Chapter 7

Gauge fixing

The general idea of this chapter can be illustrated by using a scalar-gravity toy model in four dimensions. We start with a conformally invariant action for a scalar field φ

$$\mathcal{L} = \sqrt{|g|}\left[\tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{12}R\varphi^2\right], \tag{7.1}$$

which is invariant under the following local dilatations

$$\delta\varphi = \Lambda_D\varphi\,, \qquad \delta g_{\mu\nu} = -2\Lambda_D g_{\mu\nu}\,. \tag{7.2}$$

This dilatation symmetry can be gauge fixed by choosing the gauge $\varphi = \sqrt{6}/\kappa$; this leads to the Poincaré action

$$\mathcal{L} = \tfrac{1}{2\kappa^2}\sqrt{|g|}\,R\,. \tag{7.3}$$

Therefore the actions (7.1) and (7.3) are *gauge equivalent*. Alternatively, we could have chosen new coordinates ($g'_{\mu\nu} = g_{\mu\nu}\varphi^2$), such that the resulting action is manifestly invariant under the dilatation symmetry. Although φ still transforms under dilatations, the field does not appear in the action anymore. The scalar φ has no physical degrees of freedom, and is called a "compensating scalar". Note that the scalar kinetic term has the wrong sign; this is a generic feature of compensating scalars which we will also encounter in the more complicated case of conformal supergravity.

The same mechanism will be used in this chapter to obtain five-dimensional matter coupled Poincaré supergravity. In chapter 5 the Poincaré algebra was extended to the local superconformal algebra $F^2(4)$. We constructed the minimal representation of the superconformal algebra containing the graviton, called the Standard Weyl multiplet. The fields of the Standard Weyl multiplet and their properties were given in table 5.2. Next, in chapter 6, we constructed vector-tensor multiplets and hypermultiplets in the background of this Weyl multiplet, see tables 6.1, 6.2 and 6.3 for the contents and properties of these multiplets. The corresponding actions, equations of motion and transformation rules were given in (6.110) and (6.118). As a third and final step we now want to gaugefix the extra symmetries, not belonging to the super-Poincaré algebra, and obtain Poincaré supergravity coupled to vector-tensor multiplets and hypermultiplets. As compensators we will need one hypermultiplet and one vector-tensor multiplet. Therefore, the starting point of this chapter will be local $D = 5, \mathcal{N} = 2$ conformal supergravity coupled to $(n_V + n_T + 1)$ vector-tensor multiplets and $(n_H + 1)$ hypermultiplets[1]:

$$\mathcal{L}_{total} = \mathcal{L}_{Vector-Tensor} + \mathcal{L}_{Hyper} \,. \tag{7.4}$$

This chapter is based on work to be published in [173, 174].

7.1 Conformal geometry

The superconformal tensor calculus performed in the last chapter resulted in the construction of a hypercomplex manifold spanned by $(4n_H + 4)$ hyperscalars $q^{\hat{X}}$ ($\hat{X} = 1, \ldots, 4n_H + 4$). In this chapter we will for simplicity assume the presence of a covariantly constant hermitian metric, which promotes the hypercomplex manifold to a hyperkähler manifold. This manifold includes the four scalars of the compensating hypermultiplet. In the end these compensating scalars will be removed from the manifold; therefore it is convenient to split these coordinates off by making a specific coordinate choice on the hyperkähler manifold. Details about this procedure can be found in [174].[2] Here we will skip the technical details and only give some of the relevant results. We use the hat-notation for objects that are defined on the "higher dimensional" hyperkähler manifold, spanned by the $q^{\hat{X}}$.

As we saw in chapter 6, the manifold contains three generic isometries, generated by the SU(2) Killing vectors $\hat{\vec{k}}^{\hat{X}}$. These isometries were gauged using the vectors of the vector-tensor multiplets. Using Frobenius' theorem it can be shown that the three-dimensional subspace spanned by the direction of the three SU(2) transformations can be parametrized by coordinates z^α ($\alpha = 1, 2, 3$). Furthermore, using the homothetic Killing equation (6.44), one more coordinate z^0 can be singled out, which is associated with the dilatation transformation. The other directions are indicated by q^X ($X = 1, \ldots, 4n_H$). Thus, we split the coordinates on the hyperkähler manifold as $\{q^{\hat{X}}\} = \{z^0, z^\alpha, q^X\}$. Similarly we can split the tangent space index as $\{\hat{A}\} = \{i, A\}$ ($i = 1, 2; A = 1, \ldots, 2n_H$), where i is an SU(2) index. Note that throughout this chapter we will work in this coordinate basis. In this basis the Killing vectors take on the following form

$$\hat{k}^{\hat{X}}(z^0, z^\alpha, q) = \{3z^0, 0, 0\} \,, \qquad \hat{\vec{k}}^{\hat{X}}(z^0, z^\alpha, q) = \{0, \vec{k}^\alpha(z^0, z^\alpha), 0\} \,. \tag{7.5}$$

We will choose the following metric parametrization:

$$\begin{aligned} d\hat{s}^2 &\equiv \hat{g}_{\hat{X}\hat{Y}} dq^{\hat{X}} dq^{\hat{Y}} \,. \\ &= -\frac{(dz^0)^2}{z^0} + z^0 h_{XY}(z^\alpha, q) dq^X dq^Y \\ &\quad -\hat{g}_{\alpha\beta}(z^0, z^\alpha, q)[dz^\alpha + A^\alpha_X(z^0, z^\alpha, q) dq^X][dz^\beta + A^\beta_Y(z^0, z^\alpha, q) dq^Y] \,, \end{aligned} \tag{7.6}$$

where we have chosen the signs and factors for later convenience. The object h_{XY} denotes the metric on the subspace spanned by the coordinates q^X, and $A^\alpha_X(z, q) \equiv \hat{f}^{\alpha}_{ij} \hat{f}^{ij}_X$.

[1] In comparison with chapter 6 we have: $n = n_V + 1$, $m = n_T$ and $r = n_H + 1$.

[2] In this reference we also discuss the case without a hyperscalar-metric.

Note the resemblance of (7.6) with the generic form of the Kaluza-Klein Ansatz (3.41). This is not that surprising, since we are in fact performing a "dimensional reduction" of the scalar manifold. In the above coordinate basis we find the following expressions for the vielbeins[3]:

$$\begin{array}{lll}
\hat{f}_{ij}^{0} = -\,\mathrm{i}\,\varepsilon_{ij}\sqrt{\tfrac{1}{2}z^0}\,, & \hat{f}_{ij}^{\alpha} = \sqrt{\tfrac{1}{2z^0}}\vec{k}^{\alpha}\cdot\vec{\sigma}_{ij}\,, & \hat{f}_{ij}^{X} = 0\,,\\
\hat{f}_{iA}^{0} = 0\,, & \hat{f}_{iA}^{\alpha} = -f_{iA}^{X}A_X^{\alpha}\,, & \hat{f}_{iA}^{X} = f_{iA}^{X}\,,\\
\hat{f}_{0}^{ij} = \mathrm{i}\,\varepsilon^{ij}\sqrt{\tfrac{1}{2z^0}}\,, & \hat{f}_{\alpha}^{ij} = \sqrt{\tfrac{1}{2z^0}}\vec{k}_{\alpha}\cdot\vec{\sigma}^{ij}\,, & \hat{f}_{X}^{ij} = \sqrt{\tfrac{1}{2z^0}}\vec{k}_{\alpha}\cdot\vec{\sigma}^{ij}A_X^{\alpha}\,,\\
\hat{f}_{0}^{iA} = 0\,, & \hat{f}_{\alpha}^{iA} = 0\,, & \hat{f}_{X}^{iA} = f_{X}^{iA}\,.
\end{array} \tag{7.7}$$

