Geometry of strings and branes

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Chapter 6

Matter-couplings of conformal supergravity

The basic supergravities in ten and and eleven dimensions are the low energy limits of string theory and M-theory, as we have seen in the introduction and in chapter 1. Matter-coupled supergravity theories in lower dimensions [38] have played an important role in our understanding of the low-energy limit of string theory compactifications.

For phenomenological reasons, much work has been done related to compactifications to four dimensions and the corresponding four-dimensional matter-coupled supergravities [197, 198]. The supergravities in the intermediate dimensions have also played a role in the understanding of string theory. For instance, the structure of nine-dimensional supergravity is important for the understanding of T-duality [40], whereas six-dimensional supergravity plays a crucial role in the understanding of string-string duality [199].

As we have discussed in chapter 4, a lot of attention has recently been given to five-dimensional matter-coupled supergravity theories [159, 162], thereby generalizing the earlier results of [156–158]. In this chapter, we will take the superconformal approach [167, 173–175] to construct a framework from which one can independently derive and study the possible five-dimensional matter-couplings to Poincaré supergravity.

An advantage of the superconformal construction is that, by past experience, it leads to insights into the kinematical and geometrical structure of matter-coupled Poincaré supergravity. For instance, the vector fields of superconformal vector multiplets split into the graviphoton of the Poincaré supergravity multiplet and the photons of ordinary vector multiplets by gauge-fixing the superconformal symmetries. A more recent example is the insight in relations between hyper-Kähler cones and quaternionic manifolds, based on the study of superconformal hypermultiplets [172, 200].

There is a more general interest in the $\mathcal{N} = 2\,, D = 5$ matter-coupled supergravities: they belong to the class of theories with eight supersymmetries [201]. Such theories are especially interesting since the geometry, determined by the kinetic terms of the scalars, contains undetermined functions. Theories with thirty-two supersymmetries have no matter multiplets, whereas the geometry of those with sixteen supersymmetries is completely determined by the number of matter multiplets. Theories with four supersymmetries allow for even more general geometries: continuous deformations of the manifold are allowed. Theories with eight supersymmetries have a more restricted class of geometries, which makes them more manageable.

The geometry related to supersymmetric theories with eight real supercharges is called "special geometry". Special geometry was first found in [202, 203] for local supersymmetry and in [204, 205] for rigid supersymmetry. It occurs in Calabi-Yau compactifications as the moduli space of these manifolds [206–211]. Special geometry was also a very useful tool in the investigation of supersymmetric black holes [212, 213], the work of Seiberg and Witten [214, 215], and the AdS/CFT correspondence [76].

So far, special geometry had been mainly investigated in the context of four dimensions. With the advent of the brane-world scenarios [138, 139], also the $D = 5$ variant of special geometry [156], called "very special geometry", received a lot of attention. The connection to special geometry was made in [216]. Finally, there is a connection between special geometry and and another class of geometries called "quaternionic geometry" [207], which has lead to new results on the classification of homogeneous quaternionic spaces [217, 218].

Superconformal matter multiplets with eight supersymmetries have already been introduced in [178, 179, 219]. However, there are still some ingredients missing. For instance, we will not only introduce vector multiplets in the adjoint representation but in arbitrary representations. The resulting multiplet in this case is called the "vector-tensor" multiplet. We will also construct vector-tensor multiplets in reducible, but not completely reducible representations: they are related to non-compact, non-semi-simple gaugings of Poincaré supergravity, a class of gauged supergravities that have not been considered for $\mathcal{N} = 2$ supersymmetry.

Some of the superconformal matter multiplets are on-shell: the algebra closes only modulo equations of motion. However, this does not imply that these equations of motion have to follow from an action. Indeed, this is a familiar feature of e.g. IIB supergravity and other theories with self-dual antisymmetric tensor fields. In particular for the vector-tensor multiplet, the absence of an action allows for couplings with an *odd* number of tensor multiplets, which generalizes the analysis made in [220]. For the hypermultiplets, we will introduce more general geometries than hyper-Kähler for rigid supersymmetry, or quaternionic-Kähler for local supersymmetry: we will consider hyper-Kähler manifolds without a metric: such manifolds are called "hyper-complex" manifolds.

We will start this chapter with constructing and discussing the possible matter-couplings in the absence of a Lagrangian by giving the rigid transformation rules for the vector-tensor multiplet and the hypermultiplet. We will emphasize the geometrical interpretation of the emerging algebraic structure. Next, in section 6.3, we construct the rigid superconformal Lagrangians for each of the superconformal matter multiplets. We discuss the restrictions on the possible geometries that follow from the requirement of a Lagrangian. Finally, in section 6.4, we extend the rigid superconformal symmetry to local superconformal symmetry, making use of the Weyl multiplet constructed in chapter 5.

This chapter is based on the work to be published in [17].

6.1 The vector-tensor multiplet

In this section, we will discuss superconformal vector multiplets that transform in arbitrary representations of the gauge group. From work on $\mathcal{N} = 2, D = 5$ Poincaré matter-couplings [159], it is known that vector multiplets transforming in representations other than the adjoint have to be dualized to tensor fields. We define a vector-tensor multiplet to be a vector multiplet transforming in a reducible representation that contains the adjoint representation as well as another, arbitrary, representation.

We will show that the analysis of [159] can be extended to superconformal vector multiplets. Moreover, we will generalize the gauge transformations for the tensor fields given in [159] by allowing them to transform into the field-strengths of the adjoint gauge fields. These more general gauge transformations are consistent with supersymmetry, even after breaking the conformal symmetry.

The vector-tensor multiplet contains *a priori* an arbitrary number of tensor fields. The restriction to an even number of tensor fields is not imposed by the closure of the algebra. However, one can only construct an action for an even number of tensor multiplets as we will see in section 6.3.

To make contact with other results in the literature, we will break the rigid conformal symmetry by using a vector multiplet as a compensating multiplet for the superconformal symmetry. The adjoint fields of the vector-tensor multiplet are given constant expectation values, and the scalar expectation values will play the role of a mass parameter. If one demands that the field equations do not contain tachyonic modes, an even number of tensor multiplet is required. For the case of two tensor multiplets, this will reduce the superconformal vector-tensor multiplet to the massive self-dual complex tensor multiplet of [221].

6.1.1 Adjoint representation

We will start with giving the transformation rules for a vector multiplet in the adjoint representation [179]. Such an off-shell vector multiplet has $8+8$ real degrees of freedom whose $\mathrm{SU}(2)$ labels and Weyl weights we have indicated in table 6.1. If the rank of the gauge group is n_{V}, then we have $I = 1, \ldots, n_{\mathrm{V}}$, and the scalars of the vector multiplet span a n_{V}-dimensional real vector space which is isomorphic to the manifold $\mathbb{R}^{n_{\mathrm{V}}}$.

We consider gauge fields A_μ^I and general matter fields of the vector multiplet X^I that transform under gauge transformations with parameters Λ^I according to

$$\delta_G(\Lambda^J) A_\mu^I = \partial_\mu \Lambda^I + g A_\mu^J f_{JK}{}^I \Lambda^K \,, \qquad \delta_G(\Lambda^J) X^I = -g \Lambda^J f_{JK}{}^I X^K \,, \tag{6.1}$$

Field	SU(2)	w	# d.o.f.
A_μ^I	1	0	$4\, n_{\rm V}$
Y^{ijI}	3	2	$3\, n_{\rm V}$
σ^I	1	1	$1\, n_{\rm V}$
ψ^{iI}	2	$3/2$	$8\, n_{\rm V}$

Table 6.1: The off-shell Yang-Mills multiplet.

where g is the coupling constant of the gauge group. The gauge transformations that we consider satisfy the commutation relations

$$[\delta_G(\Lambda_1^I), \delta_G(\Lambda_2^J)] = \delta_G(\Lambda_3^K)\,, \qquad \Lambda_3^K = g\Lambda_1^I\Lambda_2^J f_{IJ}{}^K\,. \tag{6.2}$$

The expression for the gauge-covariant derivative of X^I and the field-strengths are given by

$$\mathcal{D}_\mu X^I = \partial_\mu X^I + gA_\mu^J f_{JK}{}^I X^K\,, \qquad F_{\mu\nu}^I = 2\partial_{[\mu}A_{\nu]}^I + gf_{JK}{}^I A_\mu^J A_\nu^K\,. \tag{6.3}$$

The field-strength satisfies the Bianchi identity

$$\mathcal{D}_{[\mu}F_{\nu\lambda]}^I = 0\,. \tag{6.4}$$

The rigid Q- and S-supersymmetry transformation rules for the off-shell Yang-Mills multiplet are given by [179]

$$\begin{aligned}
\delta A_\mu^I &= \tfrac{1}{2}\bar\epsilon\gamma_\mu\psi^I\,,\\
\delta Y^{ijI} &= -\tfrac{1}{2}\bar\epsilon^{(i}\not{\mathcal{D}}\psi^{j)I} - \tfrac{1}{2}\,\mathrm{i}\,g\bar\epsilon^{(i}f_{JK}{}^I\sigma^J\psi^{j)K} + \tfrac{1}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi^{j)I}\,,\\
\delta\psi^{iI} &= -\tfrac{1}{4}\gamma\cdot F^I\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\not{\mathcal{D}}\sigma^I\epsilon^i - Y^{ijI}\epsilon_j + \sigma^I\eta^i\,,\\
\delta\sigma^I &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\psi^I\,.
\end{aligned} \tag{6.5}$$

The field-strength transforms according to

$$\delta F_{\mu\nu}^I = -\bar\epsilon\gamma_{[\mu}\mathcal{D}_{\nu]}\psi^I + \mathrm{i}\,\bar\eta\gamma_{\mu\nu}\psi^I\,. \tag{6.6}$$

The commutator of two Q-supersymmetry transformations yields a translation with an extra G-transformation

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_P\left(\tfrac{1}{2}\bar\epsilon_2\gamma_\mu\epsilon_1\right) + \delta_G\left(-\tfrac{1}{2}\,\mathrm{i}\,\sigma\bar\epsilon_2\epsilon_1\right)\,. \tag{6.7}$$

Note that even though we are considering rigid superconformal symmetry, the algebra (6.7) contains a field-dependent term on the right hand side. Such soft terms are commonplace in local superconformal symmetry, but here they already appear at the rigid level. In Hamiltonian language, it means that the algebra is satisfied modulo constraints.

Field	SU(2)	w	# d.o.f.
$B_{\mu\nu}^M$	1	0	$3\,n_{\rm T}$
ϕ^M	1	1	$1\,n_{\rm T}$
λ^{iM}	2	$3/2$	$4\,n_{\rm T}$

Table 6.2: The on-shell tensor multiplet.

6.1.2 Reducible representations

Instead of the set of field-strengths $F_{\mu\nu}^I$, we will now consider a more general set of tensor fields $\mathcal{H}_{\mu\nu}^{\widetilde{I}} = \{F_{\mu\nu}^I, B_{\mu\nu}^M\}$ with $\widetilde{I} = (I, M)$ $(I = 1, \cdots, n_{\rm V}; M = n_{\rm V}+1, \cdots n_{\rm V}+n_{\rm T})$. The set of $n_{\rm T}$ tensor fields $B_{\mu\nu}^M$ are accompanied by spinors λ^{iM} and scalars ϕ^M as we have indicated in table 6.2.

The representation matrices take on the form

$$(t_I)_{\widetilde{J}}{}^{\widetilde{K}} = \begin{pmatrix} (t_I)_J{}^K & (t_I)_J{}^N \\ (t_I)_M{}^K & (t_I)_M{}^N \end{pmatrix}\,, \qquad \left\{ \begin{array}{ccl} I,J,K & = & 1,\ldots,n_{\rm V} \\ M,N & = & n_{\rm V}+1,\ldots,n_{\rm V}+n_{\rm T}\,. \end{array}\right. \tag{6.8}$$

It is understood that the $(t_I)_J{}^K$ are in the adjoint representation, i.e.

$$(t_I)_J{}^K = f_{IJ}{}^K\,. \tag{6.9}$$

If $n_{\rm T} \neq 0$, then the representation is reducible. We will see that this representation can be more general than assumed so far in treatments of vector-tensor multiplet couplings. The requirement that $n_{\rm T}$ is even will only appear when we demand the existence of an action in section 6.3.2, or if we require absence of tachyonic modes. The matrices t_I satisfy commutation relations

$$[t_I, t_J] = -f_{IJ}{}^K t_K\,, \qquad \text{or} \qquad t_{I\widetilde{N}}{}^{\widetilde{M}} t_{J\widetilde{M}}{}^{\widetilde{L}} - t_{J\widetilde{N}}{}^{\widetilde{M}} t_{I\widetilde{M}}{}^{\widetilde{L}} = -f_{IJ}{}^K t_{K\widetilde{N}}{}^{\widetilde{L}}\,. \tag{6.10}$$

If the index $\widetilde{L}$ is a vector index, then this relation is satisfied using the matrices as in (6.9).

Requiring the closure of the superconformal algebra, we find Q- and S-supersymmetry transformation rules for the vector-tensor multiplet

$$\begin{aligned}
\delta A^I_\mu &= \tfrac{1}{2}\bar\epsilon\gamma_\mu\psi^I\,,\\
\delta\mathcal{H}^{\widetilde I}_{\mu\nu} &= -\bar\epsilon\gamma_{[\mu}\mathcal{D}_{\nu]}\psi^{\widetilde I} + \mathrm{i}\,g\bar\epsilon\gamma_{\mu\nu}t_{(\widetilde J\widetilde K)}{}^{\widetilde I}\sigma^{\widetilde J}\psi^{\widetilde K} + \mathrm{i}\,\bar\eta\gamma_{\mu\nu}\psi^{\widetilde I}\,,\\
\delta Y^{ij\widetilde I} &= -\tfrac{1}{2}\bar\epsilon^{(i}\not{\mathcal{D}}\psi^{j)\widetilde I} - \tfrac{1}{2}\,\mathrm{i}\,g\bar\epsilon^{(i}\left(t_{[\widetilde J\widetilde K]}{}^{\widetilde I} - 3t_{(\widetilde J\widetilde K)}{}^{\widetilde I}\right)\sigma^{\widetilde J}\psi^{j)\widetilde K} + \tfrac{1}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi^{j)\widetilde I}\,,\\
\delta\psi^{i\widetilde I} &= -\tfrac{1}{4}\gamma\cdot\mathcal{H}^{\widetilde I}\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\not{\mathcal{D}}\sigma^{\widetilde I}\epsilon^i - Y^{ij\widetilde I}\epsilon_j + \tfrac{1}{2}gt_{(\widetilde J\widetilde K)}{}^{\widetilde I}\sigma^{\widetilde J}\sigma^{\widetilde K}\epsilon^i + \sigma^{\widetilde I}\eta^i\,,\\
\delta\sigma^{\widetilde I} &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\psi^{\widetilde I}\,.
\end{aligned}\tag{6.11}$$

Note that (6.11) differs from (6.5) and (6.6) only in terms at $\mathcal{O}(g)$ in the gauge coupling constant, and that the difference is always proportional to the tensor $t_{(\widetilde J\widetilde K)}{}^{\widetilde I}$.

