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Chapter 5

Weyl multiplets of conformal supergravity

Superconformal methods are an elegant way to construct general couplings of Poincaré supergravities to matter [167, 168]. This so-called superconformal tensor calculus uses the basic superconformal multiplets as a starting point for a gauge-fixing procedure in which the superconformal symmetry is broken down to Poincaré supersymmetry.

Conformal supergravities have been constructed in various dimensions (for a review, see [38]), but not yet in five dimensions. In the five-dimensional case, these matter coupled supergravities have recently attracted renewed attention due to the important role they play in the Randall–Sundrum (RS) scenarios [138, 139] and the AdS_6/CFT_5 [169, 170] and AdS_5/CFT_4 [171] correspondences.

Moreover, it has turned out in the past that superconformal constructions lead to new insights and results in the structure of matter-couplings. A recent example is the insight in relations between hyper-Kähler cones and quaternionic manifolds, based on the study of superconformal invariant matter-couplings with hypermultiplets [172]. For all these reasons, a superconformal construction of general matter-couplings of $\mathcal{N} = 2\,, D = 5$ supergravity is useful.

The superconformal multiplet that contains all the (independent) gauge fields of the superconformal algebra is called the Weyl multiplet. In this chapter, we take the first step in the superconformal program by constructing the Weyl multiplets of $\mathcal{N} = 2\,, D = 5$ conformal supersymmetry. In our construction, we use the methods developed first for $\mathcal{N} = 1\,, D = 4$ [173, 174]. They are based on gauging the conformal superalgebra [175], which in our case is $\mathrm{F}^2(4)$.

In general, one needs to include matter fields to have an equal number of bosons and fermions. We will see that in five dimensions there are two possible sets of matter fields one can add, yielding two versions of the Weyl multiplet: the Standard Weyl multiplet and the where $\omega(x)$ is an arbitrary function, $k_\mu = g_{\mu\nu}k^\nu$, and the covariant derivative is given by $\nabla_\mu k_\nu = \partial_\mu k_\nu - \Gamma^\rho_{\mu\nu}k_\rho$. Taking the trace of (5.2), and making the restriction to flat D-dimensional Minkowski spacetime yields

$$\partial_{(\mu}k_{\nu)}(x) - \tfrac{1}{D}\eta_{\mu\nu}\partial^\rho k_\rho(x) = 0\,. \tag{5.3}$$

Taking the derivative of (5.3), we obtain

$$\Box k_\mu(x) + \left(1 - \tfrac{2}{D}\right)\partial_\mu\partial^\rho k_\rho(x) = 0\,. \tag{5.4}$$

In $D = 2$, the solutions to this are given by the infinite-dimensional group of analytic coordinate transformations[1]. In dimensions $D > 2$, the group of conformal transformations is finite-dimensional, and the most general solution to (5.3) is given by

$$k^\mu(x) = \xi^\mu + \Lambda_M^{\mu\nu}x_\nu + \Lambda_D x^\mu + (x^2\Lambda_K^\mu - 2x^\mu x\cdot\Lambda_K)\,. \tag{5.5}$$

Corresponding to the parameters ξ^μ are the translations P_μ, the parameters $\Lambda_M^{\mu\nu}$ correspond to Lorentz rotations $M_{\mu\nu}$, to Λ_D are associated the dilatations D, and Λ_K^μ are the parameters of ‘special conformal transformations’ K_μ. Thus, the full set of conformal transformations δ_C can be expressed as follows:

$$\delta_C = \xi^\mu P_\mu + \Lambda_M^{\mu\nu}M_{\mu\nu} + \Lambda_D D + \Lambda_K^\mu K_\mu\,. \tag{5.6}$$

5.1.2 Conformal Killing spinors

We next consider the extension to conformal supersymmetry. In D-dimensional Minkowski spacetime, the conformal supersymmetry transformations are defined as the supersymmetry transformations that satisfy

$$\partial_\mu\epsilon^i(x) - \tfrac{1}{D}\gamma_\mu\partial\!\!\!/\epsilon^i(x) = 0\,. \tag{5.7}$$

The solution to this equation is given by

$$\epsilon^i(x) = \epsilon^i + \mathrm{i}\,x^\mu\gamma_\mu\eta^i\,, \tag{5.8}$$

where the (constant) parameters ϵ^i correspond to “ordinary” supersymmetry transformations Q^i_α and the parameters η^i define special conformal supersymmetries generated by S^i_α.

The conformal transformations (5.5) and the supersymmetries (5.8) do not form a closed algebra. To obtain closure, one must introduce additional R-symmetry generators. In particular, in the case of 8 supercharges Q^i_α in $D = 5$, there is an additional $\mathrm{SU}(2)$ R-symmetry

[1] Recall that the Cauchy-Riemann equations for a complex function reduce to the wave equation for the real and imaginary part of that function.

Generators	P_a	$M_{[ab]}$	D	K_a	$U_{(ij)}$	$Q_{\alpha i}$	$S_{\alpha i}$
Parameters	ξ^a	$\Lambda_M^{[ab]}$	Λ_D	Λ_K^a	$\Lambda_U^{(ij)}$	ϵ^i	η^i
# symmetries	5	10	1	5	3	8	8

Table 5.1: The generators of the superconformal algebra $\mathrm{F}^2(4)$.

with generators $U_{ij} = U_{ji} (i = 1, 2)$. Thus, the full set of superconformal transformations δ_C is given by:

$$\delta_C = \xi^\mu P_\mu + \Lambda_M^{\mu\nu} M_{\mu\nu} + \Lambda_D D + \Lambda_K^\mu K_\mu + \Lambda_U^{ij} U_{ij} + \mathrm{i}\,\bar{\epsilon} Q + \mathrm{i}\,\bar{\eta} S\,. \tag{5.9}$$

The extra factor of i in the last two terms is necessary because of the reality properties of five-dimensional spinors.

We have summarized the generators and parameters of the five-dimensional superconformal algebra $\mathrm{F}^2(4)$ in table 5.1. Also indicated here are number symmetries associated to each generator: in total, there are 24+16 bosonic plus fermionic symmetries.

5.1.3 The superconformal algebra $\mathrm{F}^2(4)$

When one allows for central charges, there exist many varieties of superconformal algebras [182, 183]. However, so far a suitable superconformal Weyl multiplet has only been constructed from those superconformal algebras[2] that appear in Nahm's classification [185]. The particular real form that we need here is the five-dimensional algebra denoted by $\mathrm{F}^2(4)$, see tables 5 and 6 in [186].

The commutation relations defining the $\mathrm{F}^2(4)$ algebra are given by

$$\begin{aligned}
[M_{bc}, P_a] &= -\eta_{a[b} P_{c]}\,, & [D, P_a] &= P_a\,,\\
[M_{bc}, K_a] &= -\eta_{a[b} K_{c]}\,, & [D, K_a] &= -K_a\,,\\
\left[M_{ab}, M^{cd}\right] &= -2\delta_{[a}{}^{[c} M_{b]}{}^{d]}\,, & \left[U_{ij}, U^{kl}\right] &= 2\delta_{(i}{}^{(k} U_{j)}{}^{l)}\,,\\
[P_a, K_b] &= 2(\eta_{ab} D + 2M_{ab})\,, & &\\
\\
[M_{ab}, Q_{i\alpha}] &= -\tfrac{1}{4}(\gamma_{ab} Q_i)_\alpha\,, & [D, Q_{i\alpha}] &= \tfrac{1}{2} Q_{i\alpha}\,,\\
[M_{ab}, S_{i\alpha}] &= -\tfrac{1}{4}(\gamma_{ab} S_i)_\alpha\,, & [D, S_{i\alpha}] &= -\tfrac{1}{2} S_{i\alpha}\,,\\
[U_{kl}, Q_{i\alpha}] &= -\varepsilon_{i(k} Q_{l)\alpha}\,, & [K_a, Q_{i\alpha}] &= \mathrm{i}\,(\gamma_a S_i)_\alpha\,,\\
[U_{kl}, S_{i\alpha}] &= -\varepsilon_{i(k} S_{l)\alpha}\,, & [P_a, S_{i\alpha}] &= -\mathrm{i}\,(\gamma_a Q_i)_\alpha\,,\\
\\
\{Q_{i\alpha}, Q_{j\beta}\} &= -\tfrac{1}{2}\varepsilon_{ij}(\gamma^a)_{\alpha\beta} P_a\,, & \{S_{i\alpha}, S_{j\beta}\} &= -\tfrac{1}{2}\varepsilon_{ij}(\gamma^a)_{\alpha\beta} K_a\,,\\
\{Q_{i\alpha}, S_{j\beta}\} &= -\tfrac{1}{2}\,\mathrm{i}\left(\varepsilon_{ij} C_{\alpha\beta} D + \varepsilon_{ij}(\gamma^{ab})_{\alpha\beta} M_{ab} + 3C_{\alpha\beta} U_{ij}\right)\,. & &
\end{aligned} \tag{5.10}$$

[2]An exception is the ten-dimensional Weyl multiplet [184], which is not based on a known algebra.

The first seven commutation relations between the bosonic generators form the bosonic subalgebra $\mathrm{SO}(2,5) \times \mathrm{SU}(2)$: the conformal algebra times the R-symmetry group. In particular, as we referred to in chapter 2, the $\mathrm{SO}(2,5)$ commutation relations are obtained by substituting (2.50) into (2.51). The commutation relations of the (conformal) supercharges with the bosonic generators indicates that the supercharges are spinorial $\mathrm{SU}(2)$-doublets with dilatational weight of $\pm\frac{1}{2}$, respectively.

We will also give the form of the anti-commutators in terms of commutators of infinitesimal transformations on the fields. We can write the algebra (5.10) as

$$[T_A, T_B] = f_{AB}{}^C T_C\,, \qquad \{T_A, T_B\} = -f_{AB}{}^C T_C\,, \tag{5.11}$$

where the minus sign in the second equation is due to the factor of i in the last two terms of (5.9). We then have for all commutators of infinitesimal transformations

$$\left[\delta_A(\Lambda_1^A), \delta_B(\Lambda_2^B)\right] = \delta_C(\Lambda_3^C)\,, \qquad \Lambda_3^C = \Lambda_2^B \Lambda_1^A f_{AB}{}^C\,. \tag{5.12}$$

In particular, the anti-commutation relations for the conformal supercharges translate into the following infinitesimal commutators

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_P\left(\tfrac{1}{2}\bar{\epsilon}_2\gamma_\mu\epsilon_1\right)\,, \tag{5.13}$$

$$[\delta_S(\eta), \delta_Q(\epsilon)] = \delta_D(\tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\eta) + \delta_M(\tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\gamma^{ab}\eta) + \delta_U(-\tfrac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\eta^{j)})\,, \tag{5.14}$$

$$[\delta_S(\eta_1), \delta_S(\eta_2)] = \delta_K(\tfrac{1}{2}\bar{\eta}_2\gamma^a\eta_1)\,. \tag{5.15}$$

As a final note, we remark that the superconformal algebra is equipped with two gradings. First of all, there is a $\mathbb{Z}_2$-grading that separates the generators into bosonic and fermionic operators: this dictates the kind of bracket (commutator or anti-commutator) that has to be specified for two particular generators. There is also a $\mathbb{Z}_5$-grading given by the dilatational weights of the various generators (the numbers on the right-hand side of the commutator with the dilatational generator D): this determines what specific operators can appear on the right-hand side given the left-hand side of an algebraic relation. The coefficients in the superconformal algebra are fixed (up to an overall normalization) by imposing the generalized Jacobi-identities.

5.1.4 Representation theory

We wish to consider representations of the conformal algebra on fields $\phi^\alpha(x)$ where α stands for a collection of internal indices referring to the stability subalgebra of $x^\mu = 0$. From the expression for the conformal Killing vector (5.5), we deduce that this algebra is isomorphic to the algebra generated by $M_{\mu\nu}, D$ and K_μ.

