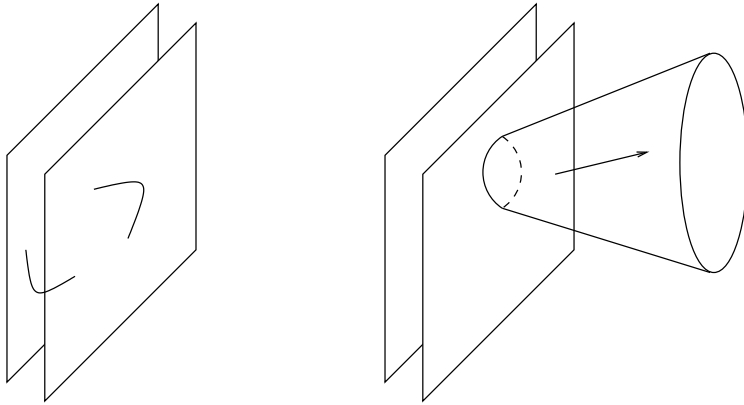# Chapter 2

The AdS/CFT correspondence

In chapter 1, we have seen that the search for a theory of Quantum Gravity has led to string theory, and that string theory has a rich structure and many exotic features, some of which do not appear in quantum field theories. Nevertheless, as we have argued in the introduction, there are many connections between theories of gravity and field theories. In the last few decades of the previous century, more, seemingly unrelated, conjectures and discoveries were made in this direction, which we will briefly summarize now.

In the nineteen seventies, it was noted by 't Hooft [68] that gauge theories like QCD behave as string theories when the gauge group becomes large: the Feynman diagram series becomes dominated by planar diagrams. Such diagrams are in a one-to-one correspondence with two-dimensional surfaces, a feature characteristic of string theories. However, the precise description of such a string theory in terms of a worldsheet action was never found.

On grounds of entropy considerations, it was argued by 't Hooft [69] and by Susskind [70] that any gravitational theory in a spacetime with length scales of the order of the Planck scale should be described by a quantum field theory living on the boundary of that spacetime. This idea is called the holographic principle; the gravitational theory is said to be holographically dual to the quantum field theory. Again, specific examples proved to be hard to find.

In many ways, the discovery of D-branes was a breakthrough for string theory. D-branes provide non-perturbative solutions to the theory. They also couple naturally to both open strings, which have gauge fields in their spectrum; and to closed strings, which have gravitons as vibration modes. We have displayed these two aspects in figure 2.1. This complementary nature of D-branes makes for a powerful framework for calculating black hole entropies [71, 72].

The connections between gauge theory and gravity described above led Maldacena to his conjecture [73] of the AdS/CFT correspondence. Inspired by the properties of D3-branes, he conjectured that Type IIB string theory on an Anti-de-Sitter (AdS) spacetime is holographically dual to a conformal field theory (CFT), namely $\mathcal{N} = 4$ supersymmetric, large N,

Figure 2.1: D-branes as open string boundary conditions and closed string sources.

$SU(N)$ Yang-Mills theory in four dimensions. Specific proposals for quantitatively checking this conjecture were soon put forward [74, 75].

In this chapter, we will start with describing the basic arguments leading to the Maldacena conjecture. After a description of some properties of Anti-de-Sitter spacetime, we will indicate how one arrives at a scheme for computing correlation functions for both the gauge theory and gravity. We will finish with a summary of the enormous body of evidence that has accumulated over the years. For more details, we refer to the Physics Report [76] and to the more elementary reviews [77–80].

2.1   The D3-brane

In this section, we will look in more detail into aspects of the D3-brane. We will start with describing the interaction between a spacetime supergravity theory and a worldvolume gauge theory. We will then take two particular limits of the system and argue that these limits are equivalent. This is the reasoning that led Maldacena to his conjecture of the AdS/CFT correspondence.

2.1.1   Interacting theories

The D3-brane is a four-dimensional BPS-solution of Type IIB string theory preserving half of the 32 supersymmetries. The form of the solution is obtained by taking $p = 3$ in the expression for the general Dp-brane solution (1.61) and implementing the self-duality constraint

(1.29)

$$
\text{D3-brane} = \begin{cases}
\mathrm{d}s^2 &= H^{-\frac{1}{2}}\mathrm{d}x_{(4)}^2 + H^{\frac{1}{2}}\left(\mathrm{d}y^2 + y^2 \mathrm{d}\Omega_{(5)}^2\right), \\
e^{\Phi} &= g_s, \\
G_{(5)} &= \mathrm{d}^4 x \wedge \mathrm{d}H^{-1} + \star \mathrm{d}^4 x \wedge \mathrm{d}H^{-1}, \\
H(r) &= 1 + \left(\frac{R}{y}\right)^4.
\end{cases}
\tag{2.1}
$$

The constant R can be determined from (1.69)

$$
R^4 = 4\pi g_s \ell_s^4. \tag{2.2}
$$

The action describing (2.1) is given by the combined system

$$
S_{\mathrm{D3}} = \frac{1}{2\kappa^2} \int_M \mathrm{d}^{10}x\, \mathcal{L}_{(10,\,3)}^{\mathrm{S}} + \int_\Sigma \mathrm{d}^4\sigma\, \mathcal{L}_{\mathrm{D3}}^{\mathrm{S}} + S_{\mathrm{int}}. \tag{2.3}
$$

For completeness, we have also indicated the action S_{int} which describes the interactions between the worldvolume and target space actions; it contains the higher-derivative and higher order α' corrections to the worldvolume and target-space actions. Hence, one can view S_{int} as the action parameterizing all the string-theory corrections to the supergravity plus worldvolume action approximation.

