Geometry of strings and branes

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Chapter 1

The string theory framework

In this chapter, we will give an overview of the string theory framework. We will start with describing several basic features of string theory, after which we we will discuss some aspects of supergravity, the low-energy effective description of string theory. In the last two sections of this chapter, we will review some recent developments in string theory: in particular, we will discuss string theory dualities and brane solutions of supergravity.

1.1 String theory

String theory was born out of attempts to explain the hadron resonance spectrum of the strong interaction. Soon after the discovery of the Veneziano scattering amplitude [18], which expressed a duality between resonances coming from the so-called s-channel and t-channel, it was realized that this amplitude described the dynamics of an open relativistic string.

Open strings have in their spectrum a massless spin-1 particle, which is reminiscent of a gauge field. However, after new experimental results were shown to be in conflict with the Veneziano amplitude, string theory as a model for the strong interaction was replaced by the gauge theory QCD. Relativistic closed strings, however, have in their spectrum a massless spin-2 particle, which corresponds precisely to the characteristic properties of a graviton. It was then argued that closed string theory could be a theory of gravity [19].

We will first explain in some detail the geometrical and dynamical setup of classical bosonic string theory. After that, we will be less detailed as we discuss the quantized and the supersymmetric versions of string theory, since most of the research described in this thesis has been performed at the level of supergravity. For more details and proper references, see the classic textbooks of [20, 21], and the more modern approach of [22, 23]. We will finish this section with a discussion of the Kaluza-Klein mechanism, and a description of interacting strings in non-trivial backgrounds.

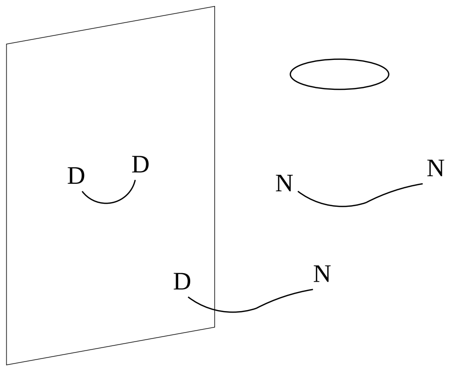

Figure 1.2: The periodic, Neumann, and Dirichlet boundary conditions for strings.

The equation of motion following from the action (1.2) is nothing else than the two-dimensional wave equation for the embedding coordinates

$$\frac{\partial}{\partial\sigma^-}\frac{\partial}{\partial\sigma^+}X^\mu(\tau,\sigma) = 0\,, \quad \sigma^\pm \equiv \tau \pm \sigma\,. \tag{1.4}$$

As can be seen from the particular form in which we have written the wave equation, there are two independent directions along which vibrations of the string can propagate, usually called left and right.

To be able to solve the equations of motion, one has to supplement them by suitable boundary conditions, as we have indicated in figure 1.2. For the closed string of length ℓ_s, one has to impose periodic boundary conditions

$$X^\mu(\tau,0) = X^\mu(\tau,\ell_s)\,, \tag{1.5}$$

but for open strings there are two different possibilities, depending on how the right-moving vibrations turn into left-moving modes at the endpoints

$$\frac{\partial}{\partial\sigma^-}X^\mu(\tau,\sigma) = \pm\frac{\partial}{\partial\sigma^+}X^\mu(\tau,\sigma)\,, \quad \sigma = 0,\ell_s\,. \tag{1.6}$$

The choice of the plus sign goes under the name of Neumann boundary conditions and corresponds to freely moving open strings. The case of the minus sign is known as Dirichlet boundary conditions, where the endpoints of the strings are actually fixed at some hyperplanes in spacetime. We will later see that these hyperplanes correspond to solitonic objects called Dirichlet-branes, or D-branes [24] for short.

The final result is that the coordinates X^μ are given as linear superpositions of all possible solutions of (1.4), subject to the boundary conditions (1.5), or (1.6). Each of the elementary vibration modes of the string worldsheet corresponds to a particle in spacetime. In particular, the energies and the polarizations of the vibration modes are related to the masses and spins of the corresponding elementary particles.

1.1.2 Quantization and superstrings

The string action (1.2) has many symmetries, including reparametrizations and rescalings of the two-dimensional worldsheet, both of which are essential in order to solve the equations of motion in full generality. The symmetry group of the worldsheet is actually infinite-dimensional [25] and goes under the name of the conformal or the Virasoro algebra.

If one tries to quantize the oscillators on the string worldsheet while retaining the conformal structure, then the string can no longer move in spacetimes of arbitrary dimension, but instead the spacetime in which the string propagates is restricted to be 26-dimensional. At first sight, this seems to rule out string theory as a realistic description of four-dimensional Quantum Gravity, but we will see in section 1.1.3 how the Kaluza-Klein mechanism solves this apparent contradiction.

There is a more severe problem with the quantized bosonic string: namely the oscillator with the lowest energy actually has an imaginary mass, meaning that it is a tachyon. The appearance of tachyons in field theory usually means that one is expanding around the wrong vacuum and that by redefining the vacuum the tachyon will disappear. Recent developments [26] indicate that this may also be the case in string theory, but a complete understanding of this will require sophisticated string field theory methods [27].

Another problem of bosonic string theory is the absence of fermions in its spectrum: if string theory is to provide a unification scheme of elementary particles and all their interactions, then one would like to have matter included as well. A modification of bosonic string theory, called superstring theory, addresses both the fermion and the tachyon problem.

There are two different approaches to superstring theory. In the Neveu-Schwarz-Ramond formulation [28, 29], one adds worldsheet fermions to the action (1.2). These world-sheet fermions have to satisfy appropriate boundary conditions. This divides the oscillators into two classes: a Neveu-Schwarz or NS-sector and a Ramond or R-sector. On the other hand, the Green-Schwarz formulation [30] starts from a spacetime supersymmetric action. These two, a priori different, formulations turn out to be equivalent in the sense that they give the same answers for scattering amplitudes.

Supersymmetry already restricts the dimension in which classical superstrings can live to $3, 4, 6$ or 10 dimensions, but in order to obtain quantum mechanical consistency, the spacetime in which superstrings move has to be ten-dimensional. The quantization of the NSR-formulation can be done in a manifestly covariant manner, but spacetime supersymmetry can only be obtained by performing the so-called GSO-projection [31] that eliminates the tachyon from the spectrum. In the GS-formulation, spacetime supersymmetry is manifest from the outset but covariance is lost and one has to resort to light-cone gauge quantization.

There are five consistent superstring theories. The first two are called Type IIA and Type IIB superstring theory. They are both theories of closed strings only, and they posses what is technically known as $\mathcal{N} = 2$ supersymmetry: the difference being that Type IIA has two spinors of opposite chirality, called $(1,1)$ supersymmetry; whereas Type IIB is a chiral theory with $(2,0)$ supersymmetry.

Then there are three theories with $\mathcal{N} = 1$ supersymmetry. First there is the Type I superstring theory of open strings, which also has a closed string sector. This theory has massless gauge fields in its spectrum that transform under the gauge group $\mathrm{SO}(32)$. Finally, there are two Heterotic string theories [32]: these are rather exotic theories of closed strings in which the right-moving and left-moving modes on the world-sheet are taken to be different. These Heterotic theories also have a gauge symmetry, and the gauge group can be $\mathrm{E}_8 \times \mathrm{E}_8$ or $\mathrm{SO}(32)$ in this case.

All these five superstring theories were shown to be free of anomalies and to give consistent quantum mechanical scattering amplitudes [33].

1.1.3 Dimensional reduction

We saw how bosonic strings and superstrings had to move in spacetimes of dimensions 26 or 10. This problem does not have to be fatal per se, since the extra dimensions of spacetime can be taken care of by a well-defined mathematical procedure called Kaluza-Klein compactification. We will illustrate this mechanism with the toy model example of a massless two-dimensional scalar field satisfying the wave equation

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right)\phi(t,x) = 0\,. \tag{1.7}$$

We now take the x-direction to be a circle of radius R, and since the scalar field has to periodic in the compact direction, we can Fourier expand the scalar field in this compact direction

$$\phi(t,x) = \phi(t, x + 2\pi n R) \rightarrow \phi(t,x) = \sum_n \phi_n(t) e^{\pi i n x/R}\,. \tag{1.8}$$

If we substitute the expansion (1.8) into the equation of motion (1.7), then we find

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi_n(t) = m^2\phi_n(t)\,, \quad m^2 = \frac{(\pi n)^2}{R^2}\,, \tag{1.9}$$

from which one observes that each Fourier-mode describes a massive particle in the remaining non-compact spacetime with a mass that is inversely proportional to the radius.

Taking the limit $R \rightarrow 0$, we see that the zero-mode decouples from all the other modes, since these become infinitely massive. The result is that, after dimensional reduction over a circle of infinitesimal radius, a massless two-dimensional scalar field is effectively described by a massless scalar field in one dimension. On the other hand, taking the limit $R \rightarrow \infty$ makes the spectrum in (1.9) continuous, and we will regain the uncompactified two-dimensional theory. In string theory, the limits $R \rightarrow 0$ and $R \rightarrow \infty$ are equivalent to each other, as we will see in section 1.3.1 when we discuss T-duality.