We denote this stability subalgebra by H, and the generators of H by $\Sigma_{\mu\nu}\,, \Delta$ and κ_μ. Denoting the complete conformal algebra by G, we can write rigid conformal transformations as

$$G_{\mathrm{rigid}} = (P \otimes H)_{\mathrm{rigid}}\,. \tag{5.16}$$

Applying the theory of induced representations, it follows that any representation $(\Sigma\,,\Delta\,,\kappa)$ of the stability subalgebra H will induce a representation $(P\,,M\,,D\,,K)$ of the full conformal algebra G with the following transformation rules (we suppress any internal indices)

$$\begin{aligned}
\delta_P(\xi)\phi(x) &= \xi^\mu\partial_\mu\phi(x)\,,\\
\delta_M(\Lambda_M)\phi(x) &= \tfrac{1}{2}\Lambda_M^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi(x) + \delta_\Sigma(\Lambda_M)\phi(x)\,,\\
\delta_D(\Lambda_D)\phi(x) &= \Lambda_D x^\lambda\partial_\lambda\phi(x) + \delta_\Delta(\Lambda_D)\phi(x)\,,\\
\delta_K(\Lambda_K)\phi(x) &= \Lambda_K^\mu(x^2\partial_\mu - 2x_\mu x^\lambda\partial_\lambda)\phi(x) +\\
&\quad \big(\delta_\delta(-2x\cdot\Lambda_K) + \delta_\Sigma(-4x_{[\mu}\Lambda_{K\nu]}) + \delta_\kappa(\Lambda_K)\big)\,\phi(x)\,.
\end{aligned} \tag{5.17}$$

Lorentz-transformations

We now look at the non-trivial representation (Σ, Δ, κ) that we use in this thesis. Concerning the Lorentz representations, we will encounter anti-symmetric tensors $\phi_{a_1\cdots a_n}(x)(n = 0, 1, 2, \cdots)$, and spinors $\psi_\alpha(x)$:

$$\delta_\Sigma(\Lambda_M)\phi_{a_1\cdots a_n}(x) = -n\,(\Lambda_M)_{[a_1}{}^b\phi_{|b|a_2\cdots a_n]}(x)\,, \tag{5.18}$$

$$\delta_\Sigma(\Lambda_M)\psi_\alpha(x) = -\tfrac{1}{4}\Lambda_M^{ab}(\gamma_{ab})_\alpha{}^\beta\psi_\beta(x)\,. \tag{5.19}$$

Dilatations

Secondly, we consider the dilatations. For most fields, the Δ-transformation is determined by a single number w, which is called the Weyl weight of ϕ^α

$$\delta_\Delta(\Lambda_D)\phi^\alpha(x) = w\Lambda_D\phi^\alpha(x)\,. \tag{5.20}$$

For scalar fields, it is often convenient to consider the set of fields $\phi^\alpha(x)$ as the coordinates of a scalar manifold with affine connection $\Gamma^\gamma_{\alpha\beta}$. With this understanding, the transformation of ϕ^α under dilatations can be characterized by

$$\delta_\Delta(\Lambda_D)\phi^\alpha = \Lambda_D k^\alpha(\phi)\,, \tag{5.21}$$

for some arbitrary function $k^\alpha(\phi)$.

Special conformal transformations

All[3] fields that we will discuss in this thesis are invariant under the internal special conformal transformations

$$\delta_\kappa(\Lambda_K)\phi^\alpha(x) = 0\,. \tag{5.22}$$

[3] An exception is formed by some of the gauge fields in the Weyl multiplet.

However, this does not mean that special conformal transformations do not play a role in constructing superconformal theories. Indeed, the derivative of a scalar field $\phi^\alpha(x)$ that transforms under Δ- and κ-transformations according to (5.21) and (5.22), will transform under special conformal transformations according to

$$\delta_\kappa(\Lambda_K)\partial_\mu\phi^\alpha(x) = -2\Lambda_{K\mu}k^\alpha(\phi)\,. \tag{5.23}$$

Scalar manifold geometry

In the next chapter, we will construct superconformal field theories. In particular, we will also construct superconformal actions. Constructing an action for a set of scalar fields $\phi^\alpha(x)$ corresponds to taking a scalar manifold with a metric $g_{\alpha\beta}(\phi)$

$$\mathcal{L} = -\tfrac{1}{2}g_{\alpha\beta}(\phi)\partial^\mu\phi^\alpha(x)\partial_\mu\phi^\beta(x)\,. \tag{5.24}$$

Such a Lagrangian describes a sigma model with the D-dimensional spacetime as “worldsheet” and the scalar manifold as target space. Requiring dilatational invariance of this kinetic term yields that the vector $k^\alpha(\phi)$ should be a homothetic Killing vector: namely it should satisfy the conformal Killing equation (5.3) for *constant* $\omega(x)$:

$$\mathfrak{D}_\alpha k_\beta + \mathfrak{D}_\beta k_\alpha = (D-2)g_{\alpha\beta}\,, \tag{5.25}$$

where D is the dimension of the “worldsheet”, and where the covariant derivative on the scalar manifold is given by $\mathfrak{D}_\alpha k_\beta = \partial_\alpha k_\beta - \Gamma^\gamma_{\alpha\beta}k_\gamma$.

Demanding invariance of (5.24) under the special conformal transformations (5.23) restricts $k^\alpha(\phi)$ even further to be an *exact* homothetic Killing vector

$$k_\alpha = \partial_\alpha\chi\,, \tag{5.26}$$

for some function $\chi(\phi)$. One can show that the restrictions (5.25) and (5.26) are equivalent to

$$\mathfrak{D}_\alpha k^\beta \equiv \partial_\alpha k^\beta + \Gamma^\beta_{\alpha\gamma}k^\gamma = w\delta^\beta_\alpha\,. \tag{5.27}$$

The Weyl weight w of ϕ^α has to be $w = \frac{1}{2}(D-2)$, or $w = \frac{3}{2}$ in $D = 5$. The proof of the necessity of (5.27) can be extracted from [187], see also [188, 189]. In these papers the conditions for conformal invariance of a sigma model with gravity are investigated. Note that the condition (5.27) can be formulated *independently* of a metric. Only an affine connection is necessary.

For the special case of a zero affine connection, the solution to (5.27) is given by

$$k^\alpha(\phi) = w\phi^\alpha\,, \tag{5.28}$$

and the transformation rule (5.21) reduces to the form (5.20). Note that the homothetic Killing vector (5.28) is indeed exact with χ given by

$$\chi = \tfrac{1}{(D-2)}g_{\alpha\beta}k^\alpha k^\beta\,. \tag{5.29}$$

Supersymmetric generalization

To construct field representations of the superconformal algebra, one can again apply the method of induced representations. In this case, one must use superfields $\Phi^\alpha(x^\mu, \theta^i_\alpha)$, where α stands for a collection of internal indices referring to the stability subalgebra of $x^\mu = \theta^i_\alpha = 0$. This algebra is isomorphic to the algebra generated by $M_{\mu\nu}\,, D\,, K_\mu\,, U_{ij}$ and S^i_α.

An additional complication, not encountered in the bosonic case, is that the representation one obtains is reducible. To obtain an irreducible representation, one must impose constraints on the superfield. It is at this point that the transformation rules become nonlinear in the fields. In this thesis, we will follow a different approach. Instead of working with superfields, we will work with the component fields. The nonlinear transformation rules are obtained by imposing the superconformal algebra.

$\mathrm{SU}(2)$-transformations

In the supersymmetric case, we must specify the $\mathrm{SU}(2)$-properties of the different fields as well as the behavior under S-supersymmetry. Concerning the $\mathrm{SU}(2)$, we will only encounter scalars ϕ, doublets ψ^i and triplets $\phi^{(ij)}$ whose transformations are given by

$$\begin{aligned} \delta_{\mathrm{SU}(2)}(\Lambda_U^{ij})\phi &= 0\,, \\ \delta_{\mathrm{SU}(2)}(\Lambda_U^{ij})\psi^i(x) &= -\Lambda_U{}^i{}_j\psi^j(x)\,, \\ \delta_{\mathrm{SU}(2)}(\Lambda_U^{ij})\phi^{ij}(x) &= -2\Lambda_U{}^{(i}{}_k\phi^{j)k}(x)\,. \end{aligned} \tag{5.30}$$

S-transformations

This leaves us with specifying how a given field transforms under the special supersymmetries generated by S^i_α. In superfield language the full S-transformation is given by a combination of an x-dependent translation in superspace, with parameter $\epsilon^i(x) = \mathrm{i}\, x^\mu\gamma_\mu\eta^i$, and an internal S-transformation. This is in perfect analogy to the bosonic case. In terms of component fields, the same holds true. The x-dependent contribution is obtained by making the substitution

$$\epsilon^i \to \mathrm{i}\, \not{x}\eta^i \tag{5.31}$$

in the Q-supersymmetry rules. The internal S-transformations can be deduced by imposing the superconformal algebra.

5.2 Local superconformal symmetry

In this section, we will discuss local superconformal symmetry. We will first introduce the various gauge fields, their transformation rules and their covariant curvatures. After that, we will discuss the emergence of curvature constraints.

Generators	P_a	$M_{[ab]}$	D	K_a	$U_{(ij)}$	$Q_{\alpha i}$	$S_{\alpha i}$
Gauge fields	$e_\mu{}^a$	$\omega_\mu{}^{[ab]}$	b_μ	$f_\mu{}^a$	$V_\mu^{(ij)}$	ψ_μ^i	ϕ_μ^i
# d.o.f.	9	$\underline{50}$	0	$\underline{25}$	12	24	$\underline{40}$

Table 5.2: The gauge fields of the superconformal algebra $\mathrm{F}^2(4)$.

5.2.1 Gauge fields and curvatures

The procedure for gauging the superconformal algebra proceeds along similar ways as for any other local symmetry algebra. We assign to every generator of the superconformal algebra T^A a gauge field $h_\mu{}^A$. We have indicated in table 5.2 the various gauge fields for the superconformal algebra $\mathrm{F}^2(4)$. Since every gauge field has an extra spacetime index μ, the gauge fields have 120+80 bosonic plus fermionic field components: five times as large as the number of gauge symmetries. The number of degrees of freedom is the difference of these two numbers: there are only 96+64 independent gauge field components.

For example, the gauge field $e_\mu{}^a$ can be acted upon by local Lorentz transformations, translations and dilatations to reduce its 25 field components to 9 degrees of freedom. Similar considerations apply to the gauge fields b_μ , V_μ^{ij} and ψ_μ^i. The three gauge fields $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and ϕ_μ^i do not have such a restriction on their field components, and they also have their degrees of freedom underlined, since they will become dependent gauge fields, as we will explain in more detail in the next section.

Gauge fields transformation rules

We can determine the transformation rules for the gauge fields using the general rules for gauge theories

$$\delta h_\mu{}^A = \partial_\mu \epsilon^A + \epsilon^C h_\mu{}^B f_{BC}{}^A \,. \tag{5.32}$$

From the algebra (5.10), we read off

$$\begin{aligned}
\delta e_\mu{}^a &= \mathcal{D}_\mu \xi^a - \Lambda_M^{ab} e_{\mu b} - \Lambda_D e_\mu{}^a + \tfrac{1}{2}\bar\epsilon\gamma^a\psi_\mu \,, \\
\delta f_\mu{}^a &= \mathcal{D}_\mu \Lambda_K^a - \Lambda_M^{ab} f_{\mu b} + \Lambda_D f_\mu{}^a + \tfrac{1}{2}\bar\eta\gamma^a\phi_\mu \,, \\
\delta\omega_\mu{}^{ab} &= \mathcal{D}_\mu\Lambda_M^{ab} - 4\xi^{[a} f_\mu{}^{b]} - 4\Lambda_K^{[a} e_\mu{}^{b]} + \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma^{ab}\phi_\mu - \tfrac{1}{2}\,\mathrm{i}\,\bar\eta\gamma^{ab}\psi_\mu \,, \\
\delta b_\mu &= \partial_\mu\Lambda_D - 2\xi^a f_{\mu a} + 2\Lambda_K^a e_{\mu a} + \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\phi_\mu + \tfrac{1}{2}\,\mathrm{i}\,\bar\eta\psi_\mu \,, \\
\delta V_\mu^{ij} &= \partial_\mu\Lambda_U^{ij} - 2\Lambda_U{}^{(i}{}_\ell V_\mu^{j)\ell} - \tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\phi_\mu^{j)} + \tfrac{3}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi_\mu^{j)} \,, \\
\delta\psi_\mu^i &= \mathcal{D}_\mu\epsilon^i - \tfrac{1}{4}\Lambda_M^{ab}\gamma_{ab}\psi_\mu^i - \tfrac{1}{2}\Lambda_D\psi_\mu^i - \Lambda_U{}^i{}_j\psi_\mu^j + \mathrm{i}\,\xi^a\gamma_a\phi_\mu^i - \mathrm{i}\,e_\mu^a\gamma_a\eta^i \,, \\
\delta\phi_\mu^i &= \mathcal{D}_\mu\eta^i - \tfrac{1}{4}\Lambda_M^{ab}\gamma_{ab}\phi_\mu^i + \tfrac{1}{2}\Lambda_D\phi_\mu^i - \Lambda_U{}^i{}_j\phi_\mu^j - \mathrm{i}\,\Lambda_K^a\gamma_a\psi_\mu^i + \mathrm{i}\,f_\mu^a\gamma_a\epsilon^i \,,
\end{aligned} \tag{5.33}$$

where $\mathcal{D}_\mu$ is the covariant derivative with respect to Lorentz rotations, dilatations, and $\mathrm{SU}(2)$-transformations:

$$\begin{aligned}
\mathcal{D}_\mu \xi^a &= \partial_\mu \xi^a + \omega_\mu{}^{ab}\xi_b + b_\mu \xi^a \,, \\
\mathcal{D}_\mu \Lambda_K^a &= \partial_\mu \Lambda_K^a + \omega_\mu{}^{ab}\Lambda_{Kb} - b_\mu \Lambda_K^a \,, \\
\mathcal{D}_\mu \Lambda_M^{ab} &= \partial_\mu \Lambda_M^{ab} + 2\omega_{\mu c}{}^{[a}\Lambda_M^{b]c} \,, \\
\mathcal{D}_\mu \epsilon^i &= \partial_\mu \epsilon^i + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon^i + \tfrac{1}{2} b_\mu \epsilon^i - V_\mu^{ij}\epsilon_j \,, \\
\mathcal{D}_\mu \eta^i &= \partial_\mu \eta^i + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\eta^i - \tfrac{1}{2} b_\mu \eta^i - V_\mu^{ij}\eta_j \,.
\end{aligned} \tag{5.34}$$