The target space action is a truncation of the Type IIB supergravity pseudo-action in the string frame (1.27) to the metric $G_{\mu\nu}$, the dilaton ϕ, and the self-dual RR-field $G_{(5)}$

$$
\mathcal{L}_{(10,\,3)}^{\mathrm{S}} = e^{-2\phi}\left(R \star \mathbb{1} - \frac{1}{2}\star \mathrm{d}\phi \wedge \mathrm{d}\phi\right) - \frac{1}{4}\star G_{(5)} \wedge G_{(5)}. \tag{2.4}
$$

The worldvolume theory is given by a Dirac-Born-Infeld (DBI) action [22]

$$
\mathcal{L}_{\mathrm{D3}}^{\mathrm{S}} = -T_{\mathrm{D3}}\, e^{-\phi} \sqrt{\det(g_{ab}^{\mathrm{S}} + 2\pi\alpha' F_{ab})} + \dots. \tag{2.5}
$$

This DBI-theory can be seen as a non-linear generalization of electromagnetism: F_{ab} is the field-strength of a vector field living on the worldvolume of the D3-brane. The complete set of degrees of freedom describing the fluctuations around the static solution (2.1) also contains spinors and scalars; they can be seen as Goldstone modes corresponding to the broken ten-dimensional supersymmetry and translational symmetry. For simplicity, we have not included them in (2.5), and we have also omitted the Wess-Zumino terms and higher order corrections.

The solution describing N overlapping D3-branes is also given by (2.1), but where the constant R in this case is given by

$$
R^4 = 4\pi g_s N \ell_s^4. \tag{2.6}
$$

For the DBI-action (2.5), no such generalization to $N > 1$ is known, but instead one has to expand (2.5) as a series in α', and generalize each term individually[1]. The lowest order terms are given by supersymmetric $\mathrm{SU}(N)$ Yang-Mills theory.

---

[1]For recent progress in finding higher-order terms in this expansion, see for instance [81].

The target space action (2.4) in the Einstein frame is given by

$$\mathcal{L}^{\mathrm{E}}_{(10,\,3)} = R \star \mathbb{1} - \frac{1}{2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{4} \star G_{(5)} \wedge G_{(5)} \,. \tag{2.7}$$

Comparing this with (1.81) and (1.85), we see that the Einstein frame, the sigma-model frame, and the dual frame all coincide for the D3-brane . Since the dilaton vanishes for the D3-brane, the form of the D3-brane solution is the same in any frame. When we will study more general p-branes, we will see that the dual frame is the preferred frame for studying the geometry.

2.1.2 Decoupling limits

In the solution (2.1), we can take the near-horizon limit

$$\frac{y}{R} \to 0 \,. \tag{2.8}$$

The metric then takes on the form

$$\begin{aligned}
\mathrm{d}s^2 &= \left(\frac{y}{R}\right)^2 \mathrm{d}x^2_{(4)} + \left(\frac{R}{y}\right)^2 \mathrm{d}y^2 + R^2 \mathrm{d}\Omega^2_{(5)} \tag{2.9} \\
&\equiv AdS_5(R) \times S^5(R) \,. \tag{2.10}
\end{aligned}$$

This geometry is a five-dimensional Anti-de-Sitter spacetime times a five-dimensional sphere. In section 2.2, we will be more detailed about the geometry of these spaces. On the other hand, if we look at the asymptotic geometry by taking

$$\frac{y}{R} \to \infty \,, \tag{2.11}$$

then the harmonic function becomes constant. The metric therefore describes Minkowski space $\mathbb{R}^{1,9}$. We have sketched the D3-brane geometry[2] in figure 2.2. In particular, the flat ten-dimensional asymptotic limit is separated from the near-horizon region by an infinitely long "throat".

Both geometries are believed to be exact vacua of string theory, which solve the full equations of motion of string theory to all orders in α'. Moreover, even though the complete D3-brane solution breaks half the supersymmetry, both the near-horizon limit (2.8) and the asymptotic limit (2.11) preserves all 32 supersymmetries of the Type IIB supergravity action [82–84]. This can be seen from taking either the near-horizon limit (2.8) or the asymptotic (2.11) directly in the supersymmetry variations (1.75): in both cases one finds that the supersymmetry variations vanish identically and that the projection condition (1.76) is not needed anymore. Hence, we can view the D3-brane as a string theory soliton that interpolates between two string theory vacua with unbroken supersymmetry.

---

[2]We have suppressed several extra dimensions, the figure only attempts to indicate the separation into two regions.
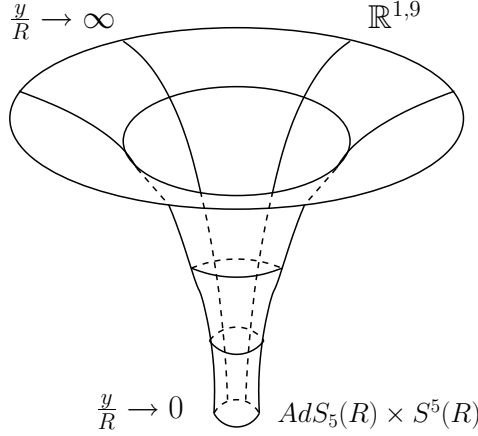
Figure 2.2: The interpolating D3-brane geometry.

The gravitational dynamics in the presence of a stack of N D3-branes separates into two regimes. Far away from the branes, the dynamics is given in terms of fluctuations around flat Minkowski spacetime, but near the branes, the dynamics is given in terms of fluctuations around an Anti-de-Sitter spacetime times sphere geometry. These two regions decouple, since a physical process of energy E_{emitted} near the brane is observed with an infinitely red-shifted energy E_{observed} far away from the brane

$$E_{\text{observed}} = \frac{\sqrt{g_{00}}\big|_{\frac{y}{R} \to 0}}{\sqrt{g_{00}}\big|_{\frac{y}{R} \to \infty}} E_{\text{emitted}} \,. \tag{2.12}$$

At the level of the actions, there is also a limit in which the near-brane and asymptotic regions decouple, namely the low-energy limit

$$\frac{E}{E_s} \to 0 \,, \quad E_s = \frac{\hbar c}{\ell_s} \,. \tag{2.13}$$

Since the massive modes of strings have energies in the order of E_s, the low-energy limit is obtained by considering processes which involve only the massless modes, which is the supergravity approximation to superstring theory.