The analog of (1.7) in supergravity is a set of ten-dimensional tensor fields satisfying non-linear differential equations in a spacetime forming a product of four-dimensional Minkowski spacetime times a compact six-dimensional manifold. After a Fourier-expansion of the tensor fields in eigenfunctions of the differential operator on the compact manifold, the higher Fourier-modes will decouple in the limit $R \to 0$, and the higher-dimensional fields are described by a set of lower-dimensional tensor fields[1].

A closely related mechanism to Kaluza-Klein reduction is called spontaneous compactification: this occurs when the zero-modes on the compact manifold do not appear as sources in the equations of motion for the higher Fourier-modes. In this rather special case, one can consistently truncate these higher modes to zero. What this means is that the solutions of the compactified lower-dimensional theory formed by the zero-modes are also solutions of the original, uncompactified, higher-dimensional theory.

For the two-dimensional scalar field example, it is consistent to truncate to the zero-mode $\phi_0(t)$, since it satisfies not only the reduced equation of motion (1.9) but also the original equation of motion (1.7). At the level of the linearized equations of motion or for compactifications on manifolds as simple as tori or group manifolds, the consistency of such truncations is always guaranteed. But for compactification on more complicated manifolds such as spheres, the zero-modes generically appear in the equations of motion of the higher Fourier-modes, and a consistent truncation is generically no longer possible: the higher Fourier-modes only decouple in the limit $R \to 0$.

1.1.4 Backgrounds and interactions

So far, we have discussed superstrings propagating in flat ten-dimensional spacetimes, and we argued that since closed strings had massless spin-2 particles in their spectrum that string theory could be a theory of gravity. General Relativity tells us that the geometry of spacetime should actually be a dynamical variable, fixed by the equations of motion. We will now discuss how to generalize the action (1.2) to strings moving in more complicated backgrounds.

In addition to a massless symmetric traceless tensor $G_{\mu\nu}$, closed superstrings have two more massless modes[2]: an anti-symmetric tensor $B_{\mu\nu}$ and a massive scalar Φ called the dilaton. The tensor $G_{\mu\nu}$ will be identified with the spacetime metric, which in (1.2) was given by the flat Minkowski spacetime metric $\eta_{\mu\nu}$. We will denote the metric on the string worldsheet Σ by γ_{ab}.

The tensor $B_{\mu\nu}$ can be interpreted as a generalized gauge field: analogously to how particles can be charged under vector fields, higher-dimensional objects such as strings can be charged under higher-rank tensor fields. Finally, the scalar Φ will couple to the string worldsheet through its curvature $R(\gamma)$. The generalization of the action (1.2) is given by a two-

[1]Tensor-components in compact directions behave as tensors of lower rank in the remaining dimensions.

[2]We will not discuss the massless fermions or Ramond-Ramond gauge fields in this section.

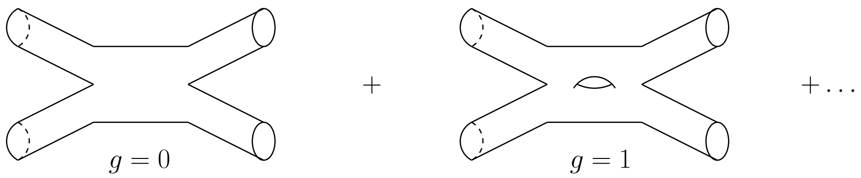

Figure 1.3: The genus expansion of string theory interactions.

dimensional non-linear sigma-model with a ten-dimensional target space

$$S = -\frac{T}{2}\int_{\Sigma} \mathrm{d}^2\sigma\,\sqrt{\gamma}\,\left[\left(\gamma^{ab}G_{\mu\nu}(X) + \varepsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu - \alpha'\Phi(X)R(\gamma)\right]\,. \quad (1.10)$$

The discussion so far only involved freely propagating strings. Scattering amplitudes for interacting particles can conveniently be calculated by the technique of Feynman diagrams in which there is a one-to-one map from a graph to a contribution to an amplitude. In string theory, the analog of this is given in terms of Riemann surfaces as we have indicated in figure 1.3.

The most convenient way to obtain scattering amplitudes is through the same path-integral methods that are used to quantize the free strings [34, 35]. In particular, the partition function corresponding to the action (1.10) is given by a series expansion over Riemann surfaces of genus g

$$\mathcal{Z}_{\text{string}} = \sum_{g=0}^{\infty}\int \mathcal{D}\gamma_{(g)}\mathcal{D}X\, e^{-S[\gamma_{(g)},X]}\,. \quad (1.11)$$

Even though the specific contribution of a Riemann surface to a string theory scattering amplitude is harder to calculate [36] than a corresponding Feynman diagram in field theory, the number of diagrams at any given genus is exactly one, whereas in field theory the number of diagrams per loop grows rapidly. The high-energy behavior of string theory scattering amplitudes is also a lot better: this is intuitively clear from the observation that vertices in Feynman diagrams are singular whereas Riemann surfaces are smooth everywhere.

In quantum electrodynamics, there is a dimensionless constant α that can be formed out of the dimensionful parameters $e^2, \hbar$, and c. The partition function can be calculated as a series expansion in Feynman diagrams with L loops: after assigning every vertex a factor α, this becomes a series expansion in α

$$\alpha = \frac{e^2}{\hbar c}\,, \quad \mathcal{Z}_{\text{QED}} = \sum_{L=0}^{\infty}\alpha^{2(L-1)}\mathcal{Z}_L\,. \quad (1.12)$$

In the expression (1.11) for the string theory partition function, we did not write any dimensionless parameter. However, string theory has as dimensionful parameters the gravitational coupling κ and the string length ℓ_s from which it is possible to form a dimensionless parameter g_s. One should therefore expect that the genus-expansion will become a series expansion in g_s

$$g_s^2 = \frac{4\pi\kappa^2}{(2\pi\ell_s)^{D-2}} \,, \qquad \mathcal{Z}_{\text{string}} = \sum_{g=0}^{\infty} g_s^{2(g-1)} \mathcal{Z}_g \,. \tag{1.13}$$

It would be disappointing if reconciling gravity with quantum mechanics involved the introduction of a new fundamental dimensionless parameter. Happily, this is not the case. What comes to rescue is that the power of g_s in (1.13) is a topological quantity: it is minus the Euler number χ of the corresponding Riemann surface. But the Euler number of a two-dimensional surface Σ is also related to an integral over its curvature through the Gauss-Bonnet theorem

$$\chi = \frac{1}{4\pi} \int_\Sigma \mathrm{d}^2\sigma \, \sqrt{\gamma}\, R(\gamma) \,, \tag{1.14}$$

which is precisely the coupling to Φ in the action (1.10). This implies that we can define the string coupling to be the expectation value of the dilaton exponential

$$g_s = \left\langle e^{\Phi} \right\rangle \,. \tag{1.15}$$

Instead of being a one-parameter family of theories labeled by a fundamental dimensionless parameter g_s, string theory is a single theory with a one-parameter family of vacuum states labeled by the expectation value of the dilaton exponential.

1.2 Supergravity

Historically, four-dimensional supergravity [10] was discovered as a gauge theory of supersymmetry, a procedure that we shall mimic for conformal supergravity and conformal supersymmetry in chapter 5. In this section, we will emphasize a different viewpoint: namely we will show that supergravity is the low-energy effective description of string theory. For each superstring theory mentioned in the previous section, we will give its supergravity action. We will also make some remarks about eleven-dimensional supergravity. For more details and an accurate historical account, we refer to [37, 38].

1.2.1 Low-energy effective actions

As we mentioned before, the worldsheet for a string propagating in a flat spacetime has an infinite-dimensional symmetry group including a two-dimensional scaling symmetry which allowed for the complete solution to the equations of motion as well as a consistent quantization.

For strings propagating in non-trivial backgrounds, this is no longer guaranteed: the action (1.10) describes a string as a non-linear sigma-model in which the spacetime fields appear as dimensionful coupling-constants on the string worldsheet. This means that the essential scale-symmetry will be broken in general.

