$\psi^i_{[I}\bar{\psi}_J\psi_{K]} = \gamma^a\psi^i_{[I}\bar{\psi}_J\gamma_a\psi_{K]}\,. \tag{A.30}$$