Appendix A

Conventions

In this appendix, we will summarize our conventions. Furthermore, we will give some useful identities that have been used in the previous chapters.

A.1 Indices

Below we will summarize the different ranges and meanings of the indices used in chapters 5 and 6. First of all, the metric that we use is mostly plus: i.e. in five dimensions, we have $g_{\mu\nu} = (- + + + +)$. In chapter 5, we have used the following notations

$$\begin{array}{ccl} \mu, \nu & 0, 1, \ldots, 4 & \text{space-time}\,, \\ a, b & 0, 1, \ldots, 4 & \text{tangent space}\,, \\ \alpha, \beta & 1, \ldots, 4 & \text{spinor}\,, \\ i, j & 1, 2 & \mathrm{SU}(2)\,, \end{array} \tag{A.1}$$

In chapter 6, we have furthermore used indices labelling the components of the matter multiplets. In particular, we have used

$$\begin{array}{ccl} \widetilde{I}, \widetilde{J} & 1, 2, \ldots n + m & \text{vector-tensor multiplet}\,, \\ I, J & 1, 2, \ldots, n & \text{vector multiplet}\,, \\ M, N & 1, 2, \ldots, m & \text{tensor multiplet}\,, \\ X, Y & 1, 2, \ldots, 4r & \text{hypermultiplet target space}\,, \\ A, B & 1, 2, \ldots, 2r & \text{hypermultiplet tangent space}\,, \\ i, j & 1, 2 & \mathrm{SU}(2)\,. \end{array} \tag{A.2}$$

In chapter 7 two compensating multiplets were introduced. The X, Y and A, B indices were replaced by hatted ones to denote the increased ranges. The other indices are as above, but with $n = n_V + 1$, $m = n_T$ and $r = n_H$, where n_V, n_T and n_H respectively are the number of Poincaré vector-, tensor- and hypermultiplets. The following indices were used[1]:

$\widetilde{I}, \widetilde{J}$	$1, 2, \dots n_V + n_T + 1$	vector-tensor multiplet ,	
I, J	$1, 2, \dots, n_V + 1$	vector multiplet ,	
M, N	$1, 2, \dots, n_T$	tensor multiplet ,	
$\tilde{x}, \tilde{y}$	$1, 2, \dots, n_V + n_T$	vector-tensor multiplet ,	
$\hat{X}, \hat{Y}$	$1, 2, \dots, 4n_H + 4$	hyperkähler hypermultiplet target space ,	
X, Y	$1, 2, \dots, 4n_H$	quaternionic-Kähler hypermultiplet target space ,	(A.3)
z^α	$\alpha = 1, 2, 3$	SU(2) subspace of the hyperkähler hypermultiplet target space ,	
$\hat{A}, \hat{B}$	$1, 2, \dots, 2n_H + 2$	hyperkähler hypermultiplet tangent space ,	
A, B	$1, 2, \dots, 2n_H$	quaternionic-Kähler hypermultiplet tangent space ,	
i, j	$1, 2$	SU(2) ,	
$\bar{\alpha}, \bar{\beta}$	$1, 2, 3$	SU(2) vector index .	

In this thesis symmetrizations are denoted with parentheses, and anti-symmetrizations with brackets around the indices. Furthermore, we (anti-)symmetrize with weight one:

$$X_{(ab)} \equiv \tfrac{1}{2}\left(X_{ab} + X_{ba}\right) , \quad X_{[ab]} \equiv \tfrac{1}{2}\left(X_{ab} - X_{ba}\right) . \tag{A.4}$$

A.2 Tensors

Our conventions for the D-dimensional Levi-Civita tensor $\varepsilon_{a_1 \dots a_D}$ are

$$\varepsilon_{01\dots(D-1)} = -\varepsilon^{01\dots(D-1)} = 1 . \tag{A.5}$$

In the local case we use the Levi-Civita tensor density as a "constant tensor". It can be obtained from the Levi-Civita tensor by using vielbeins to convert the tangent space indices to space-time indices and multiplying the result with the vielbein determinant

$$\varepsilon_{\mu_1 \dots \mu_D} = e^{-1} e_{\mu_1}{}^{a_1} \cdots e_{\mu_d}{}^{a_D} \varepsilon_{a_1 \dots a_D} , \quad \varepsilon^{\mu_1 \dots \mu_D} = e\, e^{\mu_1}{}_{a_1} \cdots e^{\mu_D}{}_{a_D} \varepsilon^{a_1 \dots a_D} , \tag{A.6}$$

where we have used the Einstein summation convention in which repeated indices are summed over.

Note that raising and lowering the indices of the Levi-Civita tensor is done with the metric, which for the Levi-Civita tensor density is done by using the definition (A.6). Contractions of the Levi-Civita tensor give products of delta-functions which are normalized as

$$\varepsilon_{a_1 \dots a_p b_1 \dots b_q} \varepsilon^{a_1 \dots a_p c_1 \dots c_q} = -p! q! \delta^{[c_1}_{[b_1} \dots \delta^{c_q]}_{b_q]} , \tag{A.7}$$

We have defined the dual of five-dimensional tensors as

$$\widetilde{A}^{a_1 \dots a_{5-n}} = \tfrac{1}{n!}\, \mathrm{i}\, \varepsilon_{a_1 \dots a_{5-n} b_1 \dots b_n} A^{b_n \dots b_1} . \tag{A.8}$$

[1] For the hypermultiplets we now assume the presence of a metric on the scalar manifold.

Using (A.7), one finds the following identities

$$\widetilde{\widetilde{A}} = A\,, \qquad \frac{1}{n!}A^{a_1\ldots a_n}B_{a_1\ldots a_n} = \frac{1}{n!}A\cdot B = \frac{1}{(n-5)!}\widetilde{A}\cdot\widetilde{B}\,, \tag{A.9}$$

where we have introduced the generalized inner product notation $A \cdot B$ that we use throughout this thesis.

We use the same conventions for the Riemann tensor and its contractions as [176]. In particular, we define the Riemann tensor as

$$R^{\mu}{}_{\nu\lambda\rho} = \partial_\lambda \Gamma^{\mu}_{\rho\nu} - \partial_\rho \Gamma^{\mu}_{\lambda\nu} + \Gamma^{\mu}_{\sigma\lambda}\Gamma^{\sigma}_{\rho\nu} - \Gamma^{\mu}_{\sigma\rho}\Gamma^{\sigma}_{\lambda\nu}\,. \tag{A.10}$$

The Ricci tensor and Ricci scalar in this thesis are given by

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}\,, \quad R = g^{\mu\nu}R_{\mu\nu}\,. \tag{A.11}$$

With these conventions, the Einstein-Hilbert action has a positive sign.

A.3 Differential forms

At several places in this thesis, we have used differential form notation. A p-form is related to a rank-p anti-symmetric tensor according to

$$F_{(p)} = \frac{1}{p!}\mathrm{d}x^{\mu_1}\ldots\mathrm{d}x^{\mu_p}F_{\mu_1\ldots\mu_p}\,. \tag{A.12}$$

The analog of the dual of an anti-symmetric tensor (A.8), is given by the Hodge-dual: i.e a differential p-form A has a $(D-p)$-form $B = \star A$ as its dual with components

$$B_{\mu_1\ldots\mu_q} = \frac{1}{p!}e\,\varepsilon_{\mu_1\ldots\mu_q\nu_1\ldots\nu_p}A^{\nu_1\ldots\nu_p}\,, \quad q = D-p\,. \tag{A.13}$$

Note in particular the different order in which the indices in (A.13) are contracted with respect to (A.8). With this definition, we have the usual identity

$$\star\star A_{(p)} = (-)^{pq+1}A_{(p)}\,, \quad q = D-p\,. \tag{A.14}$$

A.4 Spinors in five dimensions

Our five-dimensional spinors are symplectic-Majorana spinors that transform in the $(4,2)$ of $\mathrm{SO}(5)\otimes\mathrm{SU}(2)$. The generators U_{ij} of the R-symmetry group SU(2) are defined to be anti-Hermitian and symmetric, i.e.

$$(U_i{}^j)^* = -U_j{}^i\,, \qquad U_{ij} = U_{ji}\,. \tag{A.15}$$

A symmetric traceless $U_i{}^j$ corresponds to a symmetric U^{ij} since we lower or raise SU(2) indices using the ε-symbol contracting the indices in a northwest-southeast (NW–SE) convention

$$X^i = \varepsilon^{ij}X_j\,, \qquad X_i = X^j\varepsilon_{ji}\,, \qquad \varepsilon_{12} = -\varepsilon_{21} = \varepsilon^{12} = 1\,. \tag{A.16}$$

The actual value of ε is here given as an example. It is in fact arbitrary as long as it is antisymmetric, $\varepsilon^{ij} = (\varepsilon_{ij})^*$ and $\varepsilon_{jk}\varepsilon^{ik} = \delta_j{}^i$. When the SU(2) indices on spinors are omitted, NW-SE contraction is understood

$$\bar\lambda\psi = \bar\lambda^i\psi_i\,, \tag{A.17}$$

The charge conjugation matrix C and $C\gamma_a$ are antisymmetric. The matrix C is unitary and γ_a is Hermitian apart from the timelike one, which is anti-Hermitian. The bar is the Majorana bar

$$\bar\lambda^i = (\lambda^i)^T C\,. \tag{A.18}$$

We define the charge conjugation operation on spinors as

$$(\lambda^i)^C \equiv \alpha^{-1}B^{-1}\varepsilon^{ij}(\lambda^j)^*\,,\qquad \bar\lambda^{iC} \equiv \overline{(\lambda^i)^C} = \alpha^{-1}\left(\bar\lambda^k\right)^* B\varepsilon^{ki}\,, \tag{A.19}$$

where $B = C\gamma_0$, and $\alpha = \pm 1$ when one uses the convention that complex conjugation does not interchange the order of spinors, or $\alpha = \pm\,\mathrm{i}$ when it does. Symplectic Majorana spinors satisfy $\lambda = \lambda^C$. Charge conjugation acts on gamma-matrices as $(\gamma_a)^C = -\gamma_a$, does not change the order of matrices, and works on matrices in SU(2) space as $M^C = \sigma_2 M^* \sigma_2$. Complex conjugation can then be replaced by charge conjugation, if for every bi-spinor one inserts a factor -1. Then, e.g. the expressions

$$\bar\lambda^i\gamma_\mu\lambda^j\,,\qquad \mathrm{i}\,\bar\lambda^i\lambda_i \tag{A.20}$$

are real for symplectic Majorana spinors. For more details, see [137].