In order to obtain a consistent description, the coupling constants should not transform under the scale-symmetry: in other words, the beta-functions of the corresponding coupling constants should vanish. These beta-functions can be calculated as a perturbation series in α' for which the lowest-order approximation yields

$$\begin{aligned} \beta(G_{\mu\nu}) &= R_{\mu\nu} + 4\partial_\mu\Phi\partial_\nu\Phi - \tfrac{1}{4}H_\mu{}^{\lambda\rho}H_{\nu\lambda\rho} + \mathcal{O}(\alpha')\,, \\ \beta(B_{\mu\nu}) &= \nabla^\lambda\left(e^{-2\Phi}H_{\mu\nu\lambda}\right) + \mathcal{O}(\alpha')\,, \\ \beta(\Phi) &= 4\nabla^\mu\partial_\mu\Phi - 4\partial^\mu\Phi\partial_\mu\Phi + R - \tfrac{1}{12}H^{\mu\nu\lambda}H_{\mu\nu\lambda} + \mathcal{O}(\alpha')\,, \end{aligned} \tag{1.16}$$

where we have defined the field-strength of the gauge field by

$$H_{\mu\nu\lambda} = 3\partial_{[\mu}B_{\nu\lambda]}\,. \tag{1.17}$$

So, we see that demanding quantum mechanical consistency through the vanishing of the beta-functions (1.16) gives constraints on the massless modes. These constraints can be interpreted as equations of motion, since they are equivalent to the Euler-Lagrange equations for the familiar Einstein action of General Relativity in the presence of a generalized gauge field and a scalar field

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{d}^D x\,\sqrt{|G|}\,e^{-2\Phi}\left(R + 4\,(\partial\Phi)^2 - \frac{1}{12}H^2\right)\,. \tag{1.18}$$

The constant κ_0 is not fixed by the equations of motion, and in order to relate it to the gravitational coupling κ, we redefine the dilaton in such a way that is has a vanishing expectation value

$$e^\phi \equiv \frac{e^\Phi}{g_s}\,. \tag{1.19}$$

The action then takes on the form

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\,\sqrt{|G|}\,e^{-2\phi}\left(R + 4\,(\partial\phi)^2 - \frac{1}{12}H^2\right)\,. \tag{1.20}$$

where the gravitational coupling is now *defined* using (1.13)

$$\frac{1}{2\kappa^2} \equiv \frac{2\pi}{g_s^2(2\pi\ell_s)^{D-2}}\,. \tag{1.21}$$

The force coming from the dilaton exchange breaks the equivalence principle of General Relativity: free-falling frames are no longer equivalent to the absence of gravity. In particular, the beta-functions (1.16) are derived in a frame called the string frame:

$$g^{\mathrm{S}}_{\mu\nu} \equiv G_{\mu\nu}\,. \tag{1.22}$$

It is often convenient to scale the metric in such a way that the curvature term in the action has no dilatonic pre-factor. This metric is called the Einstein frame: it is related to the string frame by the transformation

$$g^{\mathrm{E}}_{\mu\nu} = e^{-\frac{4}{D-2}\phi} g^{\mathrm{S}}_{\mu\nu} \, . \tag{1.23}$$

In this frame the action (1.20) takes on the form

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{|g^{\mathrm{E}}|} \left(R - \frac{4}{D-2} (\partial\phi)^2 - \frac{1}{12} e^{-\phi} H^2 \right) . \tag{1.24}$$

To avoid cluttering actions with a lot of constants, we will often put factors of α' and g_s equal to unity: the former can always be restored by dimensional analysis; and for each term, we can find the correct factor of g_s in any metric frame from the relative scaling between that frame and the string frame, the factor of e^{Φ} for the corresponding term in the string frame, and the use of (1.19).

Any closed string theory with massless modes $G_{\mu\nu}$, $B_{\mu\nu}$, and ϕ as described by the action (1.10) has the spacetime action (1.20) as its low-energy description. We will now review how the various massless modes of the different versions of superstring theory give rise to different modifications of the action (1.20).

1.2.2 $\mathcal{N} = 2$ supergravities

The Type II superstrings each have their own version of supergravity describing their massless modes. Since closed strings have both left and right-moving modes, there are in total four different sectors for the massless modes, depending on the boundary conditions. We will only look at the bosonic sectors of the supergravity actions, since we will need their structures in chapters 2 and 3. This corresponds to keeping the massless modes of the NSNS and RR-sectors of the superstrings.

The massless modes of the NSNS-sector of all the superstrings are given by the familiar metric $g_{\mu\nu}$, the anti-symmetric tensor field $B_{\mu\nu}$, and the dilaton ϕ. To simplify the structure of the supergravity actions, we will use differential form notation in this section. In this language the tensor $B_{\mu\nu}$ is written as the two-form $B_{(2)}$, and the volume element $\sqrt{|g|}$ as $\star\mathbb{1}$. For more details on our notation, see appendix A. The RR-sector of Type IIA string theory consists of a set of two gauge potentials $\{C_{(1)}, C_{(3)}\}$. The bosonic part of the Type IIA supergravity action is given by

$$\begin{aligned} \mathcal{L}_{\mathrm{IIA}} \;=\; & e^{-2\phi}\left(R \star \mathbb{1} + 4 \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} \star H_{(3)} \wedge H_{(3)} \right) - \frac{1}{2} \star G_{(2)} \wedge G_{(2)} \\ & - \frac{1}{2} \star G_{(4)} \wedge G_{(4)} + \frac{1}{2} B_{(2)} \wedge \mathrm{d}C_{(3)} \wedge \mathrm{d}C_{(3)} \, , \end{aligned} \tag{1.25}$$

where the field-strengths of the various gauge potentials are defined as

$$H_{(3)} = \mathrm{d}B_{(2)} \, , \quad G_{(2)} = \mathrm{d}C_{(1)} \, , \quad G_{(4)} = \mathrm{d}C_{(3)} - H_{(3)} \wedge C_{(1)} \, . \tag{1.26}$$

For the Type IIB superstring one finds in the RR-sector a set of three gauge potentials $\{C_{(0)}, C_{(2)}, C_{(4)}\}$ which appear in the Type IIB supergravity action according to

$$\begin{aligned}\mathcal{L}_{\rm IIB} &= e^{-2\phi}\left(R\star\mathbb{1}+4\star{\rm d}\phi\wedge{\rm d}\phi-\frac{1}{2}\star H_{(3)}\wedge H_{(3)}\right)-\frac{1}{2}\star G_{(1)}\wedge G_{(1)}\\ &\quad -\frac{1}{2}\star G_{(3)}\wedge G_{(3)}-\frac{1}{4}\star G_{(5)}\wedge G_{(5)}-\frac{1}{2}C_{(4)}\wedge{\rm d}C_{(2)}\wedge{\rm d}B_{(2)}\,,\end{aligned} \tag{1.27}$$

where we have

$$H_{(3)}={\rm d}B_{(2)}\,,\quad G_{(1)}={\rm d}C_{(0)}\,,\quad G_{(3)}={\rm d}C_{(2)}-H_{(3)}\wedge C_{(0)}\,. \tag{1.28}$$

The five-form field-strength $G_{(5)}$ satisfies a self-duality condition

$$G_{(5)}={\rm d}C_{(4)}-\frac{1}{2}C_{(2)}\wedge{\rm d}B_{(2)}+\frac{1}{2}B_{(2)}\wedge{\rm d}C_{(2)}\,,\quad G_{(5)}\equiv\star G_{(5)}\,, \tag{1.29}$$

which does not follow from the equation of motion [39] but which has to be imposed as an extra constraint [40].

1.2.3 $\mathcal{N}=1$ supergravities

The $\mathcal{N}=1$ superstrings have $\mathcal{N}=1$ supergravities as their low-energy effective description. They share the NSNS-sector of the Type II strings, but none of the $\mathcal{N}=1$ superstrings have RR-gauge potentials, although they do have ordinary gauge fields $A^I_{(1)}$ for their respective $\mathrm{E}_8\times\mathrm{E}_8$ and $\mathrm{SO}(32)$ symmetry groups. For the Heterotic string theories, one has the following kinetic terms of the bosonic part of the supergravity actions

$$\mathcal{L}_{\rm Het}=e^{-2\phi}\left(R\star\mathbb{1}+4\star{\rm d}\phi\wedge{\rm d}\phi-\frac{1}{2}\star H_{(3)}\wedge H_{(3)}-\frac{1}{2}{\rm Tr}\,\star F_{(2)}\wedge F_{(2)}\right)\,, \tag{1.30}$$

where the trace is taken over all gauge group generators and where the field-strengths are given by

$$H_{(3)}={\rm d}B_{(2)}+\frac{1}{2}{\rm Tr}\,A_{(1)}\wedge{\rm dA}_{(1)}\,,\quad F_{(2)}={\rm d}A_{(1)}+A_{(1)}\wedge A_{(1)}\,. \tag{1.31}$$

The Type I superstring has the same field content as the Heterotic strings but a slightly different supergravity action

$$\mathcal{L}_{\rm I}=e^{-2\phi}\left(R\star\mathbb{1}+4\star{\rm d}\phi\wedge{\rm d}\phi\right)-\frac{1}{2}\star H_{(3)}\wedge H_{(3)}-\frac{1}{2}e^{-\phi}{\rm Tr}\,\star F_{(2)}\wedge F_{(2)}\,. \tag{1.32}$$

We have left out some terms in the actions (1.30) and (1.32): they are necessary to cancel the gauge and gravitational anomalies [33]; we refer to the literature for the complete expressions [21].