Using the above expressions for the vielbeins, we find the following complex structures:

$$\begin{array}{lll}
\widehat{\vec{J}}_0{}^0 = 0\,, & \widehat{\vec{J}}_\alpha{}^0 = \vec{k}_\alpha\,, & \widehat{\vec{J}}_X{}^0 = A_X^\alpha\vec{k}_\alpha\,,\\
\widehat{\vec{J}}_0{}^\beta = \tfrac{1}{z^0}\vec{k}^\beta\,, & \widehat{\vec{J}}_\alpha{}^\beta = \tfrac{1}{z^0}\vec{k}_\alpha\times\vec{k}^\beta\,, & \widehat{\vec{J}}_X{}^\beta = \tfrac{1}{z^0}A_X^\gamma\vec{k}_\gamma\times\vec{k}^\beta - \vec{J}_X{}^Z A_Z^\beta\,,\\
\widehat{\vec{J}}_0{}^Y = 0\,, & \widehat{\vec{J}}_\alpha{}^Y = 0\,, & \widehat{\vec{J}}_X{}^Y = \vec{J}_X{}^Y\,.
\end{array} \tag{7.8}$$

Covariant constancy of the vielbeins furthermore leads to the expressions for the $\mathrm{G}\ell(n_H+1,\mathbb{H})$ connections. The non-zero components are given by:

$$\begin{array}{ll}
\hat{\omega}_{\alpha i}{}^{j} = -\,\mathrm{i}\,\tfrac{1}{2z^0}\vec{k}_\alpha\cdot\vec{\sigma}_i{}^j\,, & \hat{\omega}_{\alpha A}{}^{B} = \tfrac{1}{2}f_Y^{iB}\partial_\alpha f_{iA}^Y\,,\\
\hat{\omega}_{Xi}{}^{j} = A_X^\alpha\hat{\omega}_{\alpha i}{}^{j}\,, & \hat{\omega}_{0A}{}^{B} = \tfrac{1}{2}f_Y^{iB}\partial_0 f_{iA}^Y + \tfrac{1}{2z^0}\delta_A^B\,,\\
\hat{\omega}_{Xi}{}^{A} = \mathrm{i}\,\sqrt{\tfrac{1}{2z^0}}\varepsilon_{ik}\hat{f}_X^{kA}\,, & \hat{\omega}_{XA}{}^{i} = -\,\mathrm{i}\,\sqrt{\tfrac{z^0}{2}}\varepsilon^{ij}f_{jA}^Y h_{YX}\,.
\end{array} \tag{7.9}$$

Using these results, some other relevant expressions can be derived

$$\hat{C}_{AB} = z^0 C_{AB}\,,\qquad \hat{C}_{ij} = \varepsilon_{ij}\,,\qquad \hat{C}_{iA} = 0\,, \tag{7.10}$$

$$\hat{W}_{ABC}{}^{D} = \mathcal{W}_{ABC}{}^{D}\,, \tag{7.11}$$

$$\widehat{\vec{P}}_I = \vec{P}_I\,, \tag{7.12}$$

$$\hat{k}_I^{\hat{X}} = \{0, -2\vec{k}^\alpha(\vec{\omega}_X k_I^X - \tfrac{1}{z^0}\vec{P}_I), k_I^X\}\,, \tag{7.13}$$

where $\mathcal{W}_{ABC}{}^{D}$ is the 'quaternionic Weyl tensor' defined in (C.54).

We found that for each point in the subspace $\{z^\alpha\}$, corresponding to a specific gauge fixing, the $\{q^X\}$ space describes a quaternionic-Kähler manifold. These manifolds are all related to each other by coordinate redefinitions.

We point out in appendix C.3 that the connections on a quaternionic manifold are not uniquely defined; a certain ξ-transformation can be performed to choose a convenient gauge for the connections. The gauge chosen in [174] leads to the following expressions for the $\mathrm{G}\ell(n_H,\mathbb{H})$ and SU(2) connections:

$$\hat{\omega}_{XA}{}^{B} = \omega_{XA}{}^{B}\,,\qquad \vec{\omega}_X = -\tfrac{1}{2z^0}A_X^\alpha\vec{k}_\alpha\,. \tag{7.14}$$

Note that before gauge fixing the unhatted objects are dependent on the z-coordinates.

[3] These expressions do not represent reduction Ansätze, because the fields on the right hand side still depend on $\{z^0, z^\alpha\}$.

7.2 Gauge fixing

The actions given in (6.110) and (6.118) are invariant under the full supercovariant group. In order to break the symmetries that are not present in the Poincaré algebra, we will impose the necessary gauge conditions in the following subsections.

7.2.1 Preliminaries

The first step in the gaugefixing process will be the elimination of the dependent gauge fields ϕ_μ^i and $f_\mu{}^a$, associated to S- and K-symmetry respectively. Using the relations (5.35) together with the definitions of the supercovariant curvatures, we find the following expressions for the dependent gauge fields

$$\begin{aligned}
f_a{}^a &= \tfrac{1}{16}\Big(-R(\hat\omega) - \tfrac{1}{3}\bar\psi_\rho\gamma^{\rho\mu\nu}\mathcal{D}_\mu\psi_\nu \\
&\quad + \tfrac{1}{3}\bar\psi^i_a\gamma^{abc}\psi^j_b V_{cij} + 16\bar\psi_a\gamma^a\chi - 4\,\mathrm{i}\,\bar\psi^a\psi^b T_{ab} + \tfrac{4}{3}\,\mathrm{i}\,\bar\psi^b\gamma_{abcd}\psi^a T^{cd}\Big)\,,\\
\hat\omega_\mu{}^{ab} &= \omega_\mu{}^{ab}(e) - \tfrac{1}{2}\bar\psi^{[b}\gamma^{a]}\psi_\mu - \tfrac{1}{4}\bar\psi^b\gamma_\mu\psi^a + 2e_\mu{}^{[a}b^{b]}\,,\\
\phi^i_\mu &= \tfrac{1}{2}\,\mathrm{i}\,\gamma^\nu\mathcal{D}_{[\mu}\psi^i_{\nu]} - \tfrac{1}{12}\,\mathrm{i}\,\gamma_\mu{}^{\nu\rho}\mathcal{D}_\nu\psi^i_\rho - \tfrac{1}{2}\,\mathrm{i}\,V_{[\mu}{}^{ij}\gamma^a\psi_{a]\,j} + \tfrac{1}{12}\,\mathrm{i}\,V_a{}^{ij}\gamma_\mu{}^{ab}\psi_{b\,j} \\
&\quad -T^a{}_\mu\psi^i_a - \tfrac{1}{3}T^{ab}\gamma_{b\mu}\psi^i_a - \tfrac{2}{3}T_{b\mu}\gamma^{ab}\psi^i_a - \tfrac{1}{3}T_{bc}\gamma^{abc}{}_\mu\psi_a \\
&\quad -\tfrac{1}{12}\,\mathrm{i}\,(\gamma^{ab}\gamma_\mu - \tfrac{1}{2}\gamma_\mu\gamma^{ab})b_a\psi^i_b\,,
\end{aligned} \tag{7.15}$$

with

$$\mathcal{D}_\mu = \partial_\mu + \tfrac{1}{4}\hat\omega_\mu{}^{ab}\gamma_{ab}\,. \tag{7.16}$$

We only need the contracted version of $f_\mu{}^a$ since the other components do not appear in the action or transformation rules. Also, in order to simplify notation we will choose not to eliminate $\hat\omega_\mu{}^{ab}$ in most places.