A.5 Gamma-matrices in five dimensions

The gamma-matrices γ_a are defined as matrices that satisfy the Clifford-algebra

$$\{\gamma_a,\gamma_b\} \equiv \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}\,. \tag{A.21}$$

Completely anti-symmetrized products of gamma-matrices are denoted in three different ways

$$\gamma_{(n)} = \gamma_{a_1\cdots a_n} = \gamma_{[a_1}\cdots\gamma_{a_n]}\,. \tag{A.22}$$

The product of all gamma-matrices is proportional to the unit matrix in odd dimensions. We use

$$\gamma^{abcde} = \mathrm{i}\,\varepsilon^{abcde}\,. \tag{A.23}$$

This implies that the dual of a $(5-n)$-antisymmetric gamma-matrix is the n-antisymmetric gamma-matrix given by

$$\gamma_{a_1\ldots a_n} = \tfrac{1}{(5-n)!}\,\mathrm{i}\,\varepsilon_{a_1\ldots a_n b_1\ldots b_{5-n}}\gamma^{b_{5-n}\ldots b_1}\,. \tag{A.24}$$

For convenience, we will give the values of gamma-contractions like

$$\gamma^{(m)}\gamma_{(n)}\gamma_{(m)} = c_{n,m}\gamma_{(n)}\,, \tag{A.25}$$

where the constants $c_{n,m}$ are given in table A.1. The constants for $n,m > 2$ can easily be obtained from (A.24) and table A.1.

Changing the order of spinors in a bilinear leads to the following signs

$$\bar\psi^{(1)}\gamma_{(n)}\chi^{(2)} = t_n\,\bar\chi^{(2)}\gamma_{(n)}\psi^{(1)}\qquad \begin{cases} t_n = +1 \text{ for } n = 0,1\\ t_n = -1 \text{ for } n = 2,3\end{cases} \tag{A.26}$$

where the labels (1) and (2) denote any SU(2) representation.

$c_{n,m}$	$m=1$	$m=2$
$n=0$	5	−20
$n=1$	−3	−4
$n=2$	1	4

Table A.1: Coefficients used in contractions of gamma-matrices.

A.6 Fierz-identities in five dimensions

The sixteen different gamma-matrices $\gamma_{(n)}$ for $n = 0,1,2$ form a complete basis for four-dimensional matrices. Similarly, the identity matrix $\mathbb{1}_2$ and the three Pauli-matrices σ^i for $i = 1,2,3$ form a basis for two-dimensional matrices. A change of basis in a product of two pseudo-Majorana spinors will give rise to so-called Fierz-rearrangement formulae, which in their simplest form are given by

$$\psi_j\bar{\lambda}^i = -\tfrac{1}{4}\bar{\lambda}^i\psi_j - \tfrac{1}{4}\bar{\lambda}^i\gamma^a\psi_j\gamma_a + \tfrac{1}{8}\bar{\lambda}^i\gamma^{ab}\psi_j\gamma_{ab}\,, \qquad \bar{\psi}^{[i}\lambda^{j]} = -\tfrac{1}{2}\bar{\psi}\lambda\varepsilon^{ij}\,. \tag{A.27}$$

Using such Fierz-rearrangements, other useful identities can be deduced for working with cubic fermion terms

$$\begin{aligned} &\lambda_j\bar{\lambda}^j\lambda^i = \gamma^a\lambda_j\bar{\lambda}^j\gamma_a\lambda^i = \tfrac{1}{8}\gamma^{ab}\lambda^i\bar{\lambda}\gamma_{ab}\lambda\,,\\ &\gamma^{cd}\gamma_{ab}\lambda^i\bar{\lambda}\gamma^{cd}\lambda = 4\lambda^i\bar{\lambda}\gamma^{ab}\lambda\,. \end{aligned} \tag{A.28}$$

When one multiplies three spinor doublets, one should be able to write the result in terms of $\binom{8}{3} = 56$ independent structures. From analyzing the representations, one can obtain that these are in the $(4,2)+(4,4)+(16,2)$ representations of $\overline{\mathrm{SO}(5)}\times\mathrm{SU}(2)$. They are

$$\begin{aligned} &\lambda_j\bar{\lambda}^j\lambda^i = \gamma^a\lambda_j\bar{\lambda}^j\gamma_a\lambda^i = \tfrac{1}{8}\gamma^{ab}\lambda^i\bar{\lambda}\gamma_{ab}\lambda\,,\\ &\lambda^{(k}\bar{\lambda}^i\lambda^{j)}\,,\\ &\lambda_j\bar{\lambda}^j\gamma_a\lambda^i\,. \end{aligned} \tag{A.29}$$

As a final Fierz-identity, we give a three-spinor identity which is needed to prove the invariance under supersymmetry of the action for a vector multiplet

$$\psi^i_{[I}\bar{\psi}_J\psi_{K]} = \gamma^a\psi^i_{[I}\bar{\psi}_J\gamma_a\psi_{K]}\,. \tag{A.30}$$

A similar identity was required to get the full hypermultiplet action from the [field]×[non-closure] method

$$\psi^i_{[\mu}\bar{\psi}_\nu\psi_{\rho]} = \gamma_a\psi^i_{[\mu}\bar{\psi}_\nu\gamma^a\psi_{\rho]}\,. \tag{A.31}$$

Appendix B

Reductions

B.1 Conventions

We use mostly plus signature $(- + \cdots +)$. All metrics are Einstein-frame metrics. Unless stated otherwise, doubly hatted fields and indices are eleven-dimensional, singly hatted fields and indices ten-dimensional while unhatted ones are nine-dimensional. Greek indices $\hat{\mu}, \hat{\nu}, \hat{\rho} \ldots$ denote world coordinates and Latin indices $\hat{a}, \hat{b}, \hat{c} \ldots$ represent tangent space-time. They are related by the vielbeins $\hat{e}_{\hat{\mu}}{}^{\hat{a}}$ and inverse vielbeins $\hat{e}_{\hat{a}}{}^{\hat{\mu}}$. Explicit indices x, y, z are underlined when flat and non-underlined when curved. When indices are omitted we use form notation.

B.2 Reduction of Ricci scalar

Covariant constancy of the metric translates to

$$D_\mu e_\nu{}^a = 0 = \partial_\mu e_\nu{}^a - \Gamma_{\mu\nu}{}^\rho e_\rho{}^a + \omega_\mu{}^{ab} e_{\nu b} \,. \tag{B.1}$$

Taking the antisymmetric part we obtain

$$\Omega_{\mu\nu}{}^a \equiv \omega_{[\mu}{}^{ab} e_{\nu]b} = 2\partial_\mu e_\nu{}^a \,, \qquad \Omega_{abc} = e_a{}^\mu e_b{}^\nu \Omega_{\mu\nu c} \,, \qquad \omega_{abc} = \tfrac{1}{2}(\Omega_{abc} + \Omega_{cab} - \Omega_{bca}) \,. \tag{B.2}$$

The Riemann curvature and Ricci scalar in terms of the spin connection are given by

$$R_{\mu\nu}{}^{ab} = 2D_{[\mu}\omega_{\nu]}{}^{ab} = 2\partial_{[\mu}\omega_{\nu]}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b \,, \qquad R = R_{\mu\nu}{}^{ab} e_a{}^\mu e_b{}^\nu \,. \tag{B.3}$$

Using the vielbein-Ansätze (3.8) the spin connections reduce as follows

$$\begin{aligned} &\hat{\omega}_{abc} = e^{-\alpha\phi}(\omega_{abc} + 2\alpha\eta_{a[b}\partial_{c]}\phi) \,, \qquad \hat{\omega}_{ab\underline{z}} = \tfrac{1}{2} e^{(\beta-2\alpha)\phi} F_{ab}(A) \,, \qquad \hat{\omega}_{a\underline{zz}} = 0 \,, \\ &\hat{\omega}_{\underline{z}bc} = -\tfrac{1}{2} e^{(\beta-2\alpha)\phi} F_{bc}(A) \,, \qquad \hat{\omega}_{\underline{z}b\underline{z}} = -\beta e^{-\alpha\phi}\partial_b\phi \,, \qquad \hat{\omega}_{\underline{zzz}} = 0 \,. \end{aligned} \tag{B.4}$$

The determinant of the metric reduces to

$$\hat{e} = e^{(\beta + D\alpha)\phi} e \,. \tag{B.5}$$

The Einstein-Hilbert action can now be written as

$$\begin{aligned}\hat{S} &= \frac{1}{2\kappa_{D+1}^2}\int \mathrm{d}^D x\mathrm{d}z\,\hat{e}\hat{R}(\hat{\omega}) \\ &= -\frac{1}{2\kappa_{D+1}^2}\int \mathrm{d}^D x\mathrm{d}z\,2\hat{e}\left(\hat{\omega}_{[\hat{\mu}}{}^{\hat{a}\hat{c}}\hat{\omega}_{\hat{\nu}]\hat{c}}{}^{\hat{b}}\hat{e}_{\hat{a}}{}^{\hat{\mu}}\hat{e}_{\hat{b}}{}^{\hat{\nu}}\right). \end{aligned} \tag{B.6}$$