The curly derivatives denote gauge-covariant derivatives as in (6.3) with the replacement of structure constants by general matrices t_I according to (6.9). We have extended the range of the generators from I to $\widetilde I$ in order to simplify the transformation rules with the understanding that

$$(t_M)_{\widetilde J}{}^{\widetilde K} = 0\,.\tag{6.12}$$

We find that the supersymmetry algebra (6.7) is satisfied provided the representation matrices are restricted to

$$t_{(\widetilde J\widetilde K)}{}^{I} = 0\,.\tag{6.13}$$

If $n_{\mathrm{T}} \neq 0$, then the algebra only closes provided the following two equations of motion on the fields are satisfied

$$\begin{aligned}
L^{ij\widetilde I} &\equiv t_{(\widetilde J\widetilde K)}{}^{\widetilde I}\left(2\sigma^{\widetilde J}Y^{ij\widetilde K} - \tfrac{1}{2}\,\mathrm{i}\,\bar\psi^{i\widetilde J}\psi^{j\widetilde K}\right) \qquad (6.14)\\
&= 0\,,\\
E^{\widetilde I}_{\mu\nu\lambda} &\equiv \frac{3}{g}\mathcal{D}_{[\mu}\mathcal{H}_{\nu\lambda]}{}^{\widetilde I} - \varepsilon_{\mu\nu\lambda\rho\sigma}t_{(\widetilde J\widetilde K)}{}^{\widetilde I}\left(\sigma^{\widetilde J}\mathcal{H}^{\rho\sigma\widetilde K} + \tfrac{1}{4}\,\mathrm{i}\,\bar\psi^{\widetilde J}\gamma^{\rho\sigma}\psi^{\widetilde K}\right) \qquad (6.15)\\
&= 0\,.
\end{aligned}$$

For $\widetilde I = I$, we can use (6.13) to satisfy (6.14) and to reduce (6.15) to the Bianchi identity (6.4). The tensor $F^I_{\mu\nu}$ can therefore be seen as the curl of a gauge vector A^I_μ. We conclude that the fields with indices $\widetilde I = I$ form an off-shell vector multiplet in the adjoint representation of the gauge group. In particular, this means that for $n_{\mathrm{T}} = 0$ we find back the transformation rules for the vector multiplet.

The constraints (6.14) and (6.15), with $\widetilde I = M$, do not form a supersymmetric set: they are invariant under S-supersymmetry, but under Q-supersymmetry they lead to a constraint on the spinors ψ^{iM} which we will call φ^{iM}:

$$\delta L^{ijM} = \mathrm{i}\,\bar\epsilon^{(i}\varphi^{j)M}\,,\quad \delta E^M_{\mu\nu\rho} = \bar\epsilon\gamma_{\mu\nu\rho}\varphi^M\,.\tag{6.16}$$

The expression for this constraint is given by

$$\begin{aligned}
\varphi^{iM} &\equiv t_{(\widetilde{J}\widetilde{K})}{}^{M}\Big[\mathrm{i}\,\sigma^{\widetilde{J}}\not{\mathcal{D}}\psi^{i\widetilde{K}} + \tfrac{1}{2}\,\mathrm{i}\,(\not{\mathcal{D}}\sigma^{\widetilde{J}})\psi^{i\widetilde{K}} + Y^{ik\widetilde{J}}\psi_k^{\widetilde{K}} - \tfrac{1}{4}\gamma\cdot\mathcal{H}^{\widetilde{J}}\psi^{i\widetilde{K}}\Big] \\
&\quad -g\left(\Big[t_{[\widetilde{J}\widetilde{K}]}{}^{\widetilde{L}} - 3t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{L}}\Big]\,t_{(\widetilde{I}\widetilde{L})}{}^{M} + \tfrac{1}{2}t_{\widetilde{I}\widetilde{J}}{}^{\widetilde{L}}t_{(\widetilde{K}\widetilde{L})}{}^{M}\right)\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\psi^{i\widetilde{K}} \\
&= 0\,.
\end{aligned} \tag{6.17}$$

Varying the new constraint φ^{iM} under Q-and S-supersymmetry, one finds at first sight two more constraints, E_a^M and N^M, of which the first one turns out to be dependent (see below):

$$\begin{aligned}
\delta\varphi^{i\,M} &= -\tfrac{1}{2}\,\mathrm{i}\,\not{\mathcal{D}}L^{ij\,M}\epsilon_j - \tfrac{1}{2}\,\mathrm{i}\,\gamma^a E_a^M\epsilon^i + \tfrac{1}{2}N^M\epsilon^i - \tfrac{1}{2}gt_{\widetilde{J}\widetilde{K}}{}^{M}\sigma^{\widetilde{J}}L^{ij\,\widetilde{K}}\epsilon_j \\
&\quad -\tfrac{1}{12}\,\mathrm{i}\,gt_{(\widetilde{J}\widetilde{K})}{}^{M}\gamma^{abc}\sigma^{\widetilde{J}}E_{abc}^{\widetilde{K}}\epsilon^i + 3L^{ijM}\eta_j\,.
\end{aligned} \tag{6.18}$$

The constraint N^M is given by

$$\begin{aligned}
N^M &\equiv t_{(\widetilde{J}\widetilde{K})}{}^{M}\left(\sigma^{\widetilde{J}}\Box\sigma^{\widetilde{K}} + \tfrac{1}{2}\mathcal{D}^a\sigma^{\widetilde{J}}\mathcal{D}_a\sigma^{\widetilde{K}} - \tfrac{1}{4}\mathcal{H}_{ab}^{\widetilde{J}}\mathcal{H}^{ab\widetilde{K}} - \tfrac{1}{2}\bar{\psi}^{\widetilde{J}}\not{\mathcal{D}}\psi^{\widetilde{K}} + Y^{ij\widetilde{J}}Y_{ij}{}^{\widetilde{K}}\right) \\
&\quad -\mathrm{i}\,g\Big[-\tfrac{1}{2}\,t_{[\widetilde{J}\widetilde{K}]}{}^{\widetilde{L}}t_{(\widetilde{I}\widetilde{L})}{}^{M} + 2\,t_{(\widetilde{I}\widetilde{J})}{}^{\widetilde{L}}t_{(\widetilde{K}\widetilde{L})}{}^{M}\Big]\,\sigma^{\widetilde{I}}\bar{\psi}^{\widetilde{J}}\psi^{\widetilde{K}} \\
&\quad +\tfrac{1}{2}g^2\,(t_I t_J t_K)_{\widetilde{L}}{}^{M}\sigma^I\sigma^J\sigma^K\sigma^{\widetilde{L}} \\
&= 0\,,
\end{aligned} \tag{6.19}$$

and for E_a^M we find

$$\begin{aligned}
E_a^M &\equiv t_{(\widetilde{J}\widetilde{K})}{}^{M}\left(\mathcal{D}^b\left(\sigma^{\widetilde{J}}\mathcal{H}_{ba}{}^{\widetilde{K}} + \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}^{\widetilde{J}}\gamma_{ba}\psi^{K}\right) - \tfrac{1}{8}\varepsilon_{abcde}\mathcal{H}^{bc\widetilde{J}}\mathcal{H}^{de\widetilde{K}}\right) \\
&= 0\,.
\end{aligned} \tag{6.20}$$

This last constraint is not an independent condition, but it is related to E_{abc}^M

$$E_a^M = -\tfrac{1}{12}\varepsilon_{abcde}\mathcal{D}^b E^{cdeM}\,. \tag{6.21}$$

Subsequent supersymmetry variations do not lead to any new constraints. On a technical note, we made use of identities as

$$t_{K\widetilde{I}}{}^{\widetilde{L}}t_{(\widetilde{J}\widetilde{L})}{}^{M} + t_{K\widetilde{J}}{}^{\widetilde{L}}t_{(\widetilde{I}\widetilde{L})}{}^{M} - t_{(\widetilde{I}\widetilde{J})}{}^{\widetilde{L}}t_{K\widetilde{L}}{}^{M} = 0\,, \tag{6.22}$$

which follow from the commutator relation (6.10), and the restrictions (6.12) and (6.13).

To summarize, the superconformal algebra closes on the vector-tensor multiplet modulo the set of constraints (6.14), (6.14), (6.17) and (6.19). Under Q– and S–supersymmetry, they transform to each other, but they do not form a multiplet by themselves.

6.1.3 Completely reducible representations

Using (6.13), we have reduced the representation matrices t_I to the following block-upper-triangular form:

$$(t_I)_{\widetilde{J}}{}^{\widetilde{K}} = \begin{pmatrix} f_{IJ}{}^K & (t_I)_J{}^N \\ 0 & (t_I)_M{}^N \end{pmatrix} . \tag{6.23}$$

In the case that $(t_I)_J{}^N = 0$, the representation is called *completely reducible*: the field-strengths $F^I_{\mu\nu}$ and the tensor fields $B^M_{\mu\nu}$ do not mix under gauge transformations.

Recall that every *unitary* reducible representation of a Lie group is also completely reducible, and that every representation of a *compact* Lie group is equivalent to a unitary representation. Hence, every reducible representation of a compact Lie group is also completely reducible. Non-compact Lie groups, on the other hand, have no non-trivial and finite-dimensional unitary representations. However, every reducible representation of a *connected*, *semi-simple*, non-compact Lie group or a semi-simple, non-compact Lie *algebra* is also completely reducible. See [222] for an exposition of these theorems.

Hence, we need to consider the class of non-compact Lie algebras that contain an Abelian invariant subalgebra. In $\mathcal{N} = 8$ gauged supergravity in $D = 5$ [223], the algebras $\mathrm{CSO}(p, q, r)$ were studied: they are defined as the set of matrices that leave invariant the metric

$$\eta_{IJ} = (\mathbb{1}_p, -\mathbb{1}_q, 0_r) . \tag{6.24}$$

These algebras contain the following subalgebra

$$\mathrm{SO}(p, q) \oplus \mathrm{SO}(1, 1)^{\frac{r(r-1)}{2}} \subset \mathrm{CSO}(p, q, r) . \tag{6.25}$$

In [220], a classification of possible compact gaugings of $\mathcal{N} = 2$ supergravity in five dimensions was given, but the class of non-compact gaugings was not investigated. An additional remark is that, for non-semi-simple Lie groups, we need to take the vector fields in the co-adjoint representation, rather than the adjoint representation [224]. For semi-simple Lie groups these two representations coincide.

Reducible but not completely reducible representations are then given by n_{V} Abelian vector multiplets and n_{T} tensor multiplets transforming in the following representation

$$(t_I)_{\widetilde{J}}{}^{\widetilde{K}} = \begin{pmatrix} 0 & (t_I)_J{}^N \\ 0 & 0 \end{pmatrix} . \tag{6.26}$$

The literature on five-dimensional tensor multiplets [159] states that, to write down an action, one must assume that the representation is completely reducible, meaning that gauge transformations do not mix the pure Yang–Mills field-strengths and the tensor fields. We, however, find that off-diagonal generators *are* allowed, both when requiring closure of the superconformal algebra and when writing down an action. Thus, we have found more general vector-tensor multiplets.

6.1.4 The massive self-dual tensor multiplet

To obtain the massive self-dual tensor multiplet of [221], we consider a vector-tensor multiplet for general $n_{\rm V}$ and $n_{\rm T}$. Our purpose is to use the vector multiplet as a compensating multiplet for the superconformal symmetry. Thus, we impose conditions on the fields that break the conformal symmetry and preserve Q–supersymmetry. We give the fields of the vector multiplets the following vacuum expectation values

$$F^I_{\mu\nu} = Y^{ijI} = \psi^{iI} = 0\,, \qquad \sigma^I = \frac{2m^I}{g}\,, \tag{6.27}$$

where m^I are constants. Note that these conditions break the conformal group to the Poincaré group, and break S-supersymmetry ($\eta = 0$). This is an example of a compensating multiplet in rigid supersymmetry. The breaking of conformal symmetry is characterized by the mass parameters m^I in (6.27). If we substitute (6.27) into the expression (6.14) for L^{ijM}, then we find that we can eliminate the field Y^{ijM}

$$Y^{ijM} = 0\,. \tag{6.28}$$

Moreover, we can also substitute (6.27) into the constraints $E^M_{\mu\nu\lambda}$, φ^{iM} and N^M to obtain

$$\begin{aligned} 3\partial_{[\mu}B^M_{\nu\lambda]} &= \tfrac{1}{2}\varepsilon_{\mu\nu\lambda\rho\sigma}\mathcal{M}_N{}^M B^{\rho\sigma N}\,, \\ \not{\partial}\psi^{iM} &= {\rm i}\,\mathcal{M}_N{}^M\psi^{iN}\,, \\ \Box\sigma^M &= -\left(\mathcal{M}^2\right)_N{}^M\sigma^N - \tfrac{4}{g}t_{IJ}{}^N m^I m^J \mathcal{M}_N{}^M\,. \end{aligned} \tag{6.29}$$

The mass-matrix $\mathcal{M}_N{}^M$ is defined as

$$\mathcal{M}_N{}^M \equiv g\,\sigma^I(t_I)_N{}^M = 2m^I(t_I)_N{}^M\,. \tag{6.30}$$

The last term of (6.29) can be eliminated by redefining σ^M with a constant shift. In order for the tensor fields to have no tachyonic modes, the mass-matrix needs to satisfy a symplectic condition which can only be satisfied if the number of tensor fields is even [221]. We denote the number of tensor multiplets by $n_{\rm T} = 2k$.

The exception is when the representation matrices are purely upper-diagonal: i.e when they take on the form (6.26). For that specific representation, the mass matrix vanishes identically and no tachyonic modes are present. However, in that case the self-duality condition reduces to the Bianchi identity so that we are dealing with $n_{\rm T}$ extra vector multiplets in disguise.

To obtain the massive self-dual tensor multiplet of [221] we consider the case of $n_{\rm V} = 1\,, n_{\rm T} = 2$, i.e. two (real) tensor multiplets $\{B^M_{\mu\nu}, \lambda^{iM}, \phi^M\}$ ($M, N = 2, 3$) in the background of one (Abelian) vector multiplet $\{F_{\mu\nu}\,\psi^i, \sigma\}$ that has been given the vacuum expectation value (6.27). In what follows we will use a complex notation

$$B_{\mu\nu} = B^2_{\mu\nu} + {\rm i}\,B^3_{\mu\nu}\,, \quad \overline{B}_{\mu\nu} = B^2_{\mu\nu} - {\rm i}\,B^3_{\mu\nu}\,. \tag{6.31}$$

The generators $(t_1)_{\widetilde{I}}{}^{\widetilde{J}}$ must form a representation of $\mathrm{U}(1) \simeq \mathrm{SO}(2)$. Under a $\mathrm{U}(1)$ transformation the field-strength $F_{\mu\nu}$ is invariant and the complex tensor field gets a phase

$$B'_{\mu\nu} = e^{\mathrm{i}\,\theta} B_{\mu\nu} \to \begin{pmatrix} B^2_{\mu\nu} \\ B^3_{\mu\nu} \end{pmatrix}' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} B^2_{\mu\nu} \\ B^3_{\mu\nu} \end{pmatrix} . \tag{6.32}$$

From this, we obtain the generator

$$(t_1)_{\widetilde{I}}{}^{\widetilde{J}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{6.33}$$

After substituting the conditions (6.27) into the transformation rules, we obtain

$$\begin{aligned} \delta B_{\mu\nu} &= -\bar{\epsilon}\gamma_{[\mu}\partial_{\nu]}\lambda - m\bar{\epsilon}\gamma_{\mu\nu}\lambda \,, \\ \delta\lambda^i &= -\tfrac{1}{4}\gamma \cdot B\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/\phi\epsilon^i - \mathrm{i}\,m\phi\epsilon^i \,, \\ \delta\phi &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\lambda \,, \end{aligned} \tag{6.34}$$

and

$$3\partial_{[\mu}B_{\nu\lambda]} = \mathrm{i}\,m\varepsilon_{\mu\nu\lambda\rho\sigma}B^{\rho\sigma} . \tag{6.35}$$

This reproduces the massive self-dual tensor multiplet of [221]. Note that the commutator of two Q-supersymmetries yields a translation plus a (rigid) U(1)-transformation whose parameter can be obtained from the general G-transformation in the superconformal algebra, see (6.7), by making the substitution (6.27).

From a six-dimensional point of view, the interpretation of the mass parameter m is that it is the label of the m-th Kaluza-Klein mode in the reduction of the $D = 6$ self-dual tensor multiplet. The zero-mode of the reduced tensor multiplet corresponds to a vector multiplet as can be seen from (6.35) which becomes a Bianchi identity for a field-strength when $m = 0$.

