

## Chapter 6

# Matter-couplings of conformal supergravity

### 6.1 Introduction

In the previous chapter the first step in the conformal program has been performed by constructing the Standard Weyl multiplet of  $\mathcal{N} = 2$  conformal supergravity in five dimensions. We will now take the next step in the program and introduce the different  $D = 5$  matter multiplets with eight conformal supersymmetries together with the corresponding actions (when they exist). Apart from reasons given before, there is a rather different, more general, motivation of why the  $D = 5$  matter-coupled supergravities are interesting to study. The reason is that they belong to the class of theories with eight supersymmetries [149]. Such theories are especially interesting since the geometry, determined by the kinetic terms of the scalars, contains undetermined functions. Theories with thirty-two supersymmetries have no matter multiplets while the geometry of those with sixteen supersymmetries is completely determined by the number of matter multiplets. Of course, theories with four supersymmetries allow for more general geometries. The restricted class of geometries, in the case of eight supersymmetries, makes these theories especially interesting and manageable. For instance, the work of Seiberg and Witten [150, 151] heavily relies on the presence of eight supersymmetries. Theories with eight supersymmetries are thus the maximally supersymmetric theories that, on the one hand, are not completely determined by the number of matter multiplets in the model and, on the other hand, allow arbitrary functions in their definition, i.e. continuous deformations of the metric of the manifolds.

The geometry related to supersymmetric theories with eight real supercharges is called ‘special geometry’. Special geometry was first found in [152, 153] for local supersymmetry and in [154, 155] for rigid supersymmetry. It occurs in Calabi-Yau compactifications of type II superstrings as the moduli space of these manifolds [156–161].

In the following sections we will introduce the relevant basic superconformal matter multiplets: the vector-tensor multiplet and the hypermultiplet. We will start by discussing them in a rigid superconformal context, at which level we already find all the interesting geometry. A local superconformal extension will be given in the last section.

Field	SU(2)	$w$	# d.o.f.
$A_\mu^I$	1	0	$4n$
$Y^{ijI}$	3	2	$3n$
$\sigma^I$	1	1	$1n$
$\psi^{iI}$	2	$3/2$	$8n$

**Table 6.1:** The off-shell non-Abelian vector multiplet, where  $n$  labels the number of vector multiplets.

## 6.2 The vector-tensor multiplet

In this section, we will discuss superconformal vector multiplets that transform in arbitrary representations of the gauge group. From work on  $\mathcal{N} = 2, D = 5$  Poincaré matter couplings [73] it is known that vector multiplets transforming in representations other than the adjoint have to be dualized to tensor fields. We define a vector-tensor multiplet to be a vector multiplet transforming in a reducible representation that contains the adjoint representation as well as another, arbitrary representation.

We will show that the analysis of [73] can be extended to superconformal vector multiplets. In doing this we will generalize the gauge transformations for the tensor fields [73] by allowing them to transform into the field-strengths for the adjoint gauge fields. These more general gauge transformations are consistent with supersymmetry, even after breaking the conformal symmetry.

The vector-tensor multiplet contains *a priori* an arbitrary number of tensor fields. The restriction to an even number of tensor fields is not imposed by the closure of the algebra. If one demands that the field equations do not contain tachyonic modes, an even number is required [68]. Closely related to this is the fact that one can only construct an action for an even number of tensor multiplets. But supersymmetry without an action allows the more general possibility. Note that these main results are independent of the use of superconformal or super-Poincaré algebras.

### 6.2.1 Adjoint representation

We will start with giving the transformation rules for a vector multiplet in the adjoint representation [133]. An off-shell vector multiplet has  $8 + 8$  real degrees of freedom whose SU(2) labels and Weyl weights we have indicated in table 6.1.

The gauge transformations that we consider satisfy the commutation relations ( $I = 1, \dots, n$ )

$$[\delta_G(\Lambda_1^I), \delta_G(\Lambda_2^J)] = \delta_G(\Lambda_3^K), \quad \Lambda_3^K = g\Lambda_1^I\Lambda_2^J f_{IJ}^K. \quad (6.1)$$

The gauge fields  $A_\mu^I$  ( $\mu = 0, 1, \dots, 4$ ) and general matter fields of the vector multiplet as e.g.  $X^I$

transform under gauge transformations with parameters  $\Lambda^I$  according to

$$\delta_G(\Lambda^J)A_\mu^I = \partial_\mu \Lambda^I + gA_\mu^J f_{JK}^I \Lambda^K, \quad \delta_G(\Lambda^J)X^I = -g\Lambda^J f_{JK}^I X^K, \quad (6.2)$$

where  $g$  is the coupling constant of the gauge group  $G$ . The expression for the gauge-covariant derivative of  $X^I$  and the field-strengths are given by

$$\mathcal{D}_\mu X^I = \partial_\mu X^I + gA_\mu^J f_{JK}^I X^K, \quad F_{\mu\nu}^I = 2\partial_{[\mu} A_{\nu]}^I + gf_{JK}^I A_\mu^J A_\nu^K. \quad (6.3)$$

The field-strength satisfies the Bianchi identity

$$\mathcal{D}_{[\mu} F_{\nu\lambda]}^I = 0. \quad (6.4)$$

The rigid  $Q$ - and  $S$ -supersymmetry transformation rules for the off-shell Yang-Mills multiplet are given by [133]

$$\begin{aligned} \delta A_\mu^I &= \frac{1}{2} \bar{\epsilon} \gamma_\mu \psi^I, \\ \delta Y^{ijl} &= -\frac{1}{2} \bar{\epsilon}^{(i} \mathcal{D} \psi^{j)l} - \frac{1}{2} i g \bar{\epsilon}^{(i} f_{JK}^I \sigma^J \psi^{j)K} + \frac{1}{2} i \bar{\eta}^{(i} \psi^{j)l}, \\ \delta \psi^{ij} &= -\frac{1}{4} \gamma \cdot F^I \epsilon^i - \frac{1}{2} i \mathcal{D} \sigma^I \epsilon^i - Y^{ijl} \epsilon_j + \sigma^I \eta^i, \\ \delta \sigma^I &= \frac{1}{2} i \bar{\epsilon} \psi^I. \end{aligned} \quad (6.5)$$

The commutator of two  $Q$ -supersymmetry transformations yields a translation with an extra  $G$ -transformation

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_P \left( \frac{1}{2} \bar{\epsilon}_2 \gamma_\mu \epsilon_1 \right) + \delta_G \left( -\frac{1}{2} i \sigma \bar{\epsilon}_2 \epsilon_1 \right). \quad (6.6)$$

Note that even though we are considering rigid superconformal symmetry, the algebra (6.6) contains a field-dependent term on the right-hand side. Such soft terms are commonplace in local superconformal symmetry but here they already appear at the rigid level. In Hamiltonian language, it means that the algebra is satisfied modulo constraints.

### 6.2.2 Reducible representation

Starting from  $n$  vector multiplets we now wish to consider a more general set of fields  $\mathcal{H}_{\mu\nu}^{\tilde{I}}$  ( $\tilde{I} = 1, \dots, n+m$ ). We write  $\mathcal{H}_{\mu\nu}^{\tilde{I}} = \{F_{\mu\nu}^I, B_{\mu\nu}^M\}$  with  $\tilde{I} = (I, M)$  ( $I = 1, \dots, n; M = n+1, \dots, n+m$ ). The first part of these fields corresponds to the generators in the adjoint representation. These are the fields that we used in subsection 6.2.1. The other fields form a tensor multiplet which may transform in an arbitrary, possibly reducible, representation. Properties of the tensor multiplet fields are given in table 6.2. The representation matrix can be written as

$$(t_I)_{\tilde{J}}^{\tilde{K}} = \begin{pmatrix} (t_I)_J^K & (t_I)_J^N \\ (t_I)_M^K & (t_I)_M^N \end{pmatrix}, \quad \begin{cases} I, J, K & = 1, \dots, n \\ M, N & = n+1, \dots, n+m. \end{cases} \quad (6.7)$$

It is understood that the  $(t_I)_J^K$  are in the adjoint representation, i.e.

$$(t_I)_J^K = f_{IJ}^K. \quad (6.8)$$

Field	SU(2)	$w$	# d.o.f.
$B_{\mu\nu}^M$	1	0	$3m$
$\phi^M$	1	1	$1m$
$\lambda^{iM}$	2	$3/2$	$4m$

**Table 6.2:** The on-shell tensor multiplet, where  $m$  labels the number of tensor multiplets.

If  $m \neq 0$ , then the representation  $(t_I)_{\tilde{J}}^{\tilde{K}}$  is reducible. We will see that this representation can be more general than assumed so far in treatments of vector-tensor multiplet couplings. The requirement that  $m$  is even will only appear when we demand the existence of an action in section 6.4.2, or if we require absence of tachyonic modes. The matrices  $t_I$  satisfy commutation relations

$$[t_I, t_J] = -f_{IJ}{}^K t_K, \quad \text{or} \quad t_{\tilde{I}\tilde{N}}^{\tilde{M}} t_{\tilde{J}\tilde{M}}^{\tilde{L}} - t_{\tilde{J}\tilde{N}}^{\tilde{M}} t_{\tilde{I}\tilde{M}}^{\tilde{L}} = -f_{IJ}{}^K t_{\tilde{K}\tilde{N}}^{\tilde{L}}. \quad (6.9)$$

If the index  $\tilde{L}$  is a vector index, then this relation is satisfied using the matrices as in (6.8).

Requiring the closure of the superconformal algebra, we find  $Q$ - and  $S$ -supersymmetry transformation rules for the vector-tensor multiplet and a set of constraints. The transformations are

$$\begin{aligned} \delta \mathcal{H}_{\mu\nu}^{\tilde{I}} &= -\bar{\epsilon} \gamma_{[\mu} \mathcal{D}_{\nu]} \psi^{\tilde{I}} + i g \bar{\epsilon} \gamma_{\mu\nu} t_{(\tilde{J}\tilde{K})}^{\tilde{I}} \sigma^{\tilde{J}} \psi^{\tilde{K}} + i \bar{\eta} \gamma_{\mu\nu} \psi^{\tilde{I}}, \\ \delta Y^{ij\tilde{I}} &= -\frac{1}{2} \bar{\epsilon}^{(i} \mathcal{D} \psi^{j)\tilde{I}} - \frac{1}{2} i g \bar{\epsilon}^{(i} \left( t_{[\tilde{J}\tilde{K}]}^{\tilde{I}} - 3 t_{(\tilde{J}\tilde{K})}^{\tilde{I}} \right) \sigma^{\tilde{J}} \psi^{j)\tilde{K}} + \frac{1}{2} i \bar{\eta}^{(i} \psi^{j)\tilde{I}}, \\ \delta \psi^{i\tilde{I}} &= -\frac{1}{4} \gamma \cdot \mathcal{H}^{\tilde{I}} \epsilon^i - \frac{1}{2} i \mathcal{D} \sigma^{\tilde{I}} \epsilon^i - Y^{ij\tilde{I}} \epsilon_j + \frac{1}{2} g t_{(\tilde{J}\tilde{K})}^{\tilde{I}} \sigma^{\tilde{J}} \sigma^{\tilde{K}} \epsilon^i + \sigma^{\tilde{I}} \eta^i, \\ \delta \sigma^{\tilde{I}} &= \frac{1}{2} i \bar{\epsilon} \psi^{\tilde{I}}. \end{aligned} \quad (6.10)$$

The curly derivatives denote gauge-covariant derivatives as in (6.3) with the replacement of structure constants by general matrices  $t_I$  according to (6.8). We have extended the range of the generators from  $I$  to  $\tilde{I}$  in order to simplify the transformation rules with the understanding that

$$(t_M)_{\tilde{J}}^{\tilde{K}} = 0. \quad (6.11)$$

We find that the supersymmetry algebra (6.6) is satisfied provided the representation matrices are restricted to

$$t_{(\tilde{J}\tilde{K})}^{\tilde{I}} = 0, \quad (6.12)$$

and provided the following two constraints on the fields are imposed:

$$L^{ij\tilde{I}} \equiv t_{(\tilde{J}\tilde{K})}^{\tilde{I}} \left( 2 \sigma^{\tilde{J}} Y^{ij\tilde{K}} - \frac{1}{2} i \bar{\psi}^{i\tilde{J}} \psi^{j\tilde{K}} \right) = 0, \quad (6.13)$$

$$E_{\mu\nu\lambda}^{\tilde{I}} \equiv \frac{3}{g} \mathcal{D}_{[\mu} \mathcal{H}_{\nu\lambda]}^{\tilde{I}} - \varepsilon_{\mu\nu\lambda\rho\sigma} t_{(\tilde{J}\tilde{K})}^{\tilde{I}} \left( \sigma^{\tilde{J}} \mathcal{H}^{\rho\sigma\tilde{K}} + \frac{1}{4} i \bar{\psi}^{\tilde{J}} \gamma^{\rho\sigma} \psi^{\tilde{K}} \right) = 0. \quad (6.14)$$

For  $\tilde{I} = I$ , the constraint (6.14) reduces to the Bianchi identity (6.4). The tensor  $F_{\mu\nu}^I$  can therefore be seen as the curl of a gauge vector  $A_\mu^I$ . Moreover, the constraint (6.13) is trivially satisfied for

$\widetilde{I} = I$ . We conclude that the fields with indices  $\widetilde{I} = I$  form an off-shell vector multiplet in the adjoint representation of the gauge group.