The $\mathrm{d}\omega$ term has been partially integrated, and the boundary term is assumed to be zero. Note that in the case of non-trivial boundaries some extra requirements will have to be satisfied for these terms to vanish. Substituting the expressions for the vielbeins and spin connections, we obtain

$$\begin{aligned} S &= \frac{1}{2\kappa_D^2}\int \mathrm{d}^D x\mathrm{d}z\, e^{[\beta-(D-2)\alpha]\phi} e\left\{2\omega_{[a}{}^{ac}\omega_{b]c}{}^{b} + 2\left[\beta+(D-2)\alpha\right]\omega_a{}^{ac}\partial_c\phi\right. \\ &\quad \left. + \left[\alpha^2(D-1)(D-2) + 2\alpha\beta(D-1)\right](\partial\phi)^2 - \tfrac{1}{4}e^{2(\beta-\alpha)\phi}F^2(A)\right\}. \end{aligned} \tag{B.7}$$

This action can be brought into a canonical form by choosing

$$\alpha^2 = \tfrac{1}{2(D-1)(D-2)}\,, \qquad \beta = -(D-2)\alpha\,. \tag{B.8}$$

This leads to the following scalar-gravity-Maxwell action:

$$S = \frac{1}{2\kappa_D^2}\int \mathrm{d}^D x\mathrm{d}z\, e\left\{R(\omega) - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{-2(D-1)\alpha\phi}F^2(A)\right\}. \tag{B.9}$$

B.3 Spinors and Γ-matrices in ten and nine dimensions

The Γ-matrices in ten ($\Gamma_{\hat{\mu}}$) and nine (γ_μ) dimensions can be chosen to satisfy

$$\Gamma_{\hat{\mu}}^\dagger = \eta_{\hat{\mu}\hat{\mu}}\Gamma_{\hat{\mu}} \qquad \text{and} \qquad \gamma_\mu^\dagger = \eta_{\mu\mu}\gamma_\mu\,, \tag{B.10}$$

respectively. In ten dimensions we can also choose the Γ-matrices to be real, while in nine dimensions they will be purely imaginary, which implies that

$$\Gamma_{\hat{\mu}}^T = \eta_{\hat{\mu}\hat{\mu}}\Gamma_{\hat{\mu}} \qquad \text{and} \qquad \gamma_\mu^T = -\eta_{\mu\mu}\gamma_\mu\,. \tag{B.11}$$

In ten dimensions the minimal spinor is a 32 component Majorana-Weyl spinor with 16 (real) degrees of freedom. With the choice

$$\Gamma_{11} \equiv -\Gamma_{\underline{0}\cdots\underline{9}}\,, \qquad \Gamma_{11} = \begin{pmatrix}\mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}, \tag{B.12}$$

we can write a ten-dimensional Majorana-Weyl spinor as being composed of nine-dimensional, 16 component, Majorana-Weyl spinors according to

$$\psi_+^{MW} = \begin{pmatrix}\psi_1 \\ 0\end{pmatrix}, \qquad \psi_-^{MW} = \begin{pmatrix}0 \\ \psi_2\end{pmatrix}, \tag{B.13}$$

where ψ_i are nine-dimensional Majorana-Weyl spinors and + or − denotes the chirality of the ten-dimensional spinor. The split of an arbitrary ten-dimensional spinor into two Majorana-Weyl spinors of opposite chirality can of course be done without reference to nine dimensions (through the specific choice of Γ_{11}), but each ten-dimensional Majorana-Weyl spinor will then in general have 32 non-zero components even though it only has 16 degrees of freedom. In order to reduce to nine dimensions we use

$$\Gamma_{11} = \sigma_3 \otimes \mathbb{1}\,, \qquad \Gamma_{\underline{z}} = \sigma_1 \otimes \mathbb{1}\,, \qquad \Gamma_a = \sigma_2 \otimes \gamma_a\,, \tag{B.14}$$

where z is the reduction coordinate and the Pauli matrices are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.15}$$

As mentioned above the nine-dimensional γ-matrices are purely imaginary. If we work with a reduction of type IIB, where the two spinors have the same chirality, it may be convenient to introduce complex, nine-dimensional, Weyl spinors according to

$$\begin{aligned} \psi_c &= \psi_1 + i\psi_2\,, \qquad & \lambda_c &= \lambda_2 + i\lambda_1\,, \\ \epsilon_c &= \epsilon_1 + i\epsilon_2\,, \qquad & \tilde{\lambda}_c &= \tilde{\lambda}_2 + i\tilde{\lambda}_1\,, \end{aligned} \tag{B.16}$$

which in ten-dimensional notation can be written as, e.g.

$$\psi_+^W = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} + i \begin{pmatrix} \psi_2 \\ 0 \end{pmatrix}. \tag{B.17}$$

If we instead work with a reduction of type IIA the two spinors will have opposite chirality, and can thus be composed into a ten-dimensional Majorana spinor according to

$$\psi^M = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}. \tag{B.18}$$

With the above mentioned decomposition into nine-dimensional Majorana-Weyl spinors we can write

$$\psi_\mu^M = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \epsilon^M = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \quad \lambda^M = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \tilde{\lambda}^M = \begin{pmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{pmatrix} \tag{B.19}$$

and

$$\psi_\mu^W = \begin{pmatrix} \psi_1 + i\psi_2 \\ 0 \end{pmatrix}, \quad \epsilon_\mu^W = \begin{pmatrix} \epsilon_1 + i\epsilon_2 \\ 0 \end{pmatrix}, \tag{B.20}$$

$$\lambda^W = \begin{pmatrix} 0 \\ \lambda_2 + i\lambda_1 \end{pmatrix}, \quad \tilde{\lambda}^W = \begin{pmatrix} 0 \\ \tilde{\lambda}_2 + i\tilde{\lambda}_1 \end{pmatrix}, \tag{B.21}$$

where the spinors without an M or W superscript are Majorana-Weyl spinors. Note also that it follows from the Clifford algebra and the choice of Γ_{11} that $\Gamma_{\underline{z}}$ is off-diagonal, which is crucial for this construction.

Appendix C

The geometry of scalar manifolds

In this appendix we will present the essential properties of hypercomplex manifolds, and show the relation with hyperkähler and quaternionic (Kähler) manifolds. We show how properties of the Nijenhuis tensor determine whether suitable connections for these geometries can be defined. We give the curvature relations, and finally the properties of symmetry transformations of these manifolds.

In [174] we showed that there is a map between conformal hypercomplex/hyperkähler and quaternionic(-Kähler) geometry, based on the coordinate basis chosen in section 7.1. The required geometrical properties for quaternionic manifolds were obtained by using the special coordinate basis for the identities and constraints given in chapter 6.

Hypercomplex manifolds were introduced in [177]. A very thorough paper on the subject is [178]. Examples of homogeneous hypercomplex manifolds that are not hyperkähler, can be found in [179,180], and are further discussed in appendix C of [86]. Non-compact homogeneous manifolds are dealt with in [181]. Various aspects have been treated in two workshops with mathematicians and physicists [182, 183]. To prepare this appendix, we extensively used [178]. However, in some parts we used original methods.

C.1 The family of quaternionic-like manifolds

Let V be a real vector space of dimension $4r$, whose coordinates we indicate as q^X (with $X = 1, \ldots, 4r$). We define a *hypercomplex structure* H on V to be a triple of complex structures J^α, (with $\alpha = 1, 2, 3$) which realize the algebra of quaternions,

$$J^\alpha J^\beta = -\delta^{\alpha\beta} \mathbb{1}_{4r} + \varepsilon^{\alpha\beta\gamma} J^\gamma \,. \tag{C.1}$$

A *quaternionic structure* is the space of linear combinations $a_\alpha J^\alpha$ with a_α real numbers. In this case the three-dimensional space of complex structures is globally defined, but the individual complex structures do not have to be globally defined.

Let $\mathcal{M}$ be a $4r$-dimensional manifold. An *almost hypercomplex manifold* or *almost quaternionic manifold* is defined as a manifold $\mathcal{M}$ with a field of hypercomplex or quaternionic structures.

	no preserved metric	with a preserved metric
no SU(2) curvature	hypercomplex $G\ell(r, \mathbb{H})$	hyperkähler $USp(2r)$
non-zero SU(2) curvature	quaternionic $SU(2) \cdot G\ell(r, \mathbb{H})$	quaternionic-Kähler $SU(2) \cdot USp(2r)$

Table C.1: *Quaternionic-like manifolds. These are the manifolds that have a quaternionic structure satisfying (C.1) and (C.2). The holonomy group is indicated. For the right column the metric may give another real form as e.g.* $USp(2, 2(r-1))$.

The 'almost' disappears under one extra condition. Different terminologies are used to express this condition. Sometimes it is said that the structure should be 1-integrable. The same condition is also expressed as the statement that the structure should be covariantly constant using some connections, and it is also sometimes expressed as the 'preservation of the structure' using that connection. The connection[1] here should be a symmetric (i.e. 'torsionless') connection $\Gamma_{(XY)}{}^Z$ and possibly an SU(2) connection $\omega_X{}^\alpha$. The condition is

$$0 = \mathfrak{D}_X J^\alpha{}_Y{}^Z \equiv \partial_X J^\alpha{}_Y{}^Z - \Gamma_{XY}{}^W J^\alpha{}_W{}^Z + \Gamma_{XW}{}^Z J^\alpha{}_Y{}^W + 2\varepsilon^{\alpha\beta\gamma}\omega_X{}^\beta J^\gamma{}_Y{}^Z \,. \tag{C.2}$$

If the SU(2) connection has non-vanishing curvature, the manifold is called *quaternionic*.[2] If the condition (C.2) holds with vanishing SU(2) connection, i.e.

$$0 = \mathfrak{D}_X J^\alpha{}_Y{}^Z \equiv \partial_X J^\alpha{}_Y{}^Z - \Gamma_{XY}{}^W J^\alpha{}_W{}^Z + \Gamma_{XW}{}^Z J^\alpha{}_Y{}^W \,, \tag{C.3}$$

then the manifold is *hypercomplex*. If there is a hermitian metric, i.e. a metric such that

$$J^\alpha{}_X{}^Z g_{ZY} = -J^\alpha{}_Y{}^Z g_{ZX} \,, \tag{C.4}$$

and if this metric is preserved using the connection Γ (i.e. if Γ is the Levi-Civita connection of this metric) then the hypercomplex and quaternionic manifolds are respectively promoted to hyperkähler and quaternionic-Kähler manifolds. Hence this gives rise to the scheme[3] of table C.1.