6.2 The hypermultiplet

In this section, we will discuss superconformal hypermultiplets in five dimensions. We will follow the approach of [225], which discussed four-dimensional superconformal hypermultiplets, but we will extend it to the case where an action is not needed, in the spirit explained in [201]. As for the tensor multiplets, there is no off-shell formulation with a finite number of auxiliary fields, and the supersymmetry algebra closes modulo equations of motion.

A single hypermultiplet contains four real scalars and two spinors subject to a symplectic Majorana reality condition. We take the number of hypermultiplets n_H equal to r, which means that we introduce $4r$ real scalars q^X, with $X = 1, \ldots, 4r$, and $2r$ spinors ζ^A with $A = 1, \ldots, 2r$. We have indicated these fields and their relevant properties in table 6.2.

Field	SU(2)	w	# d.o.f.
q^X	2	$\frac{3}{2}$	$4r$
ζ^A	1	2	$4r$

Table 6.3: The on-shell hypermultiplet.

Analogous to the equivalence of the real vector space $\mathbb{R}^{2n}$ to the complex vector space $\mathbb{C}^n$, the real vector space $\mathbb{R}^{4r}$ spanned by the scalars q^X is isomorphic to the quaternionic vector space $\mathbb{H}^r$. Recall that the field of quaternions $\mathbb{H}$ is defined as all four-tuples of the form $a + b\,\mathrm{i} + c\,\mathrm{j} + d\,\mathrm{k}$, with $\{a, b, c, d\} \in \mathbb{R}$ and $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{i}\,\mathrm{j}\,\mathrm{k} = -1$.

To formulate the symplectic Majorana condition, we introduce two matrices $\rho_A{}^B$ and $E_i{}^j$, with

$$\rho\rho^* = -\mathbb{1}_{2r}\,, \qquad EE^* = -\mathbb{1}_2\,. \tag{6.36}$$

This defines symplectic Majorana conditions for the fermions and the supersymmetry transformation parameters [226]:

$$\alpha\mathcal{C}\gamma_0\zeta^B\rho_B{}^A = \left(\zeta^A\right)^*\,, \qquad \alpha\mathcal{C}\gamma_0\epsilon^j E_j{}^i = (\epsilon^i)^*\,, \tag{6.37}$$

where $\mathcal{C}$ is the charge conjugation matrix, and α is an irrelevant number of modulus 1. We can always adopt the basis where $E_i{}^j = \varepsilon_{ij}$, and we will further restrict to that.

We will start this section with describing the rigid supersymmetry transformation rules and their geometrical interpretation. After that, we will realize the superconformal symmetries on the hypermultiplet. Finally, we will discuss how to gauge the isometries of the scalar manifold by coupling the hypermultiplet to a vector multiplet.

6.2.1 Rigid supersymmetry

We will show how the closure of the supersymmetry transformation rules on the scalars leads to equations defining a “hyper-complex” manifold. The scalars can then be regarded as coordinates on this hyper-complex manifold, whereas the fermions take their values in the tangent-space of the manifold. Furthermore, the closure of the algebra on the fermions leads to equations of motion.

Hyper-complex geometry

The rigid supersymmetry transformation rules for the hypermultiplet are given by

$$\begin{aligned} \delta(\epsilon)q^X &= -\mathrm{i}\,\bar{\epsilon}^i\zeta^A f_{iA}^X\,, \\ \delta(\epsilon)\zeta^A &= \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/ q^X f_X^{iA}\epsilon_i - \zeta^B\omega_{XB}{}^A\left(\delta(\epsilon)q^X\right)\,. \end{aligned} \tag{6.38}$$

The functions f_X^{iA} and $\omega_{XA}{}^B$ satisfy reality properties consistent with reality of q^X and the symplectic Majorana conditions

$$\left(f_X^{iA}\right)^* = f_X^{jB} E_j{}^i \rho_B{}^A \,, \qquad \left(\omega_{XA}{}^B\right)^* = \left(\rho^{-1}\omega_X \rho\right)_A{}^B \,. \tag{6.39}$$

A priori, the functions f_{iA}^X and f_X^{iA} are independent, but the commutator of two supersymmetries on the scalars only gives a translation if one imposes

$$\begin{aligned} f_Y^{iA} f_{iA}^X &= \delta_Y^X \,, \qquad f_X^{iA} f_{jB}^X = \delta_j^i \delta_B^A \,, \\ \mathfrak{D}_X f_Y^{iA} &\equiv \partial_X f_Y^{iA} - \Gamma_{XY}^Z f_Z^{iA} + \left(\omega_{Xj}{}^i \delta_B^A + \omega_{XB}{}^A \delta_j^i\right) f_Y^{jB} = 0 \,, \end{aligned} \tag{6.40}$$

where Γ_{XY}^Z is symmetric its lower indices.

The tensors given above have the following geometrical interpretation: f_{iA}^X and f_X^{iA} are invertible vielbeins on the scalar manifold, Γ_{XY}^Z can be interpreted as an affine torsionless connection, and $\omega_{Xj}{}^i$ and $\omega_{XB}{}^A$ are the $\mathrm{SU}(2)$-valued and $\mathrm{G}\,\ell(r,\mathbb{H})$-valued spin-connection one-forms, respectively. The constraint (6.40) then expresses that the vielbeins are covariantly constant with respect to these connections.

The scalar manifold is also endowed with a triplet of complex structures called the hypercomplex, which are constructed from the vielbeins and the Pauli-matrices σ^α

$$J_X{}^{Y\alpha} \equiv -\,\mathrm{i}\, f_X^{iA} (\sigma^\alpha)_i{}^j f_{jA}^Y \,. \tag{6.41}$$

For these complex structures, and other $\mathrm{SU}(2)$-valued quantities, we also use a doublet notation, for which

$$J_X{}^Y{}_i{}^j \equiv \mathrm{i}\, J_X{}^{Y\alpha} (\sigma^\alpha)_i{}^j = 2 f_X^{jA} f_{iA}^Y - \delta_i^j \delta_X^Y \,. \tag{6.42}$$

Using (6.40), the complex structures are covariantly constant and satisfy the quaternion algebra

$$J^\alpha J^\beta = -\mathbb{1}_{4r} \delta^{\alpha\beta} + \varepsilon^{\alpha\beta\gamma} J^\gamma \,. \tag{6.43}$$

The resulting geometry defined by the connections and complex structure goes under the name of *hyper-complex* geometry. The notion of a hyper-complex manifold appeared in the mathematics literature in [227], and various aspects have been treated in two workshops [228, 229].

Note that we do not require the existence of a metric: hyper-complex manifolds possessing a metric are called hyper-Kähler manifolds, and we will encounter them in section 6.3 when we discuss superconformal actions. Examples of (homogeneous) hyper-complex manifolds that are not hyper-Kähler were constructed in [230–232].

As a final remark, the associated curvature tensor for the $\mathrm{SU}(2)$-valued spin-connection one-form $\omega_{Xj}{}^i$ vanishes for all hyper-complex and hyper-Kähler manifolds

$$\mathcal{R}_{XYi}{}^j \equiv 2\partial_{[X}\omega_{Y]i}{}^j + 2\omega_{[X|k|}{}^j \omega_{Y]i}{}^k = 0 \,. \tag{6.44}$$

The connection is therefore always pure gauge and can be set to zero. Manifolds for which this curvature does not vanish are called quaternionic manifolds and quaternionic-Kähler manifolds, respectively. Quaternionic geometry generically arises after gauge-fixing superconformal hypermultiplets.

Reparametrizations

The supersymmetry transformation rules (6.38) are covariant with respect to two different kinds of reparametrizations. The first ones are the target space diffeomorphisms, $q^X \to \widetilde{q}^X(q)$, under which f^X_{iA} transforms as a vector, $\omega_{XA}{}^B$ as a one-form, and Γ^Z_{XY} as a connection. We can then define a variation $\widehat{\delta}$ which is covariantized with respect to these diffeomorphisms: e.g. for a quantity Δ^X we define

$$\widehat{\delta}\Delta^X = \delta\Delta^X + \Delta^Y \Gamma_{ZY}{}^X \, \delta q^Z \, . \tag{6.45}$$

Furthermore, there are reparametrizations of the tangent space, under which $f^{iA}_X(q)$ transforms as a vector, $\omega_{XA}{}^B$ as a connection,

$$\omega_{XA}{}^B \to \widetilde{\omega}_{XA}{}^B = \left[\left(\partial_X U^{-1}\right) U + U^{-1}\omega_X U\right]_A{}^B \, , \tag{6.46}$$

and the fermions as

$$\zeta^A \to \widetilde{\zeta}^A(q) = \zeta^B U_B{}^A(q) \, , \tag{6.47}$$

where $U(q)_A{}^B$ is any invertible matrix.

In general, such a transformation brings us into a basis where the fermions depend on the scalars q^X. In this sense, the hypermultiplet is written in a special basis where q^X and ζ^A are independent fields. These considerations lead us to define the covariant variation of the fermions

$$\widehat{\delta}\zeta^A \equiv \delta\zeta^A + \zeta^B \omega_{XB}{}^A \, \delta q^X \, , \tag{6.48}$$

Two models related by either target space diffeomorphisms or fermion reparametrizations of the form (6.47) are equivalent: they are different coordinate descriptions of the same system. Thus, in a covariant formalism, the fermions can be functions of the scalars. However, the expression $\partial_X \zeta^A$ makes only sense if one compares different bases. But in the same way also the expression $\zeta^B \omega_{XB}{}^A$ makes only sense if one compares different bases, as the connection has no absolute value. The only invariant object is the covariant derivative

$$\mathcal{D}_X \zeta^A \equiv \partial_X \zeta^A + \zeta^B \omega_{XB}{}^A \, . \tag{6.49}$$

Holonomy

Recall that the holonomy group of a manifold is defined as the group of transformations by which a vector can be rotated after parallel transport along a closed curve on the manifold. The holonomy group of a hyper-complex manifold is contained in $\mathrm{G}\,\ell(r,\mathbb{H}) = \mathrm{SU}^*(2r) \times \mathrm{U}(1)$, the group of transformations acting on the tangent-space.

This follows from the integrability conditions on the covariantly constant vielbeins f^{iA}_X, which relates the curvatures of the affine connection Γ^Z_{XY} and the spin-connection $\omega_{XA}{}^B$

$$R_{XYZ}{}^W = f^W_{iA} f^{iB}_Z \mathcal{R}_{XYB}{}^A \, , \qquad \delta^i_j \, \mathcal{R}_{XYB}{}^A = f^{iA}_W f^Z_{jB} \, R_{XYZ}{}^W \, , \tag{6.50}$$

	no metric	Hermitian metric
no $\mathrm{SU}(2)$ curvature	hyper-complex $\mathrm{G}\,\ell(r,\mathbb{H})$	hyper-Kähler $\mathrm{USp}(2r)$
non-zero $\mathrm{SU}(2)$ curvature	quaternionic $\mathrm{SU}(2)\cdot\mathrm{G}\,\ell(r,\mathbb{H})$	quaternionic-Kähler $\mathrm{SU}(2)\cdot\mathrm{USp}(2r)$

Table 6.4: The holonomy groups of the family of quaternionic-like manifolds.

where the curvatures are defined by

$$\begin{aligned} R^W{}_{ZXY} &\equiv 2\partial_{[X}\Gamma^W_{Y]Z} + 2\Gamma^W_{V[X}\Gamma^V_{Y]Z} \\ \mathcal{R}_{XYB}{}^A &\equiv 2\partial_{[X}\omega_{Y]B}{}^A + 2\omega_{[X|C|}{}^A\omega_{Y]B}{}^C\,. \end{aligned} \tag{6.51}$$

A consequence of (6.50) is that the Riemann curvature is purely $\mathrm{G}\,\ell(r,\mathbb{H})$-valued. Moreover, from the cyclicity properties of the Riemann tensor, it follows that

$$\begin{aligned} f^X_{Ci}f^Y_{jD}\mathcal{R}_{XYB}{}^A &= -\tfrac{1}{2}\varepsilon_{ij}W_{CDB}{}^A\,, \\ W_{CDB}{}^A &\equiv f^{iX}_C f^Y_{iD}\mathcal{R}_{XYB}{}^A \\ &= \tfrac{1}{2}f^{iX}_C f^Y_{iD}f^Z_{jB}f^{Aj}_W R_{XYZ}{}^W\,, \end{aligned} \tag{6.52}$$

where W is symmetric in all its three lower indices.

There are two possible modifications for the holonomy group of a hyper-complex manifold: when there is a metric (i.e. for hyper-Kähler manifolds), the holonomy group is reduced to $\mathrm{USp}(2r)$; and when the $\mathrm{SU}(2)$-valued curvature $\mathcal{R}_{XYi}{}^j$ is non-zero(i.e. for quaternionic manifolds), the holonomy group has an extra factor of $\mathrm{SU}(2)$. We have displayed these possibilities in table 6.2.1

As an additional remark, the Ricci tensor for hyper-complex manifolds with vanishing $\mathrm{SU}(2)$-curvature is anti-symmetric, whereas it is symmetric for hyper-complex manifolds equipped with a metric. In particular, hyper-Kähler manifolds (which fall in both classes) have a vanishing Ricci tensor. However, the Ricci-tensor for a hyper-complex manifold defines a non-vanishing but closed two-form. For a more detailed discussion on hyper-complex manifolds and their curvature relations, we refer to [17].

Nijenhuis condition

The covariant constancy condition (6.40) of the vielbein contains the affine connection Γ^Z_{XY} and the $\mathrm{G}\,\ell(r,\mathbb{H})$-valued spin-connection one-form $\omega_{XA}{}^B$. We will now indicate how these two objects can be determined from the vielbeins if and only if the ("diagonal") Nijenhuis condition

$$N_{XY}{}^Z \equiv J^\alpha{}_X{}^W\partial_{[W}J^\alpha{}_{Y]}{}^Z - (X\leftrightarrow Y) = 0\,, \tag{6.53}$$

is satisfied. In this case, the affine connection Γ^Z_{XY} is given by the Obata connection [233]

$$\Gamma^Z_{XY} = -\tfrac{1}{6}\varepsilon^{\alpha\beta\gamma} J^\alpha{}_W{}^Z J^\beta{}_{(X}{}^U \partial_{|U|} J^\gamma{}_{Y)}{}^W - \tfrac{1}{3} J^\beta{}_W{}^Z \partial_{(Y} J^\beta{}_{X)}{}^W \,, \tag{6.54}$$

which leads to covariantly constant complex structures. Moreover, one can show that any torsionless connection that leaves the complex structures invariant is equal to this Obata connection. This is similar to the way that a connection that leaves a metric invariant is the Levi-Civita connection.

With this connection one can then construct the $\mathrm{G}\,\ell(r,\mathbb{H})$ valued spin-connection

$$\omega_{XA}{}^B = \tfrac{1}{2} f^{iB}_Y \left(\partial_X f^Y_{iA} + \Gamma^Y_{XZ} f^Z_{iA}\right) \,, \tag{6.55}$$

such that the vielbeins are covariantly constant.