The effect of the low-energy limit on the action (2.3) is that the interaction part of the action S_{int} becomes negligible. Moreover, the DBI-action can be approximated by a $\mathrm{SU}(N)$ Yang-Mills theory

$$\mathcal{L}_{\text{D3}}^{\text{S}} = -\frac{1}{4g_{\text{YM}}^2} \mathrm{Tr}\, F_{ab}F^{ab} + \dots \,. \tag{2.14}$$

N D3-branes   D3-brane

W_μ

y
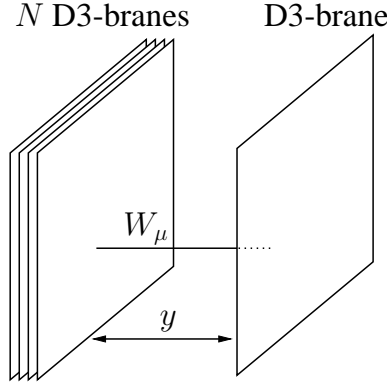
Figure 2.3: A stack of D3-branes probed by another D3-brane.

The Yang-Mills coupling constant can be obtained from the expression for the effective D3-brane tension and expanding the action (2.5)

$$g_{\mathrm{YM}}^2 = 2\pi g_s \,. \tag{2.15}$$

In the low-energy limit (2.13), the action (2.3) describes two decoupled systems. Far away from the branes, it is given by the fluctuations of Type IIB string theory around Minkowski spacetime, but the fluctuations are governed by a supersymmetric $\mathrm{SU}(N)$ Yang-Mills theory near the branes. By calculating the absorption cross-sections of scalar fields by the D3-branes, it was shown that the two systems indeed decouple in the low-energy limit [85, 86].

2.1.3 The Maldacena conjecture

From both the solution and the action perspective, the dynamics far away from the branes coincides and is given by Type IIB string theory in a Minkowski spacetime. However, near the branes, there are two different descriptions: a supersymmetric $\mathrm{SU}(N)$ gauge theory and Type IIB string theory around an Anti-de-Sitter spacetime times a sphere, respectively

To relate these two descriptions, it is useful to connect the near-horizon limit with the low-energy limit. Below the string scale E_s, a natural energy scale is given by the energy of an open string stretched between a stack of N D3-branes and a single D3-brane probe. We have indicated this setup in figure 2.3. Such a string behaves as a W-boson in the Yang-Mills theory on the D3-brane worldvolume, and its energy is given by

$$E_{\mathrm{W}} \equiv U = \frac{y}{\ell_s^2} \,. \tag{2.16}$$

If we keep g_s and N fixed and substitute the W-boson energy (2.16) into the low-energy limit (2.13), we obtain the near-horizon limit (2.8). We can then write the near-horizon metric
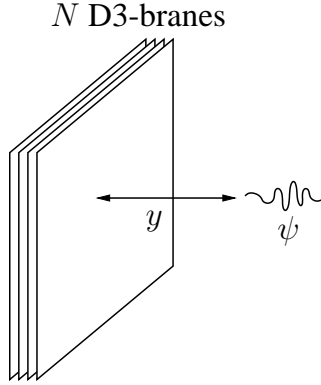
Figure 2.4: A stack of D3-brane probed by a supergravity field ψ.

(2.9) in terms of this energy scale

$$\frac{\mathrm{d}s_{\mathrm{E}}^2}{\ell_s^2} = \left(\frac{U}{(4\pi gsN)^{\frac{1}{4}}}\right)^2 \mathrm{d}x_{(4)}^2 + \left(\frac{(4\pi g_s N)^{\frac{1}{4}}}{U}\right)^2 \mathrm{d}U^2 + (4\pi g_s N)^{\frac{1}{2}} \mathrm{d}\Omega_{(5)}^2 \,. \qquad (2.17)$$

Another natural energy scale can be obtained by considering a supergravity field[3] ψ probing a stack of N D3-branes. We have indicated this setup in figure 2.4. From an analysis of the wave-equation for ψ in the background described by the metric (2.17), it was shown in [87] that this field has the characteristic energy

$$E_\psi \equiv u = \frac{y}{R^2} \,. \qquad (2.18)$$

Such a relation where the energy of a gauge theory is proportional to a distance scale in gravity is called a UV/IR-relation [87] since large energies (UV) in one theory map to low energies (IR) in the other, and vice versa.

This energy scale is a holographic energy: for a certain class of black holes, it can be shown [88] that this energy gives the same entropy as can be deduced from the holographic principle [69, 70]. The near-horizon metric (2.9) in the so-called holographic coordinates is

$$\frac{\mathrm{d}s_{\mathrm{E}}^2}{R^2} = u^2 \mathrm{d}x_{(4)}^2 + \frac{\mathrm{d}u^2}{u} + \mathrm{d}\Omega_{(5)}^2 \,. \qquad (2.19)$$

We have seen that the near-horizon limit of the D3-brane geometry corresponds to the low-energy limit in the action describing this D3-brane solution. Moreover, far away from the brane, the system describes Type IIB string theory around flat Minkowski spacetime.

---

[3]We consider an s-wave: the field ψ has no angular momentum related to the sphere S^5.

This led Maldacena to his conjecture [73] that also the near-brane descriptions should be equivalent. In other words, Type IIB string theory around $AdS_5 \times S^5$ should be equivalent to the $\mathrm{SU}(N)$ Yang-Mills theory on the four-dimensional worldvolume of the D3-branes.

This is not as absurd as it sounds. First of all, the precise worldvolume theory is $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This is a special theory: its beta-function vanishes to all orders, meaning that it is a superconformal field theory [89]. The extra superconformal symmetries correspond to the supersymmetry enhancement found in the near-horizon limit. The superconformal group in four dimensions is $\mathrm{SU}(2,2|4)$. Its bosonic subgroup is the conformal group $\mathrm{SO}(2,4)$ times the $\mathrm{SU}(4)$ R-symmetry group. These groups are isomorphic to the $\mathrm{SO}(2,4)$ isometry group of AdS_5 and to the $\mathrm{SO}(6)$ isometry group of S^5.