In addition to these local superconformal transformations, the gauge fields also transform as vectors under general coordinate transformations

$$\delta_{\mathrm{gct}}(\xi) h_\mu{}^A = \xi^\nu(x)\partial_\nu h_\mu{}^A + \partial_\mu \xi^\nu(x) h_\nu{}^A \,. \tag{5.35}$$

Covariant curvatures

The gauge fields introduced above transform to derivatives on the parameters of the superconformal algebra. A curvature for each gauge field is defined by

$$R_{\mu\nu}{}^A(T^A) = 2\partial_{[\mu} h_{\nu]}{}^A + h_\nu{}^C h_\mu{}^B f_{BC}{}^A \,. \tag{5.36}$$

Such curvatures transform covariantly according to

$$\delta R_{\mu\nu}{}^A(T^A) = \epsilon^C R_{\mu\nu}{}^B(T^B) f_{BC}{}^A \,. \tag{5.37}$$

Using the commutator expressions (5.10) we obtain the following expressions for the curvatures

$$\begin{aligned}
R_{\mu\nu}{}^a(P) &= 2\partial_{[\mu} e_{\nu]}{}^a + \underline{2\omega_{[\mu}{}^{ab} e_{\nu]b}} + \underline{2b_{[\mu} e_{\nu]}{}^a} - \tfrac{1}{2}\bar\psi_{[\mu}\gamma^a\psi_{\nu]} \,, \\
R_{\mu\nu}{}^a(K) &= 2\partial_{[\mu} f_{\nu]}{}^a + 2\omega_{[\mu}{}^{ab} f_{\nu]b} - 2b_{[\mu} f_{\nu]}{}^a - \tfrac{1}{2}\bar\phi_{[\mu}\gamma^a\phi_{\nu]} \,, \\
R_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b + \underline{8 f_{[\mu}{}^{[a} e_{\nu]}{}^{b]}} + \mathrm{i}\,\bar\phi_{[\mu}\gamma^{ab}\psi_{\nu]} \,, \\
R_{\mu\nu}(D) &= 2\partial_{[\mu} b_{\nu]} - \underline{4 f_{[\mu}{}^a e_{\nu]a}} - \mathrm{i}\,\bar\phi_{[\mu}\psi_{\nu]} \,, \\
R_{\mu\nu}{}^{ij}(V) &= 2\partial_{[\mu} V_{\nu]}{}^{ij} - 2V_{[\mu}{}^{k(i} V_{\nu]\,k}{}^{j)} - 3\,\mathrm{i}\,\bar\phi_{[\mu}^{(i}\psi_{\nu]}^{j)} \,, \\
R_{\mu\nu}{}^i(Q) &= 2\partial_{[\mu}\psi_{\nu]}^i + \tfrac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]}^i + b_{[\mu}\psi_{\nu]}^i - 2V_{[\mu}{}^{ij}\psi_{\nu]\,j} + \underline{2\,\mathrm{i}\,\gamma_a\phi_{[\mu}^i e_{\nu]}{}^a} \,, \\
R_{\mu\nu}{}^i(S) &= 2\partial_{[\mu}\phi_{\nu]}^i + \tfrac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\phi_{\nu]}^i - b_{[\mu}\phi_{\nu]}^i - 2V_{[\mu}{}^{ij}\phi_{\nu]\,j} - 2\,\mathrm{i}\,\gamma_a\psi_{[\mu}^i f_{\nu]}{}^a \,.
\end{aligned} \tag{5.38}$$

We have underlined all terms proportional to vielbeins for purposes to be explained shortly.

In addition to curvatures, we can also define a covariant derivative as the partial derivative minus the sum over all transformations with as parameter the corresponding gauge field

$$\nabla_\mu \equiv \partial_\mu - \delta_A(h_\mu{}^A) \,. \tag{5.39}$$

This definition can also be applied to the curvatures to derive the Bianchi identities

$$\nabla_{[\mu} R_{\nu\lambda]}{}^A(T^A) = 0\,. \tag{5.40}$$

5.2.2 Curvature constraints

Making the superconformal algebra G a local symmetry algebra is a subtle procedure. Taking spacetime dependent parameters in (5.5) makes it impossible to distinguish translations, Lorentz transformations, dilatations, and special conformal transformations, since they are all included in the general coordinate transformations (5.1).

Moreover, in section 5.1, we saw that global conformal transformations on fields $\phi^\alpha(x)$ could be split up as a product of translations and global transformations of $\phi^\alpha(0)$ generated by the stability sub-algebra H. The local analog of (5.16) is a product of general coordinate transformations and local transformations at $x = 0$ generated by H

$$G_{\text{local}} = (\text{GCT} \otimes H)_{\text{local}}\;. \tag{5.41}$$

However, we have so far only considered a gauge theory of general coordinate transformations and local $\text{SO}(2,5)$-transformations (and its supersymmetric extension $\text{F}^2(4)$). Somehow, we should be able to make the truncation

$$(\text{GCT} \otimes \text{SO}(2,5))_{\text{local}} \to (\text{GCT} \otimes H)_{\text{local}}\;. \tag{5.42}$$

Covariant general coordinate transformations

In particular, we would like to identify the field $e_\mu{}^a$ as the fünfbein field and not just as the gauge field for translations. To see how this can be resolved, let us rewrite (5.35) on $e_\mu{}^a$ as

$$\begin{aligned} \delta_{\text{gct}}(\xi) e_\mu{}^a &= \xi^\nu(x)\partial_\nu e_\mu{}^a + (\partial_\mu \xi^\nu(x))\, e_\nu{}^a \\ &= \partial_\mu\left(\xi^\nu(x) e_\nu{}^a\right) + \xi^\nu(x)\left(\partial_\nu e_\mu{}^a - \partial_\mu e_\nu{}^a\right) \\ &= \big(\delta_P(\xi^a) + \delta_M(\xi^\mu \omega_\mu{}^{ab}) + \delta_Q(\xi^\mu \psi^i_\mu)\big)\, e_\mu{}^a - \xi^\nu R_{\mu\nu}{}^a(P)\,. \end{aligned} \tag{5.43}$$

So, the local translations can be expressed as general coordinate transformations covariantized with respect to all symmetries except translations

$$\delta_P(\xi) \to \delta_{\text{cgct}} \equiv \delta_{\text{gct}}(\xi) - \delta_I(\xi^\mu h_\mu{}^I) \quad (I \neq P_a)\,, \tag{5.44}$$

if we impose the following curvature constraint

$$R_{\mu\nu}{}^a(P) = 0\,. \tag{5.45}$$

The curvature constraint (5.45) has as an additional effect that the gauge field $\omega_\mu{}^{ab}$ can be identified with the spin-connection since it can be solved for from (5.45). This is the reason why we underlined the second term in the first line of (5.38): it contains the spin-connection multiplied with the invertible fünfbein.

Conventional constraints

Constraints from which a gauge field can be solved are called conventional constraints, and they have been applied previously in the formulations of conformal supergravity in four dimensions for the $\mathcal{N} = 1$ [173, 175], $\mathcal{N} = 2$ [190] and $\mathcal{N} = 4$ [191], as well as in six dimensions for the $(1,0)$ [176] and $(2,0)$ [192] Weyl multiplets.

A comparison with the underlined lines in (5.38) suggests that we constrain the curvatures $R_{\mu\nu}{}^{a}(P)\,, R_{\mu\nu}{}^{ab}(M)\,, R_{\mu\nu}(D)$ and $R_{\mu\nu}{}^{i}(Q)$. However, they are not all independent: applying the Bianchi identity (5.40) to the constraint (5.45) gives

$$e_{[\mu}{}^{a} R_{\nu\lambda]}(D) = R_{[\mu\nu\lambda]}{}^{a}(M)\,. \tag{5.46}$$

We choose to impose the following constraints[4]

$$\begin{aligned} R_{\mu\nu}{}^{a}(P) &= 0 \quad (\underline{50})\,, \\ e^{\nu}{}_{b}\widehat{R}_{\mu\nu}{}^{ab}(M) &= 0 \quad (\underline{25})\,, \\ \gamma^{\mu}\widehat{R}_{\mu\nu}{}^{i}(Q) &= 0 \quad (\underline{40})\,. \end{aligned} \tag{5.47}$$

Matter fields

Before analyzing these constraints any further, we note that (5.47) contains 75+40 bosonic plus fermionic restrictions which leaves us with 21+24 degrees of freedom in the independent gauge fields. So even though we solved the problem of distinguishing local superconformal transformations from general coordinate transformations, the independent gauge fields do not form a supermultiplet with an equal number of bosonic and fermionic degrees of freedom.

The solution is to add matter fields (i.e. fields that do not gauge a superconformal symmetry) to supplement this mismatch. Which matter fields to add, and to determine their transformation rules is the subject of the next section. The effect of these extra fields will be that the transformation rules for the gauge fields (5.33) will be modified (we ignore the translations from all indices ranging over $I\,, J$)

$$\delta_J(\epsilon^J) h_\mu^I = \partial_\mu \epsilon^I + \epsilon^J h_\mu{}^A f_{AJ}{}^I + \epsilon^J M_{\mu J}{}^I\,. \tag{5.48}$$

The last term in (5.48) also modifies the definition of the curvatures (5.38) to

$$\widehat{R}_{\mu\nu}{}^{I} = 2\partial_{[\mu} h_{\nu]}{}^{I} + h_\nu{}^B h_\mu{}^A f_{AB}{}^I - 2 h_{[\mu}{}^J M_{\nu]J}{}^I\,. \tag{5.49}$$

This is the origin of the hats on the curvatures in the last two equations of (5.47): it anticipates the corrections that will come from matter terms.

In general, one can add extra matter terms to the constraints (5.47), which just amounts to redefinitions of the composite fields. By choosing suitable terms simplifications were

[4] Note that the third constraint implies that $\gamma_{[\mu\nu}\widehat{R}_{\rho\sigma]}{}^{i}(Q) = 0$.

obtained in four and six dimensions. In these cases, one could add a term to the second constraint which rendered all the constraints invariant under S-supersymmetry, but in five dimensions this turns out to be impossible. Therefore we keep the constraints as written above.

Dependent gauge fields

We underlined the number of restrictions each constraint imposes, and a comparison with table 5.2 and with the underlined lines in (5.38) shows that they have the same number of restrictions as the degrees of freedom of the gauge fields $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and ϕ^i_μ. Therefore, these fields are no longer independent.

In order to write down the explicit solutions of these constraints, it is useful to extract the terms which have been underlined in (5.38). We define $\widehat{R}'$ as the curvatures without these terms. Formally,

$$\widehat{R}'_{\mu\nu}{}^I = \widehat{R}_{\mu\nu}{}^I + 2h_{[\mu}{}^J e_{\nu]}{}^a f_{aJ}{}^I \,, \tag{5.50}$$

where $f_{aJ}{}^I$ are the structure constants in the $\mathrm{F}^2(4)$ algebra that define commutators of translations with other gauge transformations. Then, an explicit solution for them is given by

$$\begin{aligned}
\omega_\mu{}^{ab} &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}^{\;b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e_\sigma^{\;c} + 2e_\mu{}^{[a}b^{b]} - \tfrac{1}{2}\bar\psi^{[b}\gamma^{a]}\psi_\mu - \tfrac{1}{4}\bar\psi^b\gamma_\mu\psi^a \,, \\
\phi^i_\mu &= -\tfrac{1}{12}\,\mathrm{i}\left(\gamma^{ab}\gamma_\mu - \tfrac{1}{2}\gamma_\mu\gamma^{ab}\right)\widehat{R}'_{ab}{}^i(Q)\,, \\
f_\mu{}^a &= -\tfrac{1}{6}\mathcal{R}_\mu{}^a + \tfrac{1}{48}e_\mu{}^a\mathcal{R}\,, \qquad \mathcal{R}_{\mu\nu} \equiv \widehat{R}'_{\lambda\mu}{}^{ab}(M)e_a{}^\lambda e_{\nu b}\,, \qquad \mathcal{R} \equiv \mathcal{R}_\mu{}^\mu \,.
\end{aligned} \tag{5.51}$$

The non-invariance of the constraints under Q- and S-supersymmetry has to be compensated by extra Q- and S-supersymmetry transformations of the dependent gauge fields. In section 5.4, we will give these extra supersymmetry transformation rules.

5.3 The supercurrent method

We will now present an elegant method to derive the field content and transformation rules for the matter fields that have to be added to the independent gauge fields to obtain a supermultiplet.