Equations of motion

Using (6.40), (6.50) and (6.52), we compute the commutator of two supersymmetry transformations on the fermions, and find

$$[\delta(\epsilon_1),\delta(\epsilon_2)]\zeta^A = \tfrac{1}{2}\bar\epsilon_2\gamma^a\epsilon_1\partial_a\zeta^A + \tfrac{1}{4}\Gamma^A\bar\epsilon_2\epsilon_1 - \tfrac{1}{4}\gamma_a\Gamma^A\bar\epsilon_2\gamma^a\epsilon_1 \,. \tag{6.56}$$

The algebra only closes if we set the Γ^A to zero: this defines the equations of motion for the fermions,

$$\begin{aligned}\Gamma^A &\equiv \not{\mathfrak{D}}\zeta^A + \tfrac{1}{2} W_{CDB}{}^A \zeta^B \bar\zeta^D \zeta^C \\ &= 0 \,,\end{aligned} \tag{6.57}$$

where we have introduced the covariant derivative, consistent with (6.48)

$$\mathfrak{D}_\mu\zeta^A \equiv \partial_\mu\zeta^A + (\partial_\mu q^X)\zeta^B\omega_{XB}{}^A \,. \tag{6.58}$$

By varying the fermion equation of motion under supersymmetry, we derive the corresponding equation of motion for the scalar fields

$$\widehat\delta(\epsilon)\Gamma^A = \tfrac{1}{2}\,\mathrm{i}\, f^{iA}_X \epsilon_i \Delta^X \,, \tag{6.59}$$

where

$$\begin{aligned}\Delta^X &\equiv \Box q^X - \tfrac{1}{2}\bar\zeta^B\gamma_a\zeta^D\partial^a q^Y f^{iC}_Y f^X_{iA} W_{BCD}{}^A \\ &\quad -\tfrac{1}{4}\mathfrak{D}_Y W_{BCD}{}^A \bar\zeta^E\zeta^D\bar\zeta^C\zeta^B f^{iY}_E f^X_{iA} \\ &= 0 \,,\end{aligned} \tag{6.60}$$

and the covariant D'Alembertian is given by

$$\Box q^X = \partial_a\partial^a q^X + \left(\partial_a q^Y\right)\left(\partial^a q^Z\right)\Gamma_{YZ}{}^X \,. \tag{6.61}$$

There are no more constraints since Γ^A and Δ^X form a closed set under supersymmetry

$$\widehat{\delta}(\epsilon)\Delta^X = -\,\mathrm{i}\,\bar{\epsilon}^i \not{\mathcal{D}}\Gamma^A f^X_{iA} + 2\,\mathrm{i}\,\bar{\epsilon}^i \Gamma^B \bar{\zeta}^C \zeta^D f^Y_{Bi} \mathcal{R}^X{}_{YCD}\,, \tag{6.62}$$

where the covariant derivative of Γ^A is defined similar to (6.58).

To summarize, the supersymmetry algebra imposes the hyper-complex constraints (6.40) and the equations of motion (6.57) and (6.60).

6.2.2 Superconformal symmetry

We will now derive further constraints on the target space geometry from requiring the presence of superconformal symmetry. The scalars do not transform under special conformal transformations and special supersymmetry, but under dilatations and $\mathrm{SU}(2)$ transformations, we parameterize

$$\begin{aligned} \delta_D(\Lambda_D) q^X &= \Lambda_D k^X(q)\,, \\ \delta_{\mathrm{SU}(2)}(\Lambda^{ij}) q^X &= \Lambda^{ij} k^X_{ij}(q)\,, \end{aligned} \tag{6.63}$$

for some unknown functions $k^X(q)$ and $k^X_{ij}(q)$.

To derive the appropriate transformation rules for the fermions, we first note that the hyperinos should be invariant under special conformal symmetry. This is due to the fact that this symmetry changes the Weyl weight with one. The special supersymmetry transformations of the fermions are determined by calculating the commutator of special conformal and supersymmetry transformations

$$\delta_S(\eta^i)\zeta^A = -k^X f^{iA}_X \eta_i\,. \tag{6.64}$$

Next, we consider the commutator of regular and special supersymmetry (5.14). Realizing this on the scalars, we determine the expression for the generator of $\mathrm{SU}(2)$ transformations in terms of the dilatations and complex structures,

$$k^X_{ij} = \tfrac{1}{3} k^Y J_Y{}^X{}_{ij}\,. \tag{6.65}$$

Realizing (5.14) on the hyperinos, we determine the covariant variations

$$\begin{aligned} \widehat{\delta}_D(\Lambda_D)\zeta^A &= 2\Lambda_D \zeta^A\,, \\ \widehat{\delta}_{\mathrm{SU}(2)}(\Lambda^{ij})\zeta^A &= 0\,. \end{aligned} \tag{6.66}$$

Furthermore, the commutator (5.14) only closes if we impose

$$\mathfrak{D}_Y k^X = \tfrac{3}{2}\delta_Y{}^X\,, \tag{6.67}$$

which also implies

$$\mathfrak{D}_Y k^X_{ij} = \tfrac{1}{2} J_Y{}^X{}_{ij}\,. \tag{6.68}$$

We note that (6.67) determines the Weyl weight of the scalars to be $\frac{3}{2}$, as indicated in table 6.2. Note that (6.67) is imposed by supersymmetry and not, as in the usual derivations, from the dilatation invariance of an action, as we have explained in chapter 6.

The relations (6.67) and (6.65) further restrict the geometry of the target space, and it is easy to derive that the Riemann tensor has four zero eigenvectors,

$$k^X R_{XYZ}{}^W = 0\,, \qquad k_{ij}^X R_{XYZ}{}^W = 0\,. \tag{6.69}$$

Under dilatations and $\mathrm{SU}(2)$ transformations, the hyper-complex structure is scale invariant and rotated into itself,

$$\begin{aligned} &\Lambda_D \left(k^Z \partial_Z J_X^{\alpha Y} - \partial_Z k^Y J_X^{\alpha Z} + \partial_X k^Z J_Z^{\alpha Y}\right) = 0\,, \\ &\Lambda^\beta \left(k_\beta^Z \partial_Z J_X^{\alpha Y} - \partial_Z k_\beta^Y J_X^{\alpha Z} + \partial_X k_\beta^Z J_Z^{\alpha Y}\right) = -\epsilon^\alpha{}_{\beta\gamma} \Lambda^\beta J_X^{\gamma Y}\,. \end{aligned} \tag{6.70}$$

All these properties are similar to those derived from superconformal hypermultiplets in four dimensions [225, 234]. There, the $\mathrm{Sp}(1) \times \mathrm{G}\,\ell(r,\mathbb{H})$ sections, or simply, hyper-complex sections, were introduced

$$A^{iB}(q) \equiv k^X f_X^{iB}\,, \qquad (A^{iB})^* = A^{jC} E_j{}^i \rho_C{}^B\,, \tag{6.71}$$

which allow for a coordinate independent description of the target space. This means that all equations and transformation rules for the sections can be written without the occurrence of the q^X fields. For example, the hyper-complex sections are zero eigenvectors of the $\mathrm{G}\,\ell(r,\mathbb{H})$ curvature

$$A^{iB} W_{BCD}{}^E = 0\,, \tag{6.72}$$

and have supersymmetry, dilatation and $\mathrm{SU}(2)$ transformation laws given by

$$\widehat{\delta} A^{iB} = \tfrac{3}{2} f_X^{iB} \delta q^X = -\tfrac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^i \zeta^B + \tfrac{3}{2} \Lambda_D A_i{}^B - \Lambda^i{}_j A^{jB}\,, \tag{6.73}$$

where $\widehat{\delta}$ is understood as a covariant variation, in the sense of (6.48).

6.2.3 Gauging symmetries

We will now discuss how to gauge a symmetry group G of the scalar manifold by coupling the hypermultiplet to a vector multiplet. The symmetry algebra must commute with the (conformal) supersymmetry algebra. The symmetries are parametrized by

$$\delta_G q^X = -g \Lambda_G^I k_I^X(q)\,, \tag{6.74}$$

$$\widehat{\delta}_G \zeta^A = -g \Lambda_G^I t_{IB}{}^A(q) \zeta^B\,. \tag{6.75}$$

The vectors k_I^X depend on the scalars and their Poisson brackets generate the algebra of G with structure constants $f_{IJ}{}^K$,

$$k_{[I|}^Y \partial_Y k_{|J]}^X = -\tfrac{1}{2} f_{IJ}{}^K k_K^X\,. \tag{6.76}$$

and there are no new constraints from the fermions or from other commutators. Since $\mathfrak{D}_Y k_I^X$ commutes with $J_Y{}^X{}_\alpha$, the second equation in (6.85) is a consequence of the first one.

In the above analysis, we have taken the parameters Λ^I to be constants. In the following, we also allow for local gauge transformations. The gauge coupling is done by introducing vector multiplets and defining the covariant derivatives

$$\begin{aligned} \mathfrak{D}_\mu q^X &\equiv \partial_\mu q^X + g A_\mu^I k_I^X\,, \\ \mathfrak{D}_\mu \zeta^A &\equiv \partial_\mu \zeta^A + \partial_\mu q^X \omega_{XB}{}^A \zeta^B + g A_\mu^I t_{IB}{}^A \zeta^B\,. \end{aligned} \tag{6.86}$$

The commutator of two supersymmetries should now also contain a local gauge transformation, in the same way as for the multiplets of the previous sections, see (6.7). This requires an extra term in the supersymmetry transformation law of the fermions,

$$\widehat{\delta}(\epsilon)\zeta^A = \tfrac{1}{2}\,\mathrm{i}\,\not{\mathfrak{D}} q^X f_X^{iA}\epsilon_i + \tfrac{1}{2} g\sigma^I k_I^X f_{iX}^A \epsilon^i\,. \tag{6.87}$$

With this additional term, the commutator on the scalars closes. However, the fermion equation of motion Γ^A is modified with terms at $\mathcal{O}(g)$ in the gauge coupling constant

$$\begin{aligned} \Gamma^A &\equiv \not{\mathfrak{D}}\zeta^A + \tfrac{1}{2} W_{BCD}{}^A \bar{\zeta}^C \zeta^D \zeta^B - \mathrm{i}\, g (k_I^X f_{iX}^A \psi^{iI} + \zeta^B \sigma^I t_{IB}{}^A) \\ &= 0\,, \end{aligned} \tag{6.88}$$

with the same conventions as in (6.56). The subsequent variation of Γ^A under supersymmetry determines the modified equation of motion for the scalars: it receives modifications at both $\mathcal{O}(g)$ and $\mathcal{O}(g^2)$

$$\begin{aligned} \Delta^X = \; & \Box q^X - \tfrac{1}{2}\bar{\zeta}^B \gamma_a \zeta^D \mathfrak{D}^a q^Y f_Y^{iC} f_{iA}^X W_{BCD}{}^A \\ & -\tfrac{1}{4}\mathfrak{D}_Y W_{BCD}{}^A \bar{\zeta}^E \zeta^D \bar{\zeta}^C \zeta^B f_E^{iY} f_{iA}^X \\ & -g\left(2\,\mathrm{i}\,\bar{\psi}^{iI}\zeta^B t_{IB}{}^A f_{iA}^X - k_I^Y J_Y{}^X{}_{ij} Y^{ijI}\right) \\ & + g^2 \sigma^I \sigma^J \mathfrak{D}_Y k_I^X k_J^Y\,. \end{aligned} \tag{6.89}$$

The gauge-covariant D'Alembertian is given by

$$\Box q^X = \partial_a \mathfrak{D}^a q^X + g \mathfrak{D}_a q^Y \partial_Y k_I^X A^{aI} + \mathfrak{D}_a q^Y \mathfrak{D}^a q^Z \Gamma_{YZ}^X\,. \tag{6.90}$$

The equations of motions Γ^A and Δ^X still transform into each other according to (6.59) and (6.62).

6.3 Superconformal actions

In this section, we will present rigid superconformal actions for the multiplets discussed in the previous sections. We will see that demanding the existence of an action is more restrictive than only considering equations of motion. For the different multiplets, we find that new geometric objects have to be introduced.

The commutator of two gauge transformations (6.2) on the fermions requires the following constraint on the field-dependent matrices $t_I(q)$,

$$[t_I, t_J]_B{}^A = -f_{IJ}{}^K t_{KB}{}^A - 2k^X_{[I|}\mathfrak{D}_X t_{|J]B}{}^A + k^X_I k^Y_J \mathcal{R}_{XYB}{}^A \,. \tag{6.77}$$

Requiring the gauge transformations to commute with supersymmetry leads to further relations between the quantities k^X_I and $t_{IB}{}^A$. In particular, the representation matrices $t_{IB}{}^A$ are determined by the vielbeins f^{iA}_X and the vectors k^X_I

$$t_{IA}{}^B = \tfrac{1}{2} f^Y_{iA}\mathfrak{D}_Y k^X_I f^{iB}_X \tag{6.78}$$

if the following constraint on the vectors k^X_I holds

$$f^{Y(i}_A f^{j)B}_X \mathfrak{D}_Y k^X_I = 0 \,. \tag{6.79}$$

Equation (6.79) can be expressed as the vanishing of the commutator of $\mathfrak{D}_Y k^X_I$ with the complex structures

$$(\mathfrak{D}_X k^Y_I) J^\alpha{}_Y{}^Z = J^\alpha{}_X{}^Y (\mathfrak{D}_Y k^Z_I) \,. \tag{6.80}$$

This says that all the symmetries are embedded in $\mathrm{G}\,\ell(r,\mathbb{H})$. Equivalently, (6.80) can be written as the Lie derivative of the complex structure in the direction of the vector k_I

$$(\mathcal{L}_{k_I} J^\alpha)_X{}^Y \equiv k^Z_I \partial_Z J^{\alpha Y}_X - \partial_Z k^Y_I J^{\alpha Z}_X + \partial_X k^Z_I J^{\alpha Y}_Z = 0 \,. \tag{6.81}$$

Thus, this is the statement that the gauge transformations act *tri-holomorphic*, i.e. they leave the hyper-complex structure invariant.

Vanishing of the gauge-supersymmetry commutator on the fermions requires a new constraint

$$\mathfrak{D}_X \mathfrak{D}_Y k^Z_I = R_{XWY}{}^Z k^W_I \,. \tag{6.82}$$

Note that this equation is in general true for any Killing vector of a metric. As we are only considering hyper-complex manifolds without a metric so far, we could not rely on this fact, but the superconformal algebra alone imposes this equation. A consequence of (6.82) is that

$$\mathfrak{D}_Y t_{IA}{}^B = k^X_I \mathcal{R}_{YXA}{}^B \,, \tag{6.83}$$

which in turn allows for a simplification of (6.77)

$$[t_I, t_J]_B{}^A = -f_{IJ}{}^K t_{KB}{}^A - k^X_I k^Y_J \mathcal{R}_{XYB}{}^A \,. \tag{6.84}$$

The group of gauge symmetries should also commute with the superconformal algebra, in particular with dilatations and $\mathrm{SU}(2)$-transformations. This leads to

$$\begin{aligned} k^Y \mathfrak{D}_Y k^X_I &= \tfrac{3}{2} k^X_I \,, \\ k^Y_\alpha \mathfrak{D}_Y k^X_I &= \tfrac{1}{2} k^Y_I J_Y{}^X{}_\alpha \,, \end{aligned} \tag{6.85}$$

6.3.1 The Yang-Mills multiplet

The rigid superconformal invariant action describing $n_{\rm V}$ Abelian vector multiplets can be obtained by taking the cubic action of the improved vector multiplet (5.102), adding indices I, J, K on the fields, and multiplying this with a completely symmetric tensor C_{IJK}. The existence of the tensor C_{IJK} has the geometrical significance that it endows the scalar manifold $\mathbb{R}^{n_{\rm V}}$ with a metric g_{IJ}

$$g_{IJ} \equiv -\frac{1}{3}\frac{\partial^2 \ln N}{\partial\sigma^I \partial\sigma^J}\,, \qquad N \equiv C_{IJK}\sigma^I\sigma^J\sigma^K\,. \tag{6.91}$$

At leading order in the gauge coupling constant, multiplying (5.102) with C_{IJK} also gives the action for $n_{\rm V}$ non-Abelian vector multiplets. However, because the rigid transformation rules for the non-Abelian vector multiplet (6.5) differ from the transformation rules (5.102) and (5.105) of the Abelian vector multiplet at $\mathcal{O}(g)$ in the gauge coupling constant, the tensor C_{IJK} has to satisfy the following constraint

$$f_{I(J}{}^H C_{KL)H} = 0\,. \tag{6.92}$$

Furthermore, the $A\wedge F\wedge F$ Chern-Simons (CS) term has to be modified at $\mathcal{O}(g)$ and $\mathcal{O}(g^2)$. To obtain this Yang-Mills CS term, it is convenient to rewrite the CS term as an integral over a six-dimensional manifold which has a boundary given by the five-dimensional Minkowski spacetime. The six-form appearing in the integral is (in differential form notation) given by

$$I_{\rm V} = C_{IJK}F^I\wedge F^J\wedge F^K\,. \tag{6.93}$$

This six-form is both gauge-invariant and closed, by virtue of (6.92) and the Bianchi identities (6.4). It can therefore be written as the exterior derivative of a five-form which is gauge-invariant up to a total derivative. The spacetime integral over this five-form is the Yang-Mills CS-term.