First of all we observe that, after writing out all covariant derivatives, the gauge field b_μ does not appear in the action. This can be argued from K-invariance of the action. Although this prohibits us from determining its equation of motion, we will choose the conventional gauge choice for K-symmetry, namely $b_\mu = 0$.

This still leaves us with one more gauge field corresponding to a non-Poincaré symmetry: the SU(2) gauge field V_μ^{ij}. Solving for its equation of motion, corresponding to the action (7.4), gives us the following expression

$$\begin{aligned}
V_\mu^{ij} &= \tfrac{9}{2k^2}\Big(\hat g_{\hat X\hat Y}(\partial_\mu q^{\hat X} + gA^I_\mu k_I^{\hat X})k^{ij\hat Y} + \tfrac{1}{2}\,\mathrm{i}\,k^{\hat X}\hat f^{i\hat A}_{\hat X}\bar\zeta_{\hat A}\gamma_{\mu\nu}\psi^{\nu\,j} - \mathrm{i}\,k^{ij\hat X}\hat f^{\hat A}_{k\hat X}\bar\zeta_{\hat A}\gamma_\nu\gamma_\mu\psi^{k\nu} \\
&\quad -\tfrac{1}{2}C_{\widetilde I\widetilde J\widetilde K}\sigma^{\widetilde K}\bar\psi^{i\widetilde I}\gamma_\mu\psi^{j\widetilde J} + \tfrac{1}{4}\,\mathrm{i}\,C_{\widetilde I\widetilde J\widetilde K}\sigma^{\widetilde K}\sigma^{\widetilde I}\bar\psi^{i\widetilde J}\gamma_{\mu\nu}\psi^{j\nu}\Big)\,.
\end{aligned} \tag{7.17}$$

The action contains four auxiliary matter fields: D, T_{ab} and χ^i from the Weyl multiplet, and $Y^{\widetilde I}_{ij}$ from the vector-tensor multiplet. Both D and χ^i appear as Lagrange multipliers in the action, leading to the following constraints, respectively

$$D\ :\qquad C - \tfrac{1}{3}k^2 = 0\,,\qquad \text{with}\quad C \equiv C_{\widetilde I\widetilde J\widetilde K}\sigma^{\widetilde I}\sigma^{\widetilde J}\sigma^{\widetilde K}\,, \tag{7.18}$$

$$\chi^i\ :\qquad -8\,\mathrm{i}\,C_{\widetilde I\widetilde J\widetilde K}\sigma^{\widetilde I}\sigma^{\widetilde J}\psi^{\widetilde K}_i - \tfrac{4}{3}\Big(C - \tfrac{1}{3}k^2\Big)\gamma^\mu\psi_{\mu i} + \tfrac{16}{3}\,\mathrm{i}\,A^{\hat A}_i\zeta_{\hat A} = 0\,. \tag{7.19}$$

The equations of motion for $Y^{\widetilde{I}}_{ij}$ and T_{ab} are given by

$$Y^{ij\widetilde{J}} C_{\widetilde{I}\widetilde{J}\widetilde{K}} \sigma^{\widetilde{K}} = -g\, \delta^{L}_{\widetilde{I}}\, \hat{P}^{ij}_{L} + \tfrac{1}{4}\,\mathrm{i}\, C_{\widetilde{I}\widetilde{J}\widetilde{K}} \bar{\psi}^{i\widetilde{J}} \psi^{j\widetilde{K}}\,, \tag{7.20}$$

$$\begin{aligned} T_{ab} = \tfrac{9}{64k^2}\Big(& 4\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\widehat{\mathcal{H}}^{\widetilde{K}}_{ab} C_{\widetilde{I}\widetilde{J}\widetilde{K}} + \sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}^{\widetilde{K}}\gamma_{[a}\psi_{b]} C_{\widetilde{I}\widetilde{J}\widetilde{K}} + \sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}^{\widetilde{K}}\gamma_{abc}\psi^{c} C_{\widetilde{I}\widetilde{J}\widetilde{K}} \\ & + \mathrm{i}\,\sigma^{\widetilde{I}}\bar{\psi}^{\widetilde{J}}\gamma_{ab}\psi^{\widetilde{K}} C_{\widetilde{I}\widetilde{J}\widetilde{K}} + \tfrac{2}{3}k^{\hat{X}} f^{\hat{A}}_{i\hat{X}}\bar{\zeta}_{\hat{A}}\gamma_{[a}\psi^{i}_{b]} + \tfrac{2}{3}k^{\hat{X}} f^{\hat{A}}_{i\hat{X}}\bar{\zeta}_{\hat{A}}\gamma_{abc}\psi^{ic} + 2\,\mathrm{i}\,\bar{\zeta}_{\hat{A}}\gamma_{ab}\zeta^{\hat{A}}\Big)\,, \end{aligned} \tag{7.21}$$

which have been simplified by making use of (7.18).

7.2.2 Gauge choices and decomposition rules

Apart from the K-gauge we already introduced to fix the special conformal (K-)symmetry, we will have to choose gauges for the other non-Poincaré (super)symmetries as well.

D-gauge

Having written out all dependent gaugefields in the action, the kinetic terms for the graviton and the gravitino become

$$e^{-1}\mathcal{L}_{\text{EH+RS}} = \tfrac{1}{24}\left(C + k^2\right)\left(R(\hat{\omega}) + \bar{\psi}_{\mu}\gamma^{\mu\nu\rho}\mathcal{D}_{\nu}\psi_{\rho}\right)\,. \tag{7.22}$$

Similarly to the example given in (7.1)–(7.3) we can demand canonical factors for the Einstein-Hilbert and Rarita-Schwinger kinetic terms means by imposing the following D-gauge:

$$\tfrac{1}{24}\left(C + k^2\right) = -\frac{1}{2\kappa^2}\,. \tag{7.23}$$

Note that in order to get the conventional mass-dimensions for the Rarita-Schwinger term, we identify the superconformal gravitino ψ^{C}_{μ} in terms of the gravitino ψ^{P}_{μ} from the super-Poincaré multiplet as follows:

$$\psi^{C}_{\mu} \equiv \kappa\psi^{P}_{\mu}\,. \tag{7.24}$$

The index P will be suppressed in the rest of this chapter. If we combine the D-gauge (7.23) and the equation of motion for D (7.18) we obtain

$$k^2 = -\frac{9}{\kappa^2}\,, \qquad C = -\frac{3}{\kappa^2}\,. \tag{7.25}$$

The first constraint implies that $z^0 = \frac{1}{\kappa^2}$, whereas the second constraint effectively eliminates one of the vector-tensor scalars.

S-gauge

Off-diagonal kinetic terms like e.g. $\bar{\psi}_{\mu}\mathcal{D}\psi$ or $\bar{\psi}^{i}_{\mu}\mathcal{D}\zeta^{A}$ appear in the action with one overall coefficient. A canonical form of the action requires the vanishing of these terms, which can be accomplished by demanding the overall coefficient to vanish. This leads to the following constraint, called the S-gauge:

$$C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\psi^{\widetilde{K}}_{i} = 0\,. \tag{7.26}$$

This constraint effectively eliminates one of the gauginos.

SU(2)-gauge

The gauge for dilatations was chosen such that $z^0 = \frac{1}{\kappa^2}$. Similarly we may also choose a gauge for SU(2). Such a gauge would be a specific point in the three-dimensional space of the z^α. Any fixed value of these coordinates would fix a gauge, however we will leave this arbitrary. The dependence of objects on the hyperkähler manifold on the coordinates z^α thus describes the gauge dependence. By fixing the SU(2) gauge, i.e choosing z^α to be constant, all fields become particular functions of the quaternionic-Kähler coordinates q^X only. These functions may be different for different gauge choices, but once we make a choice, which is not relevant for further considerations, they are fixed.