Many more kinematic properties of both theories are in a one-to-one correspondence [76]. We saw in chapter 1 that Type IIB string theory has an $\mathrm{S}\ell(2,\mathbb{Z})$-duality symmetry; $\mathcal{N} = 4$ Yang-Mills theory also has such a duality. It is known as Montonen-Olive duality [90] in which the θ-parameter of the gauge theory is mixed with the gauge coupling constant

$$\tau \equiv \frac{\theta}{2\pi} + \frac{2\pi\,\mathrm{i}}{g_{\mathrm{YM}}^2}\,. \tag{2.20}$$

Furthermore, we can define the 't Hooft coupling constant

$$\lambda = 2g_{\mathrm{YM}}^2 N\,, \tag{2.21}$$

after which the size of the Anti-de-Sitter spacetime and the sphere becomes

$$\left(\frac{R}{\ell_s}\right)^4 = \lambda\,. \tag{2.22}$$

The string coupling constant can be expressed in terms of λ and N

$$g_s = \frac{\lambda}{4\pi N}\,. \tag{2.23}$$

The ratio of the two energy scales is also given by the 't Hooft coupling constant

$$\frac{U}{u} = \lambda^{\frac{1}{2}}\,. \tag{2.24}$$

Many computations in field theory can only be done in perturbation theory, where the dimensionless coupling constant is small. Similarly, string theory on curved spacetimes such as an AdS times sphere geometry is rather complicated, especially at the quantum level [91]. There are three regimes of the parameters N and λ for which one side of the correspondence becomes computationally feasible, which we have displayed in table 2.1.

From the above, we see that gravity and gauge theory are valid in different regimes, and if this were the whole story, the conjectured duality would be hard to verify. However,

Regime	Gravity	Gauge theory
Perturbative gauge theory	$\frac{R}{\ell_s} \ll 1$	$\lambda \ll 1$
Classical string theory	$e^{\Phi} = g_s \ll 1$	$\frac{\lambda}{N} = g_{\mathrm{YM}}^2 \ll 1$
Supergravity	$\frac{R}{\ell_s} \gg 1$	$\lambda \gg 1$

Table 2.1: Regimes of the AdS/CFT correspondence.

since $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is a conformal field theory, some correlation functions are independent of the coupling.

The Maldacena conjecture relates large N and large λ Yang-Mills theory in four dimensions to classical supergravity on a five-dimensional Anti-de-Sitter spacetime times a sphere. Such a correspondence is an example of both holography and of the string-like behavior of large N gauge theory, since the boundary of Anti-de-Sitter spacetime is Minkowski spacetime. In the following sections, we will make this more precise.

2.2 Anti-de-Sitter spacetime

In this section, we will discuss some elementary geometrical aspects of Anti-de-Sitter spacetime: we will derive several forms of its metric from an embedding equation, we will show that it solves Einstein's equations with a negative cosmological constant, and we will show that it has a projective boundary given by Minkowski spacetime. For more details, we refer to [92].

2.2.1 Embedding and metric

The $(d+1)$-dimensional Anti-Sitter spacetime[4] AdS_{d+1} may be realized as the hypersurface

$$AdS_{d+1}: \quad -X_0^2 - X_{d+1}^2 + X_1^2 + \ldots + X_d^2 = -L^2 \,, \tag{2.25}$$

in flat $\mathbb{R}^{2,d}$, where L is a parameter with dimensions of length called the Anti-de-Sitter radius. The minus sign on the right-hand side of (2.25) is essential: it ensures that AdS_{d+1} is a spacetime of negative curvature. There are two closely related spacetimes: namely the $(d+1)$-dimensional version of the sphere S^{d+1}, and of the de Sitter spacetime dS_{d+1}; they have the embeddings

$$\begin{aligned} S^{d+1}: & & X_0^2 + \ldots + X_{d+1}^2 &= L^2 \,, \\ dS_{d+1}: & & -X_0^2 + X_1^2 + \ldots + X_{d+1}^2 &= L^2 \,. \end{aligned} \tag{2.26}$$

---

[4]A subscript rather than a superscript denoting the dimension is conventional.

$AdS_{d+1} \simeq S^1 \times \mathbb{R}^d$ $\qquad\qquad\qquad\qquad$ $dS_{d+1} \simeq \mathbb{R}^1 \times S^d$
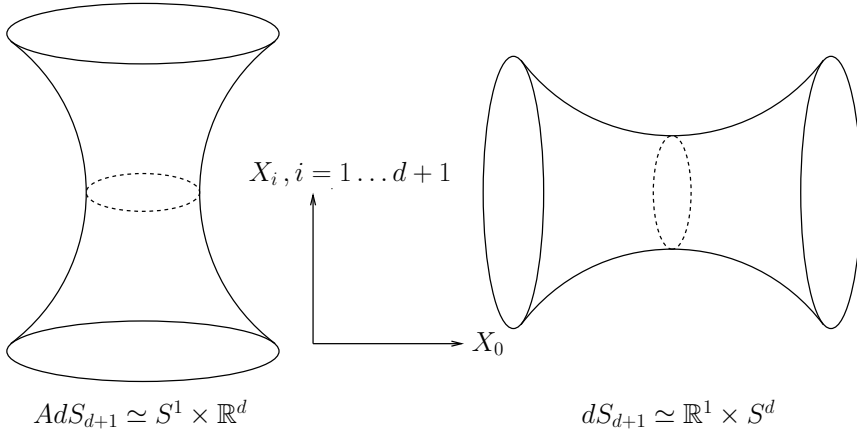
Figure 2.5: AdS_{d+1} and dS_{d+1} as hyperboloids in $\mathbb{R}^{2,d}$.

Both these spacetimes have positive curvatures, as can be inferred from the right-hand side of (2.26). The relative sign between the X_0^2 and L^2 terms determines if the spacetime has closed timelike curves. For Anti-de-Sitter spacetime, this is indeed the case: AdS_{d+1} has topology $S^1 \times \mathbb{R}^d$; de Sitter spacetime has no such closed timelike curves, and the topology is $\mathbb{R}^1 \times S^d$. In figure 2.5, we have schematically indicated the difference between the hyperbolic embeddings of AdS_{d+1} and dS_{d+1}.