On the other hand, when  $\widetilde{I} = M$ , the constraint (6.14) does not permit the fields  $B_{\mu\nu}^M$  to be written as the curl of a gauge field and they should be seen as independent tensor fields. Instead, the constraint (6.14) is a massive self-duality condition that puts the tensors  $B_{\mu\nu}^M$  *on-shell*. The constraint (6.13) implicitly allows us to eliminate the fields  $Y^{ijM}$  altogether. The general vector-tensor multiplet can then be interpreted as a set of  $m$  on-shell tensor multiplets in the background of  $n$  off-shell vector multiplets.

Using (6.12) we have reduced the representation matrices  $t_I$  to the following block-upper-triangular form:

$$(t_I)_{\widetilde{J}}^{\widetilde{K}} = \begin{pmatrix} f_{IJ}^K & (t_I)_{J^N} \\ 0 & (t_I)_{M^N} \end{pmatrix}. \quad (6.15)$$

In [73] it is mentioned that, “since terms of the form  $B^M \wedge F^I \wedge A^J$  appear to be impossible to supersymmetrize in a gauge invariant way (except possibly in very special cases) we shall also assume that  $C_{MIJ} = 0$ ”. This corresponds, as we will see in section 6.4.2, to the assumption that the representation is completely reducible, i.e.  $t_{IJ}^N = 0$ , meaning that gauge transformations do not mix the pure Yang-Mills field-strengths and the tensor fields. However, we find that off-diagonal generators *are* allowed, both when requiring closure of the superconformal algebra and when writing down an action. We thus allow reducible, but not necessarily completely reducible representations.

The constraints (6.13) and (6.14), with  $\widetilde{I} = M$ , do not yet form a supersymmetric set; successive variations under  $S$ -supersymmetry and  $Q$ -supersymmetry lead to the equations of motion for the spinors  $\psi^{\widetilde{I}}$  and scalars  $\sigma^{\widetilde{I}}$  [86]. Although this procedure generates a set of constraints, transforming to each other under  $Q$ - and  $S$ -supersymmetry, they do not seem to form a multiplet by themselves. That is, the algebra is *not* realized on this set of transformation rules.

## 6.3 The hypermultiplet

In this subsection, we discuss hypermultiplets in five dimensions. As for the tensor multiplets, there is in general no known off-shell formulation with a finite number of auxiliary fields.<sup>1</sup> Therefore, the supersymmetry algebra already leads to the equations of motion.

A single hypermultiplet contains four real scalars and two spinors subject to the symplectic Majorana reality condition. For  $r$  hypermultiplets, we introduce real scalars  $q^X(x)$ , with  $X = 1, \dots, 4r$ , and spinors  $\zeta^A(x)$  with  $A = 1, \dots, 2r$ . The properties of the hypermultiplet fields are given in table 6.3. To formulate the symplectic Majorana condition, we introduce two matrices  $\rho_A^B$  and  $E_i^j$ , with

$$\rho\rho^* = -\mathbb{1}_{2r}, \quad EE^* = -\mathbb{1}_2. \quad (6.16)$$

This defines symplectic Majorana conditions for the fermions and supersymmetry transformation parameters [163]:

$$\alpha C\gamma_0 \zeta^B \rho_B^A = (\zeta^A)^*, \quad \alpha C\gamma_0 \epsilon^j E_j^i = (\epsilon^i)^*, \quad (6.17)$$

<sup>1</sup>An off-shell tensor-formulation *can* be constructed by extending the algebra with central charges [162]. A similar procedure could also possibly be used to obtain an off-shell hypermultiplet.

Field	SU(2)	$w$	# d.o.f.
$q^X$	2	3/2	$4r$
$\zeta^A$	1	2	$4r$

**Table 6.3:** The on-shell hypermultiplet, where  $r$  labels the number of hypermultiplets.

where  $C$  is the charge conjugation matrix, and  $\alpha$  is an irrelevant number of modulus 1. We can always adopt the basis where  $E_i{}^j = \varepsilon_{ij}$ , and will further restrict to that.

The scalar fields are interpreted as coordinates of some target space, and requiring the on-shell closure of the superconformal algebra imposes certain conditions on the target space, which we derive below. Superconformal hypermultiplets in four space-time dimensions were discussed in [164]; our discussion is somehow similar, but we extend it to the case where an action is not needed, in the spirit explained in [149].

### 6.3.1 Rigid supersymmetry

We will show how the closure of the supersymmetry transformation laws leads to a ‘hypercomplex manifold’. The closure of the algebra on the bosons leads to the defining equations for this geometry, whereas the closure of the algebra on the fermions and its further consistency leads to equations of motion in this geometry, independent of an action.

The rigid supersymmetry transformations for the hypermultiplet are given by

$$\begin{aligned}\delta(\epsilon)q^X &= -i\bar{\epsilon}^i\zeta^A f_{iA}^X, \\ \delta(\epsilon)\zeta^A &= \frac{1}{2}i\bar{\phi}q^X f_X^{iA}\epsilon_i - \zeta^B\omega_{XB}{}^A(\delta(\epsilon)q^X),\end{aligned}\quad (6.18)$$

where the functions  $f_{iA}^X(q)$ ,  $f_X^{iA}(q)$  and  $\omega_{XB}{}^A(q)$  satisfy reality properties consistent with reality of  $q^X$  and the symplectic Majorana conditions, e.g.

$$(f_X^{iA})^* = f_X^{jB}E_j{}^i\rho_B{}^A, \quad (\omega_{XA}{}^B)^* = (\rho^{-1}\omega_X\rho)_A{}^B. \quad (6.19)$$

A priori the functions  $f_{iA}^X$  and  $f_X^{iA}$  are independent, but the commutator of two supersymmetries on the scalars only gives a translation if one imposes

$$\begin{aligned}f_Y{}^{iA}f_{iA}^X &= \delta_Y^X, \quad f_X^{iA}f_{jB}^X = \delta_j^i\delta_B^A, \\ \mathfrak{D}_Y f_{iB}^X &\equiv \partial_Y f_{iB}^X - \omega_{YB}{}^A f_{iA}^X + \Gamma_{ZY}{}^X f_{iB}^Z = 0,\end{aligned}\quad (6.20)$$

where  $\Gamma_{XY}{}^Z$  is some object, symmetric in the lower indices. This means that  $f_{iA}^X$  can be interpreted as vielbeins on the hyperscalar manifold, i.e.  $f_{iA}^X$  and  $f_X^{iA}$  are each others inverse and are covariantly constant with connections  $\Gamma$  and  $\omega$ . It also implies that  $\rho$  is covariantly constant. The conditions (6.20) encode all the constraints on the target space that follow from imposing the supersymmetry algebra. Below, we show that there are no further geometrical constraints coming from the fermion commutator; instead this commutator defines the equations of motion for the on-shell hypermultiplet.

### Reparametrizations

The supersymmetry transformation rules (6.18) are covariant with respect to two kinds of reparametrizations. The first ones are the target space diffeomorphisms,  $q^X \rightarrow \tilde{q}^X(q)$ , under which  $f_{iA}^X$  transforms as a vector,  $\omega_{XA}^B$  as a one-form, and  $\Gamma_{XY}^Z$  as a connection. The second set are the reparametrizations which act on the tangent space indices  $A, B, \dots$ . On the fermions, they act as

$$\zeta^A \rightarrow \tilde{\zeta}^A(q) = \zeta^B U_B^A(q), \quad (6.21)$$

where  $U(q)_A^B$  is any invertible matrix. In general, such a transformation brings us into a basis where the fermions depend on the scalars  $q^X$ . In this sense, the hypermultiplet is written in a special basis where  $q^X$  and  $\zeta^A$  are independent fields. The supersymmetry transformation rules (6.18) are covariant under (6.21) if we transform  $f_X^{iA}(q)$  as a vector and  $\omega_{XA}^B$  as a connection,

$$\omega_{XA}^B \rightarrow \tilde{\omega}_{XA}^B = [(\partial_X U^{-1})U + U^{-1}\omega_X U]_A^B. \quad (6.22)$$

These considerations lead us to define the covariant variation of the fermions:

$$\widehat{\delta}\zeta^A \equiv \delta\zeta^A + \zeta^B \omega_{XB}^A \delta q^X, \quad (6.23)$$

for any transformation  $\delta$  (supersymmetry, conformal transformations, ...). Two models related by either target space diffeomorphisms or fermion reparametrizations of the form (6.21) are equivalent; they are different coordinate descriptions of the same system. Thus, in a covariant formalism, the fermions can be functions of the scalars. However, the expression  $\partial_X \zeta^A$  only makes sense if one compares different bases. But in the same way also, the expression  $\zeta^B \omega_{XB}^A$  only makes sense if one compares different bases, as the connection has no absolute value. The only covariant object is the covariant derivative

$$\mathfrak{D}_X \zeta^A \equiv \partial_X \zeta^A + \zeta^B \omega_{XB}^A. \quad (6.24)$$

We will frequently use the covariant transformations (6.23). It can similarly be used on target-space vectors or tensors. E.g. for a quantity  $\Delta^X$ :

$$\widehat{\delta}\Delta^X = \delta\Delta^X + \Delta^Y \Gamma_{ZY}^X \delta q^Z. \quad (6.25)$$

### Geometry

The geometry of the target space is that of a *hypercomplex* manifold. It is a weakened version of hyperkähler geometry where no hermitian covariantly constant metric is defined. We refer the reader to appendix C for an introduction to these manifolds, references and the mathematical context in which they can be situated.

The crucial ingredient is a triplet of complex structures, the hypercomplex structure, defined as

$$J^\alpha{}_X{}^Y \equiv -i f_X^{iA}(\sigma^\alpha)_i{}^j f_{jA}^Y. \quad (6.26)$$

Using (6.20), they are covariantly constant and satisfy the quaternion algebra

$$J^\alpha J^\beta = -\mathbb{1}_{4r} \delta^{\alpha\beta} + \varepsilon^{\alpha\beta\gamma} J^\gamma. \quad (6.27)$$

At some places we also use a doublet notation, for which

$$J_X^Y J_i^j \equiv i J^{\alpha Y} (\sigma^\alpha)_i{}^j = 2 f_X^{jA} f_{iA}^Y - \delta_i^j \delta_X^Y. \quad (6.28)$$

The same transition between doublet and triplet notation is also used for other  $SU(2)$ -valued quantities.

The holonomy group of such a space is contained in  $Gl(r, \mathbb{H}) = SU^*(2r) \times U(1)$ , the group of transformations acting on the  $A, B$ -indices. This follows from the integrability conditions on the covariantly constant vielbeins  $f_X^{iA}$ , which relates the curvatures of the  $\omega_{XA}{}^B$  and  $\Gamma_{XY}{}^Z$  connections (see appendix C.2 for conventions on the curvatures),

$$R_{XYZ}{}^W = f_{iA}^W f_Z^{iB} \mathcal{R}_{XYB}{}^A, \quad \delta_j^i \mathcal{R}_{XYB}{}^A = f_W^{iA} f_{jB}^Z R_{XYZ}{}^W, \quad (6.29)$$

such that the Riemann curvature only lies in  $Gl(r, \mathbb{H})$ . Moreover, from the cyclicity properties of the Riemann tensor, it follows that

$$\begin{aligned} f_{Ci}^X f_{jD}^Y \mathcal{R}_{XYB}{}^A &= -\frac{1}{2} \varepsilon_{ij} W_{CDB}{}^A, \\ W_{CDB}{}^A &\equiv f_C^{iX} f_{iD}^Y \mathcal{R}_{XYB}{}^A = \frac{1}{2} f_C^{iX} f_{iD}^Y f_{jB}^Z f_W^{Aj} R_{XYZ}{}^W, \end{aligned} \quad (6.30)$$

where  $W$  is symmetric in all its three lower indices. For a more detailed discussion on hypercomplex manifolds and their curvature relations, we refer to appendix C. There we show that, in contrast with hyperkähler manifolds, hypercomplex manifolds are not necessarily Ricci flat; instead, the Ricci tensor is antisymmetric and defines a closed two-form.