Using both the S-gauge (7.26) and the equation of motion for D (7.18) in equation (7.19) we also get a constraint on the hyperino

$$A_i^{\hat{A}}\zeta_{\hat{A}} = 0\,. \tag{7.27}$$

In our coordinate basis, we obtain the following expression for the sections $A^i_{\hat{A}}$

$$A^i_{\hat{A}} \equiv \varepsilon^{ij}k_{\hat{X}}\hat{f}^{\hat{X}}_{j\hat{A}} = -3\varepsilon^{ij}\hat{f}^0_{j\hat{A}} = -3\,\mathrm{i}\,\sqrt{\tfrac{z^0}{2}}\delta^i_{\hat{A}}\,. \tag{7.28}$$

After applying the D-gauge, i.e. fixing one degree of freedom of $q^{\hat{X}}$ by (7.25), equation (7.28) plays the role of SU(2) gauge since it fixes three of the degrees of freedom contained in $A^i_{\hat{A}}$. Moreover, combining it with (7.27) one discovers that our choice of coordinates on the hyperkähler manifold is consistent with the hyperinos of the compensating multiplet carrying no physical information:

$$\zeta_i \equiv \zeta^j\varepsilon_{ji} = 0\,. \tag{7.29}$$

Decomposition rules

As a consequence of the gauge choices, the corresponding transformation parameters can be expressed in terms of the others by so-called decomposition rules. These rules will enable us to eliminate the parameters $\Lambda_D, \Lambda_K^a, \Lambda^{ij}_{\mathrm{SU}(2)}, \eta^i$ and determine the transformation rules for the remaining symmetries in section 7.3. For example, the requirement that the K-gauge should be invariant under the most general superconformal transformation, i.e. $\delta b_\mu = 0$, leads to the decomposition rule for Λ_K^a:

$$\Lambda_K^a = -\tfrac{1}{2}e^{\mu a}\left(\partial_\mu\Lambda_D + \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\phi_\mu - 2\bar{\epsilon}\gamma_\mu\chi + \tfrac{\kappa}{2}\,\mathrm{i}\,\bar{\eta}\psi_\mu\right)\,. \tag{7.30}$$

Similarly, demanding $\delta z^0 = 0$ yields

$$\Lambda_D = 0\,. \tag{7.31}$$

The decomposition rule for η^i can be found by varying the S-gauge and demanding that

$$\delta\left(C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\psi^{i\widetilde{K}}\right) = 0\,. \tag{7.32}$$

We find

$$\begin{aligned}\eta^i \;=\; & -\tfrac{\kappa^2}{12}C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{K}}\epsilon^i + \tfrac{1}{3}g\sigma^I P_I^{ij}\epsilon_j + \tfrac{1}{32\kappa^2}\,\mathrm{i}\,\gamma^{ab}\epsilon^i\bar{\zeta}_A\gamma_{ab}\zeta^A \\ & +\tfrac{\kappa^2}{16}\,\mathrm{i}\,C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\Big(\gamma^a\epsilon_j\bar{\psi}^{i\widetilde{J}}\gamma_a\psi^{j\widetilde{K}} - \tfrac{1}{16}\gamma^{ab}\epsilon^i\bar{\psi}^{\widetilde{J}}\gamma_{ab}\psi^{\widetilde{K}}\Big)\,.\end{aligned} \tag{7.33}$$

The SU(2) decomposition rule can be found by requiring that $\delta z^\alpha = 0$:

$$\Lambda^{ij}_{\mathrm{SU(2)}} = \omega_X^{ij}(\delta_Q + \delta_G)q^X + \kappa^2 g \Lambda_G^I P_I^{ij} \,. \tag{7.34}$$

7.2.3 Hypersurfaces

The gauge condition for the vector/tensor scalars (7.25), defines a $(n_V + n_T)$-dimensional hypersurface of scalars φ^x called a "very special real" manifold, embedded into a $(n_V + n_T + 1)$-dimensional space spanned by the scalars $h^{\widetilde{I}}(\varphi)$. In order to find the kinetic term for the scalars φ^x we need to identify the embedding metric g_{xy}. At this point it is convenient to rescale the $C_{\widetilde{I}\widetilde{J}\widetilde{K}}$ symbol and to redefine our scalars, in order to get a convenient normalization:

$$\begin{aligned} \sigma^{\widetilde{I}} &\equiv \alpha h^{\widetilde{I}}\,, \qquad \alpha = \sqrt{\tfrac{3}{2\kappa^2}}\,, \\ C_{\widetilde{I}\widetilde{J}\widetilde{K}} &\equiv -\tfrac{2}{\alpha} N_{\widetilde{I}\widetilde{J}\widetilde{K}}\,, \\ \mathcal{N} &\equiv N_{\widetilde{I}\widetilde{J}\widetilde{K}} h^{\widetilde{I}} h^{\widetilde{J}} h^{\widetilde{K}} = 1\,. \end{aligned} \tag{7.35}$$

The metric on the $h^{\widetilde{I}}$-manifold can be determined by substituting the equation of motion for T_{ab} (7.21) back into the action, and defining the kinetic term for the vectors/tensors as

$$\mathcal{L}_{\mathrm{kin,vec-ten}} = -\tfrac{1}{4} a_{\widetilde{I}\widetilde{J}} \widehat{\mathcal{H}}^{\widetilde{I}}_{\mu\nu} \widehat{\mathcal{H}}^{\mu\nu\widetilde{J}}\,. \tag{7.36}$$

We then find

$$a_{\widetilde{I}\widetilde{J}} = -2N_{\widetilde{I}\widetilde{J}\widetilde{K}} h^{\widetilde{K}} + 3h_{\widetilde{I}} h_{\widetilde{J}}\,, \tag{7.37}$$

where

$$h_{\widetilde{I}} \equiv a_{\widetilde{I}\widetilde{J}} h^{\widetilde{J}} = N_{\widetilde{I}\widetilde{J}\widetilde{K}} h^{\widetilde{J}} h^{\widetilde{K}} \qquad \Rightarrow \qquad h_{\widetilde{I}} h^{\widetilde{I}} = 1\,. \tag{7.38}$$

In the following we will assume that $a_{\widetilde{I}\widetilde{J}}$ is invertible; this enables us to solve (7.20) for $Y^{ij\widetilde{I}}$.

$$Y^{ij\widetilde{I}} = -\left(a^{\widetilde{I}\widetilde{J}} - \tfrac{3}{2} h^{\widetilde{I}} h^{\widetilde{J}}\right)\left(g\,\delta^L_{\widetilde{J}} \hat{P}^{ij}_L - \frac{\kappa}{\sqrt{6}}\,\mathrm{i}\, N_{\widetilde{J}\widetilde{K}\widetilde{L}} \bar{\psi}^{i\widetilde{K}} \psi^{j\widetilde{L}}\right). \tag{7.39}$$

This expression is needed to eliminate $Y^{ij\widetilde{I}}$ from the action and transformation rules. For convenience we introduce the following notation:

$$h^{\widetilde{I}}_x \equiv -\sqrt{\tfrac{3}{2}} h^{\widetilde{I}}_{,x}(\varphi)\,, \quad \rightarrow \quad h_{\widetilde{I}x} \equiv a_{\widetilde{I}\widetilde{J}} h^{\widetilde{J}}_x(\varphi) = \sqrt{\tfrac{3}{2}} h_{\widetilde{I},x}(\varphi)\,. \tag{7.40}$$