The Noether method

Consider a Lagrangian consisting of the kinetic term for a set of N spinor fields $\vec\psi$

$$\mathcal{L}_{\mathrm{matter}} = \bar{\vec\psi}\cdot\partial\!\!\!/\vec\psi \,. \tag{5.52}$$

This action is invariant under global $\mathrm{U}(N)$-rotations of the form

$$\delta_G(\Lambda)\vec\psi = \Lambda^A(T_A)\cdot\vec\psi\,, \quad \delta_G(\Lambda)\mathcal{L}_{\mathrm{matter}} = 0\,. \tag{5.53}$$

However, under local transformations the kinetic term transforms to the derivative on the gauge parameters

$$\begin{aligned} \delta_G(\Lambda(x))\mathcal{L}_{\text{matter}} &= -\tfrac{1}{2}\left(\partial^\mu \Lambda^A(x)\right)\bar{\vec{\psi}}\gamma_\mu T_A \vec{\psi} \\ &\equiv -\left(\partial_\mu \Lambda^A(x)\right) J^\mu{}_A\,, \end{aligned} \tag{5.54}$$

where $J^\mu{}_A$ is the set of Noether currents corresponding to the global symmetries. These currents are divergence-less (using the equations of motion) and transform covariantly

$$\partial^\mu J_\mu{}^A = 0\,, \quad \delta_G(\Lambda(x))J_\mu{}^A = \Lambda^C(x)J^B_\mu f_{BC}{}^A\,. \tag{5.55}$$

The standard procedure is then to introduce a gauge field $h_\mu{}^A$ in a Noether-action that compensates this extra variation

$$\delta(\Lambda)h_\mu{}^A = \partial_\mu \Lambda^A(x) + \Lambda^C(x)h_\mu{}^B f_{BC}{}^A\,, \quad \mathcal{L}_{\text{Noether}} \equiv h_\mu{}^A J^\mu{}_A\,, \tag{5.56}$$

such that the combined action is invariant under local gauge transformations

$$\mathcal{L} = \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{Noether}}\,, \quad \delta_G(\Lambda(x))\mathcal{L} = 0\,. \tag{5.57}$$

The total action can also be rewritten as a manifestly invariant action in terms of a covariant derivative

$$\mathcal{L} = \bar{\vec{\psi}}\cdot \not{\mathcal{D}}\vec{\psi}\,, \quad \mathcal{D}_\mu \equiv \partial_\mu - \delta_A(h_\mu{}^A)\,. \tag{5.58}$$

Writing out the definition of the covariant derivative, we regain (5.57).

Superconformal currents

We will now mimic this procedure for the superconformal algebra $\mathrm{F}^2(4)$. Also in this case, the currents for superconformal symmetry will themselves not be invariant under the superconformal algebra, instead they form a complete supermultiplet. To each current in this current multiplet, we can then assign a field of the Weyl multiplet. From the precise index structure and comparing with the independent gauge fields of the superconformal algebra, we derive which matter fields have to be added as well as their transformation rules.

The multiplet of currents in a superconformal context has been discussed before in the literature: the current multiplet corresponding to the vector multiplet in four dimensions for $\mathcal{N} = 1$ [174], $\mathcal{N} = 2$ [193, 194], and $\mathcal{N} = 4$ [191]; and the current multiplet of the six-dimensional $(2,0)$ self-dual tensor multiplet [192][5]. In addition to these cases, the $\mathcal{N} = 4\,, D = 5$ supercurrent [195] has also been constructed before but not in a superconformal context. Moreover, for $\mathcal{N} = 2$, the five-dimensional current found by the authors of [195] becomes reducible, as we have shown in the appendix of [16].

[5]The Weyl multiplets of $(1,0)$ in $D = 6$ [176] were derived without the use of a current multiplet.

In all these cases, after adding local improvement terms, one obtains a supercurrent multiplet containing an energy-momentum tensor $\theta_{\mu\nu} = \theta_{\nu\mu}$ and a supercurrent J^i_μ which are both conserved and (gamma-)traceless

$$\partial^\mu \theta_{\mu\nu} = \theta_\mu{}^\mu = \partial^\mu J^i_\mu = \gamma^\mu J^i_\mu = 0. \tag{5.59}$$

These improved and conserved currents correspond to the invariance under rigid superconformal symmetry for the various vector and tensor multiplets in four and six dimensions, the analog of the globally invariant $\mathcal{L}_{\mathrm{matter}}$ discussed previously.

However, the standard kinetic term of the $D = 5$ vector field

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{5.60}$$

is not scale invariant, i.e. the energy–momentum tensor is not traceless:

$$\theta_{\mu\nu} = -F_{\mu\lambda} F_\nu{}^\lambda + \tfrac{1}{4}\eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\,, \quad \theta_\mu{}^\mu = \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \neq 0\,. \tag{5.61}$$

Moreover, there do not exist gauge-invariant local improvement terms.

Fortunately, there is a remedy for this problem. Whenever there is a compensating scalar field present, i.e. a scalar with mass dimension zero but non-zero Weyl weight, then the kinetic term (5.60) can be made scale invariant by introducing a scalar coupling of the form

$$\mathcal{L} = -\tfrac{1}{4} e^\phi F_{\mu\nu} F^{\mu\nu}\,. \tag{5.62}$$

This compensating scalar is called the dilaton. In general, there are three possible origins for a dilaton coupling to a non-conformal matter multiplet: the dilaton is part of

1. the matter multiplet itself (the multiplet is then called an “improved” multiplet);
2. the conformal supergravity multiplet;
3. another matter multiplet.

The $\mathcal{N} = 2$ vector multiplet in five dimensions contains precisely such a scalar. We could therefore use it to compensate the broken scale invariance of the kinetic terms. This leads to the so-called improved vector multiplet. This is the first possibility, which will be further discussed in section 5.5.

The second possibility will be considered here (the third possibility is included for completeness). This possibility thus occurs when the Weyl multiplet itself contains a dilaton. We will see that there indeed exists a version of the Weyl multiplet containing a dilaton. This version is called the Dilaton Weyl multiplet. It turns out that there exists another version of the Weyl multiplet without a dilaton. This other version is very similar to the four- and six-dimensional Weyl multiplet and will be called the Standard Weyl multiplet.

When coupling to the Standard Weyl multiplet, one needs to add *non-local* improvement terms to the current multiplet, a feature that was first implemented for the current multiplet

Field	Equation of motion	$\mathrm{SU}(2)$	w	# d.o.f.
A_μ	$\partial_\mu F^{\mu\nu} = 0$	1	0	3
σ	$\Box\sigma = 0$	1	1	1
ψ^i	$\partial\!\!\!/\psi^i = 0$	2	$\frac{3}{2}$	4

Table 5.3: The on-shell Maxwell multiplet.

coming from the $D = 10$ vector multiplet [196]. In that case, the non-local improvement terms that were added required the use of auxiliary fields satisfying differential constraints in order to make the transformation rules local[6].

The current multiplet needs to be improved only when coupled to the Standard Weyl multiplet. In the case of the Dilaton Weyl multiplet, it is not necessary to do so, since in that case the dilaton of the Weyl multiplet can be used to compensate for the lack of scale invariance. In particular, the dilaton will couple directly to the trace of the energy-momentum tensor. We will present both the conventional and the improved current multiplet corresponding to the $\mathcal{N} = 2$ vector multiplet and in this way determine the field content and linearized transformation rules of the Dilaton and the Standard Weyl multiplet.

For matter multiplets having a traceless energy–momentum tensor, no compensating scalar is needed. To see the difference between the various cases, it is instructive to consider $(1,0)\,, D = 6$ conformal supergravity theory [176]. In that case, also a Standard and a Dilaton Weyl multiplet were found. Even though neither of those Weyl multiplets were derived using the current multiplet method, we expect that both versions can be constructed in that way: the Standard Weyl multiplet starting from the conformal $(1,0)$ tensor multiplet (being a truncation of the $(2,0)$ case [192]), and the Dilaton Weyl multiplet by starting from the non-conformal $D = 6$ vector multiplet (which upon reduction should produce our results in $D = 5$).

5.3.1 The supercurrent of the Maxwell multiplet

Our starting point is the on-shell $D = 5$ Abelian vector multiplet, also known as the Maxwell multiplet. Its field content is given by a massless vector A_μ, a symplectic Majorana spinor ψ^i in the fundamental of $\mathrm{SU}(2)$ and a real scalar σ. See table 5.3 for additional information. Our conventions are given in appendix A.

The action for the $D = 5$ Maxwell multiplet is given by

$$\mathcal{L}_{\mathrm{matter}} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2}\bar{\psi}\partial\!\!\!/\psi - \tfrac{1}{2}(\partial\sigma)^2\,. \tag{5.63}$$

[6]Note also that in $D = 10$ the trace-part and the traceless part of the energy-momentum tensor are not contained in the same multiplet which necessitates the addition of the non-local improvement terms to project out the trace-part.

Current	Noether	SU(2)	w	# d.of.
$\theta_{(\mu\nu)}$	$\partial^\mu \theta_{\mu\nu} = 0$	1	2	9
$v_\mu^{(ij)}$	$\partial^\mu v_\mu^{ij} = 0$	3	2	12
$b_{[\mu\nu]}$	$\partial^\mu b_{\mu\nu} = 0$	1	2	6
a_μ	$\partial^\mu a_\mu = 0$	1	3	4
$\theta_\mu{}^\mu$	–	1	4	1
J_μ^i	$\partial^\mu J_\mu^i = 0$	2	$\frac{5}{2}$	24
$\gamma^\mu J_\mu^i$	–	2	$\frac{7}{2}$	8

Table 5.4: The current multiplet: $\theta_\mu{}^\mu$ and $\gamma^\mu J_\mu^i$ form separate currents.

This action is invariant under the following global supersymmetries

$$\begin{aligned} \delta A_\mu &= \tfrac{1}{2}\bar{\epsilon}\gamma_\mu \psi\,, \\ \delta\psi^i &= -\tfrac{1}{4}\gamma\cdot F\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/\sigma\epsilon^i\,, \\ \delta\sigma &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\psi\,, \end{aligned} \tag{5.64}$$

as well as under the standard gauge transformation

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda\,. \tag{5.65}$$

Under local supersymmetry transformations, the action (5.63) transforms to the supercurrent J_μ^i

$$\delta(\epsilon(x))\mathcal{L}_{\text{matter}} = \mathrm{i}\,\left(\partial^\mu \bar{\epsilon}(x)\right) J_\mu\,, \tag{5.66}$$

where the explicit form of the supercurrent is given in (5.67).

The various global symmetries of the Lagrangian (5.63) lead to a number of other Noether currents: the energy–momentum tensor $\theta_{\mu\nu}$ and the $\mathrm{SU}(2)$-current v_μ^{ij}. The supersymmetry variations of these currents lead to a closed multiplet of $32+32$ degrees of freedom (see table 5.4). It is convenient to include these trace parts as separate currents since, as it turns out, they couple to independent fields of the Weyl multiplet.

We find the following expressions for the Noether currents and their supersymmetric partners in terms of bilinears of the vector multiplet fields:

$$
\begin{aligned}
\theta_{\mu\nu} &= -F_{\mu\lambda}F_{\nu}{}^{\lambda} + \tfrac{1}{4}\eta_{\mu\nu}F^2 - \partial_\mu\sigma\partial_\nu\sigma + \tfrac{1}{2}\eta_{\mu\nu}\left(\partial\sigma\right)^2 - \tfrac{1}{2}\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi\,,\\
J^i_\mu &= -\tfrac{1}{4}\,\mathrm{i}\,\gamma\cdot F\gamma_\mu\psi^i - \tfrac{1}{2}(\partial\!\!\!/\sigma)\gamma_\mu\psi^i\,,\\
v^{ij}_\mu &= \tfrac{1}{2}\bar{\psi}^i\gamma_\mu\psi^j\,,\\
a_\mu &= \tfrac{1}{8}\varepsilon_{\mu\nu\lambda\rho\sigma}F^{\nu\lambda}F^{\rho\sigma} + (\partial^\nu\sigma)F_{\nu\mu}\,,\\
b_{\mu\nu} &= \tfrac{1}{2}\varepsilon_{\mu\nu\lambda\rho\sigma}(\partial^\lambda\sigma)F^{\rho\sigma} + \tfrac{1}{2}\bar{\psi}\gamma_{[\mu}\partial_{\nu]}\psi\,,\\
\gamma^\mu J^i_\mu &= -\tfrac{1}{4}\,\mathrm{i}\,\gamma\cdot F\psi^i + \tfrac{3}{2}\partial\!\!\!/\sigma\psi^i\,,\\
\theta_\mu{}^\mu &= \tfrac{1}{4}F^2 + \tfrac{3}{2}\left(\partial\sigma\right)^2\,.
\end{aligned}
\tag{5.67}
$$

Applying the supersymmetry transformation rules (5.64) to the currents (5.67), using the Bianchi identities and equations of motion of the vector multiplet fields, one can calculate the supersymmetry transformations of the currents. A straightforward calculation yields

$$
\begin{aligned}
\delta\theta_{\mu\nu} &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\gamma_{\lambda(\mu}\partial^\lambda J_{\nu)}\,,\\
\delta J^i_\mu &= -\tfrac{1}{2}\,\mathrm{i}\,\gamma^\nu\theta_{\mu\nu}\epsilon^i - \,\mathrm{i}\,\gamma_{[\lambda}\partial^\lambda v^{ij}_{\mu]}\epsilon_j - \tfrac{1}{2}a_\mu\epsilon^i + \tfrac{1}{2}\,\mathrm{i}\,\gamma^\nu b_{\mu\nu}\epsilon^i\,,\\
\delta v^{ij}_\mu &= \mathrm{i}\,\bar{\epsilon}^{(i}J^{j)}_\mu\,,\\
\delta a_\mu &= -\bar{\epsilon}\partial^\lambda\gamma_{[\lambda}J_{\mu]} - \tfrac{1}{4}\bar{\epsilon}\gamma_{\mu\nu}\gamma^\lambda\partial^\nu J_\lambda + \tfrac{1}{4}\bar{\epsilon}\gamma_{\mu\nu}\partial^\nu\left(\gamma^\lambda J^i_\lambda\right)\,,\\
\delta b_{\mu\nu} &= \tfrac{3}{4}\,\mathrm{i}\,\bar{\epsilon}\gamma_{[\lambda\mu}\partial^\lambda J_{\nu]} - \tfrac{1}{8}\,\mathrm{i}\,\bar{\epsilon}\gamma_{\mu\nu\lambda}\gamma^\rho\partial^\lambda J_\rho + \tfrac{1}{8}\,\mathrm{i}\,\bar{\epsilon}\gamma_{\mu\nu\lambda}\partial^\lambda\left(\gamma^\rho J^i_\rho\right)\,,\\
\delta\left(\gamma^\mu J^i_\mu\right) &= -\tfrac{1}{2}\,\mathrm{i}\,\theta_\mu{}^\mu\epsilon^i + \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/ v\!\!\!/^{ij}\epsilon_j - \tfrac{1}{2}a\!\!\!/\epsilon^i + \tfrac{1}{2}\,\mathrm{i}\,\gamma\cdot b\epsilon^i\,,\\
\delta\theta_\mu{}^\mu &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\partial\!\!\!/\left(\gamma^\mu J_\mu\right)\,.
\end{aligned}
\tag{5.68}
$$

Note that we have added to the transformation rules for a_μ and $b_{\mu\nu}$ terms that are identically zero: the first term at the r.h.s. contains the divergence of the supercurrent and the last two terms are identical, but we have chosen not to explicitly evaluate the gamma-trace. Similarly, the second term in the variation of the supercurrent contains a term that is proportional to the divergence of the $\mathrm{SU}(2)$ current.