Finally, there is also an extra fermion bilinear at $\mathcal{O}(g)$ in the action called the Yukawa term. This leads to the action obtained in [219] using an intermediate linear multiplet

$$\begin{aligned}\mathcal{L}_{\rm V} \;=\; & \Big[\big(-\tfrac14 F^I_{\mu\nu}F^{\mu\nu J} - \tfrac12\bar\psi^I \not{\mathcal{D}}\psi^J - \tfrac12\mathcal{D}_a\sigma^I\mathcal{D}^a\sigma^J + Y^I_{ij}Y^{ijJ}\big)\,\sigma^K \\ & -\tfrac{1}{24}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}A^I_\mu\left(F^J_{\nu\lambda}F^K_{\rho\sigma} + \tfrac12 g[A_\nu,A_\lambda]^J F^K_{\rho\sigma} + \tfrac{1}{10}g^2[A_\nu,A_\lambda]^J[A_\rho,A_\sigma]^K\right) \\ & -\tfrac18\,\mathrm{i}\,\bar\psi^I\gamma\cdot F^J\psi^K - \tfrac12\,\mathrm{i}\,\bar\psi^{iI}\psi^{jJ}Y^K_{ij} + \tfrac14\,\mathrm{i}\,g\bar\psi^L\psi^H\sigma^I\sigma^J f_{LH}{}^K\Big]\,C_{IJK}\,. \end{aligned} \tag{6.94}$$

The equations of motion for the fields of the vector multiplet following from the action (6.94) are

$$0 = L^{ij}_I = \varphi^i_I = E^a_I = N\,, \tag{6.95}$$

where we have defined

$$
\begin{aligned}
L_I^{ij} &\equiv C_{IJK}\left(2\sigma^J Y^{ijK} - \tfrac{1}{2}\,\mathrm{i}\,\bar\psi^{iJ}\psi^{jK}\right)\,,\\
\varphi_I^i &\equiv C_{IJK}\left(\mathrm{i}\,\sigma^J \not{\mathcal{D}}\psi^{iK} + \tfrac{1}{2}\,\mathrm{i}\,(\not{\mathcal{D}}\sigma^J)\psi^{iK} + Y^{ikJ}\psi_k^K - \tfrac{1}{4}\gamma\cdot F^J\psi^{iK}\right)\\
&\quad -g\,C_{IJK}\,f_{LH}{}^K\sigma^J\sigma^L\psi^{iH}\,,\\
E_{aI} &\equiv C_{IJK}\left(D^b\left(\sigma^J F_{ba}{}^K + \tfrac{1}{4}\,\mathrm{i}\,\bar\psi^J\gamma_{ba}\psi^K\right) - \tfrac{1}{8}\varepsilon_{abcde}F^{bcJ}F^{deK}\right)\\
&\quad -\tfrac{1}{2}g\,C_{JKL}f_{IH}{}^J\sigma^K\bar\psi^L\gamma_a\psi^H - g\,C_{JKH}f_{IL}{}^J\sigma^K\sigma^L D_a\sigma^H\,,\\
N_I &\equiv C_{IJK}\left(\sigma^J\Box\sigma^K + \tfrac{1}{2}\mathcal{D}^a\sigma^J\mathcal{D}_a\sigma^K - \tfrac{1}{4}F_{ab}^J F^{abK} - \tfrac{1}{2}\bar\psi^J\not{\mathcal{D}}\psi^K + Y^{ijJ}Y_{ij}{}^K\right)\\
&\quad +\tfrac{1}{2}\,\mathrm{i}\,g\,C_{IJK}\,f_{LH}{}^K\sigma^J\bar\psi^L\psi^H\,.
\end{aligned}
\tag{6.96}
$$

We have given these equations of motion the names $L_I^{ij}, \phi_I^i, E_{aI}, N_I$ since they form a linear multiplet in the adjoint representation of the gauge group for which the transformation rules have been given in [17].

6.3.2 The vector-tensor multiplet

We will now generalize the vector action (6.94) to an action for the vector-tensor multiplets (with n_{V} vector multiplets and n_{T} tensor multiplets) discussed in section 6.1.2.

The supersymmetry transformation rules for the vector-tensor multiplet (6.11) were obtained from those for the vector multiplet (6.5) by replacing all contracted indices by the extended range of tilde indices. In addition, extra terms of $\mathcal{O}(g)$ had to be added to the transformation rules. Similar considerations apply to the generalization of the action, as we will see below.

We will first generalize the CS term (6.93) to the case of vector-tensor multiplets. It turns out that this generalization is somewhat surprising: it will involve the inclusion of derivative terms. We find the following expression for the unique closed and gauge-invariant six-form

$$
I_{\mathrm{VT}} = C_{\widetilde{I}\widetilde{J}\widetilde{K}}\mathcal{H}^{\widetilde{I}}\wedge\mathcal{H}^{\widetilde{J}}\wedge\mathcal{H}^{\widetilde{K}} - \tfrac{3}{g}\Omega_{MN}\mathcal{D}B^M\wedge\mathcal{D}B^N\,,
\tag{6.97}
$$

The tensor Ω_{MN} is antisymmetric and invertible, and it will restrict the number of tensor multiplets to be *even*: $n_{\mathrm{T}} = 2k$ and

$$
\Omega_{MN} = -\Omega_{NM}\,,\qquad \Omega_{MP}\Omega^{MR} = \delta_P{}^R\,,
\tag{6.98}
$$

The covariant derivative $\mathcal{D}B^M$ is given by

$$
\begin{aligned}
\mathcal{D}_\lambda B_{\rho\sigma}^M &= \partial_\lambda B_{\rho\sigma}^M + g\,A_\lambda^I t_{I\widetilde{J}}{}^M\mathcal{H}_{\rho\sigma}^{\widetilde{J}}\\
&= \partial_\lambda B_{\rho\sigma}^M + g\,A_\lambda^I t_{IJ}{}^M F_{\rho\sigma}^J + g\,A_\lambda^I t_{IN}{}^M B_{\rho\sigma}^N\,.
\end{aligned}
\tag{6.99}
$$

To see why the first term of (6.97) is not a closed six-form by itself, we write it out explicitly as

$$
C_{\widetilde{I}\widetilde{J}\widetilde{K}}\mathcal{H}^{\widetilde{I}}\mathcal{H}^{\widetilde{J}}\mathcal{H}^{\widetilde{K}} = C_{IJK}F^IF^JF^K + 3C_{IJM}F^IF^JB^M + 3C_{IMN}F^IB^MB^N\,.
\tag{6.100}
$$

Since the B^M tensors in (6.100) do not satisfy a Bianchi identity, we also need the second term in (6.97) to obtain a closed six-form. This leads to the following relations between the C and Ω tensors:

$$C_{IJM} = t_{(IJ)}{}^N \Omega_{NM}\,, \qquad C_{IMN} = \tfrac{1}{2} t_{IM}{}^P \Omega_{PN}\,. \tag{6.101}$$

As additional remark, the components of C can have only three different forms: C_{IJK}, C_{IJM} and C_{IMN} (and permutations). The reason is that when the first term of (6.97) is reduced to five dimensions, one of the $\mathcal{H}^{\widetilde{I}}$ factors should correspond to a vector field strength F^I. Only then can the corresponding five-form be written as $A^I \wedge \mathcal{H}^{\widetilde{J}} \wedge \mathcal{H}^{\widetilde{K}}$.

Gauge invariance of the first term of (6.97) requires that the tensor C satisfies a modified version of (6.92)

$$f_{I(J}{}^H C_{KL)H} = t_{I(J}{}^M t_{KL)}{}^N \Omega_{MN}\,. \tag{6.102}$$

In addition to this, the second term of (6.97) is only gauge invariant if the tensor Ω satisfies

$$t_{I[M}{}^P \Omega_{N]P} = 0\,, \tag{6.103}$$

such that the last one of (6.101) is consistent with the symmetry (MN).

Finally, there are extra Yukawa couplings at $\mathcal{O}(g)$ and there is a scalar potential term at $\mathcal{O}(g^2)$ in the vector-tensor multiplet action. The superconformal action for the combined system of $n_{\rm T} = 2k$ tensor multiplets in the background of $n_{\rm V}$ vector multiplets is given by

$$\begin{aligned}
\mathcal{L}_{\rm VT} \;=\;& \Big(-\tfrac{1}{4}\mathcal{H}^{\widetilde{I}}_{\mu\nu}\mathcal{H}^{\mu\nu\widetilde{J}} - \tfrac{1}{2}\bar{\psi}^{\widetilde{I}}\not{D}\psi^{\widetilde{J}} - \tfrac{1}{2}\mathcal{D}_a\sigma^{\widetilde{I}}\mathcal{D}^a\sigma^{\widetilde{J}} + Y^{\widetilde{I}}_{ij}Y^{ij\widetilde{J}}\Big)\,\sigma^{\widetilde{K}} C_{\widetilde{I}\widetilde{J}\widetilde{K}} \\
&+\tfrac{1}{16g} e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}B^M_{\mu\nu}\left(\partial_\lambda B^N_{\rho\sigma} + 2g\,t_{IJ}{}^N A^I_\lambda F^J_{\rho\sigma} + g\,t_{IP}{}^N A^I_\lambda B^P_{\rho\sigma}\right) \\
-\tfrac{1}{24} & e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}C_{IJK}A^I_\mu\left(F^J_{\nu\lambda}F^K_{\rho\sigma} + f_{FG}{}^J A^F_\nu A^G_\lambda\left(-\tfrac{1}{2}g\,F^K_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right)\right) \\
&-\tfrac{1}{8}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^M t_{FG}{}^N A^I_\mu A^F_\nu A^G_\lambda\left(-\tfrac{1}{2}g\,F^K_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right) \\
&+\Big(-\tfrac{1}{8}\,\mathrm{i}\,\bar{\psi}^{\widetilde{I}}\gamma\cdot\mathcal{H}^{\widetilde{J}}\psi^{\widetilde{K}} - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}^{i\widetilde{I}}\psi^{j\widetilde{J}}Y^{\widetilde{K}}_{ij}\Big)\,C_{\widetilde{I}\widetilde{J}\widetilde{K}} \\
&+\tfrac{1}{4}\,\mathrm{i}\,g\bar{\psi}^{\widetilde{I}}\psi^{\widetilde{J}}\sigma^{\widetilde{K}}\sigma^{\widetilde{L}}\Big(t_{[\widetilde{I}\widetilde{J}]}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{K}\widetilde{L}} - 4t_{(\widetilde{I}\widetilde{K})}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{J}\widetilde{L}}\Big) \\
&-\tfrac{1}{2}g^2\sigma^K\sigma^I\sigma^{\widetilde{L}}\sigma^J\sigma^{\widetilde{P}}C_{KMN}t_{I\widetilde{L}}{}^M t_{J\widetilde{P}}{}^N\,,
\end{aligned} \tag{6.104}$$

We stress that the tensor $C_{\widetilde{I}\widetilde{J}\widetilde{K}}$ is not a fundamental object: the essential data for the vector-tensor multiplet are the representation matrices $t_{I\widetilde{J}}{}^{\widetilde{K}}$, the Yang-Mills components C_{IJK}, and the symplectic matrix Ω_{MN}. The tensor components of the C tensor are derived quantities, and we can summarize (6.101) as

$$C_{M\widetilde{J}\widetilde{K}} = t_{(\widetilde{J}\widetilde{K})}{}^P \Omega_{PM}\,. \tag{6.105}$$

The two conditions (6.102) and (6.103) combined with the definition (6.105) imply the following generalization of (6.92)

$$t_{I(\widetilde{J}}{}^{\widetilde{M}} C_{\widetilde{K}\widetilde{L})\widetilde{M}} = 0\,. \tag{6.106}$$

To check the supersymmetry of the action (6.104), one needs all the relations between the various tensors given above. Another useful identity implied by the previous definitions is

$$t_{(\widetilde{I}\widetilde{J})}{}^{M} C_{\widetilde{K}\widetilde{L}M} = -t_{(\widetilde{K}\widetilde{L})}{}^{M} C_{\widetilde{I}\widetilde{J}M}\,. \tag{6.107}$$

The action with fields of the tensor multiplets can also be obtained from the field equations (6.17). They are now related to the action by

$$\frac{\delta \mathcal{L}_{\rm VT}}{\delta \bar{\psi}^{iM}} = {\rm i}\,\varphi_i^N \Omega_{NM}\,, \tag{6.108}$$

and the remaining bosonic terms can be obtained from a comparison with N^M in (6.19). One may then further check that also the field equations (6.15) and (6.17) follow from this action.

Note, however, that the equations of motion for the vector multiplet fields, obtained from this action, are similar to those given in (6.96), but with the contracted indices running over the extended range of vector and tensor components. Furthermore, the A_μ^I equation of motion gets corrected by a term proportional to the self-duality equation for $B_{\mu\nu}^M$

$$\frac{\delta \mathcal{L}_{\rm VT}}{\delta A_a^I} = E_I^a + \tfrac{1}{12} g \varepsilon^{abcde} A_b^J E_{cde}^M t_{JI}{}^N \Omega_{MN}\,. \tag{6.109}$$

To summarize: in order to write down a rigid superconformal action for the vector-tensor multiplet, we need to introduce a gauge-invariant, anti-symmetric, invertible tensor Ω_{MN}, which restricts the number of tensor multiplets to be even. We can still allow the transformations to have off-diagonal terms between vector and tensor multiplets, if we adapt (6.92) to (6.102).

In this way, we have constructed more general matter-couplings than were known so far: with our extension to allow for the off-diagonal term in (6.23), we also get CS-terms induced by the C_{IJM} components, which were not present in [159]. In particular, in [159] it was found that such $A \wedge F \wedge B$ terms "appear impossible to supersymmetrize (except possibly in very special cases)". However, we see that such terms appear generically in our Lagrangian by allowing for off-diagonal gauge transformations that mix the tensor fields with the Yang-Mills field-strengths.

6.3.3 The hypermultiplet

Let us recapitulate the geometrical setting for the hypermultiplet: the scalar manifold was seen to be a hyper-complex manifold possessing a triplet of complex structures that satisfied the Nijenhuis conditions (6.53). From this integrability condition, it was possible to construct an affine torsionless Obata connection Γ^Z_{XY} and the $\mathrm{G}\,\ell(r,\mathbb{H})$-valued spin-connection one-form $\omega_{XA}{}^B$. Furthermore, the $\mathrm{SU}(2)$-valued spin-connection one-form $\omega_{Xi}{}^j$ had a vanishing curvature.

Using this algebraic description of the hyper-complex geometry, the constraints that were needed to close the superconformal algebra on the on-shell hypermultiplet were seen to be equations of motion. These equations of motion, Γ^A and Δ^X, transformed covariantly with respect to diffeomorphisms on the scalar manifold and to transformations on its tangent space. However, these equations of motion were not derived from an action.

When we introduce an action, the kinetic term takes on the following generic form

$$\mathcal{L}_{\mathrm{H}} = -\tfrac{1}{2} g_{XY}(\phi)\partial_\mu\phi^X\partial^\mu\phi^Y\,, \tag{6.110}$$

where the tensor g_{XY} is interpreted as the metric on the scalar manifold. The field equations for the scalars should now also be covariant with respect to coordinate transformations on the target manifold. This implies that the connection on the tangent bundle should be the Levi-Civita connection. Only in that particular case, the field equations for the scalars will be covariant.