It follows from (7.38) that:

$$h_{\widetilde{I}} h^{\widetilde{I}}_x = h^x_{\widetilde{I}} h^{\widetilde{I}} = 0\,. \tag{7.41}$$

Let us now focus on the embedding manifold, spanned by the scalars φ^x. We define the embedding metric on this surface as

$$g_{xy} = h^{\widetilde{I}}_x h^{\widetilde{J}}_y a_{\widetilde{I}\widetilde{J}}\,. \tag{7.42}$$

This metric indeed gives the required kinetic term for φ. Apart from a metric, we can also introduce vielbeins f_x^a that are covariantly constant with respect to the spin-connection $\omega_x^{\tilde{a}\tilde{b}}$ and Levi-Civita connection Γ^z_{xy}, defined on this manifold:

$$\begin{aligned}
\Gamma^z_{xy} &= \tfrac{1}{2} g^{zw}\left(-g_{xy,w} + g_{wx,y} + g_{yw,x}\right), \qquad\qquad g_{xy;z} = 0\,, \\
g_{xy} &= \eta_{\tilde{a}\tilde{b}} f_x^{\tilde{a}} f_y^{\tilde{b}}\,, \\
f_{y;x}^{\tilde{a}} &= f_{y,x}^{\tilde{a}} + \omega_x^{\tilde{a}\tilde{b}} f_{y\tilde{b}} - \Gamma^z_{xy} f_z^{\tilde{a}} = 0\,, \qquad f_{[x,y]} = \omega_{[x}^{\tilde{a}\tilde{b}} f_{y]\tilde{b}}\,, \\
h^{\tilde{a}}_{\tilde{I}} &\equiv f_x^{\tilde{a}} h^x_{\tilde{I}}\,.
\end{aligned} \tag{7.43}$$

For future usage we also give the following useful relations, that follow from the above:

$$\begin{aligned}
h_{\tilde{I}x;y} &= h_{\tilde{I}x,y} - \Gamma^z_{xy} h_{\tilde{I}z} = \sqrt{\tfrac{2}{3}}\,(h_{\tilde{I}} g_{xy} + T_{xyz} h^z_{\tilde{I}}) \\
T_{xyz} &\equiv \sqrt{\tfrac{3}{2}}\, h_{\tilde{J}x;y} h^{\tilde{J}}_z = -\sqrt{\tfrac{3}{2}}\, h_{\tilde{J}x} h^{\tilde{J}}_{z;y} = h^{\tilde{I}}_x h^{\tilde{J}}_y h^{\tilde{K}}_z N_{\tilde{I}\tilde{J}\tilde{K}} \\
\Gamma^w_{xy} &= h^w_{\tilde{I}} h^{\tilde{I}}_{x,y} + \sqrt{\tfrac{2}{3}}\, T_{xyz} g^{zw}\,,
\end{aligned} \tag{7.44}$$

The $(n_V + n_T + 1)$ gauginos $\psi^{\tilde{I}}$ are also still constrained fields, due to the S-gauge. In order to translate these to $(n_V + n_T)$ unconstrained gauginos on the embedding space, we introduce $\lambda^{i\tilde{a}}$, which transforms as a vector in the tangent space. As we will see later, a convenient choice is given by (for agreement with the literature [76]):

$$\lambda^{i\tilde{a}} \equiv -h^a_{\tilde{I}} \psi^{i\tilde{I}}\,, \qquad \psi^{i\tilde{I}} = -h^{\tilde{I}}_{\tilde{a}} \lambda^{i\tilde{a}}\,. \tag{7.45}$$

Note that this choice for $\psi^{i\tilde{I}}$ indeed solves the S-gauge (7.26).

7.3 Results

The scalar potential. We will now determine the scalar potential like it appears in the gauge-fixed action. We will have to take into account all terms in (6.110) and (6.118) of order g^2. As the solution for Y^{ijI} (7.39) contains a term linear in g, the Y^2 and gYP terms in the actions will both contribute to the scalar potential:

$$V_{\text{scalar}} = C_{\tilde{I}\tilde{J}\tilde{K}} Y^{\tilde{I}}_{ij} Y^{ij\tilde{J}} \sigma^{\tilde{K}} - \tfrac{1}{2} g^2 \sigma^I \sigma^J \sigma^K \sigma^{\tilde{M}} \sigma^{\tilde{N}} t_{J\tilde{M}}{}^P t_{K\tilde{N}}{}^Q C_{IPQ} + 2g Y^I_{ij} \hat{P}^{ij}_I - \tfrac{1}{2} g^2 \sigma^{\tilde{I}} \sigma^{\tilde{J}} \hat{k}^{\hat{X}}_{\tilde{I}} \hat{k}_{\tilde{J}\hat{X}}\,. \tag{7.46}$$

After performing the rescaling of C and σ into N and h, substituting the expression for Y and applying the specific coordinate basis, the potential can be simplified to:

$$V_{\text{scalar}} = \frac{g^2}{\kappa^4}\left[2W^x W_x - 4\vec{P}\cdot\vec{P} + 2\vec{P}^x\cdot\vec{P}_x + 2\mathcal{N}_{iA}\mathcal{N}^{iA}\right], \tag{7.47}$$

where we defined the following quantities

$$\begin{aligned}
W^x &\equiv \tfrac{\sqrt{6}}{4} h^I K^x_I = -\tfrac{3}{4} t_{J\tilde{M}}{}^P h^J h^{\tilde{M}} h^x_P\,, \qquad K^x_I \equiv -\sqrt{\tfrac{3}{2}}\, t_{I\tilde{M}}{}^{\tilde{P}} h^{\tilde{M}} h^x_{\tilde{P}}\,, \\
\vec{P} &\equiv \kappa^2 h^I \vec{P}_I\,, \qquad \vec{P}_x \equiv \kappa^2 h^I_x \vec{P}_I\,, \qquad \mathcal{N}^{iA} \equiv \tfrac{\sqrt{6}}{4} h^I k^X_I\,.
\end{aligned} \tag{7.48}$$

The composite object $\vec{P}$, containing the moment map, also occurs with its derivative and is indeed the superpotential for this scalar potential.

The action. After applying the special coordinate basis, substituting the expressions for the dependent gauge fields and matter fields, and "reducing" the objects on the hyperkähler manifold to the quaternionic-Kähler manifold, we obtain the following action:

$$
\begin{aligned}
e^{-1}\mathcal{L} \;=\; & \tfrac{1}{2\kappa^2}R(\omega) - \tfrac14 a_{\widetilde{I}\widetilde{J}}\widehat{\mathcal{H}}^{\widetilde{I}}_{\mu\nu}\widehat{\mathcal{H}}^{\widetilde{J}\mu\nu} - \tfrac12 g_{xy}\mathcal{D}_a\varphi^x\mathcal{D}^a\varphi^y - \tfrac{1}{2\kappa^2}h_{XY}\mathcal{D}_a q^X\mathcal{D}^a q^Y \\
& + \frac{1}{16g}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}\widetilde{B}^M_{\mu\nu}\Big(\partial_\rho\widetilde{B}^N_{\sigma\tau} + 2g t_{IJ}{}^N A^I_\rho F^J_{\sigma\tau} + g\, t_{IP}{}^N A^I_\rho\widetilde{B}^P_{\sigma\tau}\Big) \\
& -\tfrac12\bar\psi_\rho\gamma^{\rho\mu\nu}\mathcal{D}_\mu\psi_\nu - \tfrac12\bar\lambda_x\not{\mathcal{D}}\lambda^x + \tfrac{1}{\kappa^2}\bar\zeta_A\not{\mathcal{D}}\zeta^A \\
& + \tfrac{g^2}{\kappa^4}\Big(-2W_xW^x + 4\vec{P}\cdot\vec{P} - 2\vec{P}^x\cdot\vec{P}_x - 2\mathcal{N}_{iA}\mathcal{N}^{iA}\Big) \\
& + \tfrac{\kappa}{12}\sqrt{\tfrac23}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}N_{IJK}A^I_\mu\Big[F^J_{\nu\lambda}F^K_{\rho\sigma} + f_{FG}{}^J A^F_\nu A^G_\lambda\Big(-\tfrac12 g\,F^K_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\Big)\Big] \\
& -\tfrac18 e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^M t_{FG}{}^N A^I_\mu A^F_\nu A^G_\lambda\Big(-\tfrac12 g\,F^K_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\Big) \\
& -\tfrac14\kappa h_{\widetilde{I}x}H^{\widetilde{I}}_{bc}\bar\psi_a\gamma^{abc}\lambda^x - \tfrac{3}{8\sqrt6}\kappa\,\mathrm{i}\,h_{\widetilde{I}}H^{cd\widetilde{I}}\bar\psi^a\gamma_{abcd}\psi^b + \tfrac14\sqrt{\tfrac23}\kappa\,\mathrm{i}\,T_{xyz}h^z_{\widetilde{I}}\bar\lambda^x\gamma\cdot H^{\widetilde{I}}\lambda^y \\
& + \tfrac{1}{8\sqrt6}\kappa\,\mathrm{i}\,h_{\widetilde{I}}\bar\lambda^x\gamma\cdot H^{\widetilde{I}}\lambda_x + \tfrac14\sqrt{\tfrac{3}{2\kappa^2}}\,\mathrm{i}\,h_{\widetilde{I}}\bar\zeta_A\gamma\cdot H^{\widetilde{I}}\zeta^A + \tfrac12\,\mathrm{i}\,\bar\psi_a\not{\mathcal{D}}\varphi^x\gamma^a\lambda_x + \mathrm{i}\,\tfrac1\kappa\bar\zeta_A\gamma^a\not{\mathcal{D}}q^X\psi^i_a f^A_{iX} \\
& -g\Big(\sqrt{\tfrac32}\tfrac{1}{\kappa^3}\,\mathrm{i}\,h^I t_{IB}{}^A\bar\zeta_A\zeta^B - 2\,\mathrm{i}\,\tfrac{1}{\kappa^2}k^X_I f^A_{iX}h^I_x\bar\zeta_A\lambda^{ix} + \sqrt{\tfrac32}\tfrac{1}{\kappa^2}h^I k^X_I f^A_{iX}\bar\zeta_A\gamma^a\psi^i_a - \kappa\bar\psi^i_a\gamma^a\lambda^{jx}h^I_x P_{Iij} \\
& -\tfrac12\sqrt{\tfrac32}\kappa\,\mathrm{i}\,h^I P_{Iij}\bar\psi^i_a\gamma^{ab}\psi^j_b + \sqrt{\tfrac23}\kappa\,\mathrm{i}\,T_{xyz}h^{Iz}P_{Iij}\bar\lambda^{ix}\lambda^{jy} + \tfrac{\kappa}{2\sqrt6}\,\mathrm{i}\,h^I P_{Iij}\bar\lambda^{ix}\lambda^j_x \\
& + \sqrt{\tfrac32}\tfrac1\kappa\,\mathrm{i}\,h^{\widetilde{I}}_x h^{\widetilde{J}}_y\bar\lambda^x\lambda^y h^{\widetilde{K}}h^{\widetilde{L}}\big(t_{\widetilde{I}\widetilde{J}}{}^{\widetilde{M}}N_{\widetilde{M}\widetilde{K}\widetilde{L}} + t_{\widetilde{K}\widetilde{I}}{}^{\widetilde{M}}N_{\widetilde{M}\widetilde{J}\widetilde{L}}\big) - \tfrac34\tfrac1\kappa\bar\psi_a\gamma^a\lambda^x h^{\widetilde{I}}_x h^{\widetilde{J}}h_{\widetilde{K}}t_{\widetilde{J}\widetilde{I}}{}^{\widetilde{K}}\Big) \\
& -\tfrac{\kappa^2}{16}\bar\psi^i_a\psi^{ja}\bar\lambda^x_i\lambda_{jx} - \tfrac{\kappa^2}{16}\bar\psi^i_a\gamma_b\psi^{ja}\bar\lambda^d_i\gamma^b\lambda_{jx} - \tfrac{\kappa^2}{64}\bar\psi_a\gamma_{bc}\psi^a\bar\lambda^x\gamma^{bc}\lambda_x - \tfrac{\kappa^2}{96}\bar\psi_a\psi_b\bar\lambda^x\gamma^{ab}\lambda_x \\
& + \tfrac{\kappa^2}{96}\bar\psi_a\gamma_b\psi_c\bar\lambda^x\gamma^{abc}\lambda_x - \tfrac{\kappa^2}{24}\bar\psi^i_a\gamma^{ab}\psi^j_b\bar\lambda^x_i\lambda_{jx} - \tfrac{\kappa^2}{24}\bar\psi^{ai}\gamma^{bc}\psi^{dj}\bar\lambda^x_i\gamma_{abcd}\lambda_{jx} + \tfrac{\kappa^2}{8}\bar\psi_a\gamma_b\psi^b\bar\psi^a\gamma_c\psi^c \\
& -\tfrac{\kappa^2}{16}\bar\psi_a\gamma_b\psi_c\bar\psi^a\gamma^c\psi^b - \tfrac{\kappa^2}{32}\bar\psi_a\gamma_b\psi_c\bar\psi^a\gamma^b\psi^c + \tfrac{\kappa^2}{32}\bar\psi_a\psi_b\bar\psi_c\gamma^{abcd}\psi_d - \tfrac{3}{16\kappa}\bar\zeta_A\gamma_{abc}\zeta^A\bar\psi^a\gamma^b\psi^c \\
& + \tfrac18\bar\psi_a\gamma^{bc}\psi^a\bar\zeta_A\gamma_{bc}\zeta^A + \tfrac{1}{16}\bar\psi^a\psi^b\bar\zeta_A\gamma_{ab}\zeta^A + \tfrac{\kappa^2}{6}\sqrt{\tfrac23}\,\mathrm{i}\,T_{xyz}\bar\psi_a\gamma_b\lambda^x\bar\lambda^y\gamma^{ab}\lambda^z \\
& + \tfrac{1}{32}\bar\lambda^x\gamma_{ab}\lambda_x\bar\zeta_A\gamma^{ab}\zeta^A + \tfrac{\kappa^2}{6}\sqrt{\tfrac23}\,\mathrm{i}\,T_{xyz}\bar\psi^i_a\gamma^a\lambda^{jx}\bar\lambda^y_i\lambda^z_j + \tfrac{9\kappa^2}{16}\bar\lambda^{ix}\gamma_a\lambda^j_x\bar\lambda^y_i\gamma_a\lambda_{jy} \\
& + \tfrac{\kappa^2}{128}\bar\lambda^x\gamma_{ab}\lambda_x\bar\lambda^y\gamma^{ab}\lambda_y + \tfrac{\kappa^2}{6}g^{zt}T_{xyz}T_{tvw}\bar\lambda^{ix}\lambda^{jy}\bar\lambda^v_i\lambda^w_j - \tfrac{\kappa^2}{48}\bar\lambda^{ix}\lambda^j_x\bar\lambda^y_i\lambda_{jy} \\
& -\tfrac{1}{4\kappa^2}\mathcal{W}_{ABCD}\bar\zeta^A\zeta^B\bar\zeta^C\zeta^D + \tfrac{1}{32\kappa^2}\bar\zeta_A\gamma_{ab}\zeta^A\bar\zeta_B\gamma^{ab}\zeta^B\,.
\end{aligned}
\tag{7.49}
$$