So far we have only used the commutator of supersymmetry on the hyperscalars, and this led us to the geometry of hypercomplex manifolds. Before continuing, we want to see what the independent objects are that determine the theory, and what the independent constraints are. We start in the supersymmetric theory from the vielbeins  $f_X^{iA}$ . They have to be real in the sense of (6.19) and invertible. With these vielbeins, we can construct the complex structures as in (6.26). In the developments above, the only remaining independent equation is the covariant constancy of the vielbein in (6.20). This equation contains the affine connection  $\Gamma_{XY}{}^Z$  and the  $Gl(r, \mathbb{H})$ -connection  $\omega_{XA}{}^B$ . These two objects can be determined from the vielbeins if and only if the ('diagonal') Nijenhuis tensor (C.24) vanishes. Indeed, for vanishing Nijenhuis tensor, the 'Obata'-connection [165]

$$\Gamma_{XY}{}^Z = -\frac{1}{6} \left( 2 \partial_{(X} J^{\alpha}{}_{Y)}{}^W + \varepsilon^{\alpha\beta\gamma} J_{(X}^{\beta U} \partial_{|U|} J^{\gamma}{}_{Y)}{}^W \right) J^{\alpha}{}_{WZ}, \quad (6.31)$$

leads to covariantly constant complex structures. Moreover, one can show that any torsionless connection that leaves the complex structures invariant is equal to this Obata connection (similar to the fact that a connection that leaves a metric invariant is the Levi-Civita connection). With this connection one can then construct the  $Gl(r, \mathbb{H})$ -connection

$$\omega_{XA}{}^B = \frac{1}{2} f_Y^{iB} \left( \partial_X f_{iA}^Y + \Gamma_{XZ}^Y f_{iA}^Z \right), \quad (6.32)$$

such that the vielbeins are covariantly constant.



### Dynamics

Now we consider the commutator of supersymmetry on the fermions, which will determine the equations of motion for the hypermultiplets. Using (6.20), (6.29) and (6.30), we compute the supersymmetry commutator on the fermions, and find

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\zeta^A = \frac{1}{2}\partial_a\zeta^A\bar{\epsilon}_2\gamma^a\epsilon_1 + \frac{1}{4}\Gamma^A\bar{\epsilon}_2\epsilon_1 - \frac{1}{4}\gamma_a\Gamma^A\bar{\epsilon}_2\gamma^a\epsilon_1. \quad (6.33)$$

On-shell closure of the algebra on the fermion requires the last two terms to vanish. The  $\Gamma^A$  are therefore called non-closure functions, and define the equations of motion for the fermions,

$$\Gamma^A = \mathfrak{D}\zeta^A + \frac{1}{2}W_{CDB}{}^A\zeta^B\bar{\zeta}^D\zeta^C = 0, \quad (6.34)$$

where we have introduced the covariant derivative with respect to the transformations (6.23)

$$\mathfrak{D}_\mu\zeta^A \equiv \partial_\mu\zeta^A + (\partial_\mu q^X)\zeta^B\omega_{XB}{}^A. \quad (6.35)$$

By varying the fermion equation of motion under supersymmetry, we derive the corresponding equations of motion for the scalar fields:

$$\widehat{\delta}(\epsilon)\Gamma^A = \frac{1}{2}i f_X^{iA}\epsilon_i\Delta^X, \quad (6.36)$$

where

$$\Delta^X = \square q^X - \frac{1}{2}\bar{\zeta}^B\gamma_a\zeta^D\partial^a q^Y f_Y^{iC} f_{iA}^X W_{BCD}{}^A - \frac{1}{4}\mathfrak{D}_Y W_{BCD}{}^A \bar{\zeta}^E \zeta^D \bar{\zeta}^C \zeta^B f_E^{iY} f_{iA}^X, \quad (6.37)$$

and the covariant Laplacian is given by

$$\square q^X = \partial_a\partial^a q^X + (\partial_a q^Y)(\partial^a q^Z)\Gamma_{YZ}{}^X. \quad (6.38)$$

In conclusion, the supersymmetry algebra imposes the hypercomplex constraints (6.20) and the equations of motion (6.34) and (6.37). These form a multiplet, as (6.36) has the counterpart

$$\widehat{\delta}(\epsilon)\Delta^X = -i\bar{\epsilon}^i\mathfrak{D}\Gamma^A f_{iA}^X + 2i\bar{\epsilon}^i\Gamma^B\bar{\zeta}^C\zeta^D f_{Bi}^Y \mathcal{R}^X{}_{YCD}, \quad (6.39)$$

where the covariant derivative of  $\Gamma^A$  is defined similar to (6.35). In the following, we will derive further constraints on the target space geometry from requiring the presence of conformal symmetry.

### 6.3.2 Superconformal symmetry

Now we define transformation rules for the hypermultiplet under the full (rigid) superconformal group. The scalars do not transform under special conformal transformations and special supersymmetry, but under dilatations and  $SU(2)$  transformations, we parametrize

$$\begin{aligned} \delta_D(\Lambda_D)q^X &= \Lambda_D k^X(q), \\ \delta_{SU(2)}(\Lambda^{ij})q^X &= \Lambda^{ij} k_{ij}^X(q), \end{aligned} \quad (6.40)$$

for some unknown functions  $k^X(q)$  and  $k_{ij}^X(q)$ .

To derive the appropriate transformation rules for the fermions, we first note that the hyperinos should be invariant under special conformal symmetry. This is due to the fact that this symmetry changes the Weyl weight with one. The special supersymmetry transformation of  $\zeta^A$  can be read off from the  $[K, Q]$ -commutator, giving rise to

$$\delta_S(\eta^j)\zeta^A = -k^X f_X^{iA} \eta_i. \quad (6.41)$$

Realizing the commutator of regular and special supersymmetry (5.21) on the scalars, we determine the expression for the generator of  $SU(2)$  transformations in terms of the dilatations and complex structures,

$$k_{ij}^X = \frac{1}{3}k^Y J_Y^X{}_{ij} \quad \text{or} \quad k^{\alpha X} = \frac{1}{3}k^Y J_Y^{\alpha X}. \quad (6.42)$$

Realizing (5.21) on the hyperinos, we determine the covariant variations

$$\widehat{\delta}_D \zeta^A = 2\Lambda_D \zeta^A, \quad \widehat{\delta}_{SU(2)} \zeta^A = 0, \quad (6.43)$$

and furthermore the commutator (5.21) only closes if we impose

$$\mathfrak{D}_Y k^X = \frac{3}{2} \delta_Y^X, \quad (6.44)$$

which also implies

$$\mathfrak{D}_Y k^{\alpha X} = \frac{1}{2} J_Y^{\alpha X}. \quad (6.45)$$

Note that (6.44) is imposed by supersymmetry. In a more usual derivation, where one considers symmetries of the Lagrangian, we would find this constraint by imposing dilatation invariance of the action, see (5.11). Our result, though, does not require the existence of an action. The relations (6.44) and (6.42) further restrict the geometry of the target space, and it is easy to derive that the Riemann tensor has four zero eigenvectors,

$$k^X R_{XYZ}{}^W = 0, \quad k^{\alpha X} R_{XYZ}{}^W = 0. \quad (6.46)$$

Also, under dilatations and  $SU(2)$  transformations, the hypercomplex structure is scale invariant and rotated into itself,

$$\begin{aligned} \Lambda_D \left( k^Z \partial_Z J^{\alpha X}{}^Y - \partial_Z k^Y J^{\alpha X}{}^Z + \partial_X k^Z J^{\alpha Z}{}^Y \right) &= 0, \\ \Lambda^\beta \left( k^{\beta Z} \partial_Z J^{\alpha X}{}^Y - \partial_Z k^{\beta Y} J^{\alpha X}{}^Z + \partial_X k^{\beta Z} J^{\alpha Z}{}^Y \right) &= -\epsilon^{\alpha\beta\gamma} \Lambda^\beta J^\gamma{}^X{}^Y. \end{aligned} \quad (6.47)$$

All properties derived above are similar to those derived from superconformal hypermultiplets in four space-time dimensions [164, 166]. There, the  $Sp(1) \times Gl(r, \mathbb{H})$  sections, or simply, hypercomplex sections, were introduced

$$A^{iB}(q) \equiv k^X f_X^{iB}, \quad (A^{iB})^* = A^{jC} E_j^i \rho_C^B, \quad (6.48)$$

which allow for a coordinate independent description of the target space. This means that all equations and transformation rules for the sections can be written without the occurrence of the  $q^X$  fields.

### 6.3.3 Symmetries

We now assume the action of a symmetry group on the hypermultiplet. We have no action, but the ‘symmetry’ operation should leave invariant the set of equations of motion. The symmetry algebra must commute with the supersymmetry algebra (and later with the full superconformal algebra). This leads to hypermultiplet couplings to a non-Abelian gauge group  $G$ . The symmetries are parametrized by

$$\begin{aligned}\delta_G q^X &= -g\Lambda_G^I k_I^X(q), \\ \widehat{\delta}_G \zeta^A &= -g\Lambda_G^I t_{IB}^A(q)\zeta^B.\end{aligned}\quad (6.49)$$

The vectors  $k_I^X$  depend on the scalars and generate the algebra of  $G$  with structure constants  $f_{IJ}^K$ ,

$$k_{[I}^Y \partial_Y k_{J]}^X = -\frac{1}{2} f_{IJ}^K k_K^X. \quad (6.50)$$

The commutator of two gauge transformations (6.1) on the fermions requires the following constraint on the field-dependent matrices  $t_I(q)$ ,

$$[t_I, t_J]_B^A = -f_{IJ}^K t_{KB}^A - 2k_{[I}^X \mathcal{D}_X t_{J]B}^A + k_I^X k_J^Y \mathcal{R}_{XYB}^A. \quad (6.51)$$

Requiring the gauge transformations to commute with supersymmetry leads to further relations between the quantities  $k_I^X$  and  $t_{IB}^A$ , allowing us to determine  $t_I(q)$  in terms of the vielbeins  $f_X^{iA}$  and the vectors  $k_I^X$

$$t_{IA}^B = \frac{1}{2} f_{iA}^Y \mathcal{D}_Y k_I^X f_X^{iB}, \quad (6.52)$$

if the vectors  $k_I^X$  satisfy the constraint

$$f_A^{Y(i} f_X^{j)B} \mathcal{D}_Y k_I^X = 0. \quad (6.53)$$

Equation (6.53) can be expressed as the vanishing of the commutator of  $\mathcal{D}_Y k_I^X$  with the complex structures:<sup>2</sup>

$$\left(\mathcal{D}_X k_I^Y\right) J^\alpha_Y{}^Z = J^\alpha_X{}^Y \left(\mathcal{D}_Y k_I^Z\right), \quad (6.54)$$

which is equivalent to the vanishing of the Lie derivative of the complex structure in the direction of the vector  $k_I$

$$\left(\mathcal{L}_{k_I} J^\alpha\right)_X{}^Y \equiv k_I^Z \partial_Z J^\alpha_X{}^Y - \partial_Z k_I^Y J^\alpha_X{}^Z + \partial_X k_I^Z J^\alpha_Z{}^Y = 0. \quad (6.55)$$

According to part C.5 of the appendix, this means that (6.55) is a special case of the statement that the vector  $k_I$  normalizes the hypercomplex structures. The vanishing of this Lie derivative, or (6.53), is expressed by saying that the gauge transformations act *triholomorphic*. Thus, it says that all the symmetries are embedded in  $\text{Gl}(r, \mathbb{H})$ .

Vanishing of the gauge-supersymmetry commutator on the fermions requires

$$\mathcal{D}_Y t_{IA}^B = k_I^X \mathcal{R}_{YXA}^B. \quad (6.56)$$

Using (6.52) this implies a new constraint,

$$\mathcal{D}_X \mathcal{D}_Y k_I^Z = R_{XWY}{}^Z k_I^W. \quad (6.57)$$

<sup>2</sup>This can be seen directly from lemma C.2.2 in appendix C.

Note that this equation is in general true for any Killing vector of a metric. As we have no metric here, we could not rely on this fact, but here the algebra imposes this equation. It turns out that (6.53) and (6.57) are sufficient for the full commutator algebra to hold.

A further identity can be derived: substituting (6.56) into (6.51) one gets

$$[t_I, t_J]_B^A = -f_{IJ}^K t_{KB}^A - k_I^X k_J^Y \mathcal{R}_{XYB}^A. \quad (6.58)$$

The group of gauge symmetries should also commute with the superconformal algebra, in particular with dilatations and SU(2) transformations. This leads to

$$k^Y \mathfrak{D}_Y k_I^X = \frac{3}{2} k_I^X, \quad k^{\alpha Y} \mathfrak{D}_Y k_I^X = \frac{1}{2} k_I^Y J^\alpha{}_{Y^X}, \quad (6.59)$$

coming from the scalars, and there are no new constraints from the fermions or from other commutators. Since  $\mathfrak{D}_Y k_I^X$  commutes with  $J^\alpha{}_{Y^X}$ , the second equation in (6.59) is a consequence of the first one.