We will show in section C.4 that the spaces in the upper row have a Ricci tensor that is antisymmetric, and those in the right column have a Ricci tensor that is symmetric (and Einstein). It follows then that the hyperkähler manifolds are Ricci-flat. The restriction of the holonomy group when one goes to the right column, just follows from the fact that the presence of a metric restricts the holonomy group further to a subgroup of $O(4r)$.[4]

[1] The word 'connection' is by mathematicians mostly used as the derivative including the 'connection coefficients'. We use here 'connection' as a word denoting these coefficients, i.e. gauge fields.

[2] For $r = 1$ there are subtleties in the definition, to which we will return below.

[3] The table is essentially taken over from [178], where there is also the terminology unimodular hypercomplex or unimodular quaternionic if the $G\ell(r)$ is reduced to $S\ell(r)$.

[4] The dot notation means that it is the product up to a common factor in both groups that does not contribute. In fact, one considers e.g. SU(2) and $USp(2r)$ on coset elements as working one from the left, and the other from the right. Then if both are -1, they do not contribute. Thus: $SU(2) \cdot USp(2r) = \frac{SU(2)\times USp(2r)}{\mathbb{Z}_2}$.

A theorem of Swann [168] shows that all quaternionic-Kähler manifolds have a corresponding hyperkähler manifold which admit a quaternionically extended homothety [a homothety extended to an SU(2) vector as in (6.42)] and which has three complex structures that rotate under an isometric SU(2) action. It has been shown in [164] that this can be implemented in superconformal tensor calculus to construct the actions of hypermultiplets in any quaternionic-Kähler manifold from a hyperkähler cone. Similarly, it has been proven in [184, 185] that any quaternionic manifold is related to a hypercomplex manifold.

Locally there is a vielbein f_X^{iA} (with $i = 1,2$ and $A = 1,\ldots,r$) with reality conditions as in (6.19). In supersymmetry we always start from these vielbeins and the integrability condition can be expressed as

$$\partial_X f_Y^{iA} - \Gamma^Z_{XY} f_Z^{iA} + f_Y^{jA}\omega_{Xj}{}^i + f_Y^{iB}\omega_{XB}{}^A = 0\,. \tag{C.5}$$

C.2 Conventions for curvatures and lemmas

We start with the notations for curvatures. The main conventions for target space curvature, fermion reparametrization curvature and SU(2) curvature are

$$\begin{aligned} R_{XYZ}{}^W &\equiv 2\partial_{[X}\Gamma_{Y]Z}{}^W + 2\Gamma_{V[X}{}^W\Gamma_{Y]Z}{}^V\,,\\ \mathcal{R}_{XYB}{}^A &\equiv 2\partial_{[X}\omega_{Y]B}{}^A + 2\omega_{[X|C|}{}^A\omega_{Y]B}{}^C\,,\\ \mathcal{R}_{XYi}{}^j &\equiv 2\partial_{[X}\omega_{Y]i}{}^j + 2\omega_{[X|k|}{}^j\omega_{Y]i}{}^k\,. \end{aligned} \tag{C.6}$$

The SU(2) curvature and connection $\omega_{Xi}{}^j$ are hermitian traceless,[5] and one can make the transition to triplet indices $\alpha = 1,2,3$ by using the sigma matrices

$$\begin{aligned} \mathcal{R}_{XYi}{}^j &= \mathrm{i}\,(\sigma^\alpha)_i{}^j\mathcal{R}_{XY}{}^\alpha\,,\\ \mathcal{R}_{XY}{}^\alpha &= -\tfrac{1}{2}\,\mathrm{i}\,(\sigma^\alpha)_i{}^j\mathcal{R}_{XYj}{}^i = 2\partial_{[X}\omega_{Y]}{}^\alpha + 2\varepsilon^{\alpha\beta\gamma}\omega_X{}^\beta\omega_Y{}^\gamma\,. \end{aligned} \tag{C.7}$$

This transition between doublet and triplet notation is valid for any triplet object, e.g. the complex structures. It is useful to know the translation of the inner product: $\mathcal{R}_i{}^j\mathcal{R}_j{}^i = -2\mathcal{R}^\alpha\mathcal{R}^\alpha$.

The curvatures by definition all satisfy the Bianchi identities that say that they are closed 2-forms, e.g.

$$\mathfrak{D}_{[X}R_{YZ]V}{}^W = 0\,. \tag{C.8}$$

Furthermore, due to the torsionless (symmetric) connection, also the cyclicity property holds.

$$R_{XYZ}{}^W + R_{ZXY}{}^W + R_{YZX}{}^W = 0\,. \tag{C.9}$$

The Ricci tensor is defined as

$$R_{XY} = R_{ZXY}{}^Z\,. \tag{C.10}$$

This is not necessarily symmetric. When Γ is the Levi-Civita connection of a metric, then one can raise and lower indices, $R_{WZXY} = R_{XYWZ}$ and the Ricci tensor is symmetric. Then one defines the scalar curvature as $R = g^{XY}R_{XY}$.

[5]This means symmetric if the indices are put at equal height using the raising or lowering tensor ε_{ij} (NW–SE convention).

We now present three lemmas that are useful in connecting scalar manifold indices with $G\ell(r,\mathbb{H})$ indices. These lemmas are used in section 6.3 and will simplify further derivations in this appendix.

Lemma C.2.1 *If a matrix $M_X{}^Y$ satisfies*

$$[J^\alpha, M] = 2\varepsilon^{\alpha\beta\gamma} J^\beta m^\gamma\,, \tag{C.11}$$

for some numbers m^γ, then the latter are given by

$$4\, r\, m^\alpha = \mathrm{Tr}\,(J^\alpha M)\,, \tag{C.12}$$

and the matrix can be written as

$$M = -m^\alpha J^\alpha + N\,, \qquad [N, J^\alpha] = 0\,. \tag{C.13}$$

A matrix M of this type is said to 'normalize the hypercomplex structure'.

Proof. The first statement is proven by taking the trace of (C.11) with J^δ. Inserting this value of m^α in (C.13), it is obvious that the remainder N commutes with the complex structures. ■

Lemma C.2.2 *If a matrix $M_X{}^Y$ commutes with the complex structures, then it can be written as*

$$M_X{}^Y = M_A{}^B f_X^{iA} f_{iB}^Y\,. \tag{C.14}$$

and vice-versa, any $M_A{}^B$ matrix can be transformed with (C.14) to a matrix commuting with the complex structures.

Proof. The vice-versa statement is easy. For the other direction, one replaces J^α with $J_i{}^j$ as in (6.28). Then multiply this equation with $f_{jA}^X f_Z^{kB}$ and consider the traceless part in AB. ■

Lemma C.2.3 *If a tensor $R_{[XY]Z}{}^W$ satisfies the cyclicity condition (C.9) and commutes with the complex structures,*

$$R_{XYZ}{}^V J^\alpha{}_V{}^W - J^\alpha{}_Z{}^V R_{XYV}{}^W = 0\,, \tag{C.15}$$

it can be written in terms of a tensor $W_{ABC}{}^D$ that is symmetric in its lower indices. If $R_{XYZ}{}^Z = 0$, then also W is traceless.

Proof. By the previous theorem, we can write

$$R_{XYW}{}^Z = f_W^{iA} f_{iB}^Z \mathcal{R}_{XYA}{}^B\,, \qquad \mathcal{R}_{XYA}{}^B = \tfrac{1}{2} f_{iA}^W f_Z^{iB} R_{XYW}{}^Z\,. \tag{C.16}$$

We can change all indices to tangent indices, defining

$$R_{ij,CDB}{}^A \equiv f_{Ci}^X f_{jD}^Y \mathcal{R}_{XYB}{}^A = -R_{ji,DCB}{}^A\,. \tag{C.17}$$

The cyclicity property of R can be used to obtain

$$0 = f_Z^{iA} R_{[WXY]}{}^Z = f_{[Y}^{iB} \mathcal{R}_{WX]B}{}^A\,. \tag{C.18}$$

We multiply this with $f^X_{iC} f^Y_{Dj} f^W_{kE}$, leading to

$$R_{kj,ECD}{}^A + R_{kj,CDE}{}^A + 2R_{jk,DEC}{}^A = 0\,. \tag{C.19}$$

The symmetric part in (jk) of this equation implies that $R_{(jk),ABC}{}^D = 0$ [multiply the equation by 3, and subtract both cyclicity rotated terms in (CDE)]. Thus we find

$$R_{ij,CDB}{}^A = -\tfrac{1}{2}\varepsilon_{ij} W_{CDB}{}^A\,, \tag{C.20}$$

with

$$W_{CDB}{}^A \equiv \varepsilon^{ij} f^X_{jC} f^Y_{iD} \mathcal{R}_{XYB}{}^A = \tfrac{1}{2}\varepsilon^{ij} f^X_{jC} f^Y_{iD} f^Z_{kB} f^{Ak}_W R_{XYZ}{}^W\,. \tag{C.21}$$

Now we prove that W is completely symmetric in the lower indices. The definition immediately implies symmetry in the first two. The $[jk]$ antisymmetric part of (C.19) gives

$$W_{ECD}{}^A + W_{DCE}{}^A - 2W_{EDC}{}^A = 0\,. \tag{C.22}$$

Antisymmetrizing this in two of the indices gives the desired result.