The covariant derivatives are given by

$$
\begin{aligned}
\mathcal{D}_\mu\varphi^x &= \partial_\mu\varphi^x + gA^I_\mu K^x_I\,, \\
\mathcal{D}_\mu h^{\widetilde{I}} &= \partial_\mu h^{\widetilde{I}} + g t_{J\widetilde{K}}{}^{\widetilde{I}}A^J_\mu h^{\widetilde{K}} = -\sqrt{\tfrac23}h^{\widetilde{I}}_x\mathcal{D}_\mu\varphi^x\,, \\
\mathcal{D}_\mu q^X &= \partial_\mu q^X + gA^I_\mu k^X_I\,, \\
\mathcal{D}_\mu\lambda^{xi} &= \partial_\mu\lambda^{xi} + \partial_\mu\phi^y\Gamma^x_{yz}\lambda^{zi} + \tfrac14\omega_\mu{}^{ab}\gamma_{ab}\lambda^{xi}
\end{aligned}
\tag{7.50}
$$

$$
\begin{aligned}
&& +\partial_\mu q^X \omega_{Xj}{}^i \lambda^{xj} + \kappa^2 g A_\mu^I P_{Ij}{}^i \lambda^{xj} + g A_\mu^I K_I^{x;y} \lambda_y^i \,, \\
\mathcal{D}_\mu \zeta^A &=& \partial_\mu \zeta^A + \partial_\mu q^X \omega_{XB}{}^A \zeta^B + \tfrac{1}{4}\omega_\mu{}^{bc}\gamma_{bc}\zeta^A + g A_\mu^I t_{IB}{}^A \zeta^B \,, \\
\mathcal{D}_\mu \psi_{\nu i} &=& (\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\psi_{\nu i} - \partial_\mu q^X \omega_{Xi}{}^j \psi_{\nu j} - \kappa^2 g A_\mu^I P_{Ii}{}^j \psi_{\nu j} \,.
\end{aligned}
$$

We chose to extract the fermionic terms from the spin connection and use $\omega_\mu{}^{ab}$ instead of $\hat{\omega}_\mu{}^{ab}$ in the covariant derivatives and the Ricci scalar, unless mentioned otherwise.

The transformation rules. The $\mathcal{N} = 2$ Poincaré supersymmetry rules that leave the above action invariant can be constructed as follows. We start from the transformation rules for the vector-tensor multiplets (6.105), for the hypermultiplets (6.112) and the two remaining transformation rules from the Weyl-multiplet (5.40) for the vielbein and gravitino. Next, the parameters corresponding to the gauge-fixed symmetries are replaced by the decomposition rules given in section 7.2.2. The remaining transformations are Poincaré supersymmetry (ϵ^i) and gauge transformations (Λ_G); they are given by:

$$
\begin{aligned}
\delta(\epsilon) e_\mu{}^a &=& \tfrac{1}{2}\kappa \bar\epsilon \gamma^a \psi_\mu \,. \\
\delta(\epsilon)\psi_\mu^i &=& \tfrac{1}{\kappa} D_\mu(\hat\omega)\epsilon^i + \tfrac{1}{4\sqrt{6}}\,\mathrm{i}\, h_{\tilde I}\widehat{\mathcal{H}}^{\tilde I ab}(\gamma_{ab\mu} - 4 g_{\mu a}\gamma_b)\epsilon^i + \delta_Q q^X \omega_X^{ij}\psi_{\mu j} - \tfrac{1}{\kappa^2\sqrt{6}}\,\mathrm{i}\, g P^{ij}\gamma_\mu \epsilon_j \\
&& -\tfrac{\kappa}{6}\bar\lambda^{ix}\gamma_\mu \lambda_x^j \epsilon_j + \tfrac{\kappa}{12}\bar\lambda^{ix}\gamma^a \lambda_x^j \gamma_{\mu a}\epsilon_j - \tfrac{\kappa}{48}\bar\lambda^{ix}\gamma^{ab}\lambda_x^j \gamma_{\mu ab}\epsilon_j + \tfrac{\kappa}{12}\bar\lambda^{ix}\gamma_{\mu a}\lambda_x^j \gamma^a \epsilon_j \\
&& +\tfrac{1}{16\kappa}\bar\zeta_A \gamma^{ab}\zeta^A \gamma_{\mu ab}\epsilon^i \,, \\
\delta(\epsilon)\varphi^x &=& \tfrac{\kappa}{2}\,\mathrm{i}\,\bar\epsilon \lambda^{\tilde a} f_{\tilde a}^x \,, \\
\delta(\epsilon) A_\mu^I &=& \vartheta_\mu^I \,, \qquad\qquad (7.51)\\
\delta(\epsilon)\lambda^{i\tilde a} &=& -\tfrac{1}{2}\,\mathrm{i}\,\widehat{\not{\mathcal{D}}}\varphi^x \epsilon^i - \delta(\epsilon)\varphi^x \omega_x^{\tilde a \tilde b} f_{x\tilde b} + \delta(\epsilon) q^X \omega_X{}^{ij}\lambda_j^{\tilde a} + \tfrac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\tilde I} h_{\tilde I}^{\tilde a}\epsilon^i \\
&& -\tfrac{1}{4\sqrt{6}} T^{\tilde a \tilde b \tilde c}[-3\bar\lambda_{\tilde b}^i \lambda_{\tilde c}^j + \bar\lambda_{\tilde b}^i \gamma_\mu \lambda_{\tilde c}^j \gamma^\mu + \tfrac{1}{2}\bar\lambda_{\tilde b}^i \gamma_{\mu\nu}\lambda_{\tilde c}^j \gamma^{\mu\nu}]\epsilon_j - \tfrac{1}{\kappa^2} g P^{\tilde a\, ij}\epsilon_j + \tfrac{1}{\kappa^2} g W^{\tilde a}\epsilon^i \,, \\
\delta(\epsilon)\widetilde{B}_{\mu\nu}^M &=& 2\mathcal{D}_{[\mu}\vartheta_{\nu]}^M - \sqrt{6} g \bar\epsilon \gamma_{[\mu}\psi_{\nu]} h_N \Omega^{MN} - \mathrm{i}\, g \bar\epsilon \gamma_{\mu\nu}\lambda^x h_{xN}\Omega^{MN} \,, \\
\delta(\epsilon) q^X &=& -\mathrm{i}\,\bar\epsilon^i \zeta^A f_{iA}^X \,, \\
\delta(\epsilon)\zeta^A &=& \tfrac{1}{2}\,\mathrm{i}\,\gamma^\mu \widehat{\mathcal{D}}_\mu q^X f_X^{iA}\epsilon^i - \delta(\epsilon) q^X \omega_{XB}{}^A \zeta^B + \tfrac{1}{\kappa} g \mathcal{N}_i^A \epsilon^i \,,
\end{aligned}
$$

with

$$
\begin{aligned}
\vartheta_\mu^{\tilde I} &\equiv& -\tfrac{1}{2}\bar\epsilon \gamma_\mu \lambda^{\tilde a} f_{\tilde a}^x h_x^{\tilde I} - \tfrac{\sqrt{6}}{4}\,\mathrm{i}\, h^{\tilde I}\bar\epsilon \psi_\mu \,, \qquad \widehat{\mathcal{D}}_\mu \varphi^x &=& \mathcal{D}_\mu \varphi^x - \tfrac{\kappa}{2}\,\mathrm{i}\,\bar\psi_\mu \lambda^x \,, \\
\mathcal{D}_\mu \vartheta_\nu^{\tilde I} &=& \partial_\mu \vartheta_\nu^{\tilde I} + g A_\mu^J t_{J\tilde K}{}^{\tilde I}\vartheta_\nu^{\tilde K} \,, \qquad \widehat{\mathcal{D}}_\mu q^X &=& \partial_\mu q^X + g A_\mu^I k_I^X + \kappa\,\mathrm{i}\,\bar\psi_\mu^i \zeta^B f_{iB}^X \,,
\end{aligned}
\qquad (7.52)
$$

and where the (gauge) covariant derivative of the Killing spinor is given by

$$
D_\mu(\hat\omega)\epsilon^i = \mathcal{D}_\mu(\hat\omega)\epsilon^i - \partial_\mu q^X \omega_X^{ij}\epsilon_j - g\kappa^2 A_\mu^I P_I^{ij}\epsilon_j \,. \qquad (7.53)
$$