In the above analysis, we have taken the parameters  $\Lambda^I$  to be constants. In the following, we also allow for local gauge transformations. The gauge coupling is done by introducing vector multiplets and defining the covariant derivatives

$$\begin{aligned} \mathfrak{D}_\mu q^X &\equiv \partial_\mu q^X + g A_\mu^I k_I^X, \\ \mathfrak{D}_\mu \zeta^A &\equiv \partial_\mu \zeta^A + \partial_\mu q^X \omega_{XB}{}^A \zeta^B + g A_\mu^I t_{IB}{}^A \zeta^B. \end{aligned} \quad (6.60)$$

The commutator of two supersymmetries should now also contain a local gauge transformation, in the same way as for the multiplets of the previous sections, see (6.6). This requires an extra term in the supersymmetry transformation law of the fermion,

$$\widehat{\delta}(\epsilon) \zeta^A = \frac{1}{2} i \mathfrak{D} q^X f_{iX}^A \epsilon_i + \frac{1}{2} g \sigma^I k_I^X f_{iX}^A \epsilon^i. \quad (6.61)$$

With this additional term, the commutator on the scalars closes, whereas on the fermions, it determines the equations of motion

$$\Gamma^A \equiv \mathfrak{D} \zeta^A + \frac{1}{2} W_{BCD}{}^A \bar{\zeta}^C \zeta^D \zeta^B - g (i k_I^X f_{iX}^A \psi^{iI} + i \zeta^B \sigma^I t_{IB}{}^A) = 0, \quad (6.62)$$

with the same conventions as in (6.33).

Acting on  $\Gamma^A$  with supersymmetry determines the equation of motion for the scalars

$$\begin{aligned} \Delta^X &= \square q^X - \frac{1}{2} \bar{\zeta}^B \gamma_a \zeta^D \mathfrak{D}^a q^Y f_Y^{iC} f_{iA}^X W_{BCD}{}^A - \frac{1}{4} \mathfrak{D}_Y W_{BCD}{}^A \bar{\zeta}^E \zeta^D \bar{\zeta}^C \zeta^B f_E^{iY} f_{iA}^X \\ &\quad - g \left( 2 i \bar{\psi}^{iI} \zeta^B t_{IB}{}^A f_{iA}^X - k_I^Y J_Y{}^X{}_{ij} Y^{ijI} \right) + g^2 \sigma^I \sigma^J \mathfrak{D}_Y k_I^X k_J^Y. \end{aligned} \quad (6.63)$$

The first line is the same as in (6.37), the second line contains the corrections due to the gauging. The gauge-covariant Laplacian is here given by

$$\square q^X = \partial_a \mathfrak{D}^a q^X + g \mathfrak{D}_a q^Y \partial_Y k_I^X A^{aI} + \mathfrak{D}_a q^Y \mathfrak{D}^a q^Z \Gamma_{YZ}^X. \quad (6.64)$$

The equations of motions  $\Gamma^A$  and  $\Delta^X$  still satisfy the same algebra with (6.36) and (6.39).

## 6.4 Rigid superconformal actions

In this section, we will present rigid superconformal actions for the multiplets discussed in the previous section. We will see that demanding the existence of an action is more restrictive than only considering equations of motion. For the different multiplets, we find that new geometric objects have to be introduced.

### 6.4.1 Vector multiplet action

The rigid superconformal invariant action describing  $n$  vector multiplets was obtained from tensor calculus using an intermediate linear multiplet in [167]. The Abelian part can be obtained by just taking the cubic action of the improved vector multiplet as given in [131], adding indices  $I, J, K$  on the fields and multiplying with the symmetric tensor  $C_{IJK}$ . For the non-Abelian case, we need conditions expressing the gauge invariance of this tensor:

$$f_{I(J^H C_{KL)H} = 0. \quad (6.65)$$

Moreover one has to add a few more terms, e.g. to complete the Chern–Simons term to its non-Abelian form. This leads to the action

$$\begin{aligned} \mathcal{L}_{\text{vector}} = & \left[ \left( -\frac{1}{4} F_{\mu\nu}^I F^{\mu\nu J} - \frac{1}{2} \bar{\psi}^I \mathcal{D} \psi^J - \frac{1}{2} \mathcal{D}_a \sigma^I \mathcal{D}^a \sigma^J + Y_{ij}^I Y^{ij J} \right) \sigma^K \right. \\ & - \frac{1}{24} \varepsilon^{\mu\nu\lambda\rho\sigma} A_\mu^I \left( F_{\nu\lambda}^J F_{\rho\sigma}^K + \frac{1}{2} g [A_\nu, A_\lambda]^J F_{\rho\sigma}^K + \frac{1}{10} g^2 [A_\nu, A_\lambda]^J [A_\rho, A_\sigma]^K \right) \\ & \left. - \frac{1}{8} i \bar{\psi}^I \gamma \cdot F^J \psi^K - \frac{1}{2} i \bar{\psi}^{iI} \psi^{jJ} Y_{ij}^K + \frac{1}{4} i g \bar{\psi}^L \psi^H \sigma^J \sigma^L f_{LH}^K \right] C_{IJK}. \quad (6.66) \end{aligned}$$

The equations of motion for the fields of the vector multiplet following from the action (6.66) are

$$0 = L_I^{ij} = \varphi_I^i = E_I^a = N_I, \quad (6.67)$$

where we have defined

$$\begin{aligned} L_I^{ij} & \equiv C_{IJK} \left( 2\sigma^J Y^{iJK} - \frac{1}{2} i \bar{\psi}^{iJ} \psi^{jK} \right), \\ \varphi_I^i & \equiv C_{IJK} \left( i \sigma^J \mathcal{D} \psi^{iK} + \frac{1}{2} i (\mathcal{D} \sigma^J) \psi^{iK} + Y^{ikJ} \psi_k^K - \frac{1}{4} \gamma \cdot F^J \psi^{iK} \right) \\ & \quad - g C_{IJK} f_{LH}^K \sigma^J \sigma^L \psi^{iH}, \\ E_{aI} & \equiv C_{IJK} \left[ \mathcal{D}^b \left( \sigma^J F_{ba}^K + \frac{1}{4} i \bar{\psi}^J \gamma_{ba} \psi^K \right) - \frac{1}{8} \varepsilon_{abcde} F^{bcJ} F^{deK} \right] \\ & \quad - \frac{1}{2} g C_{JKL} f_{IH}^J \sigma^K \bar{\psi}^L \gamma_a \psi^H - g C_{JKH} f_{IL}^J \sigma^K \sigma^L \mathcal{D}_a \sigma^H, \\ N_I & \equiv C_{IJK} \left( \sigma^J \square \sigma^K + \frac{1}{2} \mathcal{D}^a \sigma^J \mathcal{D}_a \sigma^K - \frac{1}{4} F_{ab}^J F^{abK} - \frac{1}{2} \bar{\psi}^J \mathcal{D} \psi^K + Y^{ijJ} Y_{ij}^K \right) \\ & \quad + \frac{1}{2} i g C_{IJK} f_{LH}^K \sigma^J \bar{\psi}^L \psi^H. \quad (6.68) \end{aligned}$$

These equations themselves transform as a linear multiplet in the adjoint representation of the gauge group for which the transformation rules have been given in appendix A of [86].

### 6.4.2 The vector-tensor multiplet action

We will now generalize the vector action (6.66) to an action for the vector-tensor multiplets (with  $n$  vector multiplets and  $m$  tensor multiplets) discussed in section 6.2.2.

The supersymmetry transformation rules for the vector-tensor multiplet (6.10) were obtained from those for the vector multiplet (6.5) by replacing all contracted indices by the extended range of tilde indices. In addition, extra terms of  $\mathcal{O}(g)$  had to be added to the transformation rules. Similar considerations apply to the generalization of the action, as we will see below.

To obtain the generalization of the Chern-Simons (CS) term, it is convenient to rewrite this CS-term as an integral in six dimensions which has a boundary given by the five-dimensional Minkowski space-time. The six-form appearing in the integral is given by

$$I_{\text{vector}} = C_{IJK} F^I \wedge F^J \wedge F^K, \quad (6.69)$$

where we have used form notation. This six-form is both gauge-invariant and closed, by virtue of (6.65) and the Bianchi identities (6.4). It can therefore be written as the exterior derivative of a five-form which is gauge-invariant up to a total derivative. The space-time integral over this five-form is the CS-term given in the second line of (6.66).

We now wish to generalize (6.69) to the case of vector-tensor multiplets. It turns out that the generalization of (6.69) is somewhat surprising. We find

$$I_{\text{vec-tensor}} = C_{\widetilde{IJK}} \mathcal{H}^{\widetilde{I}} \wedge \mathcal{H}^{\widetilde{J}} \wedge \mathcal{H}^{\widetilde{K}} - \frac{3}{g} \Omega_{MN} \mathcal{D}B^M \wedge \mathcal{D}B^N. \quad (6.70)$$

The tensor  $\Omega_{MN}$  is antisymmetric and invertible, and it restricts the number of tensor multiplets to be *even*

$$\Omega_{MN} = -\Omega_{NM}, \quad \Omega_{MP} \Omega^{MR} = \delta_P^R. \quad (6.71)$$

The covariant derivative of the tensor field is given by

$$\begin{aligned} \mathcal{D}_\lambda B_{\rho\sigma}^N &= \partial_\lambda B_{\rho\sigma}^N + g A_{\lambda IJ}^I t_{IJ}^N \mathcal{H}_{\rho\sigma}^{\widetilde{J}} \\ &= \partial_\lambda B_{\rho\sigma}^N + g A_{\lambda IJ}^I t_{IJ}^N F_{\rho\sigma}^J + g A_{\lambda IP}^I t_{IP}^N B_{\rho\sigma}^P. \end{aligned} \quad (6.72)$$

To see why (6.70) is a closed six-form, we write out the first term of (6.70)

$$C_{\widetilde{IJK}} \mathcal{H}^{\widetilde{I}} \wedge \mathcal{H}^{\widetilde{J}} \wedge \mathcal{H}^{\widetilde{K}} = C_{IJK} F^I \wedge F^J \wedge F^K + 3C_{IJM} F^I \wedge F^J \wedge B^M + 3C_{IMN} F^I \wedge B^M \wedge B^N. \quad (6.73)$$

Since the  $B^M$  tensors in (6.73) do not satisfy a Bianchi identity, we also need the second term in (6.70) to render it a closed six-form. This requirement of closure leads to the following relations between the  $C$  and  $\Omega$  tensors:

$$C_{IJM} = t_{(IJ)}^N \Omega_{NM}, \quad C_{IMN} = \frac{1}{2} t_{IM}^P \Omega_{PN}. \quad (6.74)$$

We stress that the tensor  $C_{\widetilde{IJK}}$  is not a fundamental object: the essential data for the vector-tensor multiplet are the representation matrices  $t_{IJ}^{\widetilde{K}}$ , the Yang-Mills components  $C_{IJK}$ , and the symplectic matrix  $\Omega_{MN}$ . The tensor components of the  $C$  tensor are derived quantities, and we can summarize (6.74) as

$$C_{M\widetilde{JK}} = t_{(\widetilde{JK})}^P \Omega_{PM}. \quad (6.75)$$

From (6.74), we deduce that the second term of (6.73) only depends on the off-diagonal (between vector and tensor multiplets) transformations. The first term of (6.73) will induce the usual five-dimensional CS-term. The generalized CS-term induced by the third term of (6.73) was given in [73]. With our extension to also allow for the off-diagonal term in (6.15), we also get CS-terms induced by the  $C_{IJM}$  components, which were not present in [73].