The reason why we added these terms is that in this way we obtain below the linearized Weyl multiplet in a conventional form. Alternatively, we could not have added these terms and later have brought the Weyl multiplet into the same conventional form by redefining the Q-transformations via a field-dependent S- and $\mathrm{SU}(2)$-transformation.

5.3.2 The improved supercurrent

We now add terms to $\theta_{\mu\nu}$ and J^i_μ such that they become (gamma-)traceless while remaining divergence-less. This requires the introduction of non-local projection operators. First of all, we add the following term to the energy-momentum tensor

$$\begin{aligned} \widehat{\theta}_{\mu\nu} &= \theta_{\mu\nu} - \tfrac{1}{4}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)\theta_\mu{}^\mu \\ &\equiv \theta_{\mu\nu} - \tfrac{1}{16}\left(\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu\right)d\,, \end{aligned} \tag{5.69}$$

where the current d has to satisfy the differential constraint

$$\Box d = 4\theta_\mu{}^\mu\,. \tag{5.70}$$

Similarly, we can add the following term to the supercurrent

$$\begin{aligned} \widehat{J}^i_\mu &= J^i_\mu - \tfrac{1}{4}\left(\gamma_\mu - \frac{\partial_\mu\not\partial}{\Box}\right)\gamma\cdot J^i \\ &\equiv J^i_\mu + \tfrac{1}{32}\gamma_{\mu\nu}\partial^\nu\lambda\,, \end{aligned} \tag{5.71}$$

where the current λ^i satisfies the differential constraint

$$\not\partial\lambda^i = -8\gamma\cdot J^i\,. \tag{5.72}$$

For the supercurrent coming from a vector multiplet in $D \neq 4$, there are no local gauge-invariant improvement terms. However, the improved energy-momentum tensor transforms to the improved supercurrent, and by varying the improved supercurrent we find a new constrained current t_{ab}. This current satisfies the following constraint in terms of the previously found currents a_μ and $b_{\mu\nu}$

$$\Box\, t_{ab} = -8\partial_{[a}a_{b]} - 4\varepsilon_{abcde}\partial^e b^{dc}\,. \tag{5.73}$$

The currents t_{ab} , λ^i and d do not generate any more currents, and we have summarized the improved current multiplet and the differential constraints in table 5.5.

We then find for the improved current multiplet the following transformation rules

$$\begin{aligned} \delta\widehat{\theta}_{\mu\nu} &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\partial^\lambda\gamma_{\lambda(\mu}\widehat{J}_{\nu)}\,, \\ \delta\widehat{J}^i_\mu &= -\tfrac{1}{2}\,\mathrm{i}\,\gamma^\nu\widehat{\theta}_{\mu\nu}\epsilon^i - \tfrac{1}{4}\,\mathrm{i}\,\left(\gamma^\lambda\gamma_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}\gamma^\lambda\right)\partial^\nu v^{ij}_\lambda\epsilon_j\,, \\ &\quad -\tfrac{3}{16}\partial^\nu\left(t_{\mu\nu} - \tfrac{1}{12}\gamma_{\mu\nu}\gamma\cdot t + \tfrac{2}{3}\gamma_{[\mu}\gamma^b t_{\nu]b}\right)\epsilon^i\,, \\ \delta v^{ij}_\mu &= \mathrm{i}\,\bar\epsilon^{(i}\widehat{J}^{j)}_\mu - \tfrac{1}{32}\,\mathrm{i}\,\bar\epsilon^{(i}\gamma_{\mu\nu}\partial^\nu\lambda^{j)}\,, \\ \delta t_{ab} &= -\bar\epsilon\gamma^\mu\gamma_{ab}\widehat{J}_\mu + \tfrac{3}{32}\bar\epsilon\not\partial\gamma_{ab}\lambda + \tfrac{1}{32}\bar\epsilon\gamma_{ab}\not\partial\lambda\,, \\ \delta\lambda^i &= \mathrm{i}\,\not\partial d\epsilon^i - 4\,\mathrm{i}\,\not v^{ij}\epsilon_j - \tfrac{1}{2}\gamma\cdot t\epsilon^i\,, \\ \delta d &= -\tfrac{1}{4}\,\mathrm{i}\,\bar\epsilon\lambda\,. \end{aligned} \tag{5.74}$$

Note that these transformations rules are perfectly well-defined, all non-localities have been absorbed in the differentially constrained currents.

Current	Restrictions	$\mathrm{SU}(2)$	w	# d.of.
$\widehat{\theta}_{(\mu\nu)}$	$\partial^\mu \widehat{\theta}_{\mu\nu} = 0\,, \quad \widehat{\theta}_\mu{}^\mu = 0$	1	2	9
$v_\mu^{(ij)}$	$\partial^\mu v_\mu^{ij} = 0$	3	2	12
$t_{[ab]}$	$\Box\, t_{ab} = -8\partial_{[a} a_{b]} - 4\varepsilon_{abcde}\partial^e b^{dc}$	1	2	10
d	$\Box d = 4\theta_\mu{}^\mu$	2	2	1
$\widehat{J}^i_\mu$	$\partial^\mu \widehat{J}^i_\mu = 0\,, \quad \gamma^\mu \widehat{J}^i_\mu = 0$	2	$\frac{5}{2}$	24
λ^i	$\partial\!\!\!/\lambda^i = -8\gamma^\mu J^i_\mu$	2	$\frac{5}{2}$	8

Table 5.5: The improved current multiplet with constrained currents.

5.3.3 The linearized Weyl multiplets

From the field content of the two current multiplets, we can immediately read off the field content of the two Weyl multiplets. They both have the same 21+24 independent gauge fields, which we have displayed in table 5.6. The dependent gauge fields do not couple to any current but are also displayed. The linearized form of the fünfbein $e_\mu{}^a$ is denoted by $h_{\mu\nu}$.

For both Weyl multiplets, the total number of degrees of freedom becomes 32+32 bosonic plus fermionic fields. The difference between the two Weyl multiplets lies in the set of matter fields. The Standard Weyl multiplet couples to the improved current multiplet and has matter fields $T_{ab}\,, \chi^i$ and D.

The Dilaton multiplet couples to the conventional current multiplet and has matter fields $B_{\mu\nu}\,, A_\mu\,, \psi^i$ and φ. This scalar field φ is the linearized form of the dilaton $\sigma \equiv e^\varphi$. There are also two extra gauge symmetries: a $\mathrm{U}(1)$ gauge symmetry and a two-form tensor gauge symmetry, the explicit form of which we have also indicated in table 5.6.

Note that some of the matter fields of the Dilaton Weyl multiplet have the same names as the fields of the vector multiplet. The reason for doing so will become clear in section 5.5 where we explain the connection between the two Weyl multiplets. From now on, until section 5.5, we will be only dealing with the Weyl multiplets and not with the vector multiplet.

The Standard Weyl multiplet

To derive the linearized transformation rules of the Standard Weyl multiplet, we introduce the following Noether term in the action

$$\mathcal{L}_{\rm Noether} = \tfrac{1}{2} h_{\mu\nu}\widehat{\theta}^{\mu\nu} + \mathrm{i}\,\bar{\psi}_\mu \widehat{J}^\mu + V_\mu^{ij} v^\mu_{ij} + T_{ab} t^{ab} + \mathrm{i}\,\bar{\chi}\lambda + D d\,. \qquad (5.75)$$

Demanding that the combined action of (5.63) and (5.75) is invariant under the supersymmetry transformations (5.64) and (5.74) results in the following linearized supersymmetry

Field	#	Gauge	SU(2)	w	Field	#	Gauge	SU(2)	w
		Elementary gauge fields					Dependent gauge fields		
${e_\mu}^a$	9	P^a	1	-1	$\omega_\mu^{[ab]}$	$-$	$M^{[ab]}$	1	0
b_μ	0	D	1	0	${f_\mu}^a$	$-$	K^a	1	1
$V_\mu^{(ij)}$	12	$\mathrm{SU}(2)$	3	0					
ψ_μ^i	24	Q_α^i	2	$-\frac{1}{2}$	ϕ_μ^i	$-$	S_α^i	2	$\frac{1}{2}$
		Dilaton Weyl multiplet					Standard Weyl multiplet		
$B_{[\mu\nu]}$	6	$\delta B_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]}$	1	0	$T_{[ab]}$	10		1	1
A_μ	4	$\delta A_\mu = \partial_\mu \Lambda$	1	0					
σ	1		1	1	D	1		1	2
ψ^i	8		2	$\frac{3}{2}$	χ^i	8		2	$\frac{3}{2}$

Table 5.6: Gauge fields and matter field of the Weyl multiplets.

transformation rules for the Standard Weyl multiplet

$$\begin{aligned}
\delta h_{\mu\nu} &= \bar\epsilon\gamma_{(\mu}\psi_{\nu)}\,,\\
\delta\psi_\mu^i &= \partial_\mu\epsilon^i + \tfrac{1}{4}{\omega_\mu}^{ab}\gamma_{ab} - V_\mu^{ij}\epsilon_j + \mathrm{i}\,\gamma\cdot T\gamma_\mu\epsilon^i\,,\\
\delta {V_\mu}^{ij} &= -\tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\phi_\mu^{j)} + 4\bar\epsilon^{(i}\gamma_\mu\chi^{j)}\,,\\
\delta T_{ab} &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma_{ab}\chi - \tfrac{3}{32}\,\mathrm{i}\,\bar\epsilon R_{ab}(Q)\,,\\
\delta\chi^i &= \tfrac{1}{4}D\epsilon^i - \frac{1}{64}\gamma\cdot R(V)^{ij}\epsilon_j + \tfrac{3}{32}\,\mathrm{i}\,\gamma\cdot T\overleftarrow{\not\partial}\epsilon^i + \tfrac{1}{32}\,\mathrm{i}\,\not\partial\gamma\cdot T\epsilon^i\,,\\
\delta D &= \bar\epsilon\not\partial\chi\,,
\end{aligned} \tag{5.76}$$

where we have used the linearized form of the expressions for the curvatures (5.38) and the dependent gauge fields (5.51)

$$\begin{aligned}
{\omega_\mu}^{ab} &= -\partial_{[a}h_{b]\mu} + \ldots\,,\\
\phi_\mu^i &= -\tfrac{1}{12}\,\mathrm{i}\,\left(\gamma^{ab}\gamma_\mu - \tfrac{1}{2}\gamma_\mu\gamma^{ab}\right)\psi_{ab}^{j)} + \ldots\,,\\
{R_{\mu\nu}}^{ij}(V) &= 2\partial_{[\mu}V_{\nu]}^{ij} + \ldots\,,\\
R_{ab}(Q) &= \left(\psi_{ab} - \tfrac{1}{12}\gamma_{ab}\gamma^{cd}\psi_{cd} + \tfrac{2}{3}\gamma_{[a}\gamma^c\psi_{b]c}\right) + \ldots\,,\\
\psi_{ab} &= 2\partial_{[a}\psi_{b]}^i\,.
\end{aligned} \tag{5.77}$$

Note that the transformation rules for ψ_μ^i and ${V_\mu}^{ij}$ differ from the original transformation rules (5.33) by matter fields as was explained in (5.48).