Notice that the fermion shifts, proportional to P^{ij}, $P^{\tilde a ij}$, $W^{\tilde a}$ and $\mathcal{N}_I{}^A$, indeed appear quadratically in the scalar potential.

The transformations under the gauge group G are given by:

$$
\begin{aligned}
\delta(\Lambda_G)A^I_\mu &= \partial_\mu\Lambda^I_G + gA^J_\mu f_{JK}{}^I\Lambda^K_G\,,\\
\delta(\Lambda_G)\widetilde{B}^M_{\mu\nu} &= -g\Lambda^J_G t_{J\widetilde{K}}{}^M H^{\widetilde{K}}_{\mu\nu}\,,\\
\delta(\Lambda_G)\varphi^x &= -g\Lambda^I_G K^x_I\,,\\
\delta(\Lambda_G)q^X &= -g\Lambda^I_G k^X_I\,,\\
\delta(\Lambda_G)\zeta^A &= -\delta(\Lambda_G)q^X\omega_{XB}{}^A\zeta^B - g\Lambda^I_G t_{IB}{}^A\zeta^B\,,\\
\delta(\Lambda_G)\lambda^{i\tilde a} &= (-\omega^{\tilde a\tilde b}_y f_{x\tilde b} + \sqrt{\tfrac{2}{3}}f^{\tilde a}_w T_{xyz}g^{zw})\lambda^x\delta(\Lambda_G)\varphi^y + \delta(\Lambda_G)q^X\omega^{ij}_X\lambda^t_j a + \kappa^2 g\Lambda^I_G P^{ij}_I\lambda^{\tilde a}_j\,,\\
\delta(\Lambda_G)\psi^i_\mu &= \delta(\Lambda_G)q^X\omega^{ij}_X\psi_{\mu j} + \kappa^2 g\Lambda^I_G P^{ij}_I\psi_{\mu j}\,.
\end{aligned}
\tag{7.54}
$$

7.4 Simplified action for domain-walls

In the previous chapter we gave the full results, including the quartic fermion couplings. However, for determining the domain-wall solutions we only need the bosonic parts of the fermionic transformation rules, and the bosonic action. In this section we have collected all relevant information, needed for such an investigation.

The bosonic parts of the fermionic transformation rules immediately lead to the BPS equations:

$$
\begin{aligned}
\delta(\epsilon)\psi^i_\mu &= 0 = \tfrac{1}{\kappa}D_\mu(\omega)\epsilon^i + \tfrac{1}{4\sqrt{6}}\,\mathrm{i}\,h_{\widetilde{I}}\mathcal{H}^{\widetilde{I}ab}(\gamma_{ab\mu} - 4g_{\mu a}\gamma_b)\epsilon^i - \tfrac{1}{\kappa^2\sqrt{6}}\,\mathrm{i}\,gP^{ij}\gamma_\mu\epsilon_j\,,\\
\delta(\epsilon)\lambda^{i\tilde a} &= 0 = -\tfrac{1}{2}\,\mathrm{i}\,\mathcal{D}\!\!\!/\varphi^x\epsilon^i + \tfrac{1}{4}\gamma\cdot\mathcal{H}^{\widetilde{I}}h^{\tilde a}_{\widetilde{I}}\epsilon^i - \tfrac{1}{\kappa^2}gP^{\tilde a\,ij}\epsilon_j + \tfrac{1}{\kappa^2}gW^{\tilde a}\epsilon^i\,,\\
\delta(\epsilon)\zeta^A &= 0 = \tfrac{1}{2}\,\mathrm{i}\,\gamma^\mu\widehat{\mathcal{D}}_\mu q^X f^{iA}_X\epsilon^i + \tfrac{1}{\kappa}g\mathcal{N}^A_i\epsilon^i\,,
\end{aligned}
\tag{7.55}
$$

with

$$
D_\mu(\omega)\epsilon^i = \mathcal{D}_\mu(\omega)\epsilon^i - \partial_\mu q^X\omega^{ij}_X\epsilon_j - g\kappa^2 A^I_\mu P^{ij}_I\epsilon_j\,. \tag{7.56}
$$

These equations can in principle be solved by choosing a specific coset manifold, i.e. specifying the constants $N_{\widetilde{I}\widetilde{J}\widetilde{K}}$ and by making the φ^x-embedding explicit. Since not every solution of the BPS equations necessarily has to satisfy the equations of motion, we will also have to give the truncated action. The bosonic equations of motion can be derived from the following truncated action:

$$
\begin{aligned}
e^{-1}\mathcal{L} = {}& \tfrac{1}{2\kappa^2}R(\omega) - \tfrac{1}{4}a_{\widetilde{I}\widetilde{J}}\mathcal{H}^{\widetilde{I}}_{\mu\nu}\mathcal{H}^{\widetilde{J}\mu\nu} - \tfrac{1}{2}g_{xy}\mathcal{D}_a\varphi^x\mathcal{D}^a\varphi^y - \tfrac{1}{2\kappa^2}h_{XY}\mathcal{D}_a q^X\mathcal{D}^a q^Y\\
&+\tfrac{1}{16g}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}\widetilde{B}^M_{\mu\nu}\Big(\partial_\rho\widetilde{B}^N_{\sigma\tau} + 2gt_{IJ}{}^N A^I_\rho F^J_{\sigma\tau} + g\,t_{IP}{}^N A^I_\rho\widetilde{B}^P_{\sigma\tau}\Big)\\
&+\tfrac{g^2}{\kappa^4}\Big(-2W_xW^x + 4\vec{P}\cdot\vec{P} - 2\vec{P}^x\cdot\vec{P}_x - 2\mathcal{N}_{iA}\mathcal{N}^{iA}\Big)\\
&+\tfrac{\kappa}{12}\sqrt{\tfrac{2}{3}}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}N_{IJK}A^I_\mu\Big[F^J_{\nu\lambda}F^K_{\rho\sigma} + f_{FG}{}^J A^F_\nu A^G_\lambda\Big(-\tfrac{1}{2}g\,F^K_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\Big)\Big]\\
&-\tfrac{1}{8}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^M t_{FG}{}^N A^I_\mu A^F_\nu A^G_\lambda\Big(-\tfrac{1}{2}g\,F^K_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\Big)\,,
\end{aligned}
\tag{7.57}
$$

where the relevant covariant derivatives are given by

$$\mathcal{D}_\mu \varphi^x = \partial_\mu \varphi^x + g A_\mu^I K_I^x \,, \qquad \mathcal{D}_\mu q^X = \partial_\mu q^X + g A_\mu^I k_I^X \,. \tag{7.58}$$