Gauge invariance of the first term of (6.70) requires that the tensor  $C$  satisfies a modified version of (6.65)

$$f_{I(J}{}^H C_{KL)H} = t_{I(J}{}^M t_{KL)}{}^N \Omega_{MN}. \quad (6.76)$$

In addition to this, the second term of (6.70) is only gauge invariant if the tensor  $\Omega$  satisfies

$$t_{I[M}{}^P \Omega_{N]P} = 0, \quad (6.77)$$

such that the second relation in (6.74) is consistent with the symmetry ( $MN$ ). The two conditions (6.76) and (6.77) combined with the definition (6.75) imply the following generalization of (6.65)

$$t_{I(J}{}^{\bar{M}} C_{\bar{K}\bar{L})\bar{M}} = 0. \quad (6.78)$$

The superconformal action for the combined system of  $m = 2k$  tensor multiplets and  $n$  vector multiplets contains the CS-term induced by (6.70) and the generalization of the vector action (6.66) to the extended range of indices. Some extra terms are necessary to complete it to an invariant action: we need mass terms and/or Yukawa coupling for the fermions at  $\mathcal{O}(g)$ , and a scalar potential at  $\mathcal{O}(g^2)$ . We thus find the following action:

$$\begin{aligned} \mathcal{L}_{\text{vec-tensor}} = & \left( -\frac{1}{4} \mathcal{H}_{\mu\nu}^{\bar{I}} \mathcal{H}^{\mu\nu\bar{J}} - \frac{1}{2} \bar{\psi}^{\bar{I}} \mathcal{D} \psi^{\bar{J}} - \frac{1}{2} \mathcal{D}_a \sigma^{\bar{I}} \mathcal{D}^a \sigma^{\bar{J}} + Y_{ij}^{\bar{I}} Y^{ij\bar{J}} \right) \sigma^{\bar{K}} C_{\bar{I}\bar{J}\bar{K}} \\ & + \frac{1}{16g} \varepsilon^{\mu\nu\lambda\rho\sigma} \Omega_{MN} B_{\mu\nu}^M \left( \partial_\lambda B_{\rho\sigma}^N + 2g t_{IJ}{}^N A_\lambda^I F_{\rho\sigma}^J + g t_{IP}{}^N A_\lambda^I B_{\rho\sigma}^P \right) \\ & - \frac{1}{24} \varepsilon^{\mu\nu\lambda\rho\sigma} C_{IJK} A_\mu^I \left( F_{\nu\lambda}^J F_{\rho\sigma}^K + f_{FG}{}^J A_\nu^F A_\lambda^G \left( -\frac{1}{2} g F_{\rho\sigma}^K + \frac{1}{10} g^2 f_{HL}{}^K A_\rho^H A_\sigma^L \right) \right) \\ & - \frac{1}{8} \varepsilon^{\mu\nu\lambda\rho\sigma} \Omega_{MN} t_{IK}{}^M t_{FG}{}^N A_\mu^I A_\nu^F A_\lambda^G \left( -\frac{1}{2} g F_{\rho\sigma}^K + \frac{1}{10} g^2 f_{HL}{}^K A_\rho^H A_\sigma^L \right) \\ & + \left( -\frac{1}{8} i \bar{\psi}^{\bar{I}} \gamma \cdot \mathcal{H}^{\bar{J}} \psi^{\bar{K}} - \frac{1}{2} i \bar{\psi}^{\bar{I}} \psi^{j\bar{J}} Y_{ij}^{\bar{K}} \right) C_{\bar{I}\bar{J}\bar{K}} + \\ & + \frac{1}{4} i g \bar{\psi}^{\bar{I}} \psi^{\bar{J}} \sigma^{\bar{K}} \sigma^{\bar{L}} \left( t_{[\bar{I}\bar{J}]}\bar{M} C_{\bar{M}\bar{K}\bar{L}} - 4 t_{(\bar{I}\bar{K})}\bar{M} C_{\bar{M}\bar{J}\bar{L}} \right) \\ & - \frac{1}{2} g^2 \sigma^K C_{KMN} t_{\bar{I}\bar{L}}{}^M \sigma^{\bar{I}} \sigma^{\bar{L}} t_{\bar{J}\bar{P}}{}^N \sigma^{\bar{J}} \sigma^{\bar{P}}. \end{aligned} \quad (6.79)$$

To check the supersymmetry of this action, one needs all the relations between the various tensors given above. Another useful identity implied by the previous definitions is

$$t_{(\bar{I}\bar{J})}\bar{M} C_{\bar{K}\bar{L}\bar{M}} = -t_{(\bar{K}\bar{L})}\bar{M} C_{\bar{I}\bar{J}\bar{M}}. \quad (6.80)$$

The terms in the action containing the fields of the tensor multiplets can also be obtained from the field equations following from the on-shell closure of the algebra in section 6.2.2. Note however that the equations of motion for the vector multiplet fields, obtained from this action, are similar to those given in (6.68), but with the contracted indices running over the extended

range of vector and tensor components. Furthermore, the  $A_\mu^I$  equation of motion gets corrected by a term proportional to the self-duality equation for  $B_{\mu\nu}^M$ .

$$\frac{\delta S_{\text{vec-tensor}}}{\delta A_a^I} = E_I^a + \frac{1}{12} g \varepsilon^{abcde} A_b^J E_{cde}^M t_{JI}{}^N \Omega_{MN}. \quad (6.81)$$

Finally, we remark that the action (6.79) is invariant under supersymmetry for the general form (6.15) of the representation matrices  $(t_I)_J^{\bar{K}}$ .

We thus conclude that in order to write down a superconformal action for the vector-tensor multiplet, we need to introduce another geometrical object, namely a gauge-invariant anti-symmetric invertible tensor  $\Omega_{MN}$ . This symplectic matrix will restrict the number of tensor multiplets to be even. We can still allow the transformations to have off-diagonal terms between vector and tensor multiplets, if we adapt (6.65) to (6.78). In this way, we have constructed more general matter couplings than were known so far. Terms of the form  $A \wedge F \wedge B$  did not appear in previous papers. We see that such terms appear generically in our Lagrangian by allowing off-diagonal gauge transformations for the tensor fields.

### 6.4.3 The hypermultiplet

Until this point, the equations of motion we derived found their origin in the fact that we had an open superconformal algebra; the non-closure functions  $\Gamma^A$ , together with their supersymmetric partners  $\Delta^X$  yielded these equations of motion. We discovered a hypercomplex scalar manifold  $\mathcal{M}$ , whose properties are described in appendix C.

Now, we will introduce an action to derive the field equations of the hypermultiplet. An important point to note is that the necessary data for the scalar manifold we had in the previous section are not sufficient anymore. This is not specific to our setting, but is a general property of nonlinear sigma models. In such models, the kinetic term for the scalars is multiplied by a scalar-dependent symmetric tensor  $g_{\alpha\beta}(\phi)$ ,

$$S = -\frac{1}{2} \int d^D x g_{\alpha\beta}(\phi) \partial_\mu \phi^\alpha \partial^\mu \phi^\beta, \quad (6.82)$$

in which  $\alpha$  and  $\beta$  run over the dimensions of the scalar manifold. The tensor  $g$  is interpreted as the metric on the target space  $\mathcal{M}$ . As the field equations for the scalars should now also be covariant with respect to coordinate transformations on the target manifold, the connection on the tangent bundle  $T\mathcal{M}$  should be the Levi-Civita connection. Only in that particular case, the field equations for the scalars will be covariant. In other words, in  $\square\phi^\alpha + \dots = 0$  the Levi-Civita connection on  $T\mathcal{M}$  will be used in the covariant box.

Therefore, in order to be able to write down an action, we will need to introduce a metric on the scalar manifold. However, this metric will also restrict the possible target spaces for the theory.

Observe that most steps in this section do not depend on the use of superconformal symmetry.<sup>3</sup> Only at the end of this section we make explicit use of this symmetry.

<sup>3</sup>Of course, the form of the field equations does reflect the superconformal symmetry.



### Without gauged isometries

To start with, we take the non-closure functions  $\Gamma^A$  to be proportional to the field equations for the fermions  $\zeta^A$ . In other words, we ask

$$\frac{\delta S_{\text{hyper}}}{\delta \bar{\zeta}^A} = 2C_{AB}\Gamma^B. \quad (6.83)$$

In general, the tensor  $C_{AB}$  could be a function of the scalars and bilinears of the fermions. If we try to construct an action with the above Ansatz, it turns out that the tensor has to be anti-symmetric in  $AB$  and

$$\frac{\delta C_{AB}}{\delta \zeta^C} = 0, \quad (6.84)$$

$$\mathfrak{D}_X C_{AB} = 0. \quad (6.85)$$

This means that the tensor does not depend on the fermions and is covariantly constant.<sup>4</sup>

This tensor  $C_{AB}$  will be used to raise and lower indices according to the NW–SE convention similar to  $\varepsilon_{ij}$ :

$$A_A = A^B C_{BA}, \quad A^A = C^{AB} A_B, \quad (6.86)$$

where  $\varepsilon^{ij}$  and  $C^{AB}$  for consistency should satisfy

$$\varepsilon_{ik}\varepsilon^{jk} = \delta_i^j, \quad C_{AC}C^{BC} = \delta_A^B. \quad (6.87)$$

We may choose  $C_{AB}$  to be constant. For this choice, the connection  $\omega_{XAB}$  is symmetric, so the structure group  $Gl(r, \mathbb{H})$  breaks to  $USp(2r - 2p, 2p)$ . The signature is the signature of  $d_{CB}$ , which is defined as  $C_{AB} = \rho_A^C d_{CB}$  where  $\rho_A^C$  was given in (6.16). However, we will allow  $C_{AB}$  also to be non-constant, but covariantly constant.

We now construct the metric on the scalar manifold as

$$g_{XY} = f_X^{iA} C_{AB} \varepsilon_{ij} f_Y^{jB}. \quad (6.88)$$

The above-mentioned requirement that the Levi-Civita connection should be used (as  $\Gamma_{XY}^Z$ ) is satisfied due to (6.85). Indeed, this guarantees that the metric is covariantly constant, such that the affine connection is the Levi-Civita one. On the other hand we have seen already that for covariantly constant complex structures we have to use the Obata connection. Hence, the Levi-Civita and Obata connection should coincide, and this is obtained from demanding (6.85) using the Obata connection. This makes us conclude that we can only write down an action for a hyperkähler scalar manifold.

We can now write down the action for the rigid hypermultiplets. It has the following form:

$$S_{\text{hyper}} = \int d^5x \left( -\frac{1}{2} g_{XY} \partial_a q^X \partial^a q^Y + \bar{\zeta}_A \mathfrak{D}_5^A - \frac{1}{4} W_{ABCD} \bar{\zeta}^A \zeta^B \bar{\zeta}^C \zeta^D \right), \quad (6.89)$$

<sup>4</sup>This can also easily be seen by using the Batalin-Vilkovisky formalism.

where the tensor  $W_{ABCD}$  can be proven to be completely symmetric in all of its indices (see appendix C). The field equations derived from this action are

$$\begin{aligned}\frac{\delta S_{\text{hyper}}}{\delta \bar{\zeta}^A} &= 2C_{AB}\Gamma^B, \\ \frac{\delta S_{\text{hyper}}}{\delta q^X} &= g_{XY}\Delta^Y - 2\bar{\zeta}_A\Gamma^B\omega_{XB}{}^A.\end{aligned}\quad (6.90)$$

Also remark that due to the introduction of the metric, the expression of  $\Delta^X$  simplifies to

$$\Delta^X = \square q^X - \bar{\zeta}^A\phi q^Y\zeta^B\mathcal{R}^X{}_{YAB} - \frac{1}{4}\mathfrak{D}^X W_{ABCD}\bar{\zeta}^A\zeta^B\zeta^C\zeta^D. \quad (6.91)$$

### Conformal invariance

Due to the presence of the metric, the condition for the homothetic Killing vector (6.44) implies that  $k_X$  is the derivative of a scalar function as in (5.10). This scalar function  $\mathcal{K}(q)$  is called the hyperkähler potential [139, 164, 168]. It determines the conformal structure, but should be restricted to

$$\mathfrak{D}_X\mathfrak{D}_Y\mathcal{K} = \frac{3}{2}g_{XY}. \quad (6.92)$$

The relation with the homothetic Killing vector is

$$k_X = \partial_X\mathcal{K}, \quad \mathcal{K} = \frac{1}{3}k_X k^X. \quad (6.93)$$

Note that this implies that, when  $\mathcal{K}$  and the complex structures are known, one can compute the metric with (6.92), using the formula for the Obata connection (6.31).

### With gauged isometries

With a metric, the symmetries of section 6.3.3 should be isometries, i.e.

$$\mathfrak{D}_X k_{YI} + \mathfrak{D}_Y k_{XI} = 0. \quad (6.94)$$

This makes the requirement (6.57) superfluous, but we still have to impose the triholomorphicity expressed by either (6.53) or (6.54) or (6.55).

In order to integrate the equations of motion to an action we have to define (locally) triples of ‘moment maps’, according to

$$\partial_X P_I^\alpha = -\frac{1}{2}J^\alpha{}_{XY}k_I^Y. \quad (6.95)$$

The integrability condition that makes this possible is the triholomorphic condition.

In the kinetic terms of the action, the derivatives should now be covariantized with respect to the new transformations. Supersymmetry invariance of the action also forces us to include some new terms proportional to  $g$  and  $g^2$

$$\begin{aligned}S_{\text{hyper}}^g &= \int d^5x \left( -\frac{1}{2}g_{XY}\mathfrak{D}_a q^X\mathfrak{D}^a q^Y + \bar{\zeta}_A\mathfrak{D}\zeta^A - \frac{1}{4}W_{ABCD}\bar{\zeta}^A\zeta^B\bar{\zeta}^C\zeta^D \right. \\ &\quad \left. - g \left( 2i k_I^X J_{iX}^A \bar{\zeta}_A \psi^{il} + i \sigma^I t_{IB}{}^A \bar{\zeta}_A \zeta^B - 2P_{Iij} Y^{lij} \right) - \frac{1}{2}g^2 \sigma^I \sigma^J k_I^X k_{JX} \right),\end{aligned}\quad (6.96)$$

[where the covariant derivatives  $\mathfrak{D}$  now also include gauge-covariantization proportional to  $g$  as in (6.60)], while the field equations have the same form as in (6.90).