The Dilaton Weyl multiplet

To derive the linearized transformation rules of the Dilaton Weyl multiplet, we introduce the following Noether term in the action (note that in particular the trace of the energy-momentum tensor couples to the dilaton φ)

$$\mathcal{L}_{\text{Noether}} = \tfrac{1}{2}h_{\mu\nu}\theta^{\mu\nu} + \mathrm{i}\,\bar{\psi}_\mu J^\mu + V_\mu^{ij} v_{ij}^\mu + B_{\mu\nu}b^{\mu\nu} + A_\mu a^\mu + \mathrm{i}\,\bar{\psi}\,(\,\mathrm{i}\,\gamma^\mu J_\mu) + \varphi\,\theta_\mu^{\ \mu}\,. \tag{5.78}$$

Demanding that the combined action of (5.63) and (5.75) is invariant under supersymmetry transformations (5.64) and (5.68) results in the following linearized supersymmetry transformation rules for the Dilaton Weyl multiplet

$$\begin{aligned}
\delta h_{\mu\nu} &= \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)}\,,\\
\delta\psi_\mu^i &= \partial_\mu\epsilon^i + \tfrac{1}{4}{\omega_\mu}^{ab}\gamma_{ab} - V_\mu^{ij}\epsilon_j + \mathrm{i}\,\gamma\cdot\underline{T}\gamma_\mu\epsilon^i\,,\\
\delta {V_\mu}^{ij} &= -\tfrac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\phi_\mu^{j)} + 4\bar{\epsilon}^{(i}\gamma_\mu\underline{\chi}^{j)}\,,\\
\delta B_{\mu\nu} &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\gamma_{\mu\nu}\psi + \tfrac{1}{2}\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]}\,,\\
\delta A_\mu &= \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\psi - \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\psi_\mu\,,\\
\delta\psi^i &= -\tfrac{1}{4}\gamma\cdot F\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/\varphi\epsilon^i + \gamma\cdot\underline{T}\epsilon^i\,,\\
\delta\varphi &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\psi\,,
\end{aligned} \tag{5.79}$$

where we have again used the expressions (5.77). The underlined fields in (5.79) are not independent fields here, but are used as a shorthand notation for the following expressions in terms of fields of the Dilaton Weyl multiplet

$$\begin{aligned}
T_{ab} &= \tfrac{1}{8}\left(F_{ab} - \tfrac{1}{6}\varepsilon_{abcde}H^{edc}\right)\,,\\
\chi^i &= \tfrac{1}{8}\,\mathrm{i}\,\partial\!\!\!/\psi^i + \frac{1}{64}\gamma^{ab}\psi_{ab}\,,\\
D &= \tfrac{1}{4}\Box\varphi - \tfrac{1}{32}\partial^\mu\partial^\nu h_{\mu\nu} + \tfrac{1}{32}\Box h_\mu^\mu\,.
\end{aligned} \tag{5.80}$$

The justification for using the expressions (5.80) is that they transform exactly as the fields T_{ab} , χ^i and D in the Standard Weyl multiplet. Finally, we have defined the curvatures for the gauge fields $B_{\mu\nu}$ and A_μ to be

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}\,, \quad H_{\mu\nu\lambda} = 3\partial_{[\mu}B_{\nu\lambda]}\,. \tag{5.81}$$

Note that the last three lines in (5.79) are similar to the transformation rules of the Maxwell multiplet (5.64). In section 5.5, we will clarify this similarity in more detail.

5.4 The Weyl multiplets

In this section, we will present the full nonlinear supersymmetry transformations rules of the two Weyl multiplets as well as the modifications to the superconformal algebra. We will first briefly outline the iterative algorithm for obtaining the complete transformation rules for the two Weyl multiplets. For more details see [176].

Covariantization

The procedure to obtain these results is straightforward. Starting from the transformation rules of the linearized Weyl multiplets (5.76) and (5.79), the first step is to replace curvatures by hatted curvatures as defined in (5.49). In the transformation rules of T_{ab} and χ^i for instance, we will need

$$\begin{aligned} \widehat{R}_{\mu\nu}{}^i(Q) &= R_{\mu\nu}{}^i(Q) + 2\,\mathrm{i}\,\gamma\cdot T\gamma_{[\mu}\psi^i_{\nu]}\,, \\ \widehat{R}_{\mu\nu}{}^{ij}(V) &= R_{\mu\nu}{}^{ij}(V) - 8\bar\psi^{(i}_{[\mu}\gamma_{\nu]}\chi^{j)} - \mathrm{i}\,\bar\psi^{(i}_{[\mu}\gamma\cdot T\psi^{j)}_{\nu]}\,, \end{aligned} \tag{5.82}$$

Similarly, we use the bosonic transformation rules given in table 5.6 as well as the supersymmetry transformations to replace derivatives on fields by derivatives which are covariantized with respect to all symmetries except translations

$$\partial_\mu \to \mathcal{D}_\mu \equiv \partial_\mu - \delta_I(h_\mu{}^I)\,. \tag{5.83}$$

Using these new transformation rules, the superconformal algebra (5.10) is imposed on all fields. This will enable us to determine the S-supersymmetry rules for all the fields. An additional effect is that the transformation rules for some fields will need nonlinear modifications in order to satisfy the algebra.

These new nonlinear transformation rules, as well as the S-supersymmetry transformations, will have to be accounted for in the curvatures and covariant derivatives, and the algebra will have to be imposed again with these modified transformation rules, until no new modifications are necessary.

Dependent gauge fields

The dependent gauge fields defined in (5.51) depend on the covariant curvatures (5.49). The non-invariance of the constraints (5.47) and the modifications to the curvatures will induce extra transformations for the dependent gauge fields. We find the following extra transformations for $\omega_\mu{}^{ab}$ and ϕ_μ^i

$$\begin{aligned}
\delta_{\rm extra}\omega_\mu{}^{ab} &= -\,\mathrm{i}\,\bar\epsilon\gamma^{[a}\gamma\cdot T\gamma^{b]}\psi_\mu - \tfrac12\bar\epsilon\gamma^{[a}\widehat{R}_\mu{}^{b]}(Q) \\
&\quad -\tfrac14\bar\epsilon\gamma_\mu\widehat{R}^{ab}(Q) - 4e_\mu{}^{[a}\bar\epsilon\gamma^{b]}\chi\,. \qquad (5.84)\\
\delta_{\rm extra}\phi_\mu^i &= \tfrac{1}{12}\,\mathrm{i}\,\left(\gamma^{ab}\gamma_\mu - \tfrac12\gamma_\mu\gamma^{ab}\right)\widehat{R}_{ab}{}^{ij}(V)\epsilon_j \\
&\quad -\tfrac16\left(\not{D}\gamma_\mu\gamma\cdot T - 2\not{D}\gamma\cdot T\gamma_\mu + \tfrac12\gamma_\mu\not{D}\gamma\cdot T + \tfrac12\gamma_\mu\gamma\cdot T\overleftarrow{\not{D}}\right)\epsilon^i \\
&\quad +\mathrm{i}\,\left(-\gamma_\mu(\gamma\cdot T)^2 + 4\gamma_c T_\mu{}^c\gamma\cdot T + 16\gamma_c T^{cd}T_{\mu d} - 4\gamma_\mu T^2\right)\epsilon^i \\
&\quad -\tfrac23\,\mathrm{i}\,\left(\gamma^{ab}\gamma_\mu - \tfrac12\gamma_\mu\gamma^{ab}\right)T_{ab}\eta^i\,. \qquad (5.85)
\end{aligned}$$

We will not need the transformations for the field $f_\mu{}^a$, except the extra transformations of $f_\mu{}^\mu$ under S-supersymmetry

$$\delta_{\rm extra,S} f_\mu{}^\mu = -5\,\mathrm{i}\,\bar\eta\chi\,. \qquad (5.86)$$

5.4.1 The modified superconformal algebra

It turns out that the commutators of the various transformation rules of the nonlinear Weyl multiplets do not lead to the original algebra (5.10). Instead, the algebra closes modulo terms that can be reorganized in terms of field dependent superconformal transformations.

For instance, the full commutator of two supersymmetry transformations acquires the following terms with respect to (5.13)

$$\begin{aligned}
[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)] &= \delta_{\rm cgct}(\xi_3^\mu) + \delta_M(\Lambda_M{}_3^{ab}) + \delta_S(\eta_3) + \delta_U(\Lambda_U{}_3^{ij}) \\
&\quad +\delta_K(\Lambda_{K3}^a) + \delta_{\rm U(1)}(\Lambda_3) + \delta_B(\Lambda_{3\mu})\,, \qquad (5.87)
\end{aligned}$$

where the covariant general coordinate transformations have been defined in (5.44). The last two terms appear only in the Dilaton Weyl multiplet formulation, where we use the notation δ_B for the two-form tensor gauge symmetry. The parameters appearing in (5.87) are

$$\begin{aligned}
\xi_3^\mu &= \tfrac12\bar\epsilon_2\gamma_\mu\epsilon_1\,,\\
\Lambda_M{}_3^{ab} &= -\,\mathrm{i}\,\bar\epsilon_2\gamma^{[a}\gamma\cdot T\gamma^{b]}\epsilon_1\,,\\
\Lambda_U{}_3^{ij} &= \mathrm{i}\,\bar\epsilon_2^{(i}\gamma\cdot T\epsilon_1^{j)}\,,\\
\eta_3^i &= -\tfrac94\,\mathrm{i}\,\bar\epsilon_2\epsilon_1\chi^i + \tfrac74\,\mathrm{i}\,\bar\epsilon_2\gamma_c\epsilon_1\gamma^c\chi^i \\
&\quad +\tfrac14\,\mathrm{i}\,\bar\epsilon_2^{(i}\gamma_{cd}\epsilon_1^{j)}\left(\gamma^{cd}\chi_j + \tfrac14\widehat{R}^{cd}{}_j(Q)\right)\,, \qquad (5.88)\\
\Lambda_{K3}^a &= -\tfrac12\bar\epsilon_2\gamma^a\epsilon_1 D + \tfrac{1}{96}\bar\epsilon_2^i\gamma^{abc}\epsilon_1^j\widehat{R}_{bcij}(V) \\
&\quad +\tfrac{1}{12}\,\mathrm{i}\,\bar\epsilon_2\left(-5\gamma^{abcd}D_bT_{cd} + 9D_bT^{ba}\right)\epsilon_1 \\
&\quad +\bar\epsilon_2\left(\gamma^{abcde}T_{bc}T_{de} - 4\gamma^cT_{cd}T^{ad} + \tfrac23\gamma^aT^2\right)\epsilon_1\,,\\
\Lambda_3 &= -\tfrac12\,\mathrm{i}\,\sigma\bar\epsilon_2\epsilon_1\,,\\
\Lambda_{3\mu} &= -\tfrac12\sigma^2\xi_{3\mu} - \tfrac12 A_\mu\Lambda_3\,.
\end{aligned}$$

The commutator between Q- and S-supersymmetry also gains modifications with respect to (5.14)

$$\begin{aligned}[\delta_S(\eta),\delta_Q(\epsilon)] &= \delta_D(\tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\eta)+\delta_M(\tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma^{ab}\eta)+\delta_U(-\tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\eta^{j)})\\ &\quad+\delta_K(\Lambda^a_{3K})\,, \end{aligned} \tag{5.89}$$

with the field-dependent special conformal transformation given by

$$\Lambda^a_{3K}=\tfrac{1}{6}\bar\epsilon\left(\gamma\cdot T\gamma_a-\tfrac{1}{2}\gamma_a\gamma\cdot T\right)\eta\,. \tag{5.90}$$

The commutator of Q- and $\mathrm{U}(1)$-transformations is given by

$$\left[\delta_Q(\epsilon),\delta_{\mathrm{U}(1)}(\Lambda)\right]=\delta_B\left(-\tfrac{1}{2}\Lambda\delta(\epsilon)A_\mu\right). \tag{5.91}$$

In the next chapter, we will discuss matter fields coupled to conformal supergravity. These matter multiplets will have to obey the modified superconformal algebra given above in (5.87), (5.89) and (5.91), apart from possible additional transformations under which the fields of the Weyl multiplets do not transform, and possibly field equations if these matter multiplets are on-shell.