We will now see what the consequences are of introducing the extra input of a metric on the geometry of the scalar manifold.

Hyper-Kähler geometry

We take the fermion equation of motion Γ^A to be proportional to the field equations following from an action

$$\frac{\delta S}{\delta\bar{\zeta}^A} = 2C_{AB}\Gamma^B\,. \tag{6.111}$$

In general, the tensor C_{AB} could be a function of the scalars and bilinears of the fermions. If we try to construct an action with the above Ansatz, it turns out that the tensor has to be anti-symmetric in AB and

$$\frac{\delta C_{AB}}{\delta\zeta^C} = 0\,, \tag{6.112}$$

$$\mathfrak{D}_X C_{AB} = 0\,. \tag{6.113}$$

In other words, the tensor does not depend on the fermions and is covariantly constant[1].

This tensor C_{AB} will be used to raise and lower tangent space indices according to the NW–SE convention similar to ε_{ij}:

$$A_A = A^B C_{BA}\,, \qquad A^A = C^{AB}A_B\,, \tag{6.114}$$

where ε^{ij} and C^{AB} are defined for consistency by

$$\varepsilon_{ik}\varepsilon^{jk} = \delta_i{}^j\,, \qquad C_{AC}C^{BC} = \delta_A{}^B\,. \tag{6.115}$$

[1]This can be derived using the Batalin-Vilkovisky formalism.

From the integrability condition for (6.113) we find

$$[\mathfrak{D}_X, \mathfrak{D}_Y]C_{AB} = 0 = -2\mathcal{R}_{XY[A}{}^C C_{B]C}\,, \tag{6.116}$$

which implies that the anti-symmetric part of the connection $\omega_{XAB} \equiv \omega_{XA}{}^C C_{CB}$ is pure gauge, and can be chosen to be zero. If we do so, the covariant constancy condition for C_{AB} reduces to the equation that C_{AB} is just constant.

We can construct the metric g_{XY} on the scalar manifold by multiplying the metric on the tangent space with the vielbeins

$$g_{XY} = f_X^{iA} f_Y^{jB} C_{AB}\varepsilon_{ij}\,. \tag{6.117}$$

Since the connection ω_{XAB} is symmetric, the original holonomy group $\mathrm{G}\,\ell(r,\mathbb{H})$ is reduced to $\mathrm{USp}(2r-2p,2p)$: its signature is the signature of d_{CB}. The tensor d_{AB} is defined as $C_{AB} = \rho_A{}^C d_{CB}$ where $\rho_A{}^C$ was given in (6.36). These restrictions on the hyper-complex geometry reduce the scalar manifold to a hyper-Kähler manifold.

Furthermore, the affine connection used in the covariant derivative in (6.113) is now given by the Levi-Civita connection constructed from the metric g_{XY}. Indeed, this guarantees that the metric is covariantly constant. On the other hand, we have already seen already that, to have covariantly constant complex structures, we have to use the Obata connection. Hence, the Levi-Civita and Obata connection coincide for hyper-Kähler manifolds.

The action for rigid hypermultiplet takes on the form

$$\mathcal{L}_{\mathrm{H}} = -\tfrac{1}{2} g_{XY}\partial_a q^X \partial^a q^Y + \bar\zeta_A \not{\mathfrak{D}}\zeta^A - \tfrac{1}{4} W_{ABCD}\bar\zeta^A\zeta^B\bar\zeta^C\zeta^D\,. \tag{6.118}$$

where the tensor W_{ABCD} can be proven to be completely symmetric in all of its indices [17]. The field equations derived from this action are

$$\begin{aligned} \frac{\delta S}{\delta\bar\zeta^A} &= 2C_{AB}\Gamma^B\,, \\ \frac{\delta S}{\delta q^X} &= g_{XY}\Delta^Y - 2\bar\zeta_A\Gamma^B\omega_{XB}{}^A\,, \end{aligned} \tag{6.119}$$

Also remark that due to the introduction of the metric, the expression of Δ^X simplifies to

$$\Delta^X = \Box q^X - \bar\zeta^A \not{\partial} q^Y \zeta^B \mathcal{R}^X{}_{YAB} - \tfrac{1}{4}\mathfrak{D}^X W_{ABCD}\bar\zeta^A\zeta^B\zeta^C\zeta^D\,. \tag{6.120}$$

Superconformal symmetry

Due to the presence of the metric, the condition for the homothetic Killing vector (6.67) implies that k_X is the derivative of a scalar function as in (5.26). This scalar function $\chi(q)$ is called the hyper-Kähler potential [188, 225, 235]. It determines the metric

$$\mathfrak{D}_X\mathfrak{D}_Y\,\chi = \tfrac{3}{2} g_{XY}\,, \tag{6.121}$$

as well as the homothetic Killing vector

$$k_X = \partial_X \chi \,, \qquad \chi = \tfrac{1}{3} k_X k^X \,. \tag{6.122}$$

Note that this implies that, when χ and the complex structures are known, one can compute the metric with (6.121), using the formula for the Obata connection (6.54).

Gauging isometries

In the presence of a metric, the symmetries of section 6.2.3 should also be symmetries of the metric, i.e they should be *isometries*. This means that the vectors ${k_I}^X$ are now Killing vectors of the metric g_{XY}

$$\mathfrak{D}_X k_{YI} + \mathfrak{D}_Y k_{XI} = 0 \,. \tag{6.123}$$

This makes the requirement (6.82) superfluous, but we will still have to impose the tri-holomorphicity expressed by either (6.79), (6.80) or (6.81).

From the tri-holomorphicity condition (6.81) we find that, in order to integrate the equations of motion to an action, we have to define (locally) triplets of "moment maps" P^α_X that satisfy

$$\partial_X P^\alpha_I = {J_{XY}}^\alpha k^Y_I \,. \tag{6.124}$$

The field equations have the same form as in (6.119), except that all derivatives are now covariantized with respect to the new transformations. The same covariantization takes place in the action but here there are now modifications at $\mathcal{O}(g)$ and $\mathcal{O}(g^2)$ in the gauge coupling constant

$$\begin{aligned} \mathcal{L}_{\rm H} \;=\; & -\tfrac{1}{2} g_{XY} \mathfrak{D}_a q^X \mathfrak{D}^a q^Y + \bar\zeta_A \not{\mathfrak{D}} \zeta^A - \tfrac{1}{4} W_{ABCD} \bar\zeta^A \zeta^B \bar\zeta^C \zeta^D \\ & -g\left(P_{Iij} Y^{Iij} + 2\,\mathrm{i}\, k^X_I f^A_{iX} \bar\zeta_A \psi^{iI} + \mathrm{i}\,\sigma^I {t_{IB}}^A \bar\zeta_A \zeta^B\right) \\ & -\tfrac{1}{2} g^2 \sigma^I \sigma^J k^X_I k_{JX} \,. \end{aligned} \tag{6.125}$$

Supersymmetry of the action leads to the constraint

$$k^X_I J^\alpha_{XY} k^Y_J = -{f_{IJ}}^K P^\alpha_K \,. \tag{6.126}$$

As only the derivative of P appears in the defining equation (6.124), one may add an arbitrary constant to P. However, this changes the right-hand side of (6.126). One should then consider whether there is a choice of these coefficients such that (6.126) is satisfied [236]. For simple groups there is always a solution[2], whereas for Abelian theories the constant remains undetermined. This free constant is the so-called Fayet–Iliopoulos term [237].

In a superconformally invariant theory, the Fayet–Iliopoulos term is not possible. Indeed, dilatation invariance of the action needs

$$3P^\alpha_I = k^X \partial_X P^\alpha_I \,. \tag{6.127}$$

[2] We thank Gary Gibbons for a discussion on this subject.

Using (6.124) or (6.85), P_{Iij} is completely determined to be

$$3P_I^\alpha = k^X J_{XY}{}^\alpha k_I^Y = -\tfrac{2}{3} k^X k^Z J_Z{}^{Y\alpha} \mathfrak{D}_Y k_{IX} \,. \tag{6.128}$$

The proof of the invariance of the action under the complete superconformal group, uses the equation obtained from (6.85) and (6.124)

$$k^{X\alpha} \mathfrak{D}_X k_I^Y = -\tfrac{1}{2} \partial^Y P_I^\alpha \,. \tag{6.129}$$

If the moment map P_I^α has the value that it takes in the conformal theory, then (6.126) is satisfied due to (6.76). Indeed, one can multiply that equation with $k_X k^Z J^\alpha{}_Z{}^W \mathfrak{D}_W$ and use (6.69), (6.80) and (6.82). Thus, in the superconformal theory, the moment maps are completely determined, and there is no further relation to be obeyed: i.e. the Fayet–Iliopoulos terms of the rigid theories are absent in this case.

6.4 Coupling to the Weyl multiplet

We are now ready to perform the last step in our program, i.e. make the extension to local superconformal supersymmetry. We will make use here of the off-shell $32+32$ Standard Weyl multiplet constructed in chapter 5. Since in the previous sections we have explained most of the subtleties concerning the possible geometrical structures, we can be brief here.

We will obtain our results in two steps. First, we require that the local superconformal commutator algebra (5.87) - (5.91) of the Weyl multiplet is also realized on the matter multiplets, keeping in mind possible additional transformations under which the fields of the standard Weyl multiplet do not transform, and possibly field equations if the matter multiplet is on-shell. Next, we apply a standard Noether procedure to extend the rigid superconformal actions to local superconformal actions.

It is important to note that we do not construct an action for conformal supergravity itself: there will be no kinetic terms for the fields of the Weyl multiplets. Instead, the Weyl multiplet is seen as a fixed background of conformal supergravity to which the various matter multiplets couple. In section 6.5.2, we will indicate how local superconformal matter multiplets nevertheless lead to dynamical theories of Poincaré supergravity coupled to matter.

6.4.1 Vector-tensor multiplet

For brevity, we will present the transformation rules for n_T tensor multiplets in the background of $n_{\rm V}$ vector multiplets. The transformation rules for the vector multiplet itself can be obtained from it by making the restriction to $n_{\rm T}=0$. The local superconformal transformations rules for the vector-tensor multiplet are the following generalization of the transformation rules (5.105) of an Abelian vector multiplet coupled to the Weyl multiplet

$$\begin{aligned}
\delta A_\mu^I &= \tfrac{1}{2}\bar\epsilon\gamma_\mu\psi^I - \tfrac{1}{2}\,\mathrm{i}\,\sigma^I\bar\epsilon\psi_\mu\,,\\
\delta B_{ab}^M &= -\bar\epsilon\gamma_{[a}D_{b]}\psi^M + \mathrm{i}\,g\bar\epsilon\gamma_{ab}t_{(\widetilde{J}\widetilde{K})}{}^M\sigma^{\widetilde{J}}\psi^{\widetilde{K}} + \mathrm{i}\,\bar\eta\gamma_{ab}\psi^M\,,\\
\delta Y^{ij\widetilde{I}} &= -\tfrac{1}{2}\bar\epsilon^{(i}\not{D}\psi^{j)\widetilde{I}} + \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\gamma\cdot T\psi^{j)\widetilde{I}} - 4\,\mathrm{i}\,\sigma^{\widetilde{I}}\bar\epsilon^{(i}\chi^{j)}\\
&\quad -\tfrac{1}{2}\,\mathrm{i}\,g\bar\epsilon^{(i}\left(t_{[\widetilde{J}\widetilde{K}]}{}^{\widetilde{I}} - 3t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\right)\sigma^{\widetilde{J}}\psi^{j)\widetilde{K}} + \tfrac{1}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi^{j)\widetilde{I}}\,,\\
\delta\psi^{i\widetilde{I}} &= -\tfrac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{I}}\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\not{D}\sigma^{\widetilde{I}}\epsilon^i - Y^{ij\widetilde{I}}\epsilon_j + \sigma^{\widetilde{I}}\gamma\cdot T\epsilon^i + \tfrac{1}{2}gt_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\epsilon^i + \sigma^{\widetilde{I}}\eta^i\,,\\
\delta\sigma^{\widetilde{I}} &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\psi^{\widetilde{I}}\,.
\end{aligned}\tag{6.130}$$

The covariant derivatives are defined by

$$\begin{aligned}
D_\mu\sigma^{\widetilde{I}} &= \mathcal{D}_\mu\sigma^{\widetilde{I}} - \tfrac{1}{2}\,\mathrm{i}\,\bar\psi_\mu\psi^{\widetilde{I}}\,,\\
\mathcal{D}_\mu\sigma^{\widetilde{I}} &= (\partial_\mu - b_\mu)\sigma^{\widetilde{I}} + gt_{J\widetilde{K}}{}^{\widetilde{I}}A_\mu^J\sigma^{\widetilde{K}}\,,\\
D_\mu\psi^{i\widetilde{I}} &= \mathcal{D}_\mu\psi^{i\widetilde{I}} + \tfrac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{I}}\psi_\mu^i + \tfrac{1}{2}\,\mathrm{i}\,\not{D}\sigma^{\widetilde{I}}\psi_\mu^i + Y^{ij\widetilde{I}}\psi_{\mu\,j} - \sigma^{\widetilde{I}}\gamma\cdot T\psi_\mu^i\\
&\quad -\tfrac{1}{2}gt_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\psi_\mu^i - \sigma^{\widetilde{I}}\phi_\mu^i\,,\\
\mathcal{D}_\mu\psi^{i\widetilde{I}} &= (\partial_\mu - \tfrac{3}{2}\,b_\mu + \tfrac{1}{4}\,\gamma_{ab}\,\omega_\mu{}^{ab})\psi^{i\widetilde{I}} - V_\mu^{ij}\psi_j^{\widetilde{I}} + gt_{J\widetilde{K}}{}^{\widetilde{I}}A_\mu^J\psi^{i\widetilde{K}}\,.
\end{aligned}\tag{6.131}$$

The covariant curvature $\widehat{\mathcal{H}}_{\mu\nu}^{\widetilde{I}}$ should be understood as having components $(\widehat{F}_{\mu\nu}^I, B_{\mu\nu}^M)$, where the covariantized Yang-Mills field-strength is given by

$$\widehat{F}_{\mu\nu}^I = 2\partial_{[\mu}A_{\nu]}^I + gf_{JK}{}^IA_\mu^JA_\nu^K - \bar\psi_{[\mu}\gamma_{\nu]}\psi^I + \tfrac{1}{2}\,\mathrm{i}\,\sigma^I\bar\psi_{[\mu}\psi_{\nu]}\,.\tag{6.132}$$

In order to close the superconformal algebra on the vector-tensor multiplet, the fields of the tensor multiplet need to satisfy equations of motion. The extensions of (6.14) and (6.15) (which are non-zero only for $\widetilde{I}$ in the tensor multiplet range) are given by

$$\begin{aligned}
L^{ijM} &\equiv t_{(\widetilde{J}\widetilde{K})}{}^M\left(2\sigma^{\widetilde{J}}Y^{ij\widetilde{K}} - \tfrac{1}{2}\,\mathrm{i}\,\bar\psi^{i\widetilde{J}}\psi^{j\widetilde{K}}\right)\\
&= 0\,,\\
E_{\mu\nu\lambda}^M &\equiv \frac{3}{g}D_{[\mu}B_{\nu\lambda]}{}^M - \varepsilon_{\mu\nu\lambda\rho\sigma}t_{(\widetilde{J}\widetilde{K})}{}^M\left(\sigma^{\widetilde{J}}\widehat{\mathcal{H}}^{\rho\sigma\widetilde{K}} - 8\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}T^{\rho\sigma} + \tfrac{1}{4}\,\mathrm{i}\,\bar\psi^{\widetilde{J}}\gamma^{\rho\sigma}\psi^{\widetilde{K}}\right)\\
&\quad -\tfrac{3}{2}\bar\psi^M\gamma_{[a}\widehat{R}_{bc]}(Q)\\
&= 0\,.
\end{aligned}\tag{6.133}$$

Analogously to section 6.1.2, subsequent variation of these constraints gives the superconformal extensions of the equations of motion for the rest of the fields of the tensor multiplet. We will not give them here explicitly, since they can be derived from the action which we will give below.