Alternatively one can use the method explained in [169] to construct the action. Since the field equations are linear in the non-closure functions and the Lagrangian should vanish on-shell, we expect that the action itself can in fact be written as a linear combination of non-closure functions, in the form of  $\Sigma[\text{field}] \times [\text{non-closure}]$ :

$$\mathcal{L} = \frac{1}{3} k_X \Delta^X + \bar{\zeta}_A \Gamma^A. \quad (6.97)$$

The two coefficients can be fixed by looking at the normalization of the kinetic terms. Substituting the non-closure functions into (6.97) and partial integrating the covariant box, we indeed find the correct action (6.96). This method is believed to be correct for any on-shell multiplet. Note however that supersymmetry is a necessary ingredient. The invariance of the hypermultiplet action under supersymmetry can easily be checked by using the following transformation rules for the non-closure functions:

$$\begin{aligned} \delta \Delta^X &= -i \bar{\epsilon}^i \mathfrak{D} \Gamma^A f_{iA}^X + 2i \bar{\epsilon}^i \Gamma^B \zeta^C \zeta^D f_{iB}^Y \mathcal{R}^X{}_{YCD} + 2\bar{\eta}^i \Gamma^A f_{iA}^X + \Delta^Y \Gamma_{ZY}{}^X \delta q^Z, \\ \delta \Gamma^A &= \frac{1}{2} i f_X^{iA} \epsilon_i \Delta^X - \delta q^X \omega_{XB}{}^A \Gamma^B, \end{aligned} \quad (6.98)$$

where the covariant derivative is given by

$$\mathcal{D}_\mu \Gamma^A = (\partial_\mu - 3b_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab}) \Gamma^A + \partial_\mu q^X \omega_{XB}{}^A \Gamma^B + g A_\mu^I t_{IB}{}^A \Gamma^B. \quad (6.99)$$

Supersymmetry of the action imposes

$$k_I^X J^\alpha{}_{XY} k_J^Y = 2 f_{IJ}{}^K P_K^\alpha. \quad (6.100)$$

As only the derivative of  $P$  appears in the defining equation (6.95), one may add an arbitrary constant to  $P$ . But that changes the right-hand side of (6.100). One should then consider whether there is a choice of these coefficients such that (6.100) is satisfied. This is the question about the center of the algebra, which is discussed in [170, 171]. For simple groups there is always a solution.<sup>5</sup> For Abelian theories the constant remains undetermined. This free constant is the so-called Fayet–Iliopoulos term.

In a conformal invariant theory, the Fayet–Iliopoulos term is not possible, since dilatation invariance of the action requires

$$3P_I^\alpha = k^X \partial_X P_I^\alpha. \quad (6.101)$$

Thus,  $P_I^\alpha$  is completely determined [using (6.95) or (6.59)] as (see also [172])

$$-6P_I^\alpha = k^X J^\alpha{}_{XY} k_I^Y = -\frac{2}{3} k^X k^Z J^\alpha{}_Z{}^Y \mathfrak{D}_Y k_{IX}. \quad (6.102)$$

<sup>5</sup>We thank Gary Gibbons for a discussion on this subject.

The proof of the invariance of the action under the complete superconformal group, uses the equation obtained from (6.59) and (6.95):

$$k^{X\alpha} \mathfrak{D}_X k_I^Y = \partial^Y P_I^\alpha . \quad (6.103)$$

If the moment map  $P_I^\alpha$  has the value that it takes in the conformal theory, then (6.100) is satisfied due to (6.50). Indeed, one can multiply that equation with  $k_X k^Z J^\alpha_Z{}^W \mathfrak{D}_W$  and use (6.54), (6.57) and (6.46). Thus, in the superconformal theory, the moment maps are determined and there is no further relation to be obeyed, i.e. the Fayet–Iliopoulos terms of the rigid theories are absent in this case.

To conclude, isometries of the scalar manifold that commute with dilatations, see (6.59), can be gauged. The resulting theory has an extra symmetry group  $G$ , whose algebra is generated by the corresponding Killing vectors.

Gathering together our results (6.79) and (6.96) the total Lagrangian describing the most general couplings of vector/tensor multiplets to hypermultiplets with rigid superconformal symmetry is given by

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{vec-tensor}} + \mathcal{L}_{\text{hyper}}^g . \quad (6.104)$$

Summarizing, in this section the actions of rigid superconformal vector/tensor-hypermultiplet couplings have been constructed. The full answer is (6.104). We found that the existence of an action requires the presence of additional tensorial objects. A review of all the independent objects needed to determine the transformation laws, or to determine the action, are given in table 6.4. Note that these objects could already be introduced at the level of rigid supersymmetry. In the next section these results will be generalized to the local case, by coupling the matter multiplets to the Weyl multiplet, but this will not introduce any new constraints.

multiplets	ALGEBRA (no action)		ACTION	
	objects	Def/restriction	objects	Def/restriction
Vect.	$f_{[IJ]}^K$	Jacobi identities	$C_{(IJK)}$	$f_{I(J}{}^H C_{KL)H} = 0 \blacktriangle$
Vect./Tensor	$(t_I)_J^{\bar{K}}$ $\bar{I} = (I, M)$	$[t_I, t_J] = -f_{IJ}{}^K t_K$ $t_{IJ}{}^K = f_{IJ}{}^K, \quad t_{IM}{}^J = 0$	$\Omega_{[MN]}$	invertible $f_{I(J}{}^H C_{KL)H} = t_{I(J}{}^M t_{KL)}{}^N \Omega_{MN}$ $t_{[I}{}^P \Omega_{N]P} = 0$
Hyper	$f_X^{iA}$	invertible and real using $\rho$ Nijenhuis condition: $N_{XY}{}^Z = 0$	$C_{[AB]}$	$\mathfrak{D}_X C_{AB} = 0$
Hyper + gauging	$k_I^X$	$\mathfrak{D}_X \mathfrak{D}_Y k_I^Z = R_{XWY}{}^Z k_I^W \blacktriangleleft$ $k_{[I}^Y \partial_Y k_{J]}^X = -\frac{1}{2} f_{IJ}{}^K k_K^X$ $\mathcal{L}_{k_I} J^\alpha = 0 \blacktriangleleft$	$P_I^\alpha \blacktriangle$	$\mathfrak{D}_X k_{YI} + \mathfrak{D}_Y k_{XI} = 0$ $\partial_X P_I^\alpha = -\frac{1}{2} J^\alpha{}_{XY} k_I^Y \blacktriangle$ $k_I^X J_{XY}^\alpha k_J^Y = 2 f_{IJ}{}^K P_K^\alpha \blacktriangle$
Hyper + conformal	$k^X \blacktriangleleft$	$\mathfrak{D}_Y k^X = \frac{3}{2} \delta_Y^X \blacktriangleleft$	$\mathcal{K}$	$\mathfrak{D}_X \mathfrak{D}_Y \mathcal{K} = \frac{3}{2} g_{XY}$
Hyper + conformal + gauged		$k^Y \mathfrak{D}_Y k_I^X = \frac{3}{2} k_I^X$		

**Table 6.4:** Various matter couplings with or without action. We indicate which are the geometrical objects that determine the theory and what are the independent constraints. The symmetries of the objects are already indicated when they appear first. In general, the equations should also be valid for the theories in the rows below (apart from the fact that ‘hyper+gauging’ and ‘hyper+conformal’ are independent, but both are used in the lowest row). However, the symbol  $\blacktriangle$  indicates that these equations are not to be taken over below. E.g. the moment map  $P_I^\alpha$  itself is completely determined in the conformal theory, and it should therefore no longer be given as an independent quantity. For the rigid theory without conformal invariance, only constant pieces can be undetermined by the given equations, and they are the Fayet–Iliopoulos terms. On the other hand, the equations indicated by  $\blacktriangleleft$  have not to be taken over for the theories with an action, as they are then satisfied due to the Killing equation or are defined by  $\mathcal{K}$ .

## 6.5 Local superconformal multiplets

In this section we will extend the supersymmetry to a local conformal supersymmetry, by making use of the off-shell 32 + 32 Standard Weyl multiplet constructed in chapter refch:weyl. We restrict ourselves here to the Standard Weyl multiplet. One may wonder whether the use of the dilaton Weyl multiplet could lead to other matter couplings. Though we can not exclude this, we do not expect a physically different result. Whether the conformal gauge-fixing program will also be insensitive to the choice of Weyl multiplet, remains to be seen.

The procedure for extending the rigid superconformal transformation rules for the various matter multiplets is to introduce covariant derivatives with respect to the superconformal symmetries. These derivatives contain the superconformal gauge fields which, in turn, will also transform to additional matter fields as explained in chapter 5.

Since in the previous sections we have explained most of the subtleties concerning the possible geometrical structures, we can be brief here. We will obtain our results in two steps. First, we require that the local superconformal commutator algebra, as it is realized on the standard Weyl multiplet (5.41)–(5.44) is also realized on the matter multiplets (with possible additional transformations under which the fields of the standard Weyl multiplet do not transform, and possibly field equations if the matter multiplet is on-shell). Note that this local superconformal algebra is a modification of the rigid superconformal algebra (5.22), (5.20) where all modifications involve the fields of the standard Weyl multiplet.

Now we will apply a standard Noether procedure to extend the rigid supersymmetric actions to a locally supersymmetric one. This will introduce the full complications of coupling the matter multiplets to conformal supergravity.

### 6.5.1 Vector-tensor multiplet

The local supersymmetry rules are given by

$$\begin{aligned}
\delta A_\mu^I &= \frac{1}{2} \bar{\epsilon} \gamma_\mu \psi^I - \frac{1}{2} i \sigma^I \bar{\epsilon} \psi_\mu, \\
\delta B_{ab}^M &= -\bar{\epsilon} \gamma_{[a} D_{b]} \psi^M - \frac{1}{2} i \sigma^M \bar{\epsilon} \widehat{R}_{ab}(Q) + i \bar{\epsilon} \gamma_{[a} \gamma \cdot T \gamma_{b]} \psi^M \\
&\quad + i g \bar{\epsilon} \gamma_{ab} t_{(\overline{JK})}^M \sigma^{\overline{J}} \psi^{\overline{K}} + i \bar{\eta} \gamma_{ab} \psi^M, \\
\delta Y^{ij\overline{I}} &= -\frac{1}{2} \bar{\epsilon}^{(i} \mathcal{D} \psi^{j)\overline{I}} + \frac{1}{2} i \bar{\epsilon}^{(i} \gamma \cdot T \psi^{j)\overline{I}} - 4 i \sigma^{\overline{I}} \bar{\epsilon}^{(i} \chi^{j)} \\
&\quad - \frac{1}{2} i g \bar{\epsilon}^{(i} \left( t_{[\overline{JK}] }^{\overline{I}} - 3 t_{(\overline{JK})}^{\overline{I}} \right) \sigma^{\overline{J}} \psi^{j)\overline{K}} + \frac{1}{2} i \bar{\eta}^{(i} \psi^{j)\overline{I}}, \\
\delta \psi^{\overline{I}} &= -\frac{1}{4} \gamma \cdot \widehat{\mathcal{H}}^{\overline{I}} \epsilon^i - \frac{1}{2} i \mathcal{D} \sigma^{\overline{I}} \epsilon^i - Y^{i\overline{I}} \epsilon_j + \sigma^{\overline{I}} \gamma \cdot T \epsilon^i + \frac{1}{2} g t_{(\overline{JK})}^{\overline{I}} \sigma^{\overline{J}} \sigma^{\overline{K}} \epsilon^i + \sigma^{\overline{I}} \eta^i, \\
\delta \sigma^{\overline{I}} &= \frac{1}{2} i \bar{\epsilon} \psi^{\overline{I}}.
\end{aligned} \tag{6.105}$$

The covariant derivatives are given by

$$\begin{aligned}
D_\mu \sigma^{\overline{I}} &= \mathcal{D}_\mu \sigma^{\overline{I}} - \frac{1}{2} i \bar{\psi}_\mu \psi^{\overline{I}}, \\
\mathcal{D}_\mu \sigma^{\overline{I}} &= (\partial_\mu - b_\mu) \sigma^{\overline{I}} + g t_{j\overline{K}}^{\overline{I}} A_\mu^j \sigma^{\overline{K}}, \\
D_\mu \psi^{\overline{I}} &= \mathcal{D}_\mu \psi^{\overline{I}} + \frac{1}{4} \gamma \cdot \widehat{\mathcal{H}}^{\overline{I}} \psi_\mu^i + \frac{1}{2} i \mathcal{D} \sigma^{\overline{I}} \psi_\mu^i + Y^{i\overline{I}} \psi_{\mu j} + \sigma^{\overline{I}} \gamma \cdot T \psi_\mu^i
\end{aligned} \tag{6.106}$$

$$\begin{aligned} & -\frac{1}{2}gt_{(\overline{JK})}^{\overline{I}}\sigma^{\overline{J}}\sigma^{\overline{K}}\psi_{\mu}^i - \sigma^{\overline{I}}\phi_{\mu}^i, \\ \mathcal{D}_{\mu}\psi^{\overline{I}} &= (\partial_{\mu} - \frac{3}{2}b_{\mu} + \frac{1}{4}\gamma_{ab}\omega_{\mu}^{ab})\psi^{\overline{I}} - V_{\mu}^{ij}\psi_j^{\overline{I}} + gt_{\overline{JK}}^{\overline{I}}A_{\mu}^J\psi^{i\overline{K}}. \end{aligned}$$