Finally, it is obvious from (C.21) that the tracelessness of R and W are equivalent. ■

The full result for such a curvature tensor is thus

$$R_{XYW}{}^Z = -\tfrac{1}{2} f^{Ai}_X \varepsilon_{ij} f^{jB}_Y f^{kC}_W f^Z_{kD} W_{ABC}{}^D\,. \tag{C.23}$$

C.3 The connections

In the definition of hypercomplex and quaternionic manifolds, the affine connection $\Gamma_{XY}{}^Z$ and an SU(2) connection $\omega_X{}^\alpha$ appear. In this subsection we will show how they can be obtained. The crucial ingredient is the Nijenhuis tensor.

C.3.1 Nijenhuis tensor

A Nijenhuis tensor $N^{\alpha\beta Z}_{XY}$ is defined for any combination of two complex structures, but we will use only the 'diagonal' Nijenhuis tensor (normalization for later convenience)

$$N_{XY}{}^Z \equiv \tfrac{1}{6} J^\alpha{}_X{}^W \partial_{[W} J^\alpha{}_{Y]}{}^Z - (X \leftrightarrow Y) = -N_{YX}{}^Z\,. \tag{C.24}$$

It satisfies a useful relation

$$N_{XY}{}^Z = J^\alpha{}_X{}^{X'} N_{X'Y}{}^{Z'} J^\alpha{}_{Z'}{}^Z\,, \tag{C.25}$$

from which one can deduce that it is traceless.

C.3.2 Obata connection and hypercomplex manifolds

The torsionless *Obata connection* [165] is defined as

$$\Gamma^{\mathrm{Ob}}{}_{XY}{}^Z = -\tfrac{1}{6}\left(2\partial_{(X} J^\alpha{}_{Y)}{}^W + \varepsilon^{\alpha\beta\gamma} J^\beta{}_{(X}{}^U \partial_{|U|} J^\gamma{}_{Y)}{}^W\right) J^\alpha{}_W{}^Z\,. \tag{C.26}$$

First, note that if a manifold is hypercomplex, i.e. if (C.3) is satisfied, then by inserting the expression for ∂J from that equation in the right-hand side of (C.26), one finds that the affine connection of the hypercomplex manifold should be the Obata connection, $\Gamma = \Gamma^{\rm Ob}$. One may thus answer the question whether an almost hypercomplex manifold [i.e. with three matrices satisfying (C.1)], defines a hypercomplex manifold [i.e. satisfies (C.3)]. As we now know that the affine connection in (C.3) should be (C.26), this can just be checked. For that purpose, the following equation is useful:

$$\partial_X J^\alpha{}_Y{}^Z - \left(\Gamma^{\rm Ob}{}_{XY}{}^W + N_{XY}{}^W\right) J^\alpha{}_W{}^Z + \left(\Gamma^{\rm Ob}{}_{XW}{}^Z + N_{XW}{}^Z\right) J^\alpha{}_Y{}^W = 0\,. \tag{C.27}$$

It shows that any hypercomplex structure can be given a torsionful connection such that the complex structures are covariantly constant. The condition for a hypercomplex manifold is thus that this connection is torsionless, i.e. that the Nijenhuis tensor vanishes. In conclusion, *a hypercomplex manifold consists of the following data: a manifold $\mathcal{M}$, with a hypercomplex structure with vanishing Nijenhuis tensor*. In the main text, we only use the Obata connection, and we thus have $\Gamma = \Gamma^{\rm Ob}$.

C.3.3 Oproiu connection and quaternionic manifolds

For the quaternionic manifolds, the affine connection and SU(2) connection can not be uniquely defined. Indeed, one can easily check that (C.2) is left invariant when we change these two connections simultaneously using an arbitrary vector ξ_W as

$$\Gamma_{XY}{}^Z \to \Gamma_{XY}{}^Z + S_{XY}^{WZ}\xi_W\,, \qquad \omega_X{}^\alpha \to \omega_X{}^\alpha + J^\alpha{}_X{}^W \xi_W\,, \tag{C.28}$$

where S is the tensor

$$S_{ZW}^{XY} \equiv 2\delta^X_{(Z}\delta^Y_{W)} - 2J^\alpha{}_Z{}^{(X} J^\alpha{}_W{}^{Y)}\,, \tag{C.29}$$

which satisfies the relation

$$S_{ZW}^{XV} J^\alpha{}_V{}^Y - J^\alpha{}_W{}^V S_{ZV}^{XY} = 2\varepsilon^{\alpha\beta\gamma} J^\beta{}_Z{}^X J^\gamma{}_W{}^Y\,. \tag{C.30}$$

Under this transformation, the $G\ell(r,\mathbb{H})$ connection transforms as

$$\omega_{XA}{}^B \to \omega_{XA}{}^B + \tfrac{1}{2} f_Y^{iB} f_{iA}^Z S_{XZ}^{YW}\xi_W\,. \tag{C.31}$$

An invariant SU(2) connection is

$$\tilde\omega_X{}^\alpha = \omega_X{}^\alpha + \tfrac{1}{3} J^\alpha{}_X{}^Y J^\beta{}_Y{}^Z \omega_Z{}^\beta = \tfrac{2}{3}\omega_X{}^\alpha - \tfrac{1}{3}\varepsilon^{\alpha\beta\gamma} J^\beta{}_X{}^Y \omega_Y{}^\gamma\,. \tag{C.32}$$

If we use (C.2) in the expression for the Nijenhuis tensor, (C.24), we find that quaternionic manifolds do not have a vanishing Nijenhuis tensor, but the latter should satisfy

$$N_{XY}{}^Z = -J^\alpha{}_{[X}{}^Z \tilde\omega_{Y]}{}^\alpha\,. \tag{C.33}$$

This condition can be solved for $\tilde\omega$. We find

$$(1-2\,r)\,\tilde\omega_X{}^\alpha = N_{XY}{}^Z J^\alpha{}_Z{}^Y\,. \tag{C.34}$$

Thus the condition for an almost quaternionic manifold to be quaternionic is that the Nijenhuis tensor satisfies

$$(1-2\,r)\,N_{XY}{}^Z = -J^\alpha{}_{[X}{}^Z N_{Y]V}{}^W J^\alpha{}_W{}^V\,. \tag{C.35}$$

On the other hand, one may also use (C.2) in the expression for the Obata connection (C.26). Then we find that the affine connection for the quaternionic manifolds is given by

$$\Gamma_{XY}{}^Z = \Gamma^{\rm Ob}{}_{XY}{}^Z - J^\alpha{}_{(X}{}^Z\omega_{Y)}{}^\alpha - \tfrac{1}{3}S_{XY}^{ZU}J^\alpha{}_U{}^V\omega_V{}^\alpha\,, \tag{C.36}$$

which exhibits the transformation (C.28).

One can take a gauge choice for the invariance. A convenient choice is to impose

$$J^\alpha{}_Y{}^Z\omega_Z{}^\alpha = 0\,. \tag{C.37}$$

With this choice $\tilde\omega_X{}^\alpha = \omega_X{}^\alpha$. The affine connection in (C.36) simplifies, and this expression is called the Oproiu connection [186]

$$\begin{aligned}\Gamma^{\rm Op}{}_{XY}{}^Z &\equiv \Gamma^{\rm Ob}{}_{XY}{}^Z - J^\alpha{}_{(X}{}^Z\omega_{Y)}{}^\alpha\\ &= \Gamma^{\rm Ob}{}_{XY}{}^Z + N_{XY}^Z - J^\alpha{}_Y{}^Z\omega_X{}^\alpha\,.\end{aligned} \tag{C.38}$$

The last expression shows that the Oproiu connection, which up to here was only proven to be necessary for solving (C.2), gives rise to covariantly constant complex structures under the condition (C.33). Indeed, the first two terms give a (torsionful) connection that gives rise to a covariantly constant hypercomplex structure, see (C.27), and the last term cancels the SU(2) connection. The condition (C.33) is now the condition that the connection $\Gamma^{\rm Op}$ is torsionless.

In conclusion, *a quaternionic manifold consists of the following data: a manifold $\mathcal{M}$, with a hypercomplex structure with Nijenhuis tensor satisfying (C.35).*

Levi-Civita connection and hyperkähler or quaternionic-Kähler manifolds. For hyperkähler manifolds, the Obata connection should coincide with the Levi-Civita connection of a metric. For quaternionic-Kähler manifolds, the connection that preserves the metric can be one of the equivalence class defined from the Oproiu connection by a transformation (C.28).