The local generalization of the action (6.104) for the vector-tensor multiplet is rather involved. A long but straightforward calculation leads us to the following expression

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm VT} \;=\; & \Big[\Big(-\tfrac14\widehat{\mathcal{H}}^{\widetilde I}_{\mu\nu}\widehat{\mathcal{H}}^{\mu\nu\widetilde J}-\tfrac12\bar\psi^{\widetilde I}\not{D}\psi^{\widetilde J}+\tfrac13\sigma^{\widetilde I}\Box^c\sigma^{\widetilde J}+\tfrac16 D_a\sigma^{\widetilde I}D^a\sigma^{\widetilde J}+Y^{\widetilde I}_{ij}Y^{ij\widetilde J}\Big)\sigma^{\widetilde K}\\
&-\tfrac43\sigma^{\widetilde I}\sigma^{\widetilde J}\sigma^{\widetilde K}\left(D+\tfrac{26}{3}T_{ab}T^{ab}\right)+4\sigma^{\widetilde I}\sigma^{\widetilde J}\widehat{\mathcal{H}}^{\widetilde I}_{ab}T^{ab}+\Big(-\tfrac18\,\mathrm{i}\,\bar\psi^{\widetilde I}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde J}\psi^{\widetilde K}\\
&-\tfrac12\,\mathrm{i}\,\bar\psi^{i\widetilde I}\psi^{j\widetilde J}Y^{\widetilde K}_{ij}+\mathrm{i}\,\sigma^{\widetilde I}\bar\psi^{\widetilde J}\gamma\cdot T\psi^{\widetilde K}-8\,\mathrm{i}\,\sigma^{\widetilde I}\sigma^{\widetilde J}\bar\psi^{\widetilde K}\chi\Big)+\tfrac16\sigma^{\widetilde I}\bar\psi_\mu\gamma^\mu\\
&\times\Big(\mathrm{i}\,\sigma^{\widetilde J}\not{D}\psi^{\widetilde K}+\tfrac12\,\mathrm{i}\,\not{D}\sigma^{\widetilde J}\psi^{\widetilde K}-\tfrac14\gamma\cdot\widehat{\mathcal{H}}^{\widetilde J}\psi^{\widetilde K}+2\sigma^{\widetilde J}\gamma\cdot T\psi^{\widetilde K}-8\sigma^{\widetilde J}\sigma^{\widetilde K}\chi\Big)\\
&-\tfrac16\bar\psi_a\gamma_b\psi^{\widetilde I}\Big(\sigma^{\widetilde J}\widehat{\mathcal{H}}^{ab\widetilde K}-8\sigma^{\widetilde J}\sigma^{\widetilde K}T^{ab}\Big)-\tfrac1{12}\sigma^{\widetilde I}\bar\psi_\lambda\gamma^{\mu\nu\lambda}\psi^{\widetilde J}\widehat{\mathcal{H}}^{\widetilde K}_{\mu\nu}\\
&+\tfrac1{12}\,\mathrm{i}\,\sigma^{\widetilde I}\bar\psi_a\psi_b\Big(\sigma^{\widetilde J}\widehat{\mathcal{H}}^{ab\widetilde K}-8\sigma^{\widetilde J}\sigma^{\widetilde K}T^{ab}\Big)+\tfrac1{48}\,\mathrm{i}\,\sigma^{\widetilde I}\sigma^{\widetilde J}\bar\psi_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\widehat{\mathcal{H}}^{\widetilde K}_{\mu\nu}\\
&-\tfrac12\sigma^{\widetilde I}\bar\psi^i_\mu\gamma^\mu\psi^{j\widetilde J}Y^{\widetilde K}_{ij}+\tfrac16\,\mathrm{i}\,\sigma^{\widetilde I}\sigma^{\widetilde J}\bar\psi^i_\mu\gamma^{\mu\nu}\psi^j_\nu Y^{\widetilde K}_{ij}-\tfrac1{24}\,\mathrm{i}\,\bar\psi_\mu\gamma_\nu\psi^{\widetilde I}\bar\psi^{\widetilde J}\gamma^{\mu\nu}\psi^{\widetilde K}\\
&+\tfrac1{12}\,\mathrm{i}\,\bar\psi^i_\mu\gamma^\mu\psi^{j\widetilde I}\bar\psi^{\widetilde J}_i\psi^{\widetilde K}_j-\tfrac1{48}\sigma^{\widetilde I}\bar\psi_\mu\psi_\nu\bar\psi^{\widetilde J}\gamma^{\mu\nu}\psi^{\widetilde K}+\tfrac1{24}\sigma^{\widetilde I}\bar\psi^i_\mu\gamma^{\mu\nu}\psi^j_\nu\bar\psi^{\widetilde J}_i\psi^{\widetilde K}_j\\
&-\tfrac1{12}\sigma^{\widetilde I}\bar\psi_\lambda\gamma^{\mu\nu\lambda}\psi^{\widetilde J}\bar\psi_\mu\gamma_\nu\psi^{\widetilde K}+\tfrac1{24}\,\mathrm{i}\,\sigma^{\widetilde I}\sigma^{\widetilde J}\bar\psi_\lambda\gamma^{\mu\nu\lambda}\psi^{\widetilde K}\bar\psi_\mu\psi_\nu\\
&+\tfrac1{48}\,\mathrm{i}\,\sigma^{\widetilde I}\sigma^{\widetilde J}\bar\psi_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\bar\psi_\mu\gamma_\nu\psi^{\widetilde K}+\tfrac1{96}\sigma^{\widetilde I}\sigma^{\widetilde J}\sigma^{\widetilde K}\bar\psi_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\bar\psi_\mu\psi_\nu\Big]C_{\widetilde I\widetilde J\widetilde K}\\
&+\tfrac1{16g}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}B^M_{\mu\nu}\left(\partial_\lambda B^N_{\rho\sigma}+2g\,t_{IJ}{}^N A^I_\lambda F^J_{\rho\sigma}+g\,t_{IP}{}^N A^I_\lambda B^P_{\rho\sigma}\right)\\
&-\tfrac18 e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^M t_{FG}{}^N A^I_\mu A^F_\nu A^G_\lambda\left(-\tfrac12 g\,F^K_{\rho\sigma}+\tfrac1{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right)\\
&-\tfrac1{24}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}C_{IJK}A^I_\mu\left(F^J_{\nu\lambda}F^K_{\rho\sigma}+f_{FG}{}^J A^F_\nu A^G_\lambda\left(-\tfrac12 g\,F^K_{\rho\sigma}\right.\right.\\
&\left.\left.+\tfrac1{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right)\right)-\tfrac12 g^2\sigma^I\sigma^J\sigma^K\sigma^{\widetilde M}\sigma^{\widetilde N}t_{J\widetilde M}{}^P t_{K\widetilde N}{}^Q C_{IPQ}\\
&+\tfrac1{10}\,\mathrm{i}\,g\bar\psi_\mu\gamma^\mu\psi^{\widetilde I}\sigma^{\widetilde J}\sigma^{\widetilde K}\sigma^{\widetilde L}\Big(\Big[t_{[\widetilde I\widetilde J]}{}^{\widetilde M}-2t_{(\widetilde I\widetilde J)}{}^{\widetilde M}\Big]C_{\widetilde M\widetilde K\widetilde L}-\tfrac12 t_{(\widetilde J\widetilde K)}{}^{\widetilde M}C_{\widetilde M\widetilde I\widetilde L}\Big)\\
&+\tfrac14\,\mathrm{i}\,g\bar\psi^{\widetilde I}\psi^{\widetilde J}\sigma^{\widetilde K}\sigma^{\widetilde L}\Big(t_{[\widetilde I\widetilde J]}{}^{\widetilde M}C_{\widetilde M\widetilde K\widetilde L}-4t_{(\widetilde I\widetilde K)}{}^{\widetilde M}C_{\widetilde M\widetilde J\widetilde L}\Big)\,, \qquad (6.134)
\end{aligned}
$$

where the superconformal D'Alembertian is defined as

$$
\begin{aligned}
\Box^c\sigma^{\widetilde I} \;=\; & D^aD_a\sigma^{\widetilde I}\\
=\; & \left(\partial^a-2b^a+\omega_b{}^{ba}\right)D_a\sigma^{\widetilde I}+gt_{\widetilde J\widetilde K}{}^{\widetilde I}A^J_aD^a\sigma^K-\frac{\mathrm{i}}{2}\bar\psi_\mu D^\mu\psi^{\widetilde I}-2\sigma^{\widetilde I}\bar\psi_\mu\gamma^\mu\chi\\
&+\frac12\bar\psi_\mu\gamma^\mu\gamma\cdot T\psi^{\widetilde I}+\frac12\bar\phi_\mu\gamma^\mu\psi^{\widetilde I}+2f_\mu{}^\mu\sigma^{\widetilde I}-\tfrac12 g\bar\psi_\mu\gamma^\mu t_{\widetilde J\widetilde K}{}^{\widetilde I}\psi^{\widetilde J}\sigma^{\widetilde K}\,. \qquad (6.135)
\end{aligned}
$$

Varying this action with respect to the fields of the tensor multiplet and the vector multiplet, we can obtain their covariant equations of motion.

6.4.2 The hypermultiplet

The local superconformal transformation rules for the hypermultiplet with gauged isometries is given by

$$\begin{aligned}\delta q^X &= -\,\mathrm{i}\,\bar\epsilon^i\zeta^A f^X_{iA}\,,\\ \widehat{\delta}\zeta^A &= \tfrac12\,\mathrm{i}\,\not{D} q^X f^{iA}_X\epsilon_i - \tfrac13\gamma\cdot T k^X f^A_{iX}\epsilon^i - \tfrac12 g\sigma^I k^X_I f^A_{iX}\epsilon^i + k^X f^A_{iX}\eta^i\,.\end{aligned} \tag{6.136}$$

The covariant derivatives are defined by

$$\begin{aligned}D_\mu q^X &= \mathcal{D}_\mu q^X + \mathrm{i}\,\bar\psi^i_\mu\zeta^A f^X_{iA}\,,\\ \mathcal{D}_\mu q^X &= \partial_\mu q^X - b_\mu k^X - V^{jk}_\mu k^X_{jk} + gA^I_\mu k^X_I\,,\\ D_\mu\zeta^A &= \mathcal{D}_\mu\zeta^A - k^X f^A_{iX}\phi^i_\mu + \tfrac12\,\mathrm{i}\,\not{D} q^X f^A_{iX}\psi^i_\mu + \tfrac13\gamma\cdot T k^X f^A_{iX}\psi^i_\mu\,,\\ &\quad +g\tfrac12\sigma^I k^X_I f^A_{iX}\psi^i_\mu \qquad\qquad (6.137)\\ \mathcal{D}_\mu\zeta^A &= \partial_\mu\zeta^A + \partial_\mu q^X\omega_{XB}{}^A\zeta^B + \tfrac14\omega_\mu{}^{bc}\gamma_{bc}\zeta^A - 2b_\mu\zeta^A + gA^I_\mu t_{IB}{}^A\zeta^B\,.\end{aligned}$$

Similar to section 6.2, requiring closure of the commutator algebra on these transformation rules yields the equation of motion for the fermions

$$\begin{aligned}\Gamma^A &= \not{D}\zeta^A + \tfrac12 W_{CDB}{}^A\zeta^B\bar\zeta^D\zeta^C - \tfrac83\,\mathrm{i}\,k^X f^A_{iX}\chi^i + 2\,\mathrm{i}\,\gamma\cdot T\zeta^A\\ &\quad -g(\,\mathrm{i}\,k^X_I f^A_{iX}\psi^{iI} + \,\mathrm{i}\,\sigma^I t_{IB}{}^A\zeta^B)\,.\end{aligned} \tag{6.138}$$

The scalar equation of motion can be obtained from varying (6.138)

$$\widehat{\delta}_Q\Gamma^A = \tfrac12\,\mathrm{i}\,f^{iA}_X\Delta^X\epsilon_i + \tfrac14\gamma^\mu\Gamma^A\bar\epsilon\psi_\mu - \tfrac14\gamma^\mu\gamma^\nu\Gamma^A\bar\epsilon\gamma_\nu\psi_\mu\,, \tag{6.139}$$

from which we obtain

$$\begin{aligned}\Delta^X &= \Box^c q^X - \bar\zeta^B\gamma^a\zeta^C D_a q^Y\mathcal{R}^X{}_{YBC} + \tfrac89 T^2 k^X\\ &\quad +\tfrac43 Dk^X + 8\,\mathrm{i}\,\bar\chi^i\zeta^A f^X_{iA} - \tfrac14\mathcal{D}^X W_{ABCD}\bar\zeta^A\zeta^B\bar\zeta^C\zeta^D\\ &\quad -g\big(2\,\mathrm{i}\,\bar\psi^{iI}\zeta^B t_{IB}{}^A f^X_{iA} - k^Y_I J_Y{}^X{}_{ij}Y^{ij}\big)\\ &\quad +g^2\sigma^I\sigma^J\mathfrak{D}_Y k^X_I k^Y_J\,,\end{aligned} \tag{6.140}$$

The superconformal D'Alembertian acting on the hyper-scalars is given by

$$\begin{aligned}\Box^c q^X &\equiv D_a D^a q^X\\ &= \partial_a D^a q^X - \tfrac52 b_a D^a q^X - \tfrac12 V^{jk}_a J_Y{}^X{}_{jk}D^a q^Y + \mathrm{i}\,\bar\psi^i_a D^a\zeta^A f^X_{iA}\\ &\quad +2f_a{}^a k^X - 2\bar\psi_a\gamma^a\chi k^X + 4\bar\psi^{(j}_a\gamma^a\chi^{k)}k^X_{jk} - \bar\psi^i_a\gamma^a\gamma\cdot T\zeta^A f^X_{iA}\\ &\quad -\bar\phi^i_a\gamma^a\zeta^A f^X_{iA} + \omega_a{}^{ab}D_b q^X - \tfrac12 g\bar\psi^a\gamma_a\psi^I k^X_I - D_a q^Y\partial_Y k^X_I A^{aI}\\ &\quad +D_a q^Y D^a q^Z\Gamma^X_{YZ}.\end{aligned} \tag{6.141}$$

The generalization of (6.125) to the case of local superconformal symmetry is given by

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm H} \;=\;& -\tfrac{1}{2}g_{XY}\mathcal{D}_a q^X\mathcal{D}^a q^Y + \bar\zeta_A \not{D}\zeta^A + \tfrac{2}{3}f_a{}^a k^2 + \tfrac{4}{9}Dk^2 + \tfrac{8}{27}T^2k^2 \\ & +2\,\mathrm{i}\,\bar\zeta_A\gamma\cdot T\zeta^A - \tfrac{16}{3}\,\mathrm{i}\,\bar\zeta_A\chi^i k^X f_{iX}^A - \tfrac{1}{4}W_{ABCD}\bar\zeta^A\zeta^B\bar\zeta^C\zeta^D \\ & -\tfrac{2}{9}\bar\psi_a\gamma^a\chi k^2 + \tfrac{1}{3}\bar\zeta_A\gamma^a\gamma\cdot T\psi_a^i k^X f_{iX}^A + \tfrac{1}{2}\,\mathrm{i}\,\bar\zeta_A\gamma^a\gamma^b\psi_a^i\mathcal{D}_b q^X f_{iX}^A \\ & -\tfrac{1}{6}\,\mathrm{i}\,\bar\psi_a\gamma^{ab}\phi_b k^2 - \bar\zeta_A\gamma^a\phi_a^i k^X f_{iX}^A \\ & +\tfrac{1}{12}\bar\psi_a^i\gamma^{abc}\psi_b^j\mathcal{D}_c q^Y J_Y{}^X{}_{ij}k_X - \tfrac{1}{9}\,\mathrm{i}\,\bar\psi^a\psi^b T_{ab}k^2 + \tfrac{1}{18}\,\mathrm{i}\,\bar\psi_a\gamma^{abcd}\psi_b T_{cd}k^2 \\ & -g\Big(P_{ij}^I Y_I^{ij} + 2\,\mathrm{i}\,k_I^X f_{iX}^A\bar\zeta_A\psi^{iI} + \mathrm{i}\,\sigma^I t_{IB}{}^A\bar\zeta_A\zeta^B \\ & +\tfrac{1}{2}\sigma^I k_I^X f_{iX}^A\bar\zeta_A\gamma^a\psi_a^i - \tfrac{1}{2}\bar\psi_a^i\gamma^a\psi^{jI}P_{Iij} + \tfrac{1}{4}\,\mathrm{i}\,\bar\psi_a^i\gamma^{ab}\psi_b^j\sigma^I P_{Iij}\Big) \\ & -\tfrac{1}{2}g^2\sigma^I\sigma^J k_I^X k_{JX}\,. \end{aligned} \tag{6.142}$$

The field equations can be obtained from this action according to

$$\begin{aligned} \frac{\delta\mathcal{S}}{\delta\bar\zeta^A} &= 2\,C_{AB}\Gamma^B\,, \\ \frac{\delta\mathcal{S}}{\delta q^X} &= g_{XY}\left(\Delta^Y - 2\bar\zeta_A\Gamma^B\omega^Y{}_B{}^A - \mathrm{i}\,\bar\psi_a^i\gamma^a\Gamma^A f_{iA}^Y\right)\,. \end{aligned} \tag{6.143}$$

6.5 Discussion and outlook

We will conclude this chapter with an overview of the results that we obtained, and a discussion of possible future research based on these results.