1.2.4 $D = 11$ supergravity

Quantum versions of superstring theory can only live in ten dimensions, and we have shown above that they all have a ten-dimensional low-energy limit. However, there also exists an eleven-dimensional supergravity theory [41]. The field content consists of a metric $g_{\mu\nu}$ and a three-form gauge potential $C_{(3)}$ described by the following bosonic Lagrangian

$$\mathcal{L}_{11} = R \star \mathbb{1} - \frac{1}{2} \star G_{(4)} \wedge G_{(4)} + \frac{1}{6} C_{(3)} \wedge G_{(4)} \wedge G_{(4)} \,, \tag{1.33}$$

where as usual

$$G_{(4)} = \mathrm{d}C_{(3)} \,. \tag{1.34}$$

For a long time, it was not clear what the meaning of this eleven-dimensional theory was, until it was discovered that if one generalizes the concept of superstrings to supermembranes, then the spacetimes in which such supermembranes can consistently move are precisely those satisfying the equations of motion of eleven-dimensional supergravity [42].

Attempts to quantize the supermembrane and to obtain its spectrum failed however, and it was shown that the supermembrane has a continuous spectrum with no discrete non-zero energy vibration modes [43]. However, the supermembrane and eleven-dimensional supergravity were a turning point in the development of string theory, since they provided many insights in the relationships between the different versions of string theory, as we will now discuss in the remainder of this chapter.

1.3 Dualities

The possibility of no less than five consistent superstring theories is an embarrassment of riches. In this section, we will sketch how all string theories are related to each other by a web of dualities.

1.3.1 T-duality

The first duality that we will discuss is a duality in which string theories compactified on circles of different radii are mapped into each other. This is possible since the embedding coordinates X^μ of a string are not ordinary scalar fields, satisfying periodic boundary conditions as in (1.8), but instead they can wrap around the compact dimension according to

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \ell_s) + 2\pi n R \,. \tag{1.35}$$

This means that the solution to the wave equation for the coordinates X^μ has momentum modes proportional to the inverse radius, but also winding terms proportional to the radius. This is also reflected in the mass formula

$$X^\mu_\pm(\tau, \sigma) \sim \left(\frac{m}{R} \pm wR \right) \sigma_\pm \,, \quad M^2 \sim \left(\frac{m^2}{R^2} + w^2 R^2 \right) \,. \tag{1.36}$$

The mass spectrum is symmetric under inversion of the radius with a simultaneous interchange of the momentum modes with the winding modes

$$R \leftrightarrow \frac{1}{R}\,, \quad m \leftrightarrow w\,: \quad M^2 \to M^2\,, \quad X^\mu_\pm(\tau,\sigma) \to \pm X^\mu_\pm(\tau,\sigma)\,. \tag{1.37}$$

For the coordinates, the effect is equivalent to a parity transformation on the right-moving modes. For the Type II superstrings this parity transformation changes the chirality of the spinors and the overall result is that the (1,1) supersymmetric Type IIA superstring theory is mapped into the (2,0) supersymmetric Type IIB superstring theory. This relation holds for any value of the radius: in particular it relates the limits $R \to 0$ and $R \to \infty$.

This has no counterpart in field theories, since particles cannot wind around a compact dimension. The effect of T-duality in supergravity is not that e.g. Type IIA supergravity and Type IIB supergravity are T-dual to each other in the sense of the Type II string theories, but rather that there is a discrete symmetry relating the two supergravity theories when both are reduced to nine dimensions over a circle of zero radius [44].

The effect of T-duality for the Heterotic superstrings is more difficult to explain, but it is related to the fact that one can see the right-moving modes as bosonic string theories compactified on sixteen-dimensional lattices, which are precisely the root lattices of the corresponding gauge groups. Under the map (1.37), the lattices of the $\mathrm{E}_8 \times \mathrm{E}_8$ and $\mathrm{SO}(32)$ are interchanged[3] making the Heterotic superstrings T-dual to each other [45].

T-duality in Type I string theory is even more astonishing, since the effect of a parity transformation on the right-moving modes interchanges the Neumann and Dirichlet boundary conditions, as can be seen from (1.6). As we argued before, and as we will show in the next section, the hyperplanes on which endpoints of open strings with Dirichlet boundary conditions end are actually solitonic solutions of string theory. The effect of T-duality on these D-branes is that it maps branes of different dimensions to each other.

The examples of T-duality which we have discussed here are only the tip of a mathematical iceberg: there are also dualities known as mirror-symmetries in which ten-dimensional string theories compactified on different six-dimensional spacetimes, known as Calabi-Yau manifolds, are related to each other. The perturbative expansions using sigma-model actions with mirror-related target spaces give the same quantum mechanical scattering amplitudes [46].

1.3.2 S-duality

String theory also possesses non-perturbative dualities in which the strong coupling regime of a string theory is related to the weak coupling regime of another theory. This class of dualities is called S-duality. These dualities are non-trivial to prove, but substantial evidence

[3] This involves breaking the gauge group in both theories to an $\mathrm{SO}(16) \times \mathrm{SO}(16)$ subgroup by turning on appropriate Wilson lines.

from string theory compactifications has been obtained. For a good review see [47]. We will now indicate how these dualities come about using the supergravity approximation.

If we transform the Heterotic $\mathrm{SO}(32)$ action (1.30) to the Einstein frame (1.23), then we find

$$\mathcal{L}^{\mathrm{E}}_{\mathrm{Het}} = R \star \mathbb{1} - \frac{1}{2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} e^{-\phi} \star H_{(3)} \wedge H_{(3)} - \frac{1}{2} e^{-\frac{1}{2}\phi}\, \mathrm{Tr}\, \star F_{(2)} \wedge F_{(2)}\,, \quad (1.38)$$

and for the rescaled Type I action (1.32) we obtain

$$\mathcal{L}^{\mathrm{E}}_{\mathrm{I}} = R \star \mathbb{1} - \frac{1}{2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} e^{\phi} \star H_{(3)} \wedge H_{(3)} - \frac{1}{2} e^{\frac{1}{2}\phi}\, \mathrm{Tr}\, \star F_{(2)} \wedge F_{(2)}\,. \quad (1.39)$$

It is clear upon inspection that the two actions (1.38) and (1.39) are transformed into each other under the discrete mapping

$$\phi \to -\phi\,. \quad (1.40)$$

Since the exponential of the dilaton corresponds to the string coupling constant, this suggests that the strong and weak coupling regimes of the Heterotic $\mathrm{SO}(32)$ and Type I superstring are mapped into each other [48]. This is a surprising result: it relates a theory of both closed and open strings to a theory of only closed strings.

Transforming the IIB supergravity action to the Einstein frame yields

$$\begin{aligned}\mathcal{L}^{\mathrm{E}}_{\mathrm{IIB}} \;=\;& R \star \mathbb{1} - \frac{1}{2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} e^{-\phi} \star H_{(3)} \wedge H_{(3)} - \frac{1}{2} e^{2\phi} \star G_{(1)} \wedge G_{(1)} \\ & - \frac{1}{2} e^{\phi} \star G_{(3)} \wedge G_{(3)} - \frac{1}{4} \star G_{(5)} \wedge G_{(5)} - \frac{1}{2} C_{(4)} \wedge \mathrm{d}C_{(2)} \wedge \mathrm{d}B_{(2)}\,. \end{aligned} \quad (1.41)$$

This action has a symmetry mixing the two scalars and two two-form potentials. In particular, it can be shown [40] that the action (1.41) remains invariant under so-called $\mathrm{S}\,\ell(2,\mathbb{R})$ transformations of the form

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \quad \begin{pmatrix} C_{(2)} \\ B_{(2)} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_{(2)} \\ B_{(2)} \end{pmatrix}\,, \quad ad - bc = 1\,, \quad (1.42)$$

where we have grouped the two real scalars together into one complex scalar τ

$$\tau = C_{(0)} + \mathrm{i}\, e^{-\phi}\,. \quad (1.43)$$

For the special case $a = d = C_{(0)} = 0$ and $b = -c = 1$, the transformation (1.42) is equivalent to (1.40). This makes it plausible that the strong coupling regime of Type IIB superstring theory[4] is actually dual to its own weak coupling regime [49].

[4]The duality symmetry is restricted to $\mathrm{S}\,\ell(2,\mathbb{Z})$ in Type IIB string theory.

1.3.3 M-theory

The strong coupling limit of Type IIA string theory is even more surprising. First we transform (1.25) to the Einstein frame to obtain

$$\begin{aligned}\mathcal{L}^{\mathrm{E}}_{\mathrm{IIA}} &= R\star\mathbb{1}-\frac{1}{2}\star\mathrm{d}\phi\wedge\mathrm{d}\phi-\frac{1}{2}e^{-\phi}\star H_{(3)}\wedge H_{(3)}-\frac{1}{2}e^{\frac{3}{2}\phi}\star G_{(2)}\wedge G_{(2)}\\ &\quad -\frac{1}{2}e^{\frac{1}{2}\phi}\star G_{(4)}\wedge G_{(4)}+\frac{1}{2}B_{(2)}\wedge\mathrm{d}C_{(3)}\wedge\mathrm{d}C_{(3)}\,.\end{aligned} \tag{1.44}$$

Then we group the various fields of this action together in the following way

$$\begin{aligned}\widehat{g}_{\mu\nu} &= e^{-\frac{1}{6}\phi}g_{\mu\nu}+e^{\frac{4}{3}\phi}C_\mu C_\nu\,,\\ \widehat{g}_{\mu z} &= e^{\frac{4}{3}\phi}C_\mu\,,\quad \widehat{g}_{zz}=e^{\frac{4}{3}\phi}\,,\\ \widehat{C}_{(3)} &= C_{(3)}+B_{(2)}\wedge\left(\mathrm{d}z+C_{(1)}\right)\,.\end{aligned} \tag{1.45}$$

Note that this is precisely the field content of $D=11$ supergravity. In fact, substituting (1.45) into the $D=11$ supergravity action (1.33), we obtain

$$\mathcal{L}_{11}=\mathcal{L}^{\mathrm{E}}_{\mathrm{IIA}}\wedge\mathrm{d}z\,, \tag{1.46}$$

meaning that Type IIA supergravity is a Kaluza-Klein reduction of $D=11$ supergravity over a circle.