The covariant curvature  $\widehat{\mathcal{H}}_{\mu\nu}^{\overline{I}}$  should be understood as having components  $(\widehat{F}_{\mu\nu}^{\overline{I}}, B_{\mu\nu})$  and

$$\widehat{F}_{\mu\nu}^{\overline{I}} = 2\partial_{[\mu}A_{\nu]}^{\overline{I}} + gf_{JK}^{\overline{I}}A_{\mu}^JA_{\nu}^{\overline{K}} - \bar{\psi}_{[\mu}\gamma_{\nu]}\psi^{\overline{I}} + \frac{1}{2}i\sigma^{\overline{I}}\bar{\psi}_{[\mu}\psi_{\nu]}. \quad (6.107)$$

The locally superconformal constraints needed to close the algebra are given by the following extensions of (6.13) and (6.14) (which are non-zero only for  $\overline{I}$  in the tensor multiplet range)

$$\begin{aligned} L^{ijM} &\equiv t_{(\overline{JK})}^M \left( 2\sigma^{\overline{J}}Y^{ij\overline{K}} - \frac{1}{2}i\bar{\psi}^{i\overline{J}}\psi^{j\overline{K}} \right) = 0, \\ E_{\mu\nu\lambda}^M &\equiv \frac{3}{g}D_{[\mu}B_{\nu\lambda]}^M - \varepsilon_{\mu\nu\lambda\rho\sigma}t_{(\overline{JK})}^M \left( \sigma^{\overline{J}}\widehat{\mathcal{H}}^{\rho\sigma\overline{K}} - 8\sigma^{\overline{J}}\sigma^{\overline{K}}T^{\rho\sigma} + \frac{1}{4}i\bar{\psi}^{\overline{J}}\gamma^{\rho\sigma}\psi^{\overline{K}} \right) \\ &\quad - \frac{3}{2g}\bar{\psi}^M\gamma_{[\mu}\widehat{R}_{\nu\lambda]}(Q) \\ &= 0. \end{aligned} \quad (6.108)$$

Here, the supercovariant derivative on the tensor is defined as

$$\begin{aligned} D_{[\mu}B_{\nu\rho]}^M &= \partial_{[\mu}B_{\nu\rho]}^M - 2b_{[\mu}B_{\nu\rho]}^M + \bar{\psi}_{[\mu}\gamma_{\nu}D_{\rho]}\psi^M + \frac{1}{2}i\sigma^M\bar{\psi}_{[\mu}\widehat{R}_{\nu\rho]}(Q) \\ &\quad - i\bar{\psi}_{[\mu}\gamma_{\nu}\gamma \cdot T\gamma_{\rho]}\psi^M - i\bar{\phi}_{[\mu}\gamma_{\nu\rho]}\psi^M \\ &\quad - ig\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\psi^{\overline{K}}\sigma^{\overline{J}}t_{(\overline{JK})}^M + gt_{\overline{JK}}^MA_{[\mu}^JA_{\nu\rho]}^{\overline{K}}. \end{aligned} \quad (6.109)$$

Analogously to subsection 6.2.2, the full set of constraints could be obtained by varying these constraints under supersymmetry.

The action, invariant under local superconformal symmetry, can be obtained by replacing the rigid covariant derivatives in (6.79) by the local covariant derivatives (6.106) and adding extra terms proportional to gravitinos or matter fields of the Weyl multiplet, determined by supersymmetry. It is convenient at this point to introduce a new S-invariant tensorfield  $\widetilde{B}_{\mu\nu}^M$  which is defined as

$$B_{\mu\nu}^M = \widetilde{B}_{\mu\nu}^M - \bar{\psi}_{[\mu}\gamma_{\nu]}\psi^M + \frac{1}{2}i\sigma^M\bar{\psi}_{[\mu}\psi_{\nu]},$$

such that the symbol  $\widehat{\mathcal{H}}_{\mu\nu}^{\overline{I}}$  can be written as

$$\widehat{\mathcal{H}}_{\mu\nu}^{\overline{I}} = H_{\mu\nu}^{\overline{I}} - \bar{\psi}_{[\mu}\gamma_{\nu]}\psi^{\overline{I}} + \frac{1}{2}i\sigma^{\overline{I}}\bar{\psi}_{[\mu}\psi_{\nu]}, \quad H_{\mu\nu}^{\overline{I}} \equiv (F_{\mu\nu}^{\overline{I}}, \widetilde{B}_{\mu\nu}^M).$$

The action then reads

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{vec-ten}}^{\text{conf}} &= \left[ \left( -\frac{1}{4}\widehat{\mathcal{H}}_{\mu\nu}^{\overline{I}}\widehat{\mathcal{H}}^{\mu\nu\overline{J}} - \frac{1}{2}\bar{\psi}^{\overline{I}}\not{D}\psi^{\overline{J}} + \frac{1}{3}\sigma^{\overline{I}}\square^c\sigma^{\overline{J}} + \frac{1}{6}D_a\sigma^{\overline{I}}D^a\sigma^{\overline{J}} + Y_{ij}^{\overline{I}}Y^{ij\overline{J}} \right) \sigma^{\overline{K}} \right. \\ &\quad - \frac{4}{3}\sigma^{\overline{I}}\sigma^{\overline{J}}\sigma^{\overline{K}} \left( D + \frac{26}{3}T_{ab}T^{ab} \right) + 4\sigma^{\overline{I}}\sigma^{\overline{J}}\widehat{\mathcal{H}}_{ab}^{\overline{K}}T^{ab} \\ &\quad \left. - \frac{1}{8}i\bar{\psi}^{\overline{I}}\gamma \cdot \widehat{\mathcal{H}}^{\overline{J}}\psi^{\overline{K}} - \frac{1}{2}i\bar{\psi}^{i\overline{I}}\psi^{j\overline{J}}Y_{ij}^{\overline{K}} + i\sigma^{\overline{I}}\bar{\psi}^{\overline{J}}\gamma \cdot T\psi^{\overline{K}} - 8i\sigma^{\overline{I}}\sigma^{\overline{J}}\bar{\psi}^{\overline{K}}\chi \right] \end{aligned}$$

$$\begin{aligned}
& +\frac{1}{6}\sigma^{\bar{I}}\bar{\psi}_{\mu}\gamma^{\mu}\left(i\sigma^{\bar{J}}\mathcal{D}\psi^{\bar{K}}+\frac{1}{2}i\mathcal{D}\sigma^{\bar{J}}\psi^{\bar{K}}-\frac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\bar{J}}\psi^{\bar{K}}+2\sigma^{\bar{J}}\gamma\cdot T\psi^{\bar{K}}-8\sigma^{\bar{J}}\sigma^{\bar{K}}\chi\right) \\
& -\frac{1}{6}\bar{\psi}_a\gamma_b\psi^{\bar{I}}\left(\sigma^{\bar{J}}\widehat{\mathcal{H}}^{ab\bar{K}}-8\sigma^{\bar{J}}\sigma^{\bar{K}}T^{ab}\right)-\frac{1}{12}\sigma^{\bar{I}}\bar{\psi}_{\lambda}\gamma^{\mu\nu\lambda}\psi^{\bar{J}}\widehat{\mathcal{H}}_{\mu\nu}^{\bar{K}} \\
& +\frac{1}{12}i\sigma^{\bar{I}}\bar{\psi}_a\psi_b\left(\sigma^{\bar{J}}\widehat{\mathcal{H}}^{ab\bar{K}}-8\sigma^{\bar{J}}\sigma^{\bar{K}}T^{ab}\right)+\frac{1}{48}i\sigma^{\bar{I}}\sigma^{\bar{J}}\bar{\psi}_{\lambda}\gamma^{\mu\nu\lambda\rho}\psi_{\rho}\widehat{\mathcal{H}}_{\mu\nu}^{\bar{K}} \\
& -\frac{1}{2}\sigma^{\bar{I}}\bar{\psi}_{\mu}^i\gamma^{\mu}\psi^{j\bar{I}}Y_{ij}^{\bar{K}}+\frac{1}{6}i\sigma^{\bar{I}}\sigma^{\bar{J}}\bar{\psi}_{\mu}^i\gamma^{\mu\nu}\psi_{\nu}^jY_{ij}^{\bar{K}}-\frac{1}{24}i\bar{\psi}_{\mu}\gamma_{\nu}\psi^{\bar{I}}\bar{\psi}^{\bar{J}}\gamma^{\mu\nu}\psi^{\bar{K}} \\
& +\frac{1}{12}i\bar{\psi}_{\mu}^i\gamma^{\mu}\psi^{j\bar{I}}\bar{\psi}_i^{\bar{J}}\psi_j^{\bar{K}}-\frac{1}{48}\sigma^{\bar{I}}\bar{\psi}_{\mu}\psi_{\nu}\bar{\psi}^{\bar{I}}\gamma^{\mu\nu}\psi^{\bar{K}}+\frac{1}{24}\sigma^{\bar{I}}\bar{\psi}_{\mu}^i\gamma^{\mu\nu}\psi_{\nu}^j\bar{\psi}_i^{\bar{J}}\psi_j^{\bar{K}} \\
& -\frac{1}{12}\sigma^{\bar{I}}\bar{\psi}_{\lambda}\gamma^{\mu\nu\lambda}\psi^{\bar{J}}\bar{\psi}_{\mu}\gamma_{\nu}\psi^{\bar{K}}+\frac{1}{24}i\sigma^{\bar{I}}\sigma^{\bar{J}}\bar{\psi}_{\lambda}\gamma^{\mu\nu\lambda}\psi^{\bar{K}}\bar{\psi}_{\mu}\psi_{\nu} \\
& +\frac{1}{48}i\sigma^{\bar{I}}\sigma^{\bar{J}}\bar{\psi}_{\lambda}\gamma^{\mu\nu\lambda\rho}\psi_{\rho}\bar{\psi}_{\mu}\gamma_{\nu}\psi^{\bar{K}}+\frac{1}{96}\sigma^{\bar{I}}\sigma^{\bar{J}}\sigma^{\bar{K}}\bar{\psi}_{\lambda}\gamma^{\mu\nu\lambda\rho}\psi_{\rho}\bar{\psi}_{\mu}\psi_{\nu}\Big]C_{\bar{I}\bar{J}\bar{K}} \\
& +\frac{1}{16g}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}\widetilde{B}_{\mu\nu}^M\left(\partial_{\rho}\widetilde{B}_{\sigma\tau}^N+2g t_{IJ}{}^N A_{\rho}^I F_{\sigma\tau}^J+g t_{IP}{}^N A_{\rho}^I \widetilde{B}_{\sigma\tau}^P\right) \\
& -\frac{1}{24}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}C_{IJK}A_{\mu}^I\left(F_{\nu\lambda}^J F_{\rho\sigma}^K+f_{FG}{}^J A_{\nu}^F A_{\lambda}^G\left(-\frac{1}{2}g F_{\rho\sigma}^K+\frac{1}{10}g^2 f_{HL}{}^K A_{\rho}^H A_{\sigma}^L\right)\right) \\
& -\frac{1}{8}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MNTIK}{}^M t_{FG}{}^N A_{\mu}^I A_{\nu}^F A_{\lambda}^G\left(-\frac{1}{2}g F_{\rho\sigma}^K+\frac{1}{10}g^2 f_{HL}{}^K A_{\rho}^H A_{\sigma}^L\right) \\
& +\frac{1}{4}i g\bar{\psi}^{\bar{I}}\psi^{\bar{J}}\sigma^{\bar{K}}\sigma^{\bar{L}}\left(t_{(\bar{I}\bar{J})}{}^{\bar{M}}C_{\bar{M}\bar{K}\bar{L}}-4t_{(\bar{I}\bar{K})}{}^{\bar{M}}C_{\bar{M}\bar{J}\bar{L}}\right) \\
& -\frac{1}{4}g\bar{\psi}_{\mu}\gamma^{\mu}\psi^{\bar{I}}\sigma^{\bar{J}}\sigma^{\bar{K}}\sigma^{\bar{L}}t_{(\bar{J}\bar{K})}{}^{\bar{M}}C_{\bar{M}\bar{I}\bar{L}}-\frac{1}{2}g^2\sigma^{\bar{I}}\sigma^{\bar{J}}\sigma^{\bar{K}}\sigma^{\bar{M}}\sigma^{\bar{N}}t_{\bar{J}\bar{M}}{}^P t_{\bar{K}\bar{N}}{}^Q C_{IPQ}, \quad (6.110)
\end{aligned}$$

where the superconformal d'Alembertian is defined as

$$\begin{aligned}
\Box\sigma^{\bar{I}} &= D^a D_a\sigma^{\bar{I}} \\
&= \left(\partial^a-2b^a+\omega_b{}^{ba}\right)D_a\sigma^{\bar{I}}+g t_{\bar{J}\bar{K}}{}^{\bar{I}}A_a^{\bar{J}}D^a\sigma^{\bar{K}}-\frac{1}{2}i\bar{\psi}_{\mu}D^{\mu}\psi^{\bar{I}}-2\sigma^{\bar{I}}\bar{\psi}_{\mu}\gamma^{\mu}\chi \\
&\quad +\frac{1}{2}\bar{\psi}_{\mu}\gamma^{\mu}\gamma\cdot T\psi^{\bar{I}}+\frac{1}{2}\bar{\phi}_{\mu}\gamma^{\mu}\psi^{\bar{I}}+2f_{\mu}{}^{\mu}\sigma^{\bar{I}}-\frac{1}{2}g\bar{\psi}_{\mu}\gamma^{\mu}t_{\bar{J}\bar{K}}{}^{\bar{I}}\psi^{\bar{J}}\sigma^{\bar{K}}. \quad (6.111)
\end{aligned}$$