5.4.2 The Standard Weyl multiplet

Applying the rules of covariantization and the extra transformations of the dependent gauge fields and imposing the modified superconformal algebra, we find the following Q- and S-supersymmetry and K-transformation rules for the independent fields of the Standard Weyl multiplet

$$\begin{aligned}
\delta e_\mu{}^a &= \tfrac{1}{2}\bar\epsilon\gamma^a\psi_\mu\,,\\
\delta\psi^i_\mu &= \mathcal{D}_\mu\epsilon^i+\,\mathrm{i}\,\gamma\cdot T\gamma_\mu\epsilon^i-\,\mathrm{i}\,\gamma_\mu\eta^i\,,\\
\delta V_\mu{}^{ij} &= -\tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\phi^{j)}_\mu+4\bar\epsilon^{(i}\gamma_\mu\chi^{j)}+\,\mathrm{i}\,\bar\epsilon^{(i}\gamma\cdot T\psi^{j)}_\mu+\tfrac{3}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi^{j)}_\mu\,,\\
\delta b_\mu &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\phi_\mu-2\bar\epsilon\gamma_\mu\chi+\tfrac{1}{2}\,\mathrm{i}\,\bar\eta\psi_\mu+2\Lambda_{K\mu}\,,\\
\delta T_{ab} &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma_{ab}\chi-\tfrac{3}{32}\,\mathrm{i}\,\bar\epsilon\widehat{R}_{ab}(Q)\,,\\
\delta\chi^i &= \tfrac{1}{4}D\epsilon^i-\tfrac{1}{64}\gamma\cdot\widehat{R}^{ij}(V)\epsilon_j+\tfrac{3}{32}\,\mathrm{i}\,\gamma\cdot T\overleftarrow{\not{D}}\epsilon^i+\tfrac{1}{32}\,\mathrm{i}\,\not{D}\gamma\cdot T\epsilon^i\\
&\quad+\tfrac{1}{24}T^2\epsilon^i-\tfrac{1}{4}\left(\gamma\cdot T\right)^2\epsilon^i+\tfrac{1}{4}\gamma\cdot T\eta^i\,,\\
\delta D &= \bar\epsilon\not{D}\chi-\tfrac{5}{3}\,\mathrm{i}\,\bar\epsilon\gamma\cdot T\chi-\,\mathrm{i}\,\bar\eta\chi\,.
\end{aligned} \tag{5.92}$$

The covariant derivative $\mathcal{D}_\mu\epsilon$ is given in (5.34). For other covariant derivatives, see the general rule (5.83). The covariant curvatures $\widehat{R}(Q)$ and $\widehat{R}(V)$ are given explicitly in (5.82). We also used the following transformations for these curvatures:

$$\begin{aligned}
\delta\widehat{R}_{ab}{}^{i}(Q) &= -\tfrac{1}{6}\left(\gamma_{ab}{}^{cd}-\gamma^{cd}\gamma_{ab}-\tfrac{1}{2}\gamma_{ab}\gamma^{cd}\right)\widehat{R}_{cd}{}^{ij}(V)\epsilon_j+\tfrac{1}{4}\widehat{R}_{ab}{}^{cd}(M)\gamma_{cd}\epsilon^i\\
&\quad+2\,\mathrm{i}\,\Big(D_{[a}\gamma\cdot T\gamma_{b]}-\tfrac{1}{3}D_{[a}\gamma_{b]}\gamma\cdot T\\
&\quad-\tfrac{1}{3}\gamma_{[a}\not{D}\gamma\cdot T\gamma_{b]}-\tfrac{1}{3}\gamma_{ab}D_c\gamma_d T^{cd}\Big)\epsilon^i\,,\\
\delta\widehat{R}_{ab}{}^{ij}(V) &= -\tfrac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\widehat{R}_{ab}{}^{j)}(S)-8\bar{\epsilon}^{(i}\gamma_{[a}D_{b]}\chi^{j)}+8\,\mathrm{i}\,\bar{\epsilon}^{(i}\gamma_{[a}\gamma\cdot T\gamma_{b]}\chi^{j)}\\
&\quad+\mathrm{i}\,\bar{\epsilon}^{(i}\gamma\cdot T\widehat{R}_{ab}{}^{j)}(Q)+\tfrac{3}{2}\,\mathrm{i}\,\bar{\eta}^{(i}\widehat{R}_{ab}{}^{j)}(Q)+8\,\mathrm{i}\,\bar{\eta}^{(i}\gamma_{ab}\chi^{j)}\,.
\end{aligned} \tag{5.93}$$

Note that we also have given the transformation rules for the gauge field of dilatations b_μ, which did not appear in the linearized Weyl multiplet. In the nonlinear Weyl multiplet it is hidden in the covariant derivatives and curvatures.

5.4.3 The Dilaton Weyl multiplet

To obtain the complete Dilaton Weyl multiplet, we first replace the scalar φ by the dilaton $\sigma = e^{\varphi}$. We then introduce appropriate powers of the dilaton in the various terms of (5.79) such that all terms have the same Weyl weight. This will make the Dilaton Weyl multiplet much more nonlinear than the Standard Weyl multiplet.

The Dilaton Weyl multiplet also contains two extra gauge transformations: the gauge transformations of A_μ with parameter Λ and those of $B_{\mu\nu}$ with parameter Λ_μ. The transformation of the fields are given by

$$\begin{aligned}
\delta e_\mu{}^a &= \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,,\\
\delta\psi_\mu^i &= \mathcal{D}_\mu\epsilon^i+\mathrm{i}\,\gamma\cdot\underline{T}\gamma_\mu\epsilon^i-\mathrm{i}\,\gamma_\mu\eta^i\,,\\
\delta V_\mu{}^{ij} &= -\tfrac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\phi_\mu^{j)}+4\bar{\epsilon}^{(i}\gamma_\mu\underline{\chi}^{j)}+\mathrm{i}\,\bar{\epsilon}^{(i}\gamma\cdot\underline{T}\psi_\mu^{j)}+\tfrac{3}{2}\,\mathrm{i}\,\bar{\eta}^{(i}\psi_\mu^{j)}\,,\\
\delta b_\mu &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\phi_\mu-2\bar{\epsilon}\gamma_\mu\underline{\chi}+\tfrac{1}{2}\,\mathrm{i}\,\bar{\eta}\psi_\mu+2\Lambda_{K\mu}\,,\\
\delta B_{\mu\nu} &= \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}\gamma_{\mu\nu}\psi+\tfrac{1}{2}\sigma^2\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]}+A_{[\mu}\delta(\epsilon)A_{\nu]}+2\partial_{[\mu}\Lambda_{\nu]}-\tfrac{1}{2}\Lambda F_{\mu\nu}\,,\\
\delta A_\mu &= \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\psi-\tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}\psi_\mu+\partial_\mu\Lambda\,,\\
\delta\psi^i &= -\tfrac{1}{4}\gamma\cdot\widehat{F}\epsilon^i-\tfrac{1}{2}\,\mathrm{i}\,\not{D}\sigma\epsilon^i+\sigma\gamma\cdot\underline{T}\epsilon^i-\tfrac{1}{4}\,\mathrm{i}\,\sigma^{-1}\bar{\psi}^i\psi^j\epsilon_j+\sigma\eta^i\,,\\
\delta\sigma &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\psi\,.
\end{aligned} \tag{5.94}$$

We have again underlined some fields to indicate that they are not independent fields but merely short-hand notations. The explicit expression for these fields in terms of fields of the

Dilaton Weyl multiplet are

$$
\begin{aligned}
T_{ab} &= \tfrac{1}{8}\sigma^{-2}\left(\sigma\widehat{F}_{ab} - \tfrac{1}{6}\varepsilon_{abcde}\widehat{H}^{edc} + \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}\gamma_{ab}\psi\right)\,, \\
\chi^i &= \tfrac{1}{8}\,\mathrm{i}\,\sigma^{-1}\not{D}\psi^i + \tfrac{1}{16}\,\mathrm{i}\,\sigma^{-2}\not{D}\sigma\psi^i - \tfrac{1}{32}\sigma^{-2}\gamma\cdot\widehat{F}\psi^i \\
&\quad + \tfrac{1}{4}\sigma^{-1}\gamma\cdot\underline{T}\psi^i + \tfrac{1}{32}\,\mathrm{i}\,\sigma^{-3}\psi_j\bar{\psi}^i\psi^j\,, \\
D &= \tfrac{1}{4}\sigma^{-1}\Box^c\sigma + \tfrac{1}{8}\sigma^{-2}(D_a\sigma)(D^a\sigma) - \tfrac{1}{16}\sigma^{-2}\widehat{F}^2 \\
&\quad - \tfrac{1}{8}\sigma^{-2}\bar{\psi}\not{D}\psi - \tfrac{1}{64}\sigma^{-4}\bar{\psi}^i\psi^j\bar{\psi}_i\psi_j - 4\,\mathrm{i}\,\sigma^{-1}\bar{\psi}\underline{\chi} \\
&\quad + \left(2\sigma^{-1}\widehat{F}_{ab} - \tfrac{26}{3}\underline{T_{ab}} + \tfrac{1}{4}\,\mathrm{i}\,\sigma^{-2}\bar{\psi}\gamma_{ab}\psi\right)\underline{T^{ab}}\,,
\end{aligned}
\tag{5.95}
$$

The conformal D'Alembertian $\Box^c$ is defined by

$$
\begin{aligned}
\Box^c\sigma \equiv D^aD_a\sigma &= \left(\partial^a - 2b^a + \omega_b{}^{ba}\right)D_a\sigma - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}_aD^a\psi - 2\sigma\bar{\psi}_a\gamma^a\underline{\chi} \\
&\quad + \tfrac{1}{2}\bar{\psi}_a\gamma^a\gamma\cdot\underline{T}\psi + \tfrac{1}{2}\bar{\phi}_a\gamma^a\psi + 2f_a{}^a\sigma\,.
\end{aligned}
\tag{5.96}
$$

The justification for using the expressions (5.95) is that they transform exactly as the fields $T_{ab}\,, \chi^i$ and D in the Standard Weyl multiplet. Finally, we have defined the curvatures for the gauge fields $B_{\mu\nu}$ and A_μ to be

$$
\begin{aligned}
\widehat{F}_{\mu\nu} &= 2\partial_{[\mu}A_{\nu]} - \bar{\psi}_{[\mu}\gamma_{\nu]}\psi + \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\psi}_{[\mu}\psi_{\nu]}\,, \\
\widehat{H}_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} - \tfrac{3}{2}\,\mathrm{i}\,\sigma\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\psi - \tfrac{3}{4}\sigma^2\bar{\psi}_{[\mu}\gamma_\nu\psi_{\rho]} + \tfrac{3}{2}A_{[\mu}F_{\nu\rho]}\,.
\end{aligned}
\tag{5.97}
$$

For the convenience of the reader we give their transformation rules

$$
\begin{aligned}
\delta\widehat{F}_{ab} &= -\bar{\epsilon}\gamma_{[a}D_{b]}\psi - \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}\widehat{R}_{ab}(Q) + \,\mathrm{i}\,\bar{\epsilon}\gamma_{[a}\gamma\cdot\underline{T}\gamma_{b]}\psi + \,\mathrm{i}\,\bar{\eta}\gamma_{ab}\psi\,, \\
\delta\widehat{H}_{abc} &= \tfrac{3}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}\gamma_{[ab}D_{c]}\psi + \tfrac{3}{2}\,\mathrm{i}\,D_{[a}\sigma\bar{\epsilon}\gamma_{bc]}\psi - \tfrac{3}{4}\sigma^2\bar{\epsilon}\gamma_{[a}\widehat{R}_{bc]}(Q) \\
&\quad - \tfrac{3}{2}\sigma\bar{\epsilon}\gamma_{[a}\gamma\cdot\underline{T}\gamma_{bc]}\psi - \tfrac{3}{2}\bar{\epsilon}\gamma_{[a}\widehat{F}_{bc]}\psi - \tfrac{3}{2}\sigma\bar{\eta}\gamma_{abc}\psi\,.
\end{aligned}
\tag{5.98}
$$

Finally, we give the Bianchi identities for these two curvatures

$$
\begin{aligned}
D_{[a}\widehat{F}_{bc]} &= \tfrac{1}{2}\bar{\psi}\gamma_{[a}\widehat{R}_{bc]}(Q)\,, \\
D_{[a}\widehat{H}_{bcd]} &= \tfrac{3}{4}\widehat{F}_{[ab}\widehat{F}_{cd]}\,.
\end{aligned}
\tag{5.99}
$$

5.5 Connection between the Weyl multiplets

In the previous section, we have shown that the Standard and Dilaton Weyl multiplets can be related to each other by expressing the fields of the Standard Weyl multiplet in terms of those of the Dilaton Weyl multiplet (see (5.95)). It is known that in six dimensions the coupling of an on-shell self-dual tensor multiplet to the Standard Weyl multiplet leads to a Dilaton Weyl multiplet [176].

Since in five dimensions a tensor multiplet is dual to a vector multiplet, it is natural to consider the coupling of a vector multiplet to the Standard Weyl multiplet. Since the Standard Weyl multiplet has no dilaton, we must consider the improved vector multiplet. We will take the vector multiplet off-shell to simplify the higher-order fermion terms.