6.5.1 Summary of geometrical objects

In table 6.5, we have collected the essential geometrical data that is needed to construct superconformal matter multiplets. We indicate which are the essential geometrical objects that determine the theory and the independent constraints imposed on them. The symmetries of the objects are indicated by brackets on their indices. All equations are also valid for the theories in the columns next to and rows below its entry, apart from the entries “hyper + gauging” and “hyper + conformal” entry, which are mutually independent.

However, the symbol ▼ indicates that these equations or symbols are not to be taken over below. E.g. the moment map P_I^α itself is completely determined in the superconformal theory, and it should thus not be given as an independent quantity anymore. For the rigid theory without conformal invariance, only constant pieces can be undetermined by the given equations, and are the Fayet–Iliopoulos terms. Furthermore, the equations and symbols indicated by ▶ are not to be taken over for the theories with an action, as they are then satisfied due to the Killing equation or are defined by χ.

	ALGEBRA (no action)		ACTION	
Multiplet	Object	Restriction	Object	Restriction
Vector	$f_{[IJ]}{}^K$	Jacobi identities	$C_{(IJK)}$	$f_{I(J}{}^H C_{KL)H} \overset{\blacktriangledown}{=} 0$
Vector-tensor	$(t_I)_{\widetilde{J}}{}^{\widetilde{K}}$ $\widetilde{I}=(I,M)$	$[t_I,t_J]=-f_{IJ}{}^K t_K$ $t_{IJ}{}^K = f_{IJ}{}^K$ $t_{IM}{}^J=0$	$\Omega_{[MN]}$	invertible $f_{I(J}{}^H C_{KL)H} = t_{I(J}{}^M t_{KL)}{}^N \Omega_{MN}$ $t_{I[M}{}^P\Omega_{N]P}=0$
Hyper	f_X^{iA}	invertible and real Nijenhuis tensor $N_{XY}{}^Z=0$	$C_{[AB]}$	$\mathfrak{D}_X C_{AB}=0$
Hyper + conformal	k^X ▶	$\mathfrak{D}_Y k^X \overset{\blacktriangleright}{=} \frac{3}{2} \delta_Y{}^X$	χ	$\mathfrak{D}_X\mathfrak{D}_Y\chi = \frac{3}{2} g_{XY}$
Hyper + gauging	k_I^X	$k^Y_{[I|}\partial_Y k^X_{|J]} = -\frac{1}{2} f_{IJ}{}^K k_K^X$ $\mathfrak{D}_X\mathfrak{D}_Y k_I^Z \overset{\blacktriangleright}{=} R_{XWY}{}^Z k_I^W$ $\mathcal{L}_{k_I} J^\alpha \overset{\blacktriangleright}{=} 0$	P_I^α ▼	$\mathfrak{D}_X k_{YI} + \mathfrak{D}_Y k_{XI}=0$ $\partial_X P_I^\alpha \overset{\blacktriangledown}{=} J^\alpha_{XY} k_I^Y$ $k_I^X J^\alpha_{XY} k_J^Y \overset{\blacktriangledown}{=} -f_{IJ}{}^K P_K^\alpha$
Hyper + conformal + gauging		$k^Y\mathfrak{D}_Y k_I^X = \frac{3}{2} k_I^X$		

Table 6.5: The superconformal matter multiplets and their essential geometrical data.

6.5.2 Gauge-fixing the conformal symmetry

In this chapter, we have discussed superconformal matter multiplets coupled to the Weyl multiplet. As mentioned in the introduction of this chapter, as well as at the end of chapter 4, the main motivation for this lengthy program was to construct matter-coupled Poincaré supergravities. We have not performed the complete gauge-fixing procedure to obtain such theories. Nevertheless, we will now briefly indicate how locally superconformal matter multiplets can lead to matter-coupled Poincaré supergravity theories.

The key observation is that the gauge field for special conformal transformations $f_\mu{}^a$ is related to the Ricci tensor $R_\mu{}^a$ of the spacetime manifold. From the constraint (5.47) and its explicit solution (5.51), we find that the trace of the special conformal gauge field is related to the Ricci scalar

$$f_a{}^a = -\tfrac{1}{16}R + \text{gravitino terms}\,. \tag{6.144}$$

This gauge field appears in the conformal D'Alembertian of scalar fields, e.g. for a five-dimensional scalar field ϕ of Weyl weight $\frac{3}{2}$, we have

$$\Box^c\phi = \left(\partial^a - \tfrac{5}{2}b^a + \omega_b{}^{ba}\right)\left(\partial_a - \tfrac{3}{2}b_a\right)\phi + 3f_a{}^a\phi\,. \tag{6.145}$$

With this definition, an action that is invariant under local conformal transformations is given by

$$e^{-1}\mathcal{L} = -\tfrac{1}{2}\phi\Box^c\phi\,. \tag{6.146}$$

We now fix the special conformal and the dilatational symmetry by imposing the Poincaré-gauge

$$b_\mu = 0\,, \qquad \phi^2 = \frac{16}{3\kappa^2}\,. \tag{6.147}$$

In this gauge, we can partially integrate (6.146) and use the solution for the spin-connection (5.51) to obtain

$$e^{-1}\mathcal{L} = \frac{1}{2\kappa^2}R\,. \tag{6.148}$$

So, we see that, in the Poincaré-gauge (6.147), the action for a local conformal scalar field (6.146) reduces to the Einstein-Hilbert action for ordinary gravity. In particular, the scale invariance of the action (6.146) is broken by the length-scale of the gravitational coupling constant κ in the dilatational gauge (6.147). Note also that the scalar field action (6.146) has the wrong sign for its kinetic term (we are using the mostly plus convention): the scalar field is therefore not a physical degree of freedom. Instead, it is a compensating scalar field for the broken conformal symmetry.

The above mechanism can also be applied to the local superconformal action for the $n_{\rm H} = r$ hypermultiplets, $n_{\rm T}$ tensor multiplets coupled to $n_{\rm V}$ vector multiplets in the background of the Standard Weyl multiplet. An additional subtlety here is that one also needs to solve the equation of motion for the scalar field D of the Weyl multiplet. In particular, we demand that all terms multiplying the gauge field $f_a{}^a$ yield a canonical Einstein-Hilbert term, and we impose the equation of motion for the scalar field D. Collecting all the relevant terms from (6.134) and (6.142), we impose the following gauge for the dilatation symmetry

$$-\tfrac{1}{24}\left(C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}} + g_{XY}k^X k^Y\right)R = \frac{1}{2\kappa^2}R\,, \qquad (6.149)$$

$$-\tfrac{4}{3}C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}} + \tfrac{4}{9}g_{XY}k^X k^Y = 0 \qquad (6.150)$$

These equations can be rewritten as

$$C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}} = -\frac{3}{\kappa^2}\,, \qquad (6.151)$$

$$g_{XY}k^X k^Y = -\frac{9}{\kappa^2}\,. \qquad (6.152)$$

We can interpret the dilatational gauge (6.152) on the scalars of the hypermultiplet as the definition of a hypersurface within the hyper-complex scalar manifold. In particular, a metric of signature $(1,r)$ on the hyper-complex manifold induces a metric of signature $(0,r)$ on the hypersurface (6.152). This follows from the resemblance of (6.152) to the embedding equation (2.25) of $(d+1)$-dimensional Anti-de-Sitter space, which is a hypersurface of signature $(1,d-1)$ in an ambient space of signature $(2,d)$.

The analysis of the dilatational gauge (6.151) on the scalars of the vector-tensor multiplets goes along similar ways. It induces a metric $g_{\widetilde{I}\widetilde{J}}$ on the vector space $\mathbb{R}^{n_{\rm V}+n_{\rm T}}$ spanned by the scalars

$$g_{\widetilde{I}\widetilde{J}} \equiv -\frac{1}{3}\frac{\partial^2 \ln C}{\partial\sigma^{\widetilde{I}}\partial\sigma^{\widetilde{J}}}\Big|_{C=-\frac{3}{\kappa^2}}\,, \qquad C \equiv C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\,. \qquad (6.153)$$

If we define scalars ϕ^x (with $x = 1,\ldots,n_{\rm V}+n_{\rm T}-1$), then the metric g_{IJ} induces a metric g_{xy} on the hypersurface (6.151) according to

$$g_{xy} \equiv g_{\widetilde{I}\widetilde{J}}\frac{\partial\sigma^{\widetilde{I}}}{\partial\phi^x}\frac{\partial\sigma^{\widetilde{J}}}{\partial\phi^y} \qquad (6.154)$$

The metric g_{xy} on the manifold spanned by the scalars ϕ^x, defines the $D=5$ variant of special geometry [156], called "very special geometry".

It will be interesting to see in what way the future analysis of the metrics $g_{\widetilde{I}\widetilde{J}}$ and g_{xy} in the case when there are non-vanishing C_{IJM} components will modify the analysis from [156]. The expected result is that, together with one hypermultiplet, one vector multiplet plays the role of a compensating multiplet for the broken conformal symmetries.

The additional conformal symmetries will also have to be gauge-fixed. The special conformal transformation can again be gauge-fixed by imposing $b_\mu = 0$, and the $\mathrm{SU}(2)$-transformations will be gauge-fixed by three of the remaining scalars of the compensating hypermultiplet. Gauge-fixing the S-supersymmetry and imposing the equation of motion for the spinor χ^i of the Weyl multiplet eliminates the spinors of the compensating vector multiplet and the hypermultiplet.

The remaining gauge field of the compensating vector multiplet will play the role of the graviphoton of the Poincaré multiplet. The equation of motion for the $\mathrm{SU}(2)$ gauge field ${V_\mu}^{ij}$ introduces a non-zero $\mathrm{SU}(2)$-valued curvature on the manifold: this promotes the hyper-complex or hyper-Kähler manifold to a quaternionic of quaternionic-Kähler manifold, respectively. Finally, the equation of motion for the tensor T_{ab} of the Weyl multiplet can be used to express T_{ab} in terms of the Yang-Mills field-strengths.

The overall result is that gauge-fixing the conformal symmetries of $n+1$ superconformal vector-tensor multiplets and $r+1$ superconformal hypermultiplets coupled to the Weyl multiplet of conformal supergravity leads to the theory of n vector-tensor multiplets and r hypermultiplets coupled to Poincaré supergravity.

6.5.3 The scalar potential

We will now present the scalar potential of the combined action for n_{T} on-shell tensor multiplets and $n_{\mathrm{H}} = r$ on-shell hypermultiplets in the background of n_{V} off-shell vector multiplets coupled to the Standard Weyl multiplet.

First, we collect all terms of $\mathcal{O}(g^2)$ in (6.134) and (6.142). However, this is not the final answer since the auxiliary field Y_I^{ij} has an algebraic equation of the form

$$2C_{I\widetilde{J}\widetilde{K}}\sigma^{\widetilde{J}}Y^{ij\widetilde{K}} = gP^{ij} + \text{fermion bilinears}\,. \tag{6.155}$$

Solving this equation and substituting the result into the term $-gP^I_{ij}Y_I^{ij}$ of (6.142) will generate an additional term of $\mathcal{O}(g^2)$.

The gauge-fixing of the superconformal symmetries has an additional effect: the corresponding parameters can be expressed in terms of the non-conformal parameters. In particular, the parameter η^i of S-transformations will be expressed in terms of the parameter ϵ^i of Q-supersymmetry. This will make the resulting Poincaré-supersymmetry transformation rules much more complicated: a complication that the conformal approach avoids until the final step in the calculations.

The expression for η^i will also involve the auxiliary field Y^{ij} of the vector multiplet, and by using (6.155) we see that e.g. the Poincaré-supersymmetry transformation for the gravitino will contain a term proportional to P_I^{ij}, and the scalar potential contains the square of that term. The other terms of the scalar potential can also be written in terms of squares of these so-called "fermion-shifts": they are defined as the terms of $\mathcal{O}(g)$ in the supersymmetry transformations of ζ^A and $\psi^{i\widetilde{I}}$, respectively

$$\begin{aligned} \delta\zeta^A &\sim -\tfrac{1}{2}g\sigma^I k_I^X f^A_{iX}\epsilon^i \equiv \mathcal{N}_i^A\epsilon^i\,, \\ \delta\psi^{i\widetilde{I}} &\sim \tfrac{1}{2}g{t_{(\widetilde{J}\widetilde{K})}}^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\epsilon^i \equiv \mathcal{P}^{\widetilde{I}}\epsilon^i\,, \end{aligned} \tag{6.156}$$

Finally, we find for the scalar potential

$$V(\sigma^{\widetilde{I}}, q^X) = g^2 C^{-1}_{IJ\widetilde{K}}P^{ijI}P^J_{ij} + 2\mathcal{N}^{iA}\mathcal{N}_{iA} + 2\sigma^I\mathcal{P}^P\mathcal{P}^Q C_{IPQ}\,, \tag{6.157}$$

where there will be some small modifications due to the constraints on C_{IJK} by the dilatational gauge-fixing. After the gauge-fixing program will have been performed, the analysis of the critical points of this potential, and the Hessian matrix of the corresponding superpotential will give more insight in its possible applications. We stress that, even though this analysis has not been performed yet, the potential (6.157) contains new ingredients which have not been discussed in the literature before.

First of all, we have also considered reducible, but not completely reducible, representations for the vector-tensor multiplet. This opens the possibility of non-compact gauge groups and the existence of new Chern-Simons terms in the action of the form $A \wedge B \wedge F$ that were not constructed before. Algebraically, this is reflected in the non-zero components of the tensor C_{IMN} that appears in the potential (6.157).

Furthermore, without an action, we also allow an odd number of tensor multiplets, which is more general than all analyses so far, which all started from an action. For the hypermultiplets, the same argument applies: here, we have considered hyper-Kähler manifold without a metric, the so-called hyper-complex manifolds.

To conclude, we expect that these new results on superconformal matter multiplets will also lead to more general matter-couplings to Poincaré supergravity. Whether such new matter-couplings will drastically modify the structure of the scalar potential in such a way that supersymmetric Randall-Sundrum scenarios become possible, remains an open question.