The metric of AdS_{d+1} can be written as

$$ \mathrm{d}s^2 = -\mathrm{d}X_0^2 - \mathrm{d}X_{d+1}^2 + \mathrm{d}X_1^2 + \ldots + \mathrm{d}X_d^2 \,, \tag{2.27} $$

which is manifestly invariant under $SO(2,d)$. Coordinate systems covering the entire hyperboloid (2.25) exactly once have a periodic timelike coordinate: in order to obtain a causal spacetime it is necessary to go to the universal covering space by unwrapping the timelike coordinate. Whenever we refer to AdS_{d+1} in the remainder of this thesis, we mean this universal covering space.

A convenient coordinate system which solves the embedding equation (2.25) is given by so-called horospherical coordinates

$$ X^\mu = \left(\frac{U}{L}\right) x^\mu \,, \quad X^\pm = \frac{-1}{\sqrt{2}}\left(X^d \pm X^{d+1}\right) \; : \quad X^- = U \,, \quad X^+ = \frac{X^\mu X_\mu + L^2}{X^-} \,. \tag{2.28} $$

Calculating the differentials of (2.28), substituting them into the line element (2.27), and using the embedding equation (2.25), one obtains the induced metric

$$ \mathrm{d}s^2 = \left(\frac{U}{L}\right)^2 \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \left(\frac{L}{U}\right)^2 \mathrm{d}U^2 \,. \tag{2.29} $$

Different forms of the metric are used to emphasize different aspects of Anti-de-Sitter space: a form of the metric convenient for studying correlation functions is [75]

$$ds^2 = \frac{\eta_{\mu\nu}dx^\mu dx^\nu + dz^2}{z^2}, \quad z = \frac{L}{U}. \tag{2.30}$$

We will frequently use another form of the metric where we take

$$ds^2 = e^{-2r/L}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2, \quad e^{-r/L} = \frac{U}{L}. \tag{2.31}$$

This particular form of the metric is known as the Poincaré coordinate-system. In this case, we can let the radial coordinate U take on only positive values: this means that the Poincaré-coordinates cover only half of the hyperboloid (2.25).

2.2.2 Curvature and cosmological constant

The physical significance of Anti-de-Sitter spacetime lies in the fact that it is a vacuum solution to the gravitational field equations with a negative cosmological constant. In $d+1$ dimensions and in the absence of matter, the Einstein-Hilbert action with a cosmological constant Λ is given by

$$S_{d+1} = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{|g|} \left(R - 2\Lambda\right). \tag{2.32}$$

The field equations that follow from the action (2.32) are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{2.33}$$

Taking the trace of this equation yields

$$\Lambda = \frac{d-1}{2(d+1)}R, \tag{2.34}$$

from which we deduce that the curvature scalar has the same sign as the cosmological constant. Substituting (2.34) back into (2.33) yields

$$R_{\mu\nu} = \frac{2\Lambda}{d-1}g_{\mu\nu}. \tag{2.35}$$

Spaces for which the Ricci tensor is proportional to the metric are called Einstein spaces. The particular class of Einstein spaces called maximally symmetric spaces satisfies a stronger constraint

$$R_{\mu\nu\lambda\rho} = \frac{2\Lambda}{d(d-1)} \left(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}\right). \tag{2.36}$$

Taking the trace of (2.36) shows that any maximally symmetric space is indeed an Einstein space.

We will now show that AdS_{d+1} is such a maximally symmetric space, and indeed a solution to (2.33). We start with writing a slightly more general Ansatz than (2.31)

$$\mathrm{d}s^2 = e^{2A(r)}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}r^2\,, \tag{2.37}$$

after which we rewrite this metric in terms of vielbeins

$$\mathrm{d}s^2 = g_{AB}\mathrm{d}x^A\mathrm{d}x^B\,, \quad g_{AB} = \eta_{ab}e_A{}^a e_B{}^b\,. \tag{2.38}$$

A convenient way of doing calculations in General Relativity is by working with differential forms: we introduce vielbein 1-forms

$$e^a = e_A{}^a\mathrm{d}x^A \rightarrow \begin{cases} e^m &=& e^{A(r)}\mathrm{d}x^m\,, \quad m = 0,\ldots d-1\,, \\ e^d &=& \mathrm{d}r\,, \end{cases} \tag{2.39}$$

and from the Cartan structure equations we obtain the spin-connection 1-form and the curvature 2-form

$$\begin{array}{rcl} \mathrm{d}e^a + \omega^a{}_b \wedge e^b &=& 0 \\ \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b &=& R^a{}_b \end{array} \rightarrow \begin{cases} \omega^m{}_d &=& A'(r)e^m\,, \\ R^m{}_n &=& -A'(r)^2 e^m \wedge e^n\,, \\ R^m{}_d &=& -\left[A''(r) + A'(r)^2\right]e^m \wedge e^d\,. \end{cases} \tag{2.40}$$

We can now read off the components of the Riemann tensor in the vielbein basis, and transform it into the standard form

$$\begin{array}{rcl} R^a{}_b &=& \tfrac{1}{2}R^a{}_{bcd}e^c \wedge e^d \\ R_{ABCD} &=& e_{Aa}R^a{}_{bcd}e_B{}^b e_C{}^c e_D{}^d \end{array} \rightarrow \begin{cases} R_{\mu\nu\mu\nu} &=& -A'(r)^2 g_{\mu\mu}g_{\nu\nu}\,, \\ R_{\mu r\mu r} &=& -\left[A''(r) + A'(r)^2\right]g_{\mu\mu}g_{rr}\,. \end{cases} \tag{2.41}$$

For future reference, we also give the Ricci tensor and the Ricci scalar

$$\begin{array}{rcl} R_{AB} &=& g^{CD}R_{CADB} \\ R &=& g^{AB}R_{AB} \end{array} \rightarrow \begin{cases} R_{\mu\mu} &=& -\left[A''(r) + dA'(r)^2\right]g_{\mu\mu}\,, \\ R_{rr} &=& -d\left[A''(r) + A'(r)^2\right]g_{rr}\,, \\ R &=& -2dA''(r) - d(d+1)A'(r)^2\,. \end{cases} \tag{2.42}$$