## 6.5.2 Hypermultiplet

Imposing the local superconformal algebra we find the following supersymmetry rules:

$$\begin{aligned}
\delta q^X &= -i\bar{\epsilon}^i\zeta^A f_{iA}^X, \\
\widehat{\delta}\zeta^A &= \frac{1}{2}i\mathcal{D}q^X f_X^{iA}\epsilon_i-\frac{1}{3}\gamma\cdot Tk^X f_{iX}^A\epsilon^i+\frac{1}{2}g\sigma^I k_I^X f_{iX}^A\epsilon^i+k^X f_{iX}^A\eta^i. \quad (6.112)
\end{aligned}$$

The covariant derivatives are given by

$$\begin{aligned}
D_{\mu}q^X &= \mathcal{D}_{\mu}q^X+i\bar{\psi}_{\mu}^i\zeta^A f_{iA}^X, \\
\mathcal{D}_{\mu}q^X &= \partial_{\mu}q^X-b_{\mu}k^X-V_{\mu}^{jk}k_{jk}^X+gA_{\mu}^I k_I^X, \\
D_{\mu}\zeta^A &= \mathcal{D}_{\mu}\zeta^A-k^X f_{iX}^A\phi_{\mu}^i+\frac{1}{2}i\mathcal{D}q^X f_{iX}^A\psi_{\mu}^i+\frac{1}{3}\gamma\cdot Tk^X f_{iX}^A\psi_{\mu}^i-g\frac{1}{2}\sigma^I k_I^X f_{iX}^A\psi_{\mu}^i \quad (6.113) \\
\mathcal{D}_{\mu}\zeta^A &= \partial_{\mu}\zeta^A+\partial_{\mu}q^X\omega_{XB}{}^A\zeta^B+\frac{1}{4}\omega_{\mu}{}^{bc}\gamma_{bc}\zeta^A-2b_{\mu}\zeta^A+gA_{\mu}^I t_{IB}{}^A\zeta^B.
\end{aligned}$$



Similar to section 6.3, requiring closure of the commutator algebra on these transformation rules yields the equation of motion for the fermions

$$\begin{aligned} \Gamma_{\text{conf}}^A &= \mathcal{D}\zeta^A + \frac{1}{2}W_{CDB}{}^A\zeta^B\bar{\zeta}^D\zeta^C - \frac{8}{3}ik^X f_{iX}^A \chi^i + 2i\gamma \cdot T\zeta^A \\ &\quad - g\left(ik_I^X f_{iX}^A \psi^{iI} + i\sigma^I t_{IB}{}^A \zeta^B\right). \end{aligned} \quad (6.114)$$

The scalar equation of motion can be obtained from varying (6.114):

$$\widehat{\delta}_Q \Gamma^A = \frac{1}{2}i f_X^{iA} \Delta^X \epsilon_i + \frac{1}{4}\gamma^\mu \Gamma^A \bar{\epsilon} \psi_\mu - \frac{1}{4}\gamma^\mu \gamma^\nu \Gamma^A \bar{\epsilon} \gamma_\nu \psi_\mu, \quad (6.115)$$

where

$$\begin{aligned} \Delta_{\text{conf}}^X &= \square^c q^X - \bar{\zeta}^B \gamma^a \zeta^C D_a q^Y \mathcal{R}^X{}_{YBC} + \frac{8}{9}T^2 k^X \\ &\quad + \frac{4}{3}Dk^X + 8i\bar{\chi}^i \zeta^A f_{iA}^X - \frac{1}{4}\mathcal{D}^X W_{ABCD} \bar{\zeta}^A \zeta^B \bar{\zeta}^C \zeta^D \\ &\quad - g(2i\bar{\psi}^{iI} \zeta^B t_{IB}{}^A f_{iA}^X - k_I^Y J_Y{}^X{}_{ij} Y^{lij}) \\ &\quad + g^2 \sigma^I \sigma^J \mathcal{D}_Y k_I^X k_J^Y, \end{aligned} \quad (6.116)$$

and the superconformal d'Alembertian is given by

$$\begin{aligned} \square^c q^X &\equiv D_a D^a q^X \\ &= \partial_a D^a q^X - \frac{5}{2}b_a D^a q^X - \frac{1}{2}V_a^{jk} J_Y{}^X{}_{jk} D^a q^Y + i\bar{\psi}_a^i D^a \zeta^A f_{iA}^X \\ &\quad + 2f_a{}^a k^X - 2\bar{\psi}_a \gamma^a \chi k^X + 4\bar{\psi}_a^{(j} \gamma^a \chi^{k)} k_{jk}^X - \bar{\psi}_a \gamma^a \gamma \cdot T \zeta^A f_{iA}^X \\ &\quad - \bar{\phi}_a^i \gamma^a \zeta^A f_{iA}^X + \omega_a{}^{ab} D_b q^X - \frac{1}{2}g\bar{\psi}^a \gamma_a \psi^I k_I^X - D_a q^Y \partial_Y k_I^X A^{aI} \\ &\quad + D_a q^Y D^a q^Z \Gamma_{YZ}^X. \end{aligned} \quad (6.117)$$

Note that so far we did not require the presence of an action. Introducing a metric, the locally conformal supersymmetric action is given by

$$\begin{aligned} e^{-1} \mathcal{L}_{\text{hyper}}^{\text{conf}} &= -\frac{1}{2}g_{XY} \mathcal{D}_a q^X \mathcal{D}^a q^Y + \bar{\zeta}_A \mathcal{D}\zeta^A + \frac{4}{9}Dk^2 + \frac{8}{27}T^2 k^2 \\ &\quad - \frac{16}{3}i\bar{\zeta}_A \chi^i k_{iX}^A + 2i\bar{\zeta}_A \gamma \cdot T \zeta^A - \frac{1}{4}W_{ABCD} \bar{\zeta}^A \zeta^B \bar{\zeta}^C \zeta^D \\ &\quad - \frac{2}{9}\bar{\psi}_a \gamma^a \chi k^2 + \frac{1}{3}\bar{\zeta}_A \gamma^a \gamma \cdot T \psi_a^i k_{iX}^A + \frac{1}{2}i\bar{\zeta}_A \gamma^a \gamma^b \psi_a^i \mathcal{D}_b q^X f_{iX}^A \\ &\quad + \frac{2}{3}f_a{}^a k^2 - \frac{1}{6}i\bar{\psi}_a \gamma^{ab} \phi_b k^2 - \bar{\zeta}_A \gamma^a \phi_a^i k_{iX}^A \\ &\quad + \frac{1}{12}\bar{\psi}_a \gamma^{abc} \psi_b^j \mathcal{D}_c q^Y J_Y{}^X{}_{ij} k_X - \frac{1}{9}i\bar{\psi}^a \psi^b T_{ab} k^2 + \frac{1}{18}i\bar{\psi}_a \gamma^{abcd} \psi_b T_{cd} k^2 \\ &\quad - g\left(i\sigma^I t_{IB}{}^A \bar{\zeta}_A \zeta^B + 2ik_I^X f_{iX}^A \bar{\zeta}_A \psi^{iI} + \frac{1}{2}\sigma^I k_I^X f_{iX}^A \bar{\zeta}_A \gamma^a \psi_a^i \right. \\ &\quad \left. + \bar{\psi}_a \gamma^a \psi^{jI} P_{Iij} - \frac{1}{2}i\bar{\psi}_a^i \gamma^{ab} \psi_b^j \sigma^I P_{Iij}\right) \\ &\quad + 2gY_{ij}^I P_I^{ij} - \frac{1}{2}g^2 \sigma^I \sigma^J k_I^X k_{JX}. \end{aligned} \quad (6.118)$$

Indeed, no further constraints other than those given in section 6.3 were necessary in this local case. In particular, the target space is still hypercomplex or, when an action exists, hyperkähler.

This action leads to the following dynamical equations

$$\begin{aligned}\frac{\delta \mathcal{S}_{\text{hyper}}^{\text{conf}}}{\delta \bar{\zeta}^A} &= 2 C_{AB} \Gamma_{\text{conf}}^B, \\ \frac{\delta \mathcal{S}_{\text{hyper}}^{\text{conf}}}{\delta q^X} &= g_{XY} \left( \Delta_{\text{conf}}^Y - 2 \bar{\zeta}_A \Gamma_{\text{conf}}^B \omega^Y{}_B{}^A - i \bar{\psi}_a^j \gamma^a \Gamma_{\text{conf}}^A f_{iA}^j \right).\end{aligned}\quad (6.119)$$

Again, this action can also be obtained by using the [field]×[non-closure] method. The transformation rules for the non-closure functions now get gravitino-corrections:

$$\begin{aligned}\delta \Delta^X &= -i \bar{\epsilon}^i \not{D} \Gamma^A f_{iA}^X + 2i \bar{\epsilon}^i \Gamma^B \zeta^C \zeta^D f_{iB}^Y \mathcal{R}^X{}_{YCD} + 2\bar{\eta}^j \Gamma^A f_{iA}^X + \Delta^Y \Gamma_{ZY}{}^X \delta q^Z \\ &\quad + \frac{1}{4} i f_{iA}^X \bar{\psi}_\mu^j \Gamma^A \bar{\epsilon} \psi^\mu - \frac{1}{4} i f_{iA}^X \bar{\psi}_\mu^j \gamma_a \Gamma^A \gamma^a \bar{\epsilon} \gamma_a \psi^\mu, \\ \delta \Gamma^A &= \frac{1}{2} i f_X^{iA} \epsilon_i \Delta^X + \frac{1}{4} \gamma^\mu \Gamma^A \bar{\epsilon} \psi_\mu - \frac{1}{4} \gamma^\mu \gamma^a \Gamma^A \bar{\epsilon} \gamma_a \psi_\mu - \delta q^X \omega_{XB}{}^A \Gamma^B,\end{aligned}\quad (6.120)$$

where the covariant derivative is given by

$$\begin{aligned}D_\mu \Gamma^A &= \mathcal{D}_\mu \Gamma^A + \frac{1}{3} i \Delta_i^A \psi_\mu^i - \frac{1}{8} \gamma^\nu \Gamma^A \bar{\psi}_\mu \psi_\nu + \frac{1}{8} \gamma^\nu \gamma^a \Gamma^A \bar{\psi}_\nu \gamma_a \psi_\mu, \\ \mathcal{D}_\mu \Gamma^A &= (\partial_\mu - 3b_\mu + \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab}) \Gamma^A + \partial_\mu q^X \omega_{XB}{}^A \Gamma^B + g A_\mu^I t_{IB}{}^A \Gamma^B.\end{aligned}\quad (6.121)$$

There are several ways to determine the coefficients of the gravitino terms; for example by trying to close the  $[Q, S]$  commutator on  $\Gamma^A$  and  $\Delta^X$ , or by using the non-closure functions in the  $[Q, Q]$  of  $\zeta^A$  and  $D_\mu A^{iA}$ , like explained in [147, p.19-21]. The extra term in the Ansatz can be obtained by requiring S-invariance of the action:

$$e^{-1} \mathcal{L} = \frac{1}{3} k_X \Delta^X + (\bar{\zeta}_A - \frac{1}{3} i \bar{\psi}_\mu^j \gamma^\mu k_X f_{iA}^j) \Gamma^A.\quad (6.122)$$

Substituting the non-closure functions into (6.122) and partial integrating the covariant box, we again find (6.118).

The Lagrangians (6.110) and (6.118) are the starting point for obtaining matter couplings to Poincaré supergravity. This involves a gauge fixing of the local scale and  $SU(2)$  symmetries, which will be studied in the next chapter.