C.4 Curvature relations

C.4.1 Splitting according to holonomy

There are two interesting possibilities of splitting the curvature on quaternionic-like manifolds. First of all, the integrability condition of (C.5) yields that the total curvature on the manifold is the sum of the SU(2) curvature and the $\mathrm{G}\ell(r,\mathbb{H})$ curvature which shows that the (restricted) holonomy splits in these two factors:

$$\begin{aligned}R_{XYW}{}^Z &= R^{\rm SU(2)}{}_{XYW}{}^Z + R^{{\rm G}\ell(r,\mathbb{H})}{}_{XYW}{}^Z\\ &= -J^\alpha{}_W{}^Z\mathcal{R}_{XY}{}^\alpha + L_W{}^Z{}_A{}^B\mathcal{R}_{XYB}{}^A\,,\qquad \text{with}\qquad L_W{}^Z{}_A{}^B \equiv f^Z_{iA}f^{iB}_W\,.\end{aligned} \tag{C.39}$$

The matrices $L_A{}^B$ and J^α commute and their mutual trace vanishes

$$J^\alpha{}_X{}^Y L_Y{}^Z{}_A{}^B = L_X{}^Y{}_A{}^B J^\alpha{}_Y{}^Z\,, \qquad J^\alpha{}_Z{}^Y L_Y{}^Z{}_A{}^B = 0\,. \tag{C.40}$$

For hypercomplex (or hyperkähler) manifolds, the SU(2) curvature vanishes. Then the Riemann tensor commutes with the complex structures and using the cyclicity, one may use lemmas C.2.2 and C.2.3 to write

$$R_{XYW}{}^Z = -\tfrac{1}{2} f_X^{Ai}\varepsilon_{ij} f_Y^{jB} f_W^{kC} f_{kD}^Z W_{ABC}{}^D\,. \tag{C.41}$$

This W is symmetric in its lower indices. The Ricci tensor is then

$$R_{XY} = \tfrac{1}{2}\varepsilon_{ij} f_X^{iB} f_Y^{jC} W_{ABC}{}^A = -R_{YX}\,. \tag{C.42}$$

Thus the Ricci tensor for hypercomplex manifolds is antisymmetric. In general, the antisymmetric part can be traced back to the curvature of the U(1) part in $\mathrm{G}\ell(r,\mathbb{H}) = \mathrm{S}\ell(r,\mathbb{H})\times \mathrm{U}(1)$. Indeed, using the cyclicity condition:

$$R_{[XY]} = R_{Z[XY]}{}^Z = -\tfrac{1}{2} R_{XYZ}{}^Z = -\mathcal{R}^{\mathrm{U}(1)}_{XY}\,, \qquad \mathcal{R}^{\mathrm{U}(1)}_{XY} \equiv \mathcal{R}_{XYA}{}^A\,. \tag{C.43}$$

C.4.2 Splitting in Ricci and Weyl curvature

The separate terms in (C.39) for quaternionic manifolds do not satisfy the cyclicity condition, and thus are not bona fide curvatures. We will now discuss another splitting

$$R = R^{\mathrm{Ric}}{}_{XYW}{}^Z + R^{(\mathrm{W})}{}_{XYW}{}^Z\,. \tag{C.44}$$

Both terms will separately satisfy the cyclicity condition. The first part only depends on the Ricci tensor of the full curvature, and is called the '*Ricci part*'. The Ricci tensor of the second part will be zero, and this part will be called the '*Weyl part*' [178]. We will prove that the second part commutes with the complex structures. The lemmas of section C.2 then imply that the second part can be written in terms of a tensor $\mathcal{W}_{ABC}{}^D$, symmetric in the lower indices and traceless. This tensor appears in supersymmetric theories, which is another reason for considering this construction. The case $r = 1$ needs a separate treatment which will be discussed afterwards.

To define the splitting (C.44), we define the first term as a function of the Ricci tensor, and $R^{(\mathrm{W})}$ is just defined as the remainder. The definition of R^{Ric} again makes use of the tensor S in (C.29):

$$\begin{aligned} R^{\mathrm{Ric}}{}_{XYZ}{}^W &\equiv 2S_{Z[X}^{WV} B_{Y]V}\,,\\ B_{XY} &\equiv \frac{1}{4r}R_{(XY)} - \frac{1}{2r(r+2)}\Pi_{(XY)}{}^{ZW}R_{ZW} + \frac{1}{4(r+1)}R_{[XY]}\,. \end{aligned} \tag{C.45}$$

Here, Π projects bilinear forms onto hermitian ones, i.e.

$$\Pi_{XY}{}^{ZW} \equiv \tfrac{1}{4}\left(\delta_X{}^Z\delta_Y{}^W + J^\alpha{}_X{}^Z J^\alpha{}_Y{}^W\right). \tag{C.46}$$

The Ricci part satisfies several properties that can be checked by a straightforward calculation:

1. The Ricci tensor of R^{Ric} is just R_{XY}.

2. The cyclicity property (C.9).
3. Considered as a matrix in its last two indices, this matrix normalizes the hypercomplex structure (see lemma C.2.1).

Especially to prove the last one, the property (C.30) can be used (multiplying it with B_{UX} and antisymmetrizing in $[ZU]$). The relation is explicitly

$$\begin{aligned} &J^\alpha{}_Z{}^W\, R^{\rm Ric}{}_{XYW}{}^V - R^{\rm Ric}{}_{XYZ}{}^W\, J^\alpha{}_W{}^V = 2\varepsilon^{\alpha\beta\gamma} J^\beta{}_Z{}^V\, \mathcal{R}^{\rm Ric}{}_{XY}{}^\gamma\,,\\ &\text{with} \qquad \mathcal{R}^{\rm Ric}{}_{XY}{}^\alpha = \frac{1}{4r} J^\alpha{}_W{}^Z R^{\rm Ric}{}_{XYZ}{}^W = 2J^\alpha{}_{[X}{}^Z B_{Y]Z}\,. \end{aligned} \tag{C.47}$$

The important information is now that the full curvature also satisfies these 3 properties. The latter one is the integrability property of (C.2):

$$0 = 2\mathfrak{D}_{[X}\mathfrak{D}_{Y]} J^\alpha{}_Z{}^V = R_{XYW}{}^V J^\alpha{}_Z{}^W - R_{XYZ}{}^W J^\alpha{}_W{}^V - 2\varepsilon^{\alpha\beta\gamma}\mathcal{R}_{XY}{}^\gamma J^\beta{}_Z{}^V\,. \tag{C.48}$$

As in general for matrices normalizing the complex structure, we can also express $\mathcal{R}_{XY}{}^\alpha$ as

$$R_{XYZ}{}^W J^\alpha{}_W{}^Z = 4r\,\mathcal{R}_{XY}{}^\alpha\,. \tag{C.49}$$

This leads to properties of the Weyl part of the curvature. First of all, it implies that this part is Ricci-flat. Secondly it also satisfies the cyclicity property. Third, it also normalizes the hypercomplex structure, defining some $\mathcal{R}^{({\rm W})\alpha}_{XY}$. We will now prove that the latter is zero for $r > 1$.

The expression for this tensor satisfies a property that can be derived, starting from its definition, by first using the cyclicity of $R^{({\rm W})}$, then the equation saying that it normalizes the hypercomplex structure, and finally that it is Ricci-flat

$$\begin{aligned} r\mathcal{R}^{({\rm W})\alpha}_{XY} &= \tfrac14 J^\alpha{}_U{}^V R^{({\rm W})}{}_{XYV}{}^U = -\tfrac12 J^\alpha{}_U{}^V R^{({\rm W})}{}_{V[XY]}{}^U\\ &= -\epsilon^{\alpha\beta\gamma}\mathcal{R}^{({\rm W})\beta}_{V[X} J^\gamma{}_{Y]}{}^V\,. \end{aligned} \tag{C.50}$$

Multiplying with $J^\alpha{}_V{}^Y$ and antisymmetrizing leads to

$$J^\alpha{}_{[V}{}^Y \mathcal{R}^{({\rm W})\alpha}_{X]Y} = 0\,. \tag{C.51}$$

Secondly, multiplying (C.50) with $J^\delta{}_Z{}^X J^\delta{}_W{}^Y$, and using (C.50) again for multiplying the complex structures at the right-hand side, leads to

$$J^\beta{}_X{}^Z J^\beta{}_Y{}^V \mathcal{R}^{({\rm W})\alpha}_{ZV} = -\mathcal{R}^{({\rm W})\alpha}_{XY} \qquad \text{or} \qquad \Pi_{XY}{}^{ZV}\mathcal{R}^{({\rm W})\alpha}_{ZV} = 0\,. \tag{C.52}$$

Finally, multiplying (C.50) with $\varepsilon^{\alpha\delta\epsilon} J^\delta{}_Z{}^Y$ leads to

$$\mathcal{R}^{({\rm W})\alpha}_{XY} = 0\,, \qquad \text{if} \quad r > 1\,. \tag{C.53}$$

Therefore $R^{({\rm W})}{}_{XYZ}{}^V$ is a tensor that satisfies all conditions of lemma C.2.3, and we can thus write

$$R_{XYZ}{}^W = R^{\rm Ric}{}_{XYZ}{}^W - \tfrac12 f_X^{Ai}\varepsilon_{ij} f_Y^{jB} f_W^{kC} f_{kD}^Z \mathcal{W}_{ABC}{}^D\,. \tag{C.54}$$

For hypercomplex manifolds, we found that the full curvature can be written in terms of a tensor $W_{ABC}{}^D$, see (C.41), which is symmetric in the lower indices, but not necessarily traceless. One can straightforwardly compute the corresponding $\mathcal{W}$, and find that this is its traceless part, the trace determining the Ricci tensor:

$$\mathcal{W}_{ABC}{}^D = W_{ABC}{}^D - \frac{3}{2(r+1)}\delta^D_{(A} W_{BC)E}{}^E\,, \qquad R_{XY} = -\mathcal{R}_{XYA}{}^A = \tfrac12 \varepsilon_{ij} f_X^{iA} f_Y^{jB} W_{ABC}{}^C\,. \tag{C.55}$$

C.4.3 The one-dimensional case

As

$$\mathrm{G}\ell(1,\mathbb{H}) = \mathrm{S}\ell(1,\mathbb{H})\times \mathrm{U}(1) = \mathrm{SU}(2)\times\mathrm{U}(1)\,, \tag{C.56}$$

we have now two SU(2) factors in the full holonomy group. This can be written explicitly by splitting L in (C.39) in a traceless and trace part:

$$L_X{}^Y{}_A{}^B = \tfrac{1}{2}\,\mathrm{i}\,(\sigma^\alpha)_A{}^B J^{-\alpha}{}_X{}^Y + \tfrac{1}{2}\delta_X^Y\delta_A^B\,. \tag{C.57}$$

This leads to the $r = 1$ form of (C.39):

$$R_{XYW}{}^Z = -J^{+\alpha}{}_W{}^Z\mathcal{R}^{+\alpha}_{XY} - J^{-\alpha}{}_W{}^Z\mathcal{R}^{-\alpha}_{XY} + \delta_W^Z\mathcal{R}^{\mathrm{U}(1)}_{XY}\,, \tag{C.58}$$

where for emphasizing the symmetry, we indicate the original complex structures as $J^{+\alpha}{}_X{}^Y$.