Comparing (2.31) with our Ansatz (2.37) we see that if we take

$$A(r) = \pm\frac{r}{L} \rightarrow A'(r) = \pm\frac{1}{L}\,, \quad A''(r) = 0\,, \tag{2.43}$$

then we regain Anti-de-Sitter spacetime in Poincaré coordinates. The choice of sign is arbitrary: comparing with the Poincaré-coordinates (2.31), we choose the minus sign here. Substituting (2.43) into the curvature expressions (2.41) and (2.42), and comparing this with (2.36), we see that AdS_{d+1} is indeed a maximally symmetric space with a negative cosmological constant given by

$$\Lambda = -\frac{d(d-1)}{2L^2}\,. \tag{2.44}$$
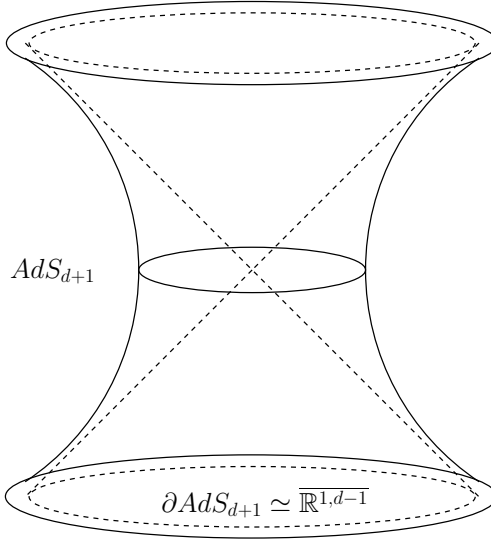
Figure 2.6: The projective boundary of Anti-de-Sitter spacetime.

2.2.3 Boundary and conformal structure

Rescaling the embedding coordinates of AdS_{d+1} by a large factor

$$X_i \to X_i' = tX_i\,, \quad t \gg 1\,, \tag{2.45}$$

changes the hyperbolic embedding equation (2.25) to

$$-X_0^2 - X_{d+1}^2 + \left(X_0^2 + \ldots + X_d^2\right) = 0\,. \tag{2.46}$$

The scaled embedding equation (2.46) describes a cone lying inside the Anti-de-Sitter space. Since all coordinates have been scaled to large values, any additional scalings have no further effect; this is expressed by the equivalence relation

$$X_i \simeq \lambda X_i\,. \tag{2.47}$$

The cone-embedding (2.46) modulo the scale equivalence relation (2.47) describes the two projective boundaries[5] of AdS_{d+1}, which are topologically equivalent to conformally compactified Minkowski space $\overline{\mathbb{R}^{1,d-1}}$. Minkowski space is conformally compactified by adding a point at infinity, analogously to how the Riemann sphere S^2 is obtained from the complex plane $\mathbb{C}$. We have indicated the projective boundary of AdS_{d+1} in figure 2.6.

---

[5]In order to consider only a single boundary, one should also mod out by a $\mathbb{Z}_2$-symmetry [75].

The $SO(2,d)$ isometry group of AdS_{d+1} is linearly realized on the coordinates of the embedding space as

$$\tfrac{1}{2}(d+2)(d+1) \quad \text{rotations} \quad : \quad X^i \to \Lambda^i{}_j X^j \,. \tag{2.48}$$

On the projective boundary $\overline{\mathbb{R}^{1,d-1}}$, the isometry group splits up into

$$
\begin{array}{llll}
\tfrac{1}{2}d(d-1) & \text{Lorentz transformations} & : & x^\mu \to \Lambda^\mu{}_\nu x^\nu \,, \\
d & \text{translations} & : & x^\mu \to x^\mu + a^\mu \,, \\
1 & \text{dilatation} & : & x^\mu \to \lambda x^\mu \,, \\
d & \text{conformal transformations} & : & \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + k^\mu \,,
\end{array}
\tag{2.49}
$$

which means that the isometry group of AdS_{d+1} acts as the conformal group on its boundary $\overline{\mathbb{R}^{1,d-1}}$. One therefore expects that if a $SO(2,d)$ invariant gravitational theory in AdS_{d+1} is to have any holographically dual description at all, then this dual theory should be given in terms of a conformal field theory on the projective boundary $\overline{\mathbb{R}^{1,d-1}}$. Moreover, since the boundary corresponds to large radial coordinate, or large energy U in the language of the previous section, the holographic dual is a conformal field theory in the UV limit.

Instead of giving the precise connection between the coordinate transformations (2.48) and (2.49), we will give the connection between the generators of the AdS_{d+1} isometry group and the d-dimensional conformal group

$$
M_{ij} = \begin{pmatrix}
M_{\mu\nu} & \tfrac{1}{4}\left(P_\mu - K_\mu\right) & \tfrac{1}{4}\left(P_\mu + K_\mu\right) \\
-\tfrac{1}{4}\left(P_\mu - K_\mu\right) & 0 & -\tfrac{1}{2}D \\
-\tfrac{1}{4}\left(P_\mu + K_\mu\right) & \tfrac{1}{2}D & 0
\end{pmatrix} \,.
\tag{2.50}
$$

The commutation relations of M_{ij} are given by

$$\left[M_{ij}, M^{kl}\right] = -2\delta_{[i}{}^{[k}M_{j]}{}^{l]} \,. \tag{2.51}$$

In chapter 5, we will come back to the algebraic structure of the conformal group and its supersymmetric extension. In particular, we will give the the commutation relations that result when one substitutes (2.50) into (2.51).

2.3 Conformal field theory

In this section, we will make more precise how the AdS/CFT correspondence is realized. We will analyze a toy model example and show that it has many of the qualitative features of more realistic models. After that, we will describe the various approximations that have to be made in practice. We will finish with a brief summary of the evidence in favor of the AdS/CFT correspondence.