From (1.45) and (1.15), we see that the exponential of the dilaton relates the string coupling constant to the radius of the eleventh dimension in units of the eleven-dimensional Planck length

$$R_{11}=g_s^{\frac{2}{3}}\,\ell_p\,. \tag{1.47}$$

The eleven-dimensional gravitational couplings constant has dimensions of ℓ_p^9: using (1.46) and (1.47), we can determine the ten-dimensional Newton's constant

$$\kappa_{11}^2\equiv\kappa_{10}^2R_{11}\to\kappa_{10}^2=\frac{\ell_p^8}{g_s^{\frac{2}{3}}}\,. \tag{1.48}$$

If we compare this with the previous expressions (1.3) and (1.13), then we obtain a relation between the ten-dimensional string length, string coupling, and Planck length. We can then express the eleven-dimensional radius in ten-dimensional quantities

$$\ell_p=g_s^{\frac{1}{3}}\,\ell_s\to R_{11}=g_s\ell_s\,. \tag{1.49}$$

This means that the strong coupling limit of ten-dimensional Type IIA string theory is an eleven-dimensional theory [50]. This theory goes under the name of M-theory [51]: it is defined to be the theory that has $D=11$ supergravity as its low-energy limit. In a similar way, there are reasons to believe that the strong coupling limit of the Heterotic $\mathrm{E}_8\times\mathrm{E}_8$ string theory is related to the same M-theory [52], but this time the extra dimension is not a circle but an interval [53].

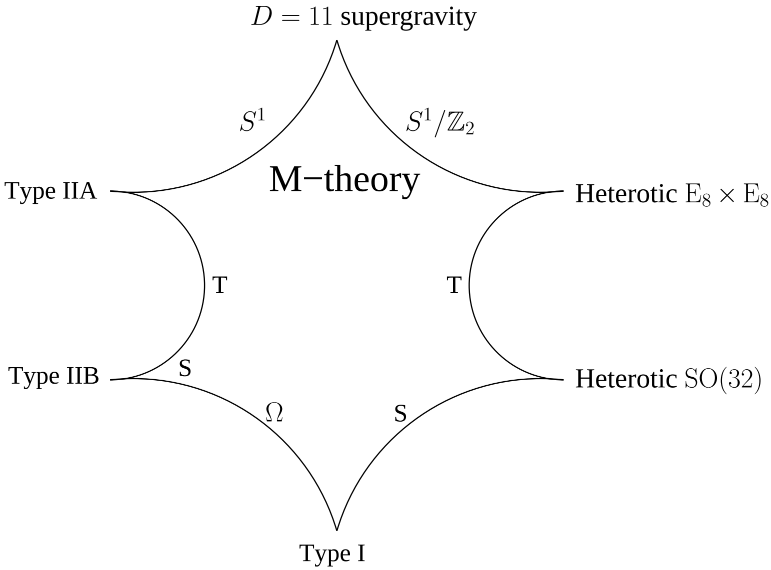

Figure 1.4: The M-theory web of string theories and their dualities.

1.3.4 The duality web

We have summarized this web of dualities in figure 1.4, where S^1 and $S^1/\mathbb{Z}_2$ indicate a circle and a line interval, respectively. There is also a duality between the Type IIB and Type I string theories which we have not discussed, but a parity operator Ω can be applied to spectrum of Type IIB string theory to obtain Type I string theory.

After this web of dualities emerged, the term M-theory was no longer used for the strong coupling limit of Type IIA string theory but for the whole framework of string theories and supergravities in ten and eleven dimensions. The overall picture is that all these theories are different vacua of a single underlying theory around each of which one can perform perturbation theory.

The various dualities interpolate between the different vacua and relate the various perturbative results. A detailed microscopic description of the full M-theory is still lacking, although there have been some attempts in this direction [54].

1.4 Branes

In this section, we will consider some aspects of branes. We will start with looking at how branes appear as solutions of the supergravity equations of motion. Then we will describe the worldvolume actions describing the fluctuations around these solutions, and we will discuss the different metric frames in which one can work. We will finish with a discussion about the tensions and charges related to the brane solutions. The geometrical aspects of these branes will be discussed in chapter 3.

1.4.1 Two-block solutions

Our approach will be to first consider a general class of solutions called two-block solutions to a generic D-dimensional supergravity action in the Einstein frame consisting of the kinetic terms for the metric, the dilaton[5] and a $p+1$-form gauge field

$$\mathcal{L}^{\mathrm{E}}_{(D,\,p)} = R\star\mathbb{1} - \frac{4}{D-2}\star\mathrm{d}\phi\wedge\mathrm{d}\phi - \frac{1}{2}e^{a\phi}g_s^{2k}\star F_{(p+2)}\wedge F_{(p+2)}\,, \tag{1.50}$$

where the exponent k of g_s is the remnant of the dilaton coupling $e^{2k\Phi}$ to the field-strength in the string frame. Using (1.19) and (1.23), we find

$$k = \frac{a}{2} + \frac{2d}{D-2}\,. \tag{1.51}$$

Anticipating that this action will describe both an electric p-brane and a magnetic $\tilde{p}$-brane, we will introduce

$$\begin{cases} d = p+1\;: & \text{worldvolume dimension of the electric } p\text{-brane} \\ \tilde{d} = \tilde{p}+1\;: & \text{worldvolume dimension of the magnetic } \tilde{p}\text{-brane} \end{cases} \tag{1.52}$$

The equations of motion for the action (1.50) in the electric formulation have as solution an

$$\text{electric } p\text{-brane} = \begin{cases} \mathrm{d}s^2_{\mathrm{E}} &=& H^{\frac{-4\tilde{d}}{(D-2)\Delta}}\mathrm{d}x^2_{(d)} + H^{\frac{4d}{(D-2)\Delta}}\mathrm{d}y^2_{(\tilde{d}+2)}\,, \\ e^{\Phi} &=& g_s H^{\frac{(D-2)a}{4\Delta}}\,, \\ F_{(p+2)} &=& g_s^{-k}\sqrt{\frac{4}{\Delta}}\,\mathrm{d}^d x\wedge\mathrm{d}H^{-1}\,, \\ H(y) &=& 1+\left(\frac{R}{y}\right)^{\tilde{d}}\,. \end{cases} \tag{1.53}$$

The parallel coordinates x^a $(a=0,\ldots,p)$ span the worldvolume of the brane, and the coordinates y^m $(m=p+1,\ldots,D-1)$ are transverse to the brane. The parameter Δ of the solution is given by

$$\Delta = \frac{(D-2)a^2}{8} + \frac{2d\tilde{d}}{(D-2)}\,. \tag{1.54}$$

The function H is a harmonic function in the transverse dimensions, depending on the transverse coordinates y^i only

$$\Delta_{(\tilde{d}+2)}H = 0 \rightarrow \quad H(y) = 1+\left(\frac{R}{y}\right)^{\tilde{d}}\,,\quad y^2\equiv\sum_m (y^m)^2\,, \tag{1.55}$$

[5]In our conventions the scalar kinetic term has a nonstandard normalization in $D<10$.

where R is an integration parameter depending on the charge of the brane, as we will see later.

Since the metric splits up into two diagonal pieces, and since the function H depends only on the transverse coordinates, such solutions are called brane solutions. Furthermore, the field strength is proportional to the worldvolume $\mathrm{d}^d x$ of the brane which corresponds to the fact that the brane couples to a gauge potential $A_{(p+1)}$.

Note that we do not consider solutions for which $\Delta = 0$: they correspond to the cases $a = d = 0$, $a = \tilde{d} = 0$, or $a \neq 0$ with $\tilde{d} < 0$. These cases correspond to a -1-brane or instanton, which is only a solution of the Wick-rotated action; the $D-3$-brane, for which the harmonic function is logarithmic; the $D-2$-brane, which falls under the class of domain-walls; and the $D-1$-brane, which is a spacetime-filling brane. We will not discuss these exotic branes, since they do not have a regular near-horizon limit, except for the domain-walls: they will be the subject of chapter 3.