We saw that for $r = 1$ we could not perform all steps to get to the decomposition (C.54). However, some authors define quaternionic and quaternionic-Kähler for $r = 1$ as a more restricted class of manifolds such that this decomposition is still valid [187]. For quaternionic-Kähler manifolds, the definition that is taken in general leads for $r = 1$ to the manifolds with holonomy $\mathrm{SU}(2)\times\mathrm{USp}(2)$, which is just SO(4). Thus with this definition all four-dimensional Riemannian manifolds would be quaternionic-Kähler. Therefore a further restriction is imposed. This further restriction is also natural in supergravity, as it is equivalent to a constraint that follows from requiring invariance of the supergravity action.

In general, as $\mathcal{R}^{(\mathrm{W})}$ normalizes the hypercomplex structure, we can by lemma C.2.1 and lemma C.2.2 write

$$R^{(\mathrm{W})}{}_{XYZ}{}^W = -\mathcal{R}^{(\mathrm{W})\,\alpha}_{XY}J^\alpha{}_Z{}^W + \mathcal{R}^{(\mathrm{W})\,B}_{XYA}L_Z{}^W{}_A{}^B = R^{(\mathrm{W})+}{}_{XYZ}{}^W + R^{(\mathrm{W})-}{}_{XYZ}{}^W\,. \tag{C.59}$$

We impose

$$\mathcal{R}^{(\mathrm{W})\,\alpha}_{XY} = 0\,, \tag{C.60}$$

as part of the definition of quaternionic manifolds with $r = 1$. This is thus the equation that is automatically valid for $r > 1$. Using lemma C.2.3, this implies that (C.54) is valid for all quaternionic manifolds.

In the one-dimensional case, we can see that a possible metric is already fixed up to a multiplicative function. Indeed, the C_{AB} that is used in (6.88) can only be proportional to ε_{AB}. Therefore, it is said that there is a *conformal metric*, i.e. a metric determined up to a (local) scale function $\lambda(q)$:

$$g_{XY} \equiv \lambda(q) f_X^{iA} f_Y^{jB}\varepsilon_{ij}\varepsilon_{AB}\,. \tag{C.61}$$

One can check that this metric is hermitian for any $\lambda(q)$, i.e. $J^\alpha{}_{XY} = J^\alpha{}_X{}^Z g_{ZY}$ is antisymmetric. The remaining question is whether this metric is covariantly constant, which boils down to the covariant constancy of C_{AB}. This condition can be simplified using the Schouten identity:

$$\mathfrak{D}_X C_{AB} = \partial_X C_{AB} + 2\omega_{X[A}{}^C C_{|C|B]} = \partial_X C_{AB} + \omega_{XC}{}^C C_{AB} = \varepsilon_{AB}\left(\partial_X\lambda(q) + \omega_{XC}{}^C\lambda(q)\right). \tag{C.62}$$

We can choose a function $\lambda(q)$ such that C is covariantly constant iff $\omega_{XC}{}^C$ is a total derivative, i.e. if the U(1) curvature vanishes. Thus in the one-dimensional case hypercomplex manifolds become hyperkähler, and quaternionic manifolds become quaternionic-Kähler if and only if the U(1) factor in the curvature part $G\ell(1,\mathbb{H})$ vanishes.

C.4.4 The curvature of Quaternionic-Kähler manifolds

In quaternionic-Kähler manifolds, the affine connection is the Levi-Civita connection of a metric. Therefore, the Ricci tensor is symmetric. As we have already proven that in the hypercomplex case the symmetric part vanishes, hyperkähler manifolds have vanishing Ricci tensor. Now we will prove that the quaternionic-Kähler spaces are Einstein, and that moreover the SU(2) curvatures are proportional to the complex structures with a proportionality factor that is dependent on the scalar curvature.

We start again from the integrability property (C.48). Multiplying with $J^{\delta}{}_V{}^X$ gives

$$\begin{aligned} R_{YZ}\delta^{\alpha\delta} - \varepsilon^{\alpha\delta\beta}R_{XYZ}{}^W J^\beta{}_W{}^X + J^\alpha{}_Z{}^W R_{XYW}{}^V J^\delta{}_V{}^X - \\ -2\varepsilon^{\alpha\beta\delta}\mathcal{R}_{ZY}{}^\beta + 2\delta^{\alpha\delta}\mathcal{R}_{XY}{}^\beta J^\beta{}_Z{}^X - 2\mathcal{R}_{XY}{}^\delta J^\alpha{}_Z{}^X = 0\,. \end{aligned} \quad (C.63)$$

The second and third term can be rewritten

$$\begin{aligned} R_{XYW}{}^V J^\delta{}_V{}^X &= -R_{YWX}{}^V J^\delta{}_V{}^X - R_{WXY}{}^V J^\delta{}_V{}^X \\ &= -R_{YWX}{}^V J^\delta{}_V{}^X + R_{YXW}{}^V J^\delta{}_V{}^X\,, \\ 2R_{XYW}{}^V J^\delta{}_V{}^X &= -4r\mathcal{R}_{YW}{}^\delta\,. \end{aligned} \quad (C.64)$$

In the first line, the cyclicity property of the Riemann tensor is used. Then, the symmetry in interchanging the first two and last two indices (here we use that the curvature originates from a Levi-Civita connection) and finally interchanging the indices on the last complex structure, using its antisymmetry (Hermiticity of the metric). This leads to

$$R_{YZ}\delta^{\alpha\delta} + \varepsilon^{\alpha\delta\beta}2(r-1)\mathcal{R}_{YZ}{}^\beta - 2(r-1)\mathcal{R}_{YX}{}^\delta J^\alpha{}_Z{}^X + 2\delta^{\alpha\delta}\mathcal{R}^\beta_{XY} J^\beta{}_Z{}^X = 0\,. \quad (C.65)$$

Multiplying with $\delta^{\alpha\delta}$ gives

$$R_{YZ} = -\tfrac{2}{3}(r+2)J^\beta{}_Z{}^X\mathcal{R}_{XY}{}^\beta\,. \quad (C.66)$$

On the other hand, multiplying (C.65) with $\varepsilon^{\alpha\delta\gamma}$ gives only a non-trivial result for $r \neq 1$, in which case we find

$$\text{for} \quad r > 1 \;: \qquad 2\mathcal{R}_{YZ}{}^\alpha = \varepsilon^{\alpha\beta\gamma} J^\beta{}_Y{}^X \mathcal{R}_{XZ}{}^\gamma\,. \quad (C.67)$$

We impose the same equation for $r = 1$. We will connect this equation to another requirement below.

By replacing $\varepsilon^{\alpha\beta\gamma}J^\beta{}_Y{}^X$ by $-(J^\alpha J^\gamma)_Y{}^X - \delta^X_Y\delta^{\alpha\gamma}$ we get

$$\mathcal{R}_{XY}{}^\alpha = -\tfrac{1}{3}J^\alpha{}_X{}^Z J^\beta{}_Z{}^V \mathcal{R}_{VY}{}^\beta = \frac{1}{2(r+2)}J^\alpha{}_X{}^Z R_{ZY}\,. \quad (C.68)$$

We also have

$$J^\alpha{}_X{}^Z\mathcal{R}_{ZY}{}^\beta = \varepsilon^{\alpha\beta\gamma}\mathcal{R}_{XY}{}^\gamma - \frac{1}{2(r+2)}\delta^{\alpha\beta}R_{XY}\,. \quad (C.69)$$

The final step is obtained by using (C.48) once more. Now multiply this equation with $\varepsilon^{\alpha\beta\gamma}J^{\beta YX}J^\gamma{}_V{}^U$, and use for the contraction of the Riemann curvature tensor with $J^{\beta YX}$ that we may interchange pairs of indices such that (C.49) can be used. Then everywhere $J^\alpha\mathcal{R}^\beta$ appears, for which we can use (C.69). This leads to the equation expressing that the manifold is Einstein:

$$R_{XY} = \frac{1}{4r}g_{XY}R\,. \quad (C.70)$$

With (C.68), the SU(2) curvature is proportional to the complex structure:

$$\mathcal{R}_{XY}{}^{\alpha} = \tfrac{1}{2}\nu J^{\alpha}{}_{XY}\,, \qquad \nu \equiv \frac{1}{4r(r+2)}R\,. \tag{C.71}$$

The Einstein property drastically simplifies the expression for B in (C.45) to

$$B_{XY} = \tfrac{1}{4}\nu g_{XY}\,. \tag{C.72}$$

The Ricci part of the curvature then becomes proportional to the curvature of a quaternionic projective space of the same dimension:

$$\left(R^{\mathbb{H}P^n}\right)_{XYWZ} = \tfrac{1}{2}g_{Z[X}g_{Y]W} + \tfrac{1}{2}J^{\alpha}_{XY}J^{\alpha}_{ZW} - \tfrac{1}{2}J^{\alpha}_{Z[X}J^{\alpha}_{Y]W} = \tfrac{1}{2}J^{\alpha}_{XY}J^{\alpha}_{ZW} + L_{[ZW]}{}^{AB}L_{[XY]AB}\,. \tag{C.73}$$

The full curvature decomposition is then

$$R_{XYWZ} = \nu(R^{\mathbb{H}P^n})_{XYWZ} + \tfrac{1}{2}L_{ZW}{}^{AB}\mathcal{W}_{ABCD}L_{XY}{}^{CD}\,, \tag{C.74}$$

with $\mathcal{W}_{ABCD}$ completely symmetric. The constraint appearing in supergravity fixes the value of ν to $-\kappa^2$. The quaternionic-Kähler manifolds appearing in supergravity thus have negative scalar curvature, and this implies that all such manifolds that have at least one isometry are non-compact.