2.3.1 A toy model example

In order to clarify how a conformal field theory can give a holographically dual description to a gravitational theory, we will consider a toy model example of a $d+1$-dimensional scalar field and its potential coupled to gravity

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^{d+1}x \sqrt{|g|} \left(R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right) . \tag{2.52}$$

The equations of motion for the scalar and the metric in this model are given by

$$
\begin{aligned}
\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi \right) &= \frac{\partial V}{\partial \phi} , \\
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \left(\frac{1}{4}(\partial\phi)^2 + \frac{1}{2} V(\phi) \right) g_{\mu\nu} .
\end{aligned}
\tag{2.53}
$$

For generic values of the scalar field, the equations of motion are complicated to solve: some simplifications occur if we look for perturbations around certain critical points of the potential

$$\varphi = \phi - \phi_c , \quad \frac{\partial V}{\partial \phi}\big|_{\phi=\phi_c} = 0 , \quad V(\phi_c) < 0 , \tag{2.54}$$

and introduce new parameters Λ and M^2

$$\Lambda = \frac{1}{2} V(\phi_c) , \quad M^2 = \frac{\partial^2 V}{\partial \phi^2}\big|_{\phi=\phi_c} . \tag{2.55}$$

The equations of motion then take on the form

$$\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \varphi \right) = M^2 \varphi , \tag{2.56}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 . \tag{2.57}$$

So, we see that each critical point of the scalar potential in (2.54) corresponds to an Anti-de-Sitter spacetime with a negative cosmological constant given in terms of the value of the potential; the masses of the fluctuations around such critical points are given in terms of the Hessian matrix of the potential[6]. We can associate a length scale L to the Anti-de-Sitter spacetime and express the mass in units of this length scale

$$\Lambda = -\frac{d(d-1)}{2L^2} , \quad \frac{m}{L} = M . \tag{2.58}$$

Since we expect that the dynamics of a gravitational theory in Anti-de-Sitter spacetime is described by a conformal field theory on its boundary, we are particularly interested in the

---

[6]A slightly negative m^2 does not imply instability, as long as the bound $m^2 \geq -\frac{d^2}{4}$ is satisfied [93] .

boundary conditions of the scalar field. If we take the metric of the form (2.30) then scaling arguments determine the behavior of solutions of the wave equation (2.56) near the boundary $z = 0$

$$\varphi(\vec{x}, z) = z^{d - \Delta^+} \varphi_+(\vec{x}) + z^{d - \Delta^-} \varphi_-(\vec{x}), \quad \Delta^\pm = \frac{d}{2} \pm \frac{\sqrt{d^2 + 4m^2}}{2}. \tag{2.59}$$

In general, only one of the two solutions will give a finite-energy solution[7]: in particular, if the mass does not saturate the Breitenlohner-Freedman bound [93], we can only select Δ^+

$$m^2 > 1 - \frac{d^2}{4}, \quad \Delta \equiv \Delta^+, \quad \varphi_0(\vec{x}) \equiv \varphi_+(\vec{x}). \tag{2.60}$$

The bound (2.60) corresponds to the unitarity bound on scaling dimensions of operators in a conformal field theory

$$\Delta \geq \frac{d - 2}{2}. \tag{2.61}$$

Standard Green's functions techniques then determine the complete solution to the equations of motion (2.56) in terms of the bulk-to-boundary propagator $K_\Delta(z, \vec{x}, \vec{x}')$

$$\varphi(\vec{x}, z) = \int \mathrm{d}^d x' K_\Delta(z, \vec{x}, \vec{x}') \varphi_0(\vec{x}'), \quad K_\Delta(z, \vec{x}, \vec{x}') \simeq \frac{z^\Delta}{(z^2 + |\vec{x} - \vec{x}'|^2)^\Delta}. \tag{2.62}$$

Substituting the solution (2.62) into the Euclidean version of the action (2.52) and performing the z-integral yields the on-shell action up to a constant factor

$$S[\varphi_0] \simeq \int \mathrm{d}\vec{x} \int \mathrm{d}\vec{x}' \frac{\varphi_0(\vec{x}) \varphi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2\Delta}}. \tag{2.63}$$

If we now view this action as a functional of the boundary data and differentiate its exponential with respect to the scalar fields

$$\frac{\delta^2}{\delta \varphi_0(\vec{x}) \delta \varphi_0(\vec{x}')} e^{-S[\varphi_0]} \quad \simeq \quad \frac{1}{|\vec{x} - \vec{x}'|^{2\Delta}} \tag{2.64}$$

$$\equiv \quad \langle \mathcal{O}_\Delta(\vec{x}) \mathcal{O}_\Delta(\vec{x}') \rangle_{\mathrm{CFT}}, \tag{2.65}$$

then we observe that we have obtained the two-point correlation function for a conformal field theory operator $\mathcal{O}_\Delta$ of scaling dimension Δ. Analogous formulae exist for the higher-point correlation functions. One can view the scalar field $\phi_0(\vec{x})$ as a source term or generalized coupling to the operator $\mathcal{O}_\Delta$

$$e^{-S[\varphi_0]} = \left\langle e^{\int \mathrm{d}^d \vec{x} \varphi_0(\vec{x}) \mathcal{O}_\Delta(\vec{x})} \right\rangle_{\mathrm{CFT}}. \tag{2.66}$$

---

[7] For $-\frac{d^2}{4} \leq m^2 \leq 1 - \frac{d^2}{4}$, both Δ^+ and Δ^- are admissible [94].
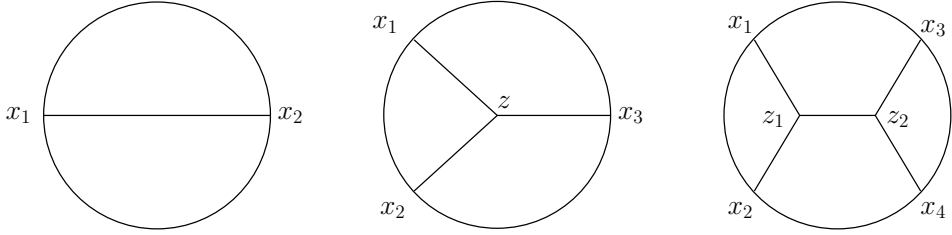
Figure 2.7: Witten diagrams of 2-, 3- and 4-point correlation functions.

The above equation can be made more precise: in particular, the precise regularizations which need to be performed on both sides of (2.66) and the connection with the conformal anomaly were discussed in [95].