We can transform the field strength to its magnetic dual

$$g_s^{2-k} F_{(\tilde{p}+2)} \equiv e^{a\phi} g_s^k \star F_{(p+2)} \,, \quad \tilde{p} \equiv D - p - 4. \tag{1.56}$$

This gives for the action

$$\mathcal{L}^{\mathrm{E}}_{(D,\,\tilde{p})} = R \star \mathbb{1} - \frac{4}{D-2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} e^{-a\phi} g_s^{2-4k} \star F_{(\tilde{p}+2)} \wedge F_{(\tilde{p}+2)} \,. \tag{1.57}$$

The magnetic dual formulation (1.57) supports a

$$\text{magnetic } \tilde{p}\text{-brane} = \begin{cases} \mathrm{d}s^2_{\mathrm{E}} &= H^{\frac{-4d}{(D-2)\Delta}} \mathrm{d}x^2_{(\tilde{d})} + H^{\frac{4\tilde{d}}{(D-2)\Delta}} \mathrm{d}y^2_{(d+2)} \,, \\ e^{\Phi} &= g_s H^{\frac{-(D-2)a}{4\Delta}} \,, \\ F_{(\tilde{p}+2)} &= g_s^{k-2} \sqrt{\frac{4}{\Delta}}\, \mathrm{d}^{\tilde{d}} x \wedge \mathrm{d}H^{-1} \,, \\ H(y) &= 1 + \left(\frac{R}{y}\right)^d \,, \end{cases} \tag{1.58}$$

where the function H is now harmonic on the $d+2$ transverse directions.

We will now give some explicit brane solutions of supergravities in ten and eleven dimensions. We will start with giving solutions of the Type IIA and Type IIB supergravity equations of motion following from the actions (1.44) and (1.41), after which we will give the eleven-dimensional two-block brane solutions.

Strings and five-branes

Since all $D = 10$ supergravities are low-energy limits of superstring theories, one expects that they should have string-like solutions. We can obtain such solutions if we truncate the actions (1.41) and (1.44) to only their first three terms. This gives the action (1.50) with

$D = 10$, $p = 1$ and $a = -1$. If we substitute this into the general two-block solutions (1.53), we obtain the fundamental string solution [55]

$$\mathrm{F1} = \begin{cases} \mathrm{d}s^2 &= H^{-\frac{3}{4}}\mathrm{d}x^2_{(2)} + H^{\frac{1}{4}}\mathrm{d}y^2_{(8)}\,, \\ e^{\Phi} &= g_s H^{-\frac{1}{2}}\,, \\ H_{(3)} &= \mathrm{d}^2x \wedge \mathrm{d}H^{-1}\,, \\ H(y) &= 1 + \left(\frac{R}{y}\right)^6\,. \end{cases} \tag{1.59}$$

The magnetic dual of this is a magnetic five-brane, also known as the Neveu-Schwarz five-brane [56, 57].

$$\mathrm{NS5} = \begin{cases} \mathrm{d}s^2 &= H^{-\frac{1}{4}}\mathrm{d}x^2_{(6)} + H^{\frac{3}{4}}\mathrm{d}y^2_{(4)}\,, \\ e^{\Phi} &= g_s H^{\frac{1}{2}}\,, \\ \widetilde{H}_{(7)} &= \mathrm{d}^6x \wedge \mathrm{d}H^{-1}\,, \\ H(y) &= 1 + \left(\frac{R}{y}\right)^2\,. \end{cases} \tag{1.60}$$

Dp-branes

The Type II string theories have RR-potentials $C_{(p)}$ in their massless spectrum, where $p = 1, 3$ for Type IIA and $p = 0, 2, 4$ for Type IIB. We notice that the kinetic terms of the gauge potentials have a factor of $a = \frac{3-p}{2}$ in their dilaton exponential. The branes coupling to these gauge potentials are called Dp-branes [58], with solutions given by

$$\mathrm{D}p = \begin{cases} \mathrm{d}s^2 &= H^{\frac{p-7}{8}}\mathrm{d}x^2_{(p+1)} + H^{\frac{p+1}{8}}\mathrm{d}y^2_{(D-p-1)}\,, \\ e^{\Phi} &= g_s H^{\frac{3-p}{2}}\,, \\ G_{(p+2)} &= g_s^{-1}\mathrm{d}^{p+1}x \wedge \mathrm{d}H^{-1}\,, \\ H(y) &= 1 + \left(\frac{R}{y}\right)^{D-p-3}\,, \end{cases} \tag{1.61}$$

where for $p > 3$ the magnetic field strength has been given[6]. In Type IIA one finds D0 and D2-branes as well as their magnetic duals, the D4 and D6-branes. Type IIB contains D1 and self-dual D3-branes as well as the magnetic D5-branes. The gauge field $C_{(0)}$ supports a D(-1)-brane called the D-instanton and a D7-brane.

It was a major breakthrough in string theory when it was realized that these Dp-branes could be identified as the hyperplanes on which open strings can end [24]. In chapter 2, we will look in more detail at the implications of the different aspects of D-branes.

Membranes and five-branes

The action of $D = 11$ supergravity given in (1.33) can be truncated to only its first two terms, giving the form (1.50) with $D = 11$ and $p = 2$. This should come as no surprise since we

[6] The D3-brane solution will be given in (2.1).

mentioned before that there exists an eleven-dimensional supermembrane. The solution of this M2-brane is given by [59]

$$\text{M2} = \begin{cases} \mathrm{d}s^2 &= H^{-\frac{2}{3}}\mathrm{d}x_{(3)}^2 + H^{\frac{1}{3}}\mathrm{d}y_{(8)}^2\,, \\ G_{(4)} &= \mathrm{d}^3x \wedge \mathrm{d}H^{-1}\,, \\ H(y) &= 1 + \left(\frac{R}{y}\right)^6\,. \end{cases} \tag{1.62}$$

The magnetic dual of the M2-brane is the M5-brane which has as solution [60]

$$\text{M5} = \begin{cases} \mathrm{d}s^2 &= H^{-\frac{1}{3}}\mathrm{d}x_{(6)}^2 + H^{\frac{2}{3}}\mathrm{d}y_{(5)}^2\,, \\ \widetilde{G}_{(7)} &= \mathrm{d}^6x \wedge \mathrm{d}H^{-1}\,, \\ H(y) &= 1 + \left(\frac{R}{y}\right)^3\,. \end{cases} \tag{1.63}$$

Dualities between branes

In the previous section, we have discussed dualities in string theory. In particular, we saw how S-duality mixes the different scalar fields and gauge fields of Type IIB supergravity and how T-duality changes the boundary condition of open strings. We also saw how $D = 11$ supergravity could be dimensionally reduced to Type IIA supergravity in ten dimensions. Since branes couple to the various gauge fields, and since Dp-branes are also the hyperplanes on which open strings can end, dualities will relate the various branes to one another. In addition to that, the eleven-dimensional branes reduce to solutions of Type IIA supergravity.

In figure 1.5, we have indicated the various branes in ten and eleven dimensions and the dualities between them. Dimensional reduction is given by arrows, T-duality by solid lines, and S-duality by dashed lines. Electric/magnetic duality and some T-dualities are not indicated. Also depicted are some branes which do not fall in our two-block solutions, such as the waves (W) and Kaluza-Klein monopoles (KK), but which are related to ordinary branes by duality or reduction. Another exotic brane, the D-instanton, is left out altogether. The 8-branes and 9-branes on the right of figure 1.5 are special branes: they correspond to domain-walls and spacetime-filling branes. For more details, see [61]. In chapters 3 and 4, we will discuss domain-walls in more detail.

1.4.2 Worldvolume actions

Branes are not just static solutions but dynamical objects, since they couple to gravity and to gauge potentials. The fluctuations around the static solutions are described by worldvolume actions, which are generalizations of the actions for the particle and the string given in (1.1) and (1.2). An additional remark is that some of the brane solutions we gave in (1.53) are singular, which means that the target space action (1.50) needs to be supplemented with a source term in these cases

$$S_{\text{total}} = \frac{1}{2\kappa^2}\int_{\mathcal{M}} \mathrm{d}^D x\, \mathcal{L}_{(D,\,p)} + \int_{\Sigma} \mathrm{d}^d\sigma\, \mathcal{L}_{p-\text{brane}}\,. \tag{1.64}$$

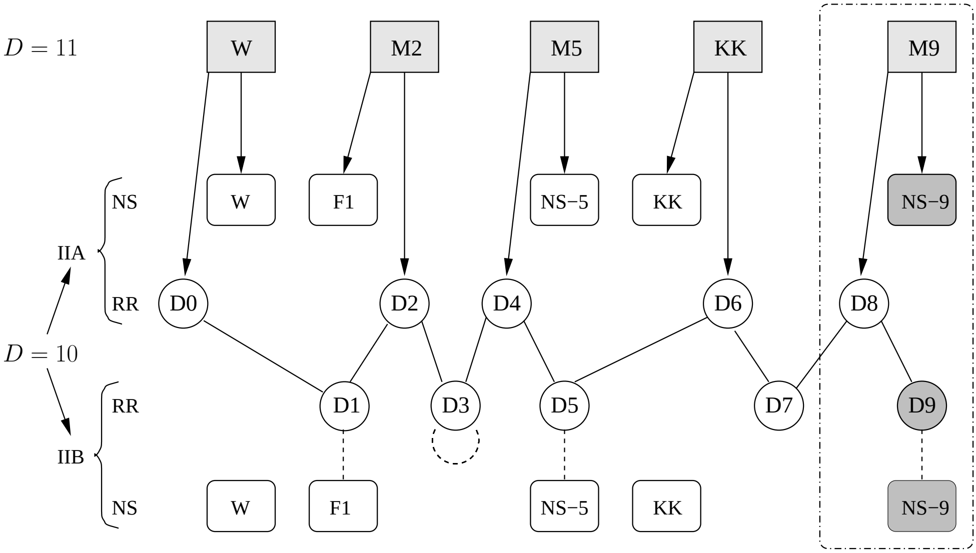

Figure 1.5: The various branes in $D=10$ and $D=11$ and their dualities.