5.5.1 The improved Maxwell multiplet

We will first consider the improved vector multiplet in a flat background, i.e. no coupling to conformal supergravity. Our starting point is the Lagrangian corresponding to an off-shell vector multiplet:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}(\partial\sigma)^2 + Y^{ij}Y_{ij}\,. \tag{5.100}$$

The action corresponding to this Lagrangian is invariant under the off-shell supersymmetries

$$\begin{aligned} \delta A_\mu &= \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\psi\,, \\ \delta Y^{ij} &= -\tfrac{1}{2}\bar{\epsilon}^{(i}\partial\!\!\!/\psi^{j)}\,, \\ \delta\psi^i &= -\tfrac{1}{4}\gamma\cdot F\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/\sigma\epsilon^i - Y^{ij}\epsilon_j\,, \\ \delta\sigma &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\psi\,. \end{aligned} \tag{5.101}$$

The action has the wrong Weyl weight to be scale invariant. We therefore improve it by multiplying all terms with the dilaton. This requires additional cubic terms in the action to keep it invariant under supersymmetry. We thus obtain the Lagrangian for the improved vector multiplet:

$$\begin{aligned} \mathcal{L} &= \left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\bar{\psi}\partial\!\!\!/\psi - \tfrac{1}{2}(\partial\sigma)^2 + Y^{ij}Y_{ij}\right)\sigma \\ &\quad -\tfrac{1}{24}\varepsilon^{\mu\nu\lambda\rho\sigma}A_\mu F_{\nu\lambda}F_{\rho\sigma} - \tfrac{1}{8}\,\mathrm{i}\,\bar{\psi}\gamma\cdot F\psi - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}^i\psi^j Y_{ij}\,. \end{aligned} \tag{5.102}$$

The equations of motion and the Bianchi identity corresponding to this Lagrangian are given by

$$0 = L^{ij} = \varphi^i = E_a = N = G_{abc}\,. \tag{5.103}$$

where we have defined

$$\begin{aligned} L^{ij} &\equiv 2\sigma Y^{ij} - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}^i\psi^j\,, \\ \varphi^i &\equiv \mathrm{i}\,\sigma\partial\!\!\!/\psi^i + \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/\sigma\psi^i - \tfrac{1}{4}\gamma\cdot F\psi^i + Y^{ij}\psi_j\,, \\ E_a &\equiv \partial^b\left(\sigma F_{ba} + \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}\gamma_{ba}\psi\right) - \tfrac{1}{8}\varepsilon_{abcde}F^{bc}F^{de}\,, \\ N &\equiv -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\bar{\psi}\partial\!\!\!/\psi + \sigma\Box\sigma + \tfrac{1}{2}(\partial\sigma)^2 + Y^{ij}Y_{ij}\,, \\ G_{abc} &\equiv \partial_{[a}F_{bc]}\,. \end{aligned} \tag{5.104}$$

5.5.2 Coupling to the Standard Weyl multiplet

Next, we consider the coupling of the improved vector multiplet to the Standard Weyl multiplet. The transformation rules for the fields of the off-shell vector multiplet can be found by imposing the superconformal algebra (5.87). We thus find the following Q- and S-transformation rules:

$$\begin{aligned} \delta A_\mu &= \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\psi - \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}\psi_\mu\,, \\ \delta Y^{ij} &= -\tfrac{1}{2}\bar{\epsilon}^{(i}\not{D}\psi^{j)} + \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\gamma\cdot T\psi^{j)} - 4\,\mathrm{i}\,\sigma\bar{\epsilon}^{(i}\chi^{j)} + \tfrac{1}{2}\,\mathrm{i}\,\bar{\eta}^{(i}\psi^{j)}\,, \\ \delta\psi^i &= -\tfrac{1}{4}\gamma\cdot\widehat{F}\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\not{D}\sigma\epsilon^i + \sigma\gamma\cdot T\epsilon^i - Y^{ij}\epsilon_j + \sigma\eta^i\,, \\ \delta\sigma &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\psi\,, \end{aligned} \tag{5.105}$$

where the covariant curvature is

$$\widehat{F}_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} - \bar{\psi}_{[\mu}\gamma_{\nu]}\psi + \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\psi}_{[\mu}\psi_{\nu]}\,. \tag{5.106}$$

The supercovariant extension of the Bianchi identity reads

$$0 = G_{abc} \;=\; D_{[a}\widehat{F}_{bc]} - \tfrac{1}{2}\bar{\psi}\gamma_{[a}\widehat{R}_{bc]}(Q)\,. \tag{5.107}$$

The second term in the transformation of A_μ, reflected also in the curvature, signals a modification of the supersymmetry algebra, as can be seen by comparing with the general rule (5.87)

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \ldots + \delta_{\mathrm{U}(1)}\left(\Lambda_3 = -\tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}_2\epsilon_1\right), \tag{5.108}$$

where the dots indicate all the terms present for the fields of the Standard Weyl multiplet and where the last term is the gauge transformation of A_μ.

Our next goal is to find the equations of motion for the vector multiplet coupled to conformal supergravity. These equations of motion should be an extension of the flat spacetime results given in (5.105). One way to proceed is to first find the curved background extension of the flat spacetime action defined by (5.102) and next derive the equations of motion from this action. However, for our present purposes, it is sufficient to find the equations of motion only.

We want to identify the spinor ψ^i of the vector multiplet with the spinor ψ^i of the Dilaton Weyl multiplet. This is why we have given these two spinors the same name in the first place. Comparing the $\mathrm{SU}(2)$ triplet term in the supersymmetry transformations of the two spinors, see (5.94) and (5.105), we deduce that the constraint L^{ij} does not get any corrections, and we must have

$$L^{ij} = 2\sigma Y^{ij} - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}^i\psi^j\,. \tag{5.109}$$

There are now two ways to proceed. One way is to make the transition to an on-shell vector multiplet by using (5.109) to eliminate the auxiliary field Y^{ij} from the transformation rules (5.105). The commutator of two supersymmetry transformations would then only close modulo the equations of motion.

A more elegant way is to note that the equations of motion must transform into each other. By varying (5.109) under (5.105) we find

$$\delta L^{ij} = \mathrm{i}\,\bar{\epsilon}^{(i}\varphi^{j)}\,, \tag{5.110}$$

where the supercovariant extension of φ^i is now given by

$$\begin{aligned}\varphi^i \;=\;& \mathrm{i}\,\sigma \not{D}\psi^i + \tfrac{1}{2}\,\mathrm{i}\,\not{D}\sigma\psi^i - \tfrac{1}{4}\gamma\cdot\widehat{F}\psi^i + Y^{ij}\psi_j \\ &+2\sigma\gamma\cdot T\psi^i - 8\sigma^2\chi^i\,.\end{aligned} \tag{5.111}$$

Varying this expression under (5.105) and using (5.107) leads to the other equations of motion. We find

$$\delta\varphi^i = -\tfrac{1}{2}\,\mathrm{i}\,\not{D}L^{ij}\epsilon_j - \tfrac{1}{2}\,\mathrm{i}\,\gamma^a E_a\epsilon^i + \tfrac{1}{2}N\epsilon^i - \gamma\cdot TL^{ij}\epsilon_j\,. \tag{5.112}$$

The supercovariant generalizations of (5.105) are given by

$$\begin{aligned}E_a \;=\;& D^b\left(\sigma\widehat{F}_{ba} - 8\sigma^2 T_{ba} + \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}\gamma_{ba}\psi\right) - \tfrac{1}{8}\varepsilon_{abcde}\widehat{F}^{bc}\widehat{F}^{de}\,,\\ N \;=\;& -\tfrac{1}{4}\widehat{F}_{ab}\widehat{F}^{ab} - \tfrac{1}{2}\bar{\psi}\not{D}\psi + \sigma\Box^c\sigma + \tfrac{1}{2}D^a\sigma D_a\sigma + Y^{ij}Y_{ij}\\ &+\mathrm{i}\,\bar{\psi}\gamma\cdot T\psi - 16\,\mathrm{i}\,\sigma\bar{\psi}\chi - \tfrac{104}{3}\sigma^2 T_{ab}T^{ab} + 8\sigma\widehat{F}_{ab}T^{ab} - 4\sigma^2 D\,,\end{aligned} \tag{5.113}$$

where we have used the expression for the conformal D'Alembertian given in (5.96). The supercovariant equations of motion and Bianchi identity are then given by

$$0 = L^{ij} = \varphi^i = E_a = N = G_{abc}\,. \tag{5.114}$$

5.5.3 Solving the equations of motion

In six dimensions, the equations of motion for an on-shell tensor multiplet coupled to the Standard Weyl multiplet can be used to eliminate the matter fields of the latter in terms of the matter fields of the Dilaton Weyl multiplet.

Precisely the same happens here. First of all, the equations of motion for Y^{ij} can be used to eliminate this auxiliary field. Next, the equations of motion for ψ^i and σ can be used to solve for the fields χ^i and D, respectively. The expressions for these fields exactly coincide with the ones we found in (5.95).

The solution for the matter field T_{ab} in terms of the fields of the Dilaton Weyl multiplet is more subtle. It requires that we first reinterpret the equation of motion for the vector field as the Bianchi identity for a two-form antisymmetric tensor gauge field $B_{\mu\nu}$. To be precise, we rewrite $E_a = 0$ from (5.113) as a Bianchi identity

$$D_{[a}\widehat{H}_{bcd]} \;=\; \tfrac{3}{4}\widehat{F}_{[ab}\widehat{F}_{cd]}\,, \tag{5.115}$$

where the three-form curvature $\widehat{H}_{abc}$ is defined by

$$-\tfrac{1}{6}\varepsilon_{abcde}\widehat{H}^{edc} = 8\sigma^2 T_{ab} - \sigma\widehat{F}_{ab} - \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}\gamma_{ab}\psi\,. \tag{5.116}$$

Note that the latter equation is just a rewriting of the relation (5.95).

The Bianchi identity (5.115) can be solved in terms of an antisymmetric two-form gauge field $B_{\mu\nu}$. The superconformal algebra (5.87) imposes that such a field transforms under supersymmetry as follows:

$$\delta_Q B_{\mu\nu} = \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}\gamma_{\mu\nu}\psi + \tfrac{1}{2}\sigma^2\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} + A_{[\mu}\delta(\epsilon)A_{\nu]}\,. \tag{5.117}$$

In addition, one finds that the field $B_{\mu\nu}$ transforms under a $\mathrm{U}(1)$ and a vector gauge transformation as follows

$$\delta B_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]} - \tfrac{1}{2}\Lambda F_{\mu\nu}\,. \tag{5.118}$$

Furthermore, the commutator of two Q-transformations picks up a vector gauge transformation δ_B for the field $B_{\mu\nu}$:

$$\begin{aligned} &[\delta(\epsilon_1),\delta(\epsilon_2)] = \ldots + \delta_{\mathrm{U}(1)}\left(\Lambda_3\right) + \delta_B\left(\Lambda_{3\mu}\right)\,, \\ &\Lambda_3 = -\tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}_2\epsilon_1\,, \qquad \Lambda_{3\mu} = -\tfrac{1}{4}\sigma^2\bar{\epsilon}_2\gamma_\mu\epsilon_1 - \tfrac{1}{2}A_\mu\Lambda_3\,. \end{aligned} \tag{5.119}$$

From the transformation rules (5.118) for $B_{\mu\nu}$ it follows that the supercovariant field strength $\widehat{H}_{\mu\nu\rho}$ is given by

$$\widehat{H}_{\mu\nu\rho} \;=\; 3\partial_{[\mu}B_{\nu\rho]} - \tfrac{3}{2}\,\mathrm{i}\,\sigma\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\psi - \tfrac{3}{4}\sigma^2\bar{\psi}_{[\mu}\gamma_\nu\psi_{\rho]} + \tfrac{3}{2}A_{[\mu}F_{\nu\rho]}\,. \tag{5.120}$$

This field strength indeed satisfies the Bianchi identity (5.115).

We conclude that the connection between the Standard and Dilaton Weyl multiplets can be obtained by first coupling an improved vector multiplet to the Standard Weyl multiplet and, next, solving the equations of motion. To solve the equation of motion for the vector field in terms of the matter field T_{ab}, one must first reinterpret this equation of motion as the Bianchi identity for an antisymmetric two-form gauge field.

Dilaton Weyl multiplet. This result is similar to what was found for the $(1,0)\,, D=6$ Weyl multiplet [176].

In [169], the field content and transformation rules for the Standard Weyl multiplet were constructed from the $\mathrm{F}(4)$-gauged six-dimensional supergravity [177] using the AdS_6/CFT_5 correspondence. Another attempt was undertaken in [178] by reducing the six-dimensional result [176] to five dimensions. However, by gauge-fixing some symmetries of the superconformal algebra during the reduction process, they found a multiplet that is larger than the Weyl multiplet that we will construct.

We will start this chapter by giving an introduction to the algebraic structure of rigid conformal (super)symmetry. In section 5.2, we will discuss local conformal supersymmetry and the gauging of the superconformal algebra. In section 5.3, we will construct the supercurrent as well as the improved supercurrent in order to determine the field content and the linearized transformation rules of the two Weyl multiplets. The linearized results will be used in section 5.4 to construct the fully non-linear transformation rules of the two Weyl multiplets as well as the modified superconformal algebra. Finally, in section 5.5, we will clarify the connection between the Weyl multiplets by showing that the coupling of an off-shell vector multiplet to the Standard Weyl multiplet gives rise to the Dilaton Weyl multiplet.

This chapter is based on the work published in [16]. A similar paper with overlapping results appeared somewhat later [179]. For a more extensive background on conformal supergravity, we refer to the reviews [180, 181].

5.1 Rigid superconformal symmetry

In this section, we will start with deriving the rigid superconformal transformations. After that, we will clarify the algebraic structure of the superconformal transformations by giving the (anti-)commutation relations of the superconformal algebra. Finally, we discuss some aspects of the corresponding representation theory.

5.1.1 Conformal Killing vectors

We will first introduce conformal symmetry, and in a second step we will extend this to conformal supersymmetry. Given a spacetime with a metric tensor $g_{\mu\nu}(x)$, the conformal transformations are defined as the class of general coordinate transformations that leaves "angles" invariant. The parameters of these coordinate transformations define a conformal Killing vector $k^\mu(x)$

$$\delta_{\rm gct} x^\mu = -k^\mu(x)\,. \tag{5.1}$$

The defining equation for this conformal Killing vector is given by

$$\delta_{\rm gct}(k) g_{\mu\nu}(x) \equiv \nabla_\mu k_\nu(x) + \nabla_\nu k_\mu(x) = \omega(x) g_{\mu\nu}(x)\,, \tag{5.2}$$