There is also a diagrammatic way of displaying the equations involved, as we have indicated in figure 2.7. The points in the Witten diagram labeled x_i are positioned at the boundary of AdS, whereas the points denoted by z_i are located in the bulk of the Anti-de-Sitter space-time. To each vertex one assigns a propagator for the corresponding field, and one integrates the bulk coordinates z_i over the entire Anti-de-Sitter spacetime, which can be quite complicated in practice [96].

2.3.2 Approximations of the correspondence

In its strongest form, the AdS/CFT correspondence relates the partition function of a gravitational theory on a manifold $\mathcal{M}$ to the partition function for a conformal field theory on the boundary $\partial\mathcal{M}$

$$\mathcal{Z}_{\text{gravity}}(\mathcal{M}) = \mathcal{Z}_{\text{CFT}}(\partial\mathcal{M}), \qquad (2.67)$$

the canonical example being the equivalence of IIB string theory on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in $3 + 1$-dimensions

$$\mathcal{Z}_{\text{IIB}}(AdS_5 \times S^5) = \mathcal{Z}_{\text{SYM}}(\overline{\mathbb{R}^{1,3}}). \qquad (2.68)$$

Since string theory on Anti-de-Sitter spaces is not well enough understood even at the classical level, a weaker but more manageable form of the correspondence is to approximate the full quantum string theory on $AdS_5 \times S^5$ in (2.68) with its effective classical supergravity action which translates in the field theory to the regime of large gauge group and large 't Hooft coupling

$$e^{-S_{\text{IIB}}(AdS_5 \times S^5)} = \mathcal{Z}_{\text{SYM}}(\overline{\mathbb{R}^{1,3}}), \quad N, \lambda \gg 1. \qquad (2.69)$$

Since the classical supergravity computations have to be compared with strong coupling results for the field theory, and since only for conformal field theory such calculations can be

performed, the AdS/CFT correspondence has not yet been applied to theories without super-conformal symmetry such as pure QCD in four dimensions.

Combining the original ten-dimensional equations of motion [39] with the complete Kaluza-Klein mass-spectrum of Type IIB supergravity on S^5 [97, 98] would in principle give the complete dynamics for fluctuations around the $AdS_5 \times S^5$ background. Since the expansion in spherical harmonics on S^5 is quite complicated, one would like to eliminate the higher Kaluza-Klein modes.

Since the radius of the sphere is proportional to the Anti-de-Sitter radius, a normal Kaluza-Klein reduction (i.e. taking the radius to zero) will not solve this problem: instead one needs to make a consistent truncation to the zero-modes. However, finding the correct reduction Ansatz is already complicated at the linearized level, and the non-linear interactions are even more daunting: they have been worked out in only a few sectors of the theory [99].

On the other hand, there is a known complete non-linear five-dimensional supergravity theory, the $SO(6)$ gauged $\mathcal{N} = 8$ theory of [100, 101], which has the same graviton multiplet as Type IIB supergravity and is invariant under the same superalgebra. It is widely believed, but nevertheless still unproven, that this $D = 5$, $\mathcal{N} = 8$ theory is a consistent truncation of Type IIB theory on $AdS_5 \times S^5$, meaning that any classical solution of the five-dimensional theory can be lifted to ten dimensions.

For practical calculations, the form of the AdS/CFT correspondence is therefore

$$e^{-S_{D=5}^{\mathcal{N}=8}(AdS_5)} = \mathcal{Z}_{\text{SYM}}(\overline{\mathbb{R}^{1,3}}), \quad N, \lambda \gg 1, \tag{2.70}$$

which is similar to the relation (2.66) for our toy model example (2.52) if we keep in mind that every fluctuating field on the Anti-de-Sitter side appears as a term in the Lagrangian on the conformal field theory side as a coupling to some composite conformal operator.

2.3.3 Evidence for the AdS/CFT correspondence

We can summarize the kinematic evidence for the AdS/CFT correspondence with the dictionary given in table 2.2.

There is also a large body of dynamical evidence in favor of the AdS/CFT correspondence. Soon after Maldacena's conjecture [73] and the concrete proposals [74, 75] for calculating amplitudes, a whole class of correlators was calculated [102] from the supergravity side. In most cases, perfect agreement with the known field theory results was found. In other cases, computations from the supergravity point of view yielded new and unexpected non-renormalization theorems for certain classes of field theory correlators [103, 104].

Many other calculations have been performed: instanton corrections to perturbative results [105, 106], relations between Wilson loops in gauge theory [107] and minimal surfaces in string theory [108], and thermal properties of black holes [109] in relation with the field theory free energy [110]. In chapter 3, we will discuss some of the generalizations of the AdS/CFT correspondence.

Concept	Gravity	Gauge theory
UV/IR	length y	energy $U = \frac{y}{\ell_s^2}$
Decoupling	near-horizon	low-energy
Regime	curvature radius $\frac{R}{\ell_s}$	't Hooft coupling λ
Coupling constant	g_s	g_{YM}^2
Stringy corrections	$\mathcal{O}(\alpha')$	$\mathcal{O}(\frac{1}{\lambda})$
Quantum corrections	$\mathcal{O}(g_s)$	$\mathcal{O}(\frac{1}{N})$
Isometry/symmetry	$SO(2,4) \times SO(6)$	$SU(2,2\vert4)$
$S\ell(2,\mathbb{Z})$-duality	$\tau = C_{(0)} + \frac{i}{g_s}$	$\tau = \frac{\theta}{2\pi} + \frac{2\pi i}{g_{\mathrm{YM}}^2}$
Scalar	field $\varphi(\vec{x}, z)$	coupling $\varphi_0(\vec{x})$
Dimension	mass m	scaling Δ
Bound	$m^2 \geq 1 - \frac{d^2}{4}$	$\Delta \geq \frac{(d-2)}{2}$

Table 2.2: A gravity/gauge theory dictionary.

We conclude by remarking that it is fair to say that the AdS/CFT correspondence is no longer a mere conjecture, but that it is a firmly established gravity/gauge theory correspondence. For more details, we refer to the review [76].