In order to obtain the worldvolume action, we first assign coordinates σ^a , $a = 0 \ldots p$ to the brane. This gives as induced metric g_{ab} on the brane worldvolume Σ the pull-back of the metric $G_{\mu\nu}$ on the target space $\mathcal{M}$

$$g_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \,. \tag{1.65}$$

A general Ansatz for the action is then given by

$$\begin{aligned} \mathcal{L}_{p-\text{brane}} &= \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{WZ}} \\ &= -\tau_p\sqrt{|g|} + \mu_p C_{(p+1)} + \ldots \,. \end{aligned} \tag{1.66}$$

Here, τ_p is the energy-density, or tension, and μ_p is the charge-density. The last term is the generalization of the coupling of a particle to a gauge field called the Wess-Zumino action. The kinetic term modifies the condition on the harmonic function

$$\Delta_{(\tilde{d}+2)} H(y^m) = \kappa^2 \tau_p \, \delta(y^m) \,. \tag{1.67}$$

Generically, the embedding coordinates X^μ can be identified with the Goldstone modes corresponding to the translational symmetries that are broken by the brane. This means that the modes form a scalar multiplet; this is the case for the branes such as the M2-brane. However, other branes, such as the D3-brane or the M5-brane, can have a vector multiplet [62] or a tensor multiplet [63] as their massless modes. In chapter 2, we will look in more detail at the worldvolume action of the D3-brane.

1.4.3 Tensions and charges

For the general two-block solutions we gave before, we can actually calculate the tension and charge-density. We first define the deviation of the flat metric $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,, \tag{1.68}$$

and then we substitute the two-block solution (1.53) into the ADM-formula which expresses the tension as a spatial surface integral over a sphere surrounding the brane

$$\begin{aligned}\tau_p &= \frac{1}{2\kappa^2}\int_{S^{\tilde{d}+1}} \mathrm{d}^{\tilde{d}+1}\Sigma^m \left(\partial^n h_{mn} - \partial_m h^a{}_a\right) \\ &= \frac{4}{\Delta}\frac{\tilde{d}R^{\tilde{d}}\Omega_{(\tilde{d}+1)}}{2\kappa^2}\,,\end{aligned} \tag{1.69}$$

where the indices m, n run over the transverse coordinates, and where the index a runs over all spatial coordinates $1 \ldots D-1$. The surface area of the sphere is given by

$$\Omega_{(\tilde{d}+1)} = \frac{2\pi^{\frac{\tilde{d}+2}{2}}}{\Gamma\left(\frac{\tilde{d}+2}{2}\right)}\,. \tag{1.70}$$

The charge of the brane is given by a generalized Gauss-law

$$\begin{aligned}\mu_p &= \frac{1}{2\kappa^2}\int_{S^{\tilde{d}+1}} e^{a\phi} \star F_{(p+2)} \\ &= \sqrt{\frac{4}{\Delta}}\frac{\tilde{d}R^{\tilde{d}}\Omega_{(\tilde{d}+1)}}{2\kappa^2}\,.\end{aligned} \tag{1.71}$$

Comparing (1.69) with (1.71), we obtain what is known as the BPS-condition

$$\tau_p = \sqrt{\frac{4}{\Delta}}|\mu_p|\,. \tag{1.72}$$

The BPS-condition (1.72) is the limiting case of a more general condition called the Bogomol'nyi bound

$$\tau_p \geq \sqrt{\frac{4}{\Delta}}|\mu_p|\,. \tag{1.73}$$

This bound is valid for solutions of supersymmetric theories such as the branes of supergravities we discussed. In eleven dimensions, the superalgebra is generated by 32-component supercharges Q_α

$$\{Q_\alpha, Q_\beta\} = (\gamma^\mu \mathcal{C})_{\alpha\beta} P_\mu + \frac{1}{2!}(\gamma^{\mu\nu}\mathcal{C})_{\alpha\beta} Z_{\mu\nu} + \frac{1}{5!}\left(\gamma^{\mu\nu\lambda\rho\sigma}\mathcal{C}\right)_{\alpha\beta} Z_{\mu\nu\lambda\rho\sigma}\,, \tag{1.74}$$

where $\gamma^{\mu\cdots}$ are Dirac-matrices and where $\mathcal{C}$ is the charge conjugation matrix. For our conventions on gamma-matrices see appendix A. The operator P_μ represents the generator of translations, and the tensors $Z_{(2)}$ and $Z_{(5)}$ are called central charges, since they commute with the supersymmetry charges.

All the operators in (1.74) can be thought of as charges: they can be expressed as integrals over conserved currents which, using Noether's theorem, correspond to the symmetries of $D = 11$ supergravity such as supersymmetry, general coordinate invariance, and gauge invariance. Since the left-hand side of (1.74) is a positive operator, the sum of the momentum operator and the central charges are bounded from below. This bound is equivalent to (1.73), and it has to be satisfied by every solution of a supersymmetric theory.

A generic solution of a supersymmetric theory will itself not be completely supersymmetric, but when the BPS-bound (1.73) is exactly saturated, as it is in (1.72), then the corresponding solution preserves some of the supersymmetries of the underlying theory. In particular, for the branes in $D = 10$ and $D = 11$, we have $\Delta = 4$ which means that all these branes preserve half of the 32 spacetime supersymmetries.

So far, we have neglected the fermions in the supergravity action, since their structure is rather complicated in general. For the brane solutions that we have presented, only the bosonic fields have a non-vanishing value. In order for this to be consistent with supersymmetry, the fermions that are set to zero also need to have a vanishing supersymmetry variation.

For instance for the Dp-branes, the supersymmetry variations for the supersymmetric partners of the graviton and the dilaton, the gravitino ψ_μ and the dilatino λ, are given by [64]

$$\begin{aligned}
\delta\psi_\mu &= \partial_\mu\epsilon - \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon + \frac{(-)^p}{8(p+2)!}e^{\phi}\,\gamma^{(p+2)}\cdot F_{(p+2)}\gamma_\mu\epsilon'_p\,, \\
\delta\lambda &= \partial\!\!\!/\phi\epsilon + \frac{3-p}{4(p+2)!}e^{\phi}\,\gamma^{(p+2)}\cdot F_{(p+2)}\epsilon'_p\,,
\end{aligned} \tag{1.75}$$

where $\omega_\mu{}^{ab}$ denotes the spin-connection corresponding to the metric of the Dp-brane solution (1.61), and where we have used the notation of appendix A. The spinor ϵ'_p is equal to ϵ up to a chirality projection matrix, depending on the value of p [64]. Substituting the solutions for the metric, dilaton, and field-strength of the Dp-brane solution (1.61), we find the following condition on the spinor ϵ

$$\epsilon + \gamma_{01\ldots p}\epsilon'_p = 0\,. \tag{1.76}$$

Spinors for which the supersymmetry variation of the gravitino vanishes are called Killing spinors. The condition (1.76) implies that the Killing spinor ϵ is projected to only half of its original degrees of freedom, in other words, only half of the supersymmetries of the Type IIA and Type IIB supergravity actions are realized on the Dp-brane solutions.

There are also more general brane solutions, which can be thought of as intersections of the elementary branes and which have different values for Δ. If $\Delta = 4/n$ the corresponding brane preserves $32/2^n$ supersymmetries. The importance of the BPS-bound lies in the fact that it is independent of any coupling constant, meaning that the brane tension is stable for quantum corrections. However, recent developments show that there are also stable objects in string theory which are non-BPS [65].

The supersymmetry algebra (1.74) contains much more information than we have room to discuss here. As an example, we remark that the complete spectrum of branes can be deduced from it [66]. The spatial components of every central charge $Z_{(p)}$ correspond to a p-brane, giving the wave, the M2-brane and M5-brane in $D = 11$. One can also dualize the timelike component to a $Z_{(D-p)}$ charge[7] which should correspond to a $D - p$-brane. In $D = 11$ this gives a Kaluza-Klein monopole and a M9-brane. This explains the extra branes appearing in figure 1.5.