Finally, we should still comment on the extra constraint (C.67) for $r = 1$. In the mathematics literature [187] the extra constraint is that the quaternionic structure annihilates the curvature tensor, which is the vanishing of

$$\begin{aligned}(J^{\alpha}\cdot R)_{XYWZ} &\equiv J^{\alpha}{}_X{}^V R_{VYWZ} + J^{\alpha}{}_Y{}^V R_{XVWZ} + J^{\alpha}{}_Z{}^V R_{XYWV} + J^{\alpha}{}_W{}^V R_{XYVZ} \\ &= \varepsilon^{\alpha\beta\gamma}\left(\mathcal{R}_{XY}{}^{\beta}J^{\gamma}_{ZW} + \mathcal{R}_{ZW}{}^{\beta}J^{\gamma}_{XY}\right),\end{aligned} \tag{C.75}$$

where the second expression is obtained using once more (C.48). We have proven that (C.67) was sufficient extra input to have $\mathcal{R}^{\alpha}_{XY}$ proportional to J^{α}_{XY} implying $J^{\alpha}\cdot R = 0$. Vice versa: multiplying (C.75) with $\varepsilon^{\alpha\delta\epsilon}J^{\epsilon}{}_{YZ}$ leads to (C.67) if $J^{\alpha}\cdot R = 0$. Thus indeed the vanishing of (C.75) is an equivalent condition that can be imposed for $r = 1$ and that is automatically satisfied for $r > 1$.

C.5 Symmetries

Symmetries of manifolds are most known as isometries for Riemannian manifolds (i.e. when there is a metric). They are transformations $\delta q^X = k_I^X(q)\Lambda^I$, where Λ^I are infinitesimal parameters. They are determined by the Killing equation[6]

$$\mathfrak{D}_{(X}k_{Y)I} = 0\,, \qquad k_{XI} \equiv g_{XY}k^Y{}_I\,. \tag{C.76}$$

[6] See also 'conformal Killing vectors' in section 5.1.

This definition can only be used when there is a metric. However, there is a weaker equation that can be used for defining symmetries also in the absence of a metric, but when parallel transport is defined. Indeed, the Killing equation implies that

$$-R_{YZX}{}^{W}k_{WI} = \mathfrak{D}_Y\mathfrak{D}_Z k_{XI} - \mathfrak{D}_Z\mathfrak{D}_Y k_{XI} = \mathfrak{D}_Y\mathfrak{D}_Z k_{XI} + \mathfrak{D}_Z\mathfrak{D}_X k_{YI}\,. \tag{C.77}$$

Using the cyclicity condition on the left-hand side to write

$$R_{YZX}{}^{W} = \tfrac{1}{2}\left(R_{YZX}{}^{W} - R_{ZXY}{}^{W} - R_{XYZ}{}^{W}\right)\,, \tag{C.78}$$

we obtain

$$\mathfrak{D}_X\mathfrak{D}_Y k_I^Z = R_{XWY}{}^{Z}k_I^W\,. \tag{C.79}$$

This equation does not need a metric any more. We will use it as definition of symmetries when there is no metric available. We will see that it leads to the group structure that is known from the Riemannian case.

Of course, we will require also that the symmetries respect the quaternionic structure. This is the statement that the vector k_I^X normalizes the quaternionic structure:

$$\mathcal{L}_{k_I}J^\alpha{}_X{}^Y \equiv k_I^Z\partial_Z J^\alpha{}_X{}^Y + \left(\partial_X k_I^Z\right)J^\alpha{}_Z{}^Y - J^\alpha{}_X{}^Z\left(\partial_Z k_I^Y\right) = b_I^{\alpha\beta}J^\beta{}_X{}^Y\,, \tag{C.80}$$

for some functions $b_I^{\alpha\beta}(q)$. This b_I is antisymmetric, as can be seen by multiplying the equation with $J^\gamma{}_Y{}^X$.

Thus we define symmetries in quaternionic-like manifolds as those $\delta q^X = k_I^X(q)\Lambda^I$, such that the vectors k_I^X satisfy (C.79) and (C.80).

We first consider (C.80). One can add an affine torsionless connection to the derivatives, because they cancel. As a total covariant derivative on J vanishes, we add in case of quaternionic manifolds the SU(2) connection to the first derivative. This addition is of the form of the right-hand side. Thus defining P_I^γ by $b_I^{\alpha\beta} - 2\varepsilon^{\alpha\beta\gamma}\omega_X{}^\gamma k_I^X = 2\varepsilon^{\alpha\beta\gamma}\nu P_I^\gamma$, the remaining statement is that there is a $P_I^\alpha(q)$ (possibly zero) such that[7]

$$J^\alpha{}_X{}^Z\left(\mathfrak{D}_Z k_I^Y\right) - \left(\mathfrak{D}_X k_I^Z\right)J^\alpha{}_Z{}^Y = -2\varepsilon^{\alpha\beta\gamma}J^\beta{}_X{}^Y\nu P_I^\gamma\,. \tag{C.81}$$

The equation now takes on the form of (C.11) in lemma C.2.1. Thus, using this lemma, as well as lemma C.2.2, we have

$$\mathfrak{D}_X k_I^Y = \nu J^\alpha{}_X{}^Y\,P_I^\alpha + L_X{}^Y{}_A{}^B t_{IB}{}^A\,. \tag{C.82}$$

$t_{IB}{}^A$ is the matrix that we saw in the fermion gauge transformation law (6.49). The rule (C.12) gives an expression for P_I^α, which is called the *moment map*:

$$4r\,\nu\,P_I^\alpha = -J^\alpha{}_X{}^Y\left(\mathfrak{D}_Y k_I^X\right). \tag{C.83}$$

Using the second equation, (C.79) we now find

$$R_{ZWX}{}^{Y}k_I^W = \mathfrak{D}_Z\mathfrak{D}_X k_I^Y = \nu J^\alpha{}_X{}^Y\left(\mathfrak{D}_Z P_I^\alpha\right) + L_X{}^Y{}_A{}^B\left(\mathfrak{D}_Z t_{IB}{}^A\right). \tag{C.84}$$

[7] Here we introduce in fact νP. The factor ν is included for agreement with other papers and allows a smooth limit $\nu = 0$ to the hypercomplex or hyperkähler case. In fact, we have seen in (6.55) that supersymmetry in the setting of hypercomplex manifolds demands that the right-hand side of (C.80) is zero. We will see below that this is unavoidable for hypercomplex manifolds even outside the context of supersymmetry.

Using the curvature decomposition (C.39) and projecting onto the complex structures and L, we find two equations

$$\mathcal{R}_{ZW}{}^{\alpha} k_I^W = -\nu \mathfrak{D}_Z P_I^{\alpha}\,, \qquad \mathcal{R}_{ZWB}{}^{A} k_I^W = \mathfrak{D}_Z t_{IB}{}^{A}\,. \tag{C.85}$$

The algebra that the vectors k_I^X define is

$$2k_{[I}^Y \mathfrak{D}_Y k_{J]}^X + f_{IJ}{}^K k_K^X = 0\,, \tag{C.86}$$

where $f_{IJ}{}^K$ are structure constants. Multiplying this relation with $J^{\alpha}{}_X{}^Z \mathfrak{D}_Z$, and using (C.79), and (C.83) gives

$$2J^{\alpha}{}_X{}^Z (\mathfrak{D}_Z k_{[I}^Y)(\mathfrak{D}_Y k_{J]}^X) + 2J^{\alpha}{}_X{}^Z R_{ZWY}{}^X k_{[I}^Y k_{J]}^W - 4r\nu f_{IJ}{}^K P_K^{\alpha} = 0\,. \tag{C.87}$$

The trace that appears in the first term can be evaluated by using (C.81) and once more (C.83), while in the second term we can use the cyclicity condition of the curvature and (C.49) to obtain

$$-2\nu^2 \varepsilon^{\alpha\beta\gamma} P_I^{\beta} P_J^{\gamma} + \mathcal{R}_{YW}{}^{\alpha} k_I^Y k_J^W - \nu f_{IJ}{}^K P_K^{\alpha} = 0\,. \tag{C.88}$$

We thus found that the moment maps, defined in (C.83) satisfy (C.85) and (C.88). The first of these shows that we can take $\nu = 0$ for the hypercomplex or hyperkähler manifolds. Both these two relations vanish identically in this case. However, for quaternionic-Kähler and hyperkähler manifolds, we can use (C.71), and dividing by ν leads to

$$J^{\alpha}{}_{ZW} k_I^W = -2\mathfrak{D}_Z P_I^{\alpha}\,, \tag{C.89}$$

$$-2\nu \varepsilon^{\alpha\beta\gamma} P_I^{\beta} P_J^{\gamma} + \tfrac{1}{2} J^{\alpha}{}_{YW} k_I^Y k_J^W - f_{IJ}{}^K P_K^{\alpha} = 0\,. \tag{C.90}$$

These equations are thus equivalent to the previous ones for $\nu \neq 0$ if there is a metric. This is thus the quaternionic-Kähler case, for which these relations appear already in [188]. But we did not *derive* these equations for the $\nu = 0$ (hyperkähler) case. Rather, the first one is taken as the definition of P for this case. This equation also follows from supersymmetry requirements, where the moment map P_I^{α} is an object that is needed to define the action, see (6.95). The moment map is then determined up to constants. As we saw in section 6.3, the constants are fixed when conformal symmetry is imposed. Similarly, the second equation appears in supersymmetry as a requirement, see (6.100). For a conformal invariant theory, the constants in P_I^{α} are determined and the moment map again satisfies (C.90) automatically due to a similar calculation as the one that we did above for $\nu \neq 0$. Note, however, that for the quaternionic manifolds that are not quaternionic-Kähler, we can only use (C.85) and (C.88), as (C.89) and (C.90) need a metric. For hypercomplex manifolds, on the other hand, the moment maps are not defined.