1.4.4 Metric frames

We have seen that the dilaton can be used to rescale the metric: this enabled us to go from the string frame, in which supergravities are derived, to the Einstein frame, where the curvature term has no dilatonic pre-factor. In this section, we will discuss two other metric frames that we will need later.

The dilaton dependence of the effective tension τ_p will in general depend on the frame being used. We define the sigma-model frame $g^\sigma_{\mu\nu}$ as the frame in which a particular brane tension is independent of the dilaton. We denote this intrinsic tension with T_p. The worldvolume action of a BPS p-brane in the sigma-model frame couples to the induced metric g^σ_{ab} and is given by

$$\mathcal{L}^\sigma_{p-\mathrm{brane}} = -T_p \left(\sqrt{|g^\sigma|} + C_{(p+1)} \right) . \tag{1.77}$$

It scales homogeneously under the scale transformations

$$g^\sigma_{ab} \to \lambda^2 g^\sigma_{ab} \,, \quad C_{(p+1)} \to \lambda^{p+1} C_{(p+1)} \,. \tag{1.78}$$

This so-called trombone symmetry is a symmetry of the p-brane equations of motion [67], which implies that the combined system (1.64) of the target space action and the worldvolume action has to scale homogeneously as λ^{p+1}. For this to happen, we have to let the dilaton scale as

$$e^\phi \to \lambda^\alpha e^\phi \,, \quad \alpha = -\frac{2d\tilde{d}}{(D-2)a} \,. \tag{1.79}$$

Using the expressions (1.52) for the worldvolume dimensions of the electric and magnetic brane, we can relate the sigma-model frame to the Einstein frame by

$$g^\sigma_{\mu\nu} = e^{\omega_\sigma \phi} g^{\mathrm{E}}_{\mu\nu} \,, \quad \omega_\sigma = -\frac{a}{d} \,. \tag{1.80}$$

[7]Except for $p = 0$ which has no timelike components and for P_μ which has the Hamiltonian as its timelike component. Self-dual charges $Z_{(D/2)}$ also correspond to only a single brane.

The target space action (1.50) in the sigma-model frame (1.80) has an overall dilaton factor and a modified kinetic term for the dilaton

$$\mathcal{L}^{\sigma}_{(D,\,p)} = e^{\delta_\sigma \phi}\left(R \star \mathbb{1} + \gamma_\sigma \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} \star F_{(p+2)} \wedge F_{(p+2)}\right) . \tag{1.81}$$

which are given by

$$\delta_\sigma = \frac{(D-2)a}{2d} , \quad \gamma_\sigma = \frac{D-1}{D-2}\delta_\sigma^2 - \frac{4}{D-2} . \tag{1.82}$$

We have already seen an example of such a sigma-model frame: the string frame defined in (1.23) which is in agreement with (1.80) if we substitute the string solution parameters $a = -1$ and $p = 1$. The worldvolume action (1.10) has no dilaton dependence, and the target space action (1.20) has an overall $e^{-2\phi}$ factor. This guarantees that the combined action transforms homogeneously as λ^2 under

$$g^{\mathrm{S}}_{\mu\nu} \to \lambda^2 g^{\mathrm{S}}_{\mu\nu} , \quad B_{\mu\nu} \to \lambda^2 B_{\mu\nu} , \quad e^{\phi} \to \lambda^3 e^{\phi} . \tag{1.83}$$

Finally, there is another important frame – the dual frame. This frame is defined as the sigma-model frame of the magnetically dual brane. The dual frame is related to the Einstein frame by

$$g^{\mathrm{D}}_{\mu\nu} = e^{\omega_{\mathrm{D}} \phi} g^{\mathrm{E}}_{\mu\nu} , \quad \omega_{\mathrm{D}} = \frac{a}{\tilde{d}} . \tag{1.84}$$

In the dual frame, the magnetically dual brane's tension is independent of the dilaton, and the magnetic formulation of the target space action (1.57) has an overall dilaton factor and a modified kinetic term for the dilaton

$$\mathcal{L}^{\mathrm{D}}_{(D,\,\tilde{p})} = e^{\delta_{\mathrm{D}} \phi}\left(R \star \mathbb{1} + \gamma_{\mathrm{D}} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} \star F_{(\tilde{p}+2)} \wedge F_{(\tilde{p}+2)}\right) , \tag{1.85}$$

which are given by

$$\delta_{\mathrm{D}} = -\frac{(D-2)a}{2\tilde{d}} , \quad \gamma_{\mathrm{D}} = \frac{D-1}{D-2}\delta_{\mathrm{D}}^2 - \frac{4}{D-2} . \tag{1.86}$$

In chapter 3, we will see that the near-horizon geometry of a brane takes on a simplified form in the dual frame.

We can now calculate the dilaton-dependence of a p-brane in the string frame from

$$\mathcal{L}^{\mathrm{S}}_{p-\mathrm{brane}} = -\tau_p \left(\sqrt{|g^{\mathrm{S}}|} + C_{(p+1)}\right) , \tag{1.87}$$

and using the connections between the various frames given in (1.23) and (1.80) we find

$$\tau_p = e^{-k\phi} T_p , \quad k = \frac{a}{2} + \frac{2d}{D-2} . \tag{1.88}$$

Comparing with (1.51), we see that the dilaton coupling for the p-brane worldvolume action in the string frame is equal to the dilaton coupling for the electric field-strength in the string frame.

Remembering that the exponential of the dilaton corresponds to the string coupling, we can compute the coupling dependence of the various brane tensions in the string frame. For the string, we take $a = -1$ and $p = 1$; for the Neveu-Schwarz five-brane, we take $a = 1$ and $p = 5$; and finally for the Dp-branes, we have $a = \frac{3-p}{2}$

$$\frac{\tau_{\mathrm{F1}}}{T_{\mathrm{F1}}} = 1\,, \qquad \frac{\tau_{\mathrm{D}p}}{T_{\mathrm{D}p}} = \frac{1}{g_s}\,, \qquad \frac{\tau_{\mathrm{NS5}}}{T_{\mathrm{NS5}}} = \frac{1}{g_s^2}\,. \tag{1.89}$$

So, we see that the NS5-brane and the Dp-branes are really solitonic objects: they become massive compared to fundamental strings for small values of the string couplings.

In the next chapters, we will need expressions for the tension of Dp-branes. In [58] this was calculated by comparing the worldsheet and the target space calculations of RR-field and gravitational interactions. This gave the same ℓ_s-dependence as derived from dimensional analysis, but the precise numerical factor can also be fixed by demanding that

$$\frac{\tau_{F1}}{\tau_{D1}} = g_s\,. \tag{1.90}$$

Using (1.3) and (1.89) gives as the answer

$$T_{\mathrm{D}p} = \frac{2\pi}{(2\pi\ell_s)^{p+1}}\,. \tag{1.91}$$

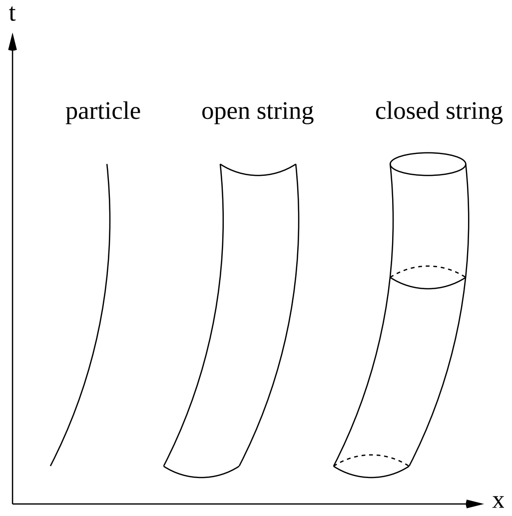

Figure 1.1: A particle worldline and string worldsheets.

1.1.1 Free bosonic string theory

The mathematical formulation of string theory proceeds along similar ways as the relativistic motion of particles, as we have indicated in figure 1.1. Consider a particle with mass m, moving through a flat D-dimensional spacetime with coordinates X^{μ}. Here, one assigns a parameter τ to the worldline Λ that the particle sweeps out in spacetime, and the action is simply the length of the worldline

$$S_{\text{particle}} = -m \int_{\Lambda} \mathrm{d}\tau \, \sqrt{|\partial_\tau X^{\mu} \partial_\tau X_{\mu}|} \, . \tag{1.1}$$

The dynamics of a relativistic string with tension T can likewise be formulated by assigning coordinates $\sigma^a = (\tau, \sigma)$ to the two-dimensional world-sheet Σ. The action is given by the surface that the string worldsheet sweeps out in spacetime

$$S_{\text{string}} = -T \int_{\Sigma} \mathrm{d}^2\sigma \, \sqrt{|\det(\partial_a X^{\mu} \partial_b X_{\mu})|} \, . \tag{1.2}$$

For historical reasons, the tension of the string is often expressed in terms of the Regge-slope parameter α', which is related to the length of the string ℓ_s by

$$T = \frac{1}{2\pi\alpha'} \, , \quad \alpha' = \frac{\ell_s^2}{\hbar} \, . \tag{